# Backstepping Active Disturbance Rejection Control for Lower Triangular Nonlinear Systems With Mismatched Stochastic Disturbances

Ze-Hao Wu, *Member, IEEE*, Feiqi Deng, *Senior Member, IEEE*, Bao-Zhu Guo, *Senior Member, IEEE*, Chufen Wu, and Qiaomin Xiang

***Abstract*—In this article, we apply the active disturbance rejection control (ADRC) to the output tracking of a class of lower triangular nonlinear systems subject to mismatched bounded stochastic disturbances of unknown statistic characteristics and nonvanishing at the origin. A major obstacle is that the paths of the stochastic disturbances are nowhere differentiable almost surely which causes that the stochastic disturbances cannot be refined into the control input channel by the usual way of state transformation. To overcome this obstacle, a set of second-order extended state observers is first designed to estimate, in real time, the disturbance in each channel, and then a backstepping ADRC based on feedforward compensation and a constructive backstepping procedure is developed, guaranteeing that the closed-loop output tracks a time-varying reference signal in practically mean square and the closed-loop states are practically bounded in probability first defined in this article. Finally, some numerical simulations are presented to validate the effectiveness of the proposed backstepping ADRC approach.**

***Index Terms*—Active disturbance rejection control (ADRC), backstepping control, extended state observer (ESO), lower triangular nonlinear systems, mismatched bounded stochastic disturbances, output tracking.**

## I. Introduction

UNKNOWN disturbances coming from various sources exist widely in many practical control systems, which often leads to severe negative effects of control precision. For this reason, analysis and synthesis for control systems subject to disturbances has been becoming one of the most animate research fields in the past few decades [1]. On the one hand, there have been many control approaches attributed to disturbance attenuation for systems subject to unknown disturbances, such as the stochastic control approaches [2]–[8] and the robust control approaches [9]–[12], where the former is often feasible for attenuating noise disturbances with known statistical characteristics and the latter can be applied in coping with more general bounded disturbance and uncertainty without statistical characteristics. However, the most robust control approaches are in the worst case scenario, which makes the controller design rather conservative. On the other hand, with the new challenges in controller design for systems subject to more general disturbance and improving anti-disturbance performance and robustness of the closed loop, some novel disturbance rejection control methods like the disturbance observer-based control (DOBC) [13]–[15], based on disturbance estimation and feedforward compensation, have been developed. The core idea of these disturbance rejection approaches is that an observer is designed to estimate disturbance, and then an observer-based controller which uses the state of the observer can be designed to compensate the disturbance and guarantees satisfactory performance of the closed loop. The active disturbance rejection control (ADRC) is a representative control approach for disturbance rejection for systems subject to disturbance and uncertainty in a large scale, proposed by Han in the late 1980s [16].

The extended state observer (ESO) is the key part of ADRC which aims at the online estimation of the "total disturbance" which includes internal unmodeled dynamics and external disturbance affecting system performance, where the internal uncertainty is regarded as the "internal disturbance" of the system. Based on estimation obtained by ESO, an ESO-based controller can be designed to cancel the total disturbance in the feedback loop and hence the desired control performance can be ensured. This ESO-based estimation/cancellation nature makes ADRC be capable of eliminating the total disturbance before it causes a negative effect to the plant, and at the same time reduces significantly the control energy in practice [17]. In the past two decades, the effectiveness and practicality of ADRC have been validated in many engineering applications, such as dc–dc power converter [18], flight vehicles control [19], synchronous motors [20], gasoline engines [21], and power plants [22], to name just a few.

Manuscript received January 29, 2020; revised May 23, 2020 and September 4, 2020; accepted January 5, 2021. Date of publication January 28, 2021; date of current version March 17, 2022. This work was supported in part by the National Natural Science Foundation of China under Grant 61903087, Grant 62073144, Grant 61733008, Grant 61873260, Grant 12071074, and Grant 11901091; and in part by the Natural Science Foundation of Guangdong Province under Grant 2018A030310357 and Grant 2018A1660005. This article was recommended by Associate Editor Z. Wang. *(Corresponding author: Feiqi Deng.)*

Ze-Hao Wu, Chufen Wu, and Qiaomin Xiang are with the School of Mathematics and Big Data, Foshan University, Foshan 528000, China (e-mail: zehaowu@amss.ac.cn; chufenwu@126.com; qmxiangncu@163.com).

Feiqi Deng is with the Systems Engineering Institute, South China University of Technology, Guangzhou 510640, China (e-mail: aufqdeng@scut.edu.cn).

Bao-Zhu Guo is with the Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and also with the School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China (e-mail: bzguo@iss.ac.cn).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TSMC.2021.3050820.

Digital Object Identifier 10.1109/TSMC.2021.3050820

On the other hand, the theoretical foundation of ADRC has also attracted increasing attention over the years (see [23]–[28]). Nevertheless, most of these researchers mainly address ADRC for systems subject to disturbance and uncertainty to be matched with the control input, and very few are known for the stochastic counterpart. In reality, however, the mismatched disturbances are more general and exist widely in practical systems [29], and some theoretical developments of ADRC for nonlinear systems subject to mismatched disturbances have been achieved recently (see [30]–[33]). In addition, as is known, disturbances of stochastic characteristics are more common in practical systems. The ADRC approach has been applied to output-feedback stabilization for nonlinear systems subject to bounded stochastic disturbance matched with the control input [34], [35].

Although there have been some theoretical developments concerning the ADRC for nonlinear systems with mismatched disturbances without stochastic characteristics or bounded stochastic disturbance matched with the control input, the theoretical foundation of ADRC for nonlinear systems subject to mismatched stochastic disturbances has scarcely been considered up to present. In this article, we develop the ADRC approach to the output tracking problem for a class of lower triangular nonlinear systems subject to mismatched bounded stochastic disturbances of unknown statistic characteristics and nonvanishing at the origin. There exist evident difficulties in applying the conventional ADRC to such systems. This is because the paths of bounded stochastic disturbances are nowhere differentiable almost surely, leading to the fact that they cannot be refined into the control input channel by the way of state transformation like the aforementioned literature [31], [32]. To overcome this obstacle, a backstepping control strategy which is a constructive procedure is adopted in this article on the basis of the real-time estimation of bounded stochastic disturbances by a set of second-order ESOs. The backstepping control strategy has been proposed and generalized for stochastic nonlinear systems in [4] and [5]. Since then, remarkable progresses have been made in the backstepping control designs for stochastic nonlinear systems (see [6], [8]).

The main contributions of this article can be summed up as follows.

1) From a theoretical perspective, the applicability of ADRC is expanded to the nonlinear systems subject to mismatched bounded stochastic disturbances of unknown statistic characteristics without differentiability assumptions for the time-varying disturbances as required in most existing literature like [16], [23]–[28], and [30]–[33] and nonvanishing at the origin.
2) By combining ESO with the backstepping control technique, a backstepping ADRC is designed to guarantee that the closed-loop output can track the reference signal with satisfactory transient performance not just the steady one obtained in most existing literature, and the closed-loop states are practically bounded in probability, which is first defined in this article, where the theoretical proof for the latter is novel to some certain extent.
3) For the concerned systems, a set of second-order ESOs is designed for real-time estimation of the mismatched stochastic disturbances with observer gain only to be $r^2$ much smaller than $r^{n+1}$ in many existing literature like [23]–[27] and [30]–[32], which is more likely to meet the engineering application requirements of low bandwidth and with much smaller peaking values than the conventional ESO.

We proceed as follows. In the next section, Section II, the problem is formulated. The backstepping ADRC design and the theoretical analysis are given in detail in Section III. Some numerical simulations are performed to demonstrate the effectiveness of the proposed backstepping ADRC approach in Section IV, followed up the conclusion in Section V.

We use the following notations throughout this article. The $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space; and $\mathbb{E}X$ denotes the mathematical expectation of a random variable $X$. For a scalar $K$, $|K|$ denotes its absolute value. For a vector or matrix $K$, $K^\top$ represents its transpose; $\|K\|$ represents the Euclidean norm of a vector $K$; $\lambda_{\min}(K)$ and $\lambda_{\max}(K)$ are the minimal and maximal eigenvalues of the symmetric real matrix $K$, respectively; and $I_{2\times 2}$ denotes the $2\times 2$ unit matrix. In addition, $\bar{x}_i \triangleq (x_1,\dots,x_i)^\top$, $\hat{x} \triangleq (\hat{x}_1,\dots,\hat{x}_n)^\top$, $\hat{w} \triangleq (\hat{w}_1,\dots,\hat{w}_n)^\top$, $\varrho = (\varrho_1,\dots,\varrho_n)^\top$, and $\eta = (\eta_1,\dots,\eta_n)^\top$. For any function $\phi_i:[0,\infty)\times\mathbb{R}^\kappa \to \mathbb{R}$, $(\partial\phi_i)/(\partial\vartheta) \triangleq ((\partial\phi_i)/(\partial\vartheta_1),\dots,(\partial\phi_i)/(\partial\vartheta_\kappa))$ $\forall\vartheta = (\vartheta_1,\dots,\vartheta_\kappa)^\top \in \mathbb{R}^\kappa$. For any subset $A\subset\Omega$, $A^c \triangleq \Omega - A$. To simplify the notation, most obvious domains for time variables will be dropped hereafter when there is no confusion.

## II. Problem Formulation

Although many stochastic control approaches have been adopted to deal with stochastic disturbance, they are mostly addressed to disturbance attenuation for stochastic systems, based on passive control design ideology instead of active disturbance rejection studied in this article. We point out that ADRC is a novel active control approach to reject disturbance not just disturbance attenuation for a class of uncertain stochastic systems, for which the most advantage is its estimation/cancellation strategy where the mismatched stochastic disturbances are estimated by a set of second-order ESOs in real time and are canceled in the feedback loop. The estimation and compensation are all in real time so that the mismatched stochastic disturbances cannot be accumulated to cause damage of the systems to some considerable extent.

The system that is considered in this article is a class of lower triangular nonlinear systems subject to mismatched bounded stochastic disturbances described by

$$\begin{cases} \dot{x}_1(t) = x_2(t) + h_1(x_1(t)) + w_1(t) \\ \dot{x}_2(t) = x_3(t) + h_2(x_1(t), x_2(t)) + w_2(t) \\ \quad\vdots \\ \dot{x}_n(t) = h_n(x(t)) + w_n(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{1}$$

where $x(t) = (x_1(t),\dots,x_n(t))^\top \in \mathbb{R}^n$, $u(t)\in\mathbb{R}$, and $y(t)\in\mathbb{R}$ are the measurable state, input, and output of system, respectively. The known functions $h_i:\mathbb{R}^i\to\mathbb{R}$ $(i=1,2,\dots,n-1)$ are continuously differentiable for $(n-i)$ times with respect to their arguments $\bar{x}_i \in \mathbb{R}^i$ and $h_n$ is a known locally Lipschitz continuous function; $w_i(t) \triangleq \phi_i(t, B(t))$ $(i = 1, 2, \ldots, n)$ for some unknown functions $\phi_i(t, \vartheta)$:$[0, \infty) \times \mathbb{R}^\kappa \to \mathbb{R}$ are the bounded stochastic disturbances not necessarily in the control input channel, where $B(t) = (B_1(t), \ldots, B_\kappa(t))^\top$ is a $\kappa$-dimensional independent standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ being a $\sigma$-field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration, and $P$ being a probability measure.

We emphasize that this article is devoted to find out what kinds of stochastic noises can be estimated and rejected by the ADRC approach. This is an important problem, very different from most existing literature where the disturbance attenuation for stochastic systems driven by the white noise is addressed. Thus, only the mismatched bounded stochastic noise is taken into account in the process yet the measurement noise is beyond the scope of the objective of this article. For uncertain systems with both measurement and process noise, the effects of the measurement noise on state estimation are attenuated by a filter design would be another important problem. A recent study on this regard can be found in [36].

The functions $\phi_i$ defining the bounded stochastic disturbances satisfy Assumption (A1).

*Assumption 1 (A1):* The $\phi_i(t, \vartheta) : [0, \infty) \times \mathbb{R}^\kappa \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ are continuously differentiable and twice continuously differentiable with respect to $t$ and $\vartheta$, respectively, and there exist some known constants $\beta_i > 0$, such that for all $t \geq 0$, $\vartheta = (\vartheta_1, \ldots, \vartheta_\kappa)^\top \in \mathbb{R}^\kappa$

$$|\phi_i(t, \vartheta)| + \left|\frac{\partial \phi_i(t, \vartheta)}{\partial t}\right| + \sum_{j=1}^{\kappa}\left|\frac{\partial \phi_i(t, \vartheta)}{\partial \vartheta_j}\right|^2 + \frac{1}{2}\sum_{j=1}^{\kappa}\left|\frac{\partial^2 \phi_i(t, \vartheta)}{\partial \vartheta_j^2}\right| \leq \beta_i. \tag{2}$$

*Remark 1:* The bounded stochastic disturbances $w_i(t)$ $(i = 1, 2, \ldots, n)$ will be estimated by a set of second-order ESOs and canceled by an ESOs-based controller, so both the bounded stochastic disturbances and their "variations" reasonably need to be bounded which is guaranteed by the condition (2) in Assumption (A1). This is because unbounded disturbances will make the controller unbounded as well which is not physically implementable. In addition, the existence of partial derivatives in Assumption (A1) is required for the deterministic functions $\phi_i(t, \vartheta) : [0, \infty) \times \mathbb{R}^\kappa \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ with respect to their arguments $t$ and $\vartheta$, not for the stochastic disturbances themselves, where the functions $\phi_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are used to define the stochastic disturbances by $w_i(t) \triangleq \phi_i(t, B(t))$.

*Remark 2:* The mismatched bounded stochastic disturbances $w_i(t)$ $(i = 1, 2, \ldots, n)$ are quite general with strong practical background. First, such disturbances cover those disturbances without stochastic characteristics which have been widely addressed by the ADRC approaches in the existing literature like [16], [23]–[28], and [30]–[33] where $\phi_i(\cdot)$ are the functions of the time variable $t$ only, i.e., $w_i(t) \triangleq \phi_i(t)$, and in this case, the condition (2) in Assumption (A1) is reduced to be the usual one that both the disturbances and their derivatives with respect to the time are bounded. Second, mismatched disturbances of stochastic characteristics are more natural and common in practical systems and their classical derivatives are even not required, which is novel in the theory of ADRC and is more suitable for the requirements in engineering practice. Finally, the statistical characteristics of the mismatched bounded stochastic disturbances are allowed to be unknown because the functions $\phi_i(\cdot)$ are unknown and the disturbances are nonvanishing at the origin.

Let $\nu(t)$ be a time-varying reference signal which is supposed to be continuously differentiable for $n$ times, and we denote $(\nu_1(t), \ldots, \nu_{n+1}(t))^\top = (\nu(t), \ldots, \nu^{(n)}(t))^\top$. The reference signal $\nu(t)$ is supposed to satisfy Assumption (A2).

*Assumption 2 (A2):* There exists a known positive constant $M$ such that

$$\sup_{t\geq 0}\left(|\nu(t)| + \sum_{i=1}^{n}|\nu^{(i)}(t)|\right) \leq M. \tag{3}$$

This article focuses on the output tracking problem of the system (1). The control objective is to design an anti-disturbance controller based on ADRC and the backstepping strategy, guaranteeing that the closed-loop output $y(t)$ can track the reference signal $\nu(t)$ in practically mean-square sense with satisfactory transient performance and good robustness, and at the same time, the closed-loop states are practically bounded in probability. Particularly, it should be noticed that engineers usually require the output $y(t)$ to be tracking the reference signal $\nu(t)$ not only in the steady state but also more importantly in the transient process. That is, the output $y(t)$ can track the reference signal $\nu(t)$ uniformly in $t \in [T, \infty)$ for any positive constant $T$ despite various kinds of bounded stochastic disturbances.

## III. Backstepping ADRC Design and Convergence Analysis

To overcome the obstacle that the mismatched bounded stochastic disturbances cannot be refined into a stochastic total disturbance in the control input channel by the state transformation, system (1) is regarded as a system of $n$ connected first-order subsystems. The stochastic disturbance in each first-order subsystem can be regarded as an extended state variable to be estimated by a second-order ESO. By applying the Itô differentiation rule to the mismatched stochastic disturbances $w_i(t) \triangleq \phi_i(t, B(t))$ $(i = 1, 2, \ldots, n)$, each first-order subsystem is regarded as a second-order Itô-type stochastic one, that is, system (1) is regarded as the one composed of $n$ second-order Itô-type stochastic systems as follows:

$$\begin{cases} dx_i(t) = [x_{i+1}(t) + h_i(\bar{x}_i(t)) + w_i(t)]dt \\ dw_i(t) = \left[\dfrac{\partial \phi_i(t, B(t))}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{j=1}^{\kappa}\dfrac{\partial^2 \phi_i(t, B(t))}{\partial \vartheta_j^2}\right]dt \\ \qquad\qquad + \dfrac{\partial \phi_i(t, B(t))}{\partial \vartheta}dB(t) \\ \qquad \triangleq \psi_i(t)dt + \psi_i^*(t)dB(t), \quad 1 \leq i \leq n-1 \end{cases} \tag{4}$$

$$\begin{cases} dx_n(t) = [h_n(x(t)) + w_n(t) + u(t)]dt \\ dw_n(t) = \left[\frac{\partial \phi_n(t, B(t))}{\partial t} + \frac{1}{2}\sum_{j=1}^{\kappa}\frac{\partial^2 \phi_n(t, B(t))}{\partial \vartheta_j^2}\right]dt \\ \qquad + \frac{\partial \phi_n(t, B(t))}{\partial \vartheta}dB(t) \\ \qquad \triangleq \psi_n(t)dt + \psi_n^*(t)dB(t). \end{cases} \tag{5}$$

For each second-order Itô-type stochastic system, a second-order ESO will be designed to estimate in real time the extended state $w_i(t)$. Based on the estimates of the extended state (bounded stochastic disturbances), the known system functions, and the reference signal, a series of virtual control variables are constructed recursively to derive a Lyapunov function specified in (36) and the actual backstepping ADRC controller (37) later.

For (4) and (5), a set of second-order ESOs is designed as follows:

$$\begin{cases} \dot{\hat{x}}_i(t) = x_{i+1}(t) + h_i(\bar{x}_i(t)) + \hat{w}_i(t) + a_{i1}r(x_i(t) - \hat{x}_i(t)) \\ \dot{\hat{w}}_i(t) = a_{i2}r^2(x_i(t) - \hat{x}_i(t)) \quad 1 \le i \le n-1 \end{cases} \tag{6}$$

$$\begin{cases} \dot{\hat{x}}_n(t) = h_n(x(t)) + \hat{w}_n(t) + u(t) + a_{n1}r(x_n(t) - \hat{x}_n(t)) \\ \dot{\hat{w}}_n(t) = a_{n2}r^2(x_n(t) - \hat{x}_n(t)) \end{cases} \tag{7}$$

where $\hat{x}_i(t)$ and $\hat{w}_i(t)$ are, respectively, the estimates of states $x_i(t)$ and the disturbances $w_i(t)$, $r$ is a tuning gain parameter, and $a_{ij}$ $(j = 1, 2)$ are the parameters guaranteeing the matrices

$$A_i = \begin{pmatrix} -a_{i1} & 1 \\ -a_{i2} & 0 \end{pmatrix} \tag{8}$$

to be Hurwitz.

It should be pointed out that ESOs (6), (7) are known and hence cannot contain unknown diffusion terms although all the variables involved are stochastic processes. Set

$$\begin{cases} \eta_{i1} = r(x_i - \hat{x}_i), \ \ \eta_{i2} = w_i - \hat{w}_i \\ \eta_i = (\eta_{i1}, \eta_{i2})^\top, \quad 1 \le i \le n \\ \varrho_i = x_i - x_i^*, \quad 1 \le i \le n \end{cases} \tag{9}$$

where $x_i^*$ are the virtual control variable designed recursively later.

By (4)–(7) and the Itô differentiation rule, a direct computation shows that $\eta_i(t) = (\eta_{i1}(t), \eta_{i2}(t))^\top$ satisfy the following Itô-type stochastic differential equations $(1 \le i \le n)$:

$$\begin{cases} d\eta_{i1}(t) = r(\eta_{i2}(t) - a_{i1}\eta_{i1}(t))dt \\ d\eta_{i2}(t) = -ra_{i2}\eta_{i1}(t)dt + \psi_i(t)dt + \psi_i^*(t)dB(t) \end{cases} \tag{10}$$

where $\psi_i(\cdot)$ and $\psi_i^*(\cdot)$ are defined as those in (4) and (5).

Next, we proceed the following steps for the backstepping ADRC controller design based on the ESOs.

*Step 1:* Set

$$x_1^* = \nu. \tag{11}$$

Choose the Lyapunov function

$$V_1(t) = \frac{1}{2}\varrho_1^2(t) + \eta_1^\top(t)Q_1\eta_1(t) \tag{12}$$

where $Q_1$ is a positive-definite matrix solution satisfying $Q_1A_1 + A_1^\top Q_1 = -I_{2\times 2}$ with $A_1$ given in (8).

By Assumption (A1) and Itô's formula, we have

$$\begin{aligned} dV_1 \le& \big[\varrho_1(x_2 + h_1(x_1) + w_1 - \nu_2) - r\|\eta_1\|^2 \\ &+ 2\lambda_{\max}(Q_1)\|\eta_1\| \cdot |\psi_1| + \lambda_{\max}(Q_1)\|\psi_1^*\|^2\big]dt \\ &+ \frac{\partial \eta_1^\top Q_1 \eta_1}{\partial \eta_{12}}\psi_1^* dB(t) \\ \le& \big[-r\|\eta_1\|^2 + \varrho_1 x_2^* + \varrho_1(x_2 - x_2^*) + \varrho_1 h_1(x_1) + \varrho_1 w_1 \\ &- \varrho_1\nu_2 + 2\lambda_{\max}(Q_1)\beta_1\|\eta_1\| + \lambda_{\max}(Q_1)\beta_1\big]dt \\ &+ \frac{\partial \eta_1^\top Q_1 \eta_1}{\partial \eta_{12}}\psi_1^* dB(t) \end{aligned} \tag{13}$$

where $\psi_1$ and $\psi_1^*$ are defined in (4). Design a virtual ADRC controller $x_2^*$ based on a second-order ESO as

$$x_2^* = -\left(r + \frac{1}{2\varepsilon_1}\right)\varrho_1 - h_1(x_1) - \hat{w}_1 + \nu_2 \tag{14}$$

where $r$ is a tuning gain parameter, $\varepsilon_1$ is a positive parameter to be chosen later, and "$-\hat{w}_1$" is a feedforward compensation term designed to cancel the stochastic disturbance $w_1$ in real time.

By (14) and Young's inequality, (13) can be estimated further as

$$\begin{aligned} dV_1 \le& \left[-r\|\eta_1\|^2 - \left(r + \frac{1}{2\varepsilon_1}\right)\varrho_1^2 + \varrho_1\eta_{12} + \varrho_1\varrho_2 \right. \\ &\left. + 2\lambda_{\max}(Q_1)\beta_1\|\eta_1\| + \lambda_{\max}(Q_1)\beta_1\right]dt \\ &+ \frac{\partial \eta_1^\top Q_1 \eta_1}{\partial \eta_{12}}\psi_1^* dB(t) \\ \le& \left[-r\|\eta_1\|^2 - r\varrho_1^2 + \frac{\varepsilon_1}{2}|\eta_{12}|^2 + \varrho_1\varrho_2 + \frac{\varepsilon_1}{2}\|\eta_1\|^2 \right. \\ &\left. + \frac{2}{\varepsilon_1}\lambda_{\max}^2(Q_1)\beta_1^2 + \lambda_{\max}(Q_1)\beta_1\right]dt \\ &+ \frac{\partial \eta_1^\top Q_1 \eta_1}{\partial \eta_{12}}\psi_1^* dB(t) \\ \le& \big[-(r - \varepsilon_1)\|\eta_1\|^2 - r\varrho_1^2 + \varrho_1\varrho_2 + M_1\big]dt \\ &+ \frac{\partial \eta_1^\top Q_1 \eta_1}{\partial \eta_{12}}\psi_1^* dB(t) \end{aligned} \tag{15}$$

where we set

$$M_1 = \lambda_{\max}(Q_1)\beta_1\left(1 + \frac{2}{\varepsilon_1}\lambda_{\max}(Q_1)\beta_1\right). \tag{16}$$

*Step 2:* Choose the Lyapunov function

$$V_2(t) = \frac{1}{2}\sum_{j=1}^{2}\varrho_j^2(t) + \sum_{j=1}^{2}\eta_j^\top(t)Q_j\eta_j(t) \tag{17}$$

where $Q_2$ is the positive-definite matrix solution satisfying $Q_2A_2 + A_2^\top Q_2 = -I_{2\times 2}$ with $A_2$ given in (8). By Assumption (A1) and Itô's formula, we can obtain

$$dV_2 \le \Big\{ -(r - \varepsilon_1)\|\eta_1\|^2 - r\varrho_1^2 + \varrho_1\varrho_2 + M_1 \\ + \varrho_2(x_3 + h_2(\bar{x}_2) + w_2)$$

$$-\varrho_2\left[\frac{\partial x_2^*}{\partial x_1}(x_2+h_1(x_1)+w_1)\right.$$
$$\left.+\ \frac{\partial x_2^*}{\partial \hat{w}_1}a_{12}r\eta_{11}+\sum_{j=1}^{2}\frac{\partial x_2^*}{\partial \nu_j}\nu_{j+1}\right]-r\|\eta_2\|^2$$
$$\left.+\ 2\lambda_{\max}(Q_2)\|\eta_2\|\cdot|\psi_2|+\lambda_{\max}(Q_2)\|\psi_2^*\|^2\right\}dt$$
$$+\sum_{j=1}^{2}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t)$$
$$\leq\left\{-(r-\varepsilon_1)\|\eta_1\|^2-r\varrho_1^2+\varrho_1\varrho_2+M_1\right.$$
$$+\ \varrho_2(x_3^*+x_3-x_3^*+h_2(\bar{x}_2)+w_2)$$
$$-\ \varrho_2\left[\frac{\partial x_2^*}{\partial x_1}(x_2+h_1(x_1)+w_1)+\frac{\partial x_2^*}{\partial \hat{w}_1}a_{12}r\eta_{11}\right.$$
$$\left.+\ \sum_{j=1}^{2}\frac{\partial x_2^*}{\partial \nu_j}\nu_{j+1}\right]dt-r\|\eta_2\|^2$$
$$\left.+2\lambda_{\max}(Q_2)\beta_2\|\eta_2\|+\ \lambda_{\max}(Q_2)\beta_2\right\}dt$$
$$+\sum_{j=1}^{2}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t) \tag{18}$$

where $\psi_2$ and $\psi_2^*$ are defined in (4). Design a virtual ADRC controller based on two second-order ESOs as

$$x_3^*=-\Lambda_{2r}\varrho_2-\varrho_1-h_2(\bar{x}_2)-\hat{w}_2$$
$$+\ \frac{\partial x_2^*}{\partial x_1}(x_2+h_1(x_1)+\hat{w}_1)+\sum_{j=1}^{2}\frac{\partial x_2^*}{\partial \nu_j}\nu_{j+1} \tag{19}$$

where

$$\Lambda_{2r}\triangleq\frac{1}{2\varepsilon_2}+\frac{1}{4\varepsilon_2}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2+\left(1+\frac{a_{12}^2}{4\varepsilon_2}\left(\frac{\partial x_2^*}{\partial \hat{w}_1}\right)^2\right)r \tag{20}$$

with $r$ being a tuning gain parameter and $\varepsilon_2$ being a positive parameter to be chosen later, and "$-\hat{w}_2$" is a feedforward compensation term designed to cancel the stochastic disturbance $w_2$ in real time. By (18) and (19), we have

$$dV_2\leq\left\{-(r-\varepsilon_1)\|\eta_1\|^2-r\varrho_1^2+M_1-\Lambda_{2r}\varrho_2^2+\varrho_2\eta_{22}\right.$$
$$-\ \varrho_2\frac{\partial x_2^*}{\partial x_1}\eta_{12}+\varrho_2\varrho_3-\varrho_2\frac{\partial x_2^*}{\partial \hat{w}_1}a_{12}r\eta_{11}-r\|\eta_2\|^2$$
$$\left.+\ 2\lambda_{\max}(Q_2)\beta_2\|\eta_2\|+\lambda_{\max}(Q_2)\beta_2\right\}dt$$
$$+\ \sum_{j=1}^{2}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t). \tag{21}$$

By Young's inequality

$$\varrho_2\eta_{22}\leq q\frac{1}{2\varepsilon_2}\varrho_2^2+\frac{\varepsilon_2}{2}\|\eta_2\|^2$$
$$-\ \varrho_2\frac{\partial x_2^*}{\partial x_1}\eta_{12}\leq\frac{1}{4\varepsilon_2}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\varrho_2^2+\varepsilon_2\|\eta_1\|^2$$
$$-\ \varrho_2\frac{\partial x_2^*}{\partial \hat{w}_1}a_{12}r\eta_{11}\leq\frac{a_{12}^2}{4\varepsilon_2}\left(\frac{\partial x_2^*}{\partial \hat{w}_1}\right)^2 r\varrho_2^2+\varepsilon_2 r\|\eta_1\|^2$$
$$2\lambda_{\max}(Q_2)\beta_2\|\eta_2\|\leq\frac{2}{\varepsilon_2}\lambda_{\max}^2(Q_2)\beta_2^2+\frac{\varepsilon_2}{2}\|\eta_2\|^2. \tag{22}$$

These, together with (21), yield

$$dV_2\leq\left[-(r-\varepsilon_2 r-\varepsilon_1-\varepsilon_2)\|\eta_1\|^2-(r-\varepsilon_2)\|\eta_2\|^2\right.$$
$$\left.-\ r\varrho_1^2-r\varrho_2^2+\varrho_2\varrho_3+M_2\right]dt$$
$$+\ \sum_{j=1}^{2}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t) \tag{23}$$

where we set

$$M_2=\sum_{j=1}^{2}\lambda_{\max}(Q_j)\beta_j\left(1+\frac{2}{\varepsilon_j}\lambda_{\max}(Q_j)\beta_j\right). \tag{24}$$

Suppose that the above procedures are repeated through to step $i$, that is, the virtual ADRC controllers $x_j^*(j=1,2,\ldots,i+1)$ have been designed, and the Lyapunov function

$$V_i(t)=\sum_{j=1}^{i}\frac{1}{2}\varrho_j^2(t)+\sum_{j=1}^{i}\eta_j^\top(t)Q_j\eta_j(t) \tag{25}$$

satisfies

$$dV_i\leq\left\{-\sum_{k=1}^{i-1}\left[\left(1-\sum_{j=k+1}^{i}\varepsilon_j\right)r-\sum_{j=k}^{i}\varepsilon_j\right]\|\eta_k\|^2\right.$$
$$\left.-(r-\varepsilon_i)\|\eta_i\|^2-r\sum_{j=1}^{i}\varrho_j^2+\varrho_i\varrho_{i+1}+M_i\right\}dt$$
$$+\ \sum_{j=1}^{i}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t) \tag{26}$$

where for each $1\leq j\leq i$, $Q_j$ is the positive-definite matrix solution satisfying $Q_jA_j+A_j^\top Q_j=-I_{2\times 2}$ with $A_j$ given in (8), $r$ is a tuning gain parameter, and $\varepsilon_j$ is a positive parameter to be chosen later. Set

$$x_{i+1}^*=-\Lambda_{ir}\varrho_i-\varrho_{i-1}-h_i(\bar{x}_i)-\hat{w}_i$$
$$+\ \sum_{j=1}^{i-1}\frac{\partial x_i^*}{\partial x_j}(x_{j+1}+h_j(\bar{x}_j)+\hat{w}_j)+\sum_{j=1}^{i}\frac{\partial x_i^*}{\partial \nu_j}\nu_{j+1} \tag{27}$$

$$M_i=\sum_{j=1}^{i}\lambda_{\max}(Q_j)\beta_j\left(1+\frac{2}{\varepsilon_j}\lambda_{\max}(Q_j)\beta_j\right). \tag{28}$$

*Step $i+1$:* Choose the Lyapunov function

$$V_{i+1}(t) = \frac{1}{2}\sum_{j=1}^{i+1}\varrho_j^2(t) + \sum_{j=1}^{i+1}\eta_j^\top(t)Q_j\eta_j(t). \tag{29}$$

By Assumption (A1), (26), and Itô's formula, it can be obtained that

$$\begin{aligned} dV_{i+1} \le \Bigg\{&-\sum_{k=1}^{i-1}\Bigg[\Bigg(1-\sum_{j=k+1}^{i}\varepsilon_j\Bigg)r - \sum_{j=k}^{i}\varepsilon_j\Bigg]\|\eta_k\|^2 \\ &-(r-\varepsilon_i)\|\eta_i\|^2 - r\sum_{j=1}^{i}\varrho_j^2 + \varrho_i\varrho_{i+1} + M_i \\ &+\varrho_{i+1}(x_{i+2}^* + x_{i+2} - x_{i+2}^* + h_{i+1}(\bar{x}_{i+1}) + w_{i+1}) \\ &-\varrho_{i+1}\sum_{j=1}^{i}\frac{\partial x_{i+1}^*}{\partial x_j}(x_{j+1} + h_j(\bar{x}_j) + w_j) \\ &-\varrho_{i+1}\sum_{j=1}^{i}\frac{\partial x_{i+1}^*}{\partial \hat{w}_j}a_{j2}r\eta_{j1} - \varrho_{i+1}\sum_{j=1}^{i+1}\frac{\partial x_{i+1}^*}{\partial \nu_j}\nu_{j+1} \\ &-r\|\eta_{i+1}\|^2 + 2\lambda_{\max}(Q_{i+1})\beta_{i+1}\|\eta_{i+1}\| \\ &+\lambda_{\max}(Q_{i+1})\beta_{i+1}\Bigg\}dt + \sum_{j=1}^{i+1}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t). \end{aligned} \tag{30}$$

Design a virtual ADRC controller based on $i+1$ second-order ESOs as follows:

$$\begin{aligned} x_{i+2}^* = &-\Lambda_{(i+1)r}\varrho_{i+1} - \varrho_i - h_{i+1}(\bar{x}_{i+1}) - \hat{w}_{i+1} \\ &+\sum_{j=1}^{i}\frac{\partial x_{i+1}^*}{\partial x_j}\big(x_{j+1} + h_j(\bar{x}_j) + \hat{w}_j\big) + \sum_{j=1}^{i+1}\frac{\partial x_{i+1}^*}{\partial \nu_j}\nu_{j+1} \end{aligned} \tag{31}$$

where

$$\begin{aligned} \Lambda_{(i+1)r} \triangleq &\frac{1}{2\varepsilon_{i+1}} + \frac{1}{4\varepsilon_{i+1}}\sum_{j=1}^{i}\bigg(\frac{\partial x_{i+1}^*}{\partial x_j}\bigg)^2 \\ &+\Bigg(1 + \frac{1}{4\varepsilon_{i+1}}\sum_{j=1}^{i}a_{j2}^2\bigg(\frac{\partial x_{i+1}^*}{\partial \hat{w}_j}\bigg)^2\Bigg)r \end{aligned} \tag{32}$$

with $r$ being a tuning gain parameter and $\varepsilon_{i+1}$ being a positive parameter to be chosen later, and "$-\hat{w}_{i+1}$" is a feedforward compensation term designed to cancel the stochastic disturbance $w_{i+1}$ in real time.

For all $1 \le j \le i$, it follows from Young's inequality that:

$$\begin{aligned} -\varrho_{i+1}\eta_{(i+1)2} &\le \frac{1}{2\varepsilon_{i+1}}\varrho_{i+1}^2 + \frac{\varepsilon_{i+1}}{2}\|\eta_{i+1}\|^2 \\ -\varrho_{i+1}\frac{\partial x_{i+1}^*}{\partial x_j}\eta_{j2} &\le \frac{1}{4\varepsilon_{i+1}}\bigg(\frac{\partial x_{i+1}^*}{\partial x_j}\bigg)^2\varrho_{i+1}^2 + \varepsilon_{i+1}\|\eta_j\|^2 \\ -\varrho_{i+1}r\frac{\partial x_{i+1}^*}{\partial \hat{w}_j}a_{j2}\eta_{j1} &\le \frac{a_{j2}^2}{4\varepsilon_{i+1}}\bigg(\frac{\partial x_{i+1}^*}{\partial \hat{w}_j}\bigg)^2 r\varrho_{i+1}^2 + \varepsilon_{i+1}r\|\eta_j\|^2 \\ 2\lambda_{\max}(Q_{i+1})\beta_{i+1}\|\eta_{i+1}\| &\le \frac{2}{\varepsilon_{i+1}}\lambda_{\max}^2(Q_{i+1})\beta_{i+1}^2 + \frac{\varepsilon_{i+1}}{2}\|\eta_{i+1}\|^2. \end{aligned} \tag{33}$$

By (30), (31), and (33), we have

$$\begin{aligned} dV_{i+1} \le \Bigg\{&-\sum_{k=1}^{i-1}\Bigg[\Bigg(1-\sum_{j=k+1}^{i}\varepsilon_j\Bigg)r - \sum_{j=k}^{i}\varepsilon_j\Bigg]\|\eta_k\|^2 \\ &-(r-\varepsilon_i)\|\eta_i\|^2 - r\sum_{j=1}^{i}\varrho_j^2 + M_i - \Lambda_{(i+1)r}\varrho_{i+1}^2 \\ &+\varrho_{i+1}\varrho_{i+2} - \varrho_{i+1}\eta_{(i+1)2} - \varrho_{i+1}\sum_{j=1}^{i}\frac{\partial x_{i+1}^*}{\partial x_j}\eta_{j2} \\ &-\varrho_{i+1}r\sum_{j=1}^{i}\frac{\partial x_{i+1}^*}{\partial \hat{w}_j}a_{j2}\eta_{j1} - r\|\eta_{i+1}\|^2 \\ &+2\lambda_{\max}(Q_{i+1})\beta_{i+1}\|\eta_{i+1}\| + \lambda_{\max}(Q_{i+1})\beta_{i+1}\Bigg\}dt \\ &+\sum_{j=1}^{i+1}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t) \\ \le \Bigg\{&-\sum_{k=1}^{i-1}\Bigg[\Bigg(1-\sum_{j=k+1}^{i}\varepsilon_j\Bigg)r - \sum_{j=k}^{i}\varepsilon_j\Bigg]\|\eta_k\|^2 \\ &-(r-\varepsilon_i)\|\eta_i\|^2 - r\sum_{j=1}^{i}\varrho_j^2 + M_i - r\varrho_{i+1}^2 \\ &+\varrho_{i+1}\varrho_{i+2} + \varepsilon_{i+1}\sum_{j=1}^{i}\|\eta_j\|^2 + \varepsilon_{i+1}r\sum_{j=1}^{i}\|\eta_j\|^2 \\ &+\varepsilon_{i+1}\|\eta_{i+1}\|^2 - r\|\eta_{i+1}\|^2 \\ &+\lambda_{\max}(Q_{i+1})\beta_{i+1}\bigg(1 + \frac{2}{\varepsilon_{i+1}}\lambda_{\max}(Q_{i+1})\beta_{i+1}\bigg)\Bigg\}dt \\ &+\sum_{j=1}^{i+1}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t) \\ = \Bigg\{&-\sum_{k=1}^{i}\Bigg[\Bigg(1-\sum_{j=k+1}^{i+1}\varepsilon_j\Bigg)r - \sum_{j=k}^{i+1}\varepsilon_j\Bigg]\|\eta_k\|^2 \\ &-(r-\varepsilon_{i+1})\|\eta_{i+1}\|^2 \\ &-r\sum_{j=1}^{i+1}\varrho_j^2 + \varrho_{i+1}\varrho_{i+2} + M_{i+1}\Bigg\}dt \\ &+\sum_{j=1}^{i+1}\frac{\partial \eta_j^\top Q_j\eta_j}{\partial \eta_{j2}}\psi_j^* dB(t) \end{aligned} \tag{34}$$

where

$$M_{i+1} = \sum_{j=1}^{i+1}\lambda_{\max}(Q_j)\beta_j\bigg(1 + \frac{2}{\varepsilon_j}\lambda_{\max}(Q_j)\beta_j\bigg). \tag{35}$$

*Step n:* In the final step, the actual ADRC controller $u$ appears. Choose the Lyapunov function

$$V_n(t) = \frac{1}{2}\sum_{j=1}^{n}\varrho_j^2(t) + \sum_{j=1}^{n}\eta_j^\top(t)Q_j\eta_j(t). \tag{36}$$

The actual ADRC controller $u$ based on $n$ second-order ESOs is designed as follows:

$$\begin{aligned} u = x_{n+1}^* &= -\Lambda_{nr}\varrho_n - \varrho_{n-1} - h_n(\bar{x}_n) - \hat{w}_n \\ &+ \sum_{j=1}^{n-1}\frac{\partial x_n^*}{\partial x_j}(x_{j+1} + h_j(\bar{x}_j) + \hat{w}_j) + \sum_{j=1}^{n}\frac{\partial x_n^*}{\partial v_j}v_{j+1} \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Lambda_{nr} &\triangleq \frac{1}{2\varepsilon_n} + \frac{1}{4\varepsilon_n}\sum_{j=1}^{n-1}\left(\frac{\partial x_n^*}{\partial x_j}\right)^2 \\ &+ \left(1 + \frac{1}{4\varepsilon_n}\sum_{j=1}^{n-1}a_{j2}^2\left(\frac{\partial x_n^*}{\partial \hat{w}_j}\right)^2\right)r \end{aligned} \tag{38}$$

with $r$ being a tuning gain parameter and $\varepsilon_n$ being a positive parameter to be chosen later, and "$-\hat{w}_n$" is a feedforward compensation term designed to cancel the stochastic disturbance $w_n$ in real time.

Compared with the existing literature addressing the backstepping ADRC for uncertain systems, a major difficulty in this article is that the states of the closed loop are difficult or impossible to be bounded almost surely because they are the states of an essentially Itô-type stochastic nonlinear system. This makes many available techniques in nonlinear system control not applicable to arrive at the convergence of the closed-loop system. Some novel techniques are adopted here to show practical boundedness in the probability of the closed loop first defined in this article. The convergence of the closed loop of system (1) under control (37) can be summed up as the following theorem (Theorem 1).

*Theorem 1:* Suppose that Assumptions (A1) and (A2) hold. Then, there exist positive constant $C$ independent of the tuning gain parameter $r$ and a positive constant $r^*$, such that for any $r > r^*$, any initial values $x(0) \in \mathbb{R}^n$, $\hat{x}(0) \in \mathbb{R}^n$, $\hat{w}(0) \in \mathbb{R}^n$ and any positive constant $T$, there exists globally a unique strong solution $x(t)$, $\hat{w}(t)$ to the closed loop of system (1) under control (37) and it has the following convergence.

1) The estimation errors satisfy

$$\mathbb{E}|w_i(t) - \hat{w}_i(t)|^2 \leq \frac{C}{r}, \quad i = 1, 2, \ldots, n \tag{39}$$

which is uniformly in $t \in [T, \infty)$.

2) The closed-loop output $y(t)$ tracks the reference signal $v(t)$ in practically mean square in the sense that

$$\mathbb{E}|y(t) - v(t)|^2 \leq \frac{C}{r} \tag{40}$$

which is uniformly in $t \in [T, \infty)$.

3) For each $i = 1, 2, \ldots, n$, the closed-loop state $x_i(t)$ is practically bounded in probability in the sense that there exists a monotonically decreasing non-negative function $\gamma_i$ satisfying $\lim_{r\to\infty}\gamma_i(r) = 0$ and a non-negative function $N_i$, such that

$$P\{|x_i(t)| < N_i(r)\} \geq 1 - \gamma_i(r) \tag{41}$$

which is uniformly in $t \in [T, \infty)$.

*Proof:* It follows from the above recursive steps that:

$$\begin{aligned} dV_n \leq &\left\{-\sum_{k=1}^{n-1}\left[\left(1 - \sum_{j=k+1}^{n}\varepsilon_j\right)r - \sum_{j=k}^{n}\varepsilon_j\right]\|\eta_k\|^2 \right. \\ &\left. - (r - \varepsilon_n)\|\eta_n\|^2 - r\sum_{j=1}^{n}\varrho_j^2 + M_n\right\}dt \\ &+ \sum_{j=1}^{n}\frac{\partial\eta_j^\top Q_j\eta_j}{\partial\eta_{j2}}\psi_j^* dB(t) \end{aligned} \tag{42}$$

with

$$M_n = \sum_{j=1}^{n}\lambda_{\max}(Q_j)\beta_j\left(1 + \frac{2}{\varepsilon_j}\lambda_{\max}(Q_j)\beta_j\right). \tag{43}$$

The above positive parameters $\varepsilon_j$ $(j = 1, 2, \ldots, n)$ are chosen such that $\varepsilon^* \triangleq \sum_{j=1}^{n}\varepsilon_j < (1)/(2)$, and the gain parameter $r$ is chosen such that $r > r^* \triangleq \max\{(\varepsilon^*)/(1 - 2\varepsilon^*), 1\}$ in what follows. It then follows from (42) that:

$$\begin{aligned} dV_n \leq &\left\{-\sum_{k=1}^{n-1}\left[\left(1 - \varepsilon^*\right)r - \varepsilon^*\right]\|\eta_k\|^2 - (r - \varepsilon^*)\|\eta_n\|^2 \right. \\ &\left. - r\sum_{j=1}^{n}\varrho_j^2 + M_n\right\}dt + \sum_{j=1}^{n}\frac{\partial\eta_j^\top Q_j\eta_j}{\partial\eta_{j2}}\psi_j^* dB(t) \\ \leq &\left\{-[(1 - \varepsilon^*)r - \varepsilon^*]\sum_{j=1}^{n}\|\eta_j\|^2 - r\sum_{j=1}^{n}\varrho_j^2 + M_n\right\}dt \\ &+ \sum_{j=1}^{n}\frac{\partial\eta_j^\top Q_j\eta_j}{\partial\eta_{j2}}\psi_j^* dB(t) \\ \leq &\left\{-\frac{\varepsilon^* r}{\max_{1\leq j\leq n}\lambda_{\max}(Q_j)}\sum_{j=1}^{n}\eta_j^\top Q_j\eta_j - r\sum_{j=1}^{n}\varrho_j^2 \right. \\ &\left. + M_n\right\}dt + \sum_{j=1}^{n}\frac{\partial\eta_j^\top Q_j\eta_j}{\partial\eta_{j2}}\psi_j^* dB(t) \\ \leq &\, [-rC_1V_n + M_n]dt + \sum_{j=1}^{n}\frac{\partial\eta_j^\top Q_j\eta_j}{\partial\eta_{j2}}\psi_j^* dB(t) \end{aligned} \tag{44}$$

where $C_1 \triangleq \min\{[(\varepsilon^*)/(\max_{1\leq j\leq n}\lambda_{\max}(Q_j))], 2\}$. It is easy to conclude that there exists globally a unique strong solution $(\varrho^\top(t), \eta^\top(t))^\top$ and hence there exists globally a unique strong solution $x(t)$, $\hat{w}(t)$ to the closed loop of system (1) under the ADRC controller (37). By (44)

$$d[e^{rC_1t}V_n(t)] \leq e^{rC_1t}M_n dt + e^{rC_1t}\sum_{j=1}^{n}\frac{\partial\eta_j^\top Q_j\eta_j}{\partial\eta_{j2}}\psi_j^* dB(t) \tag{45}$$

and so

$$V_n(t) \leq e^{-rC_1 t}V_n(0) + \int_0^t e^{-rC_1(t-s)}M_n ds + \int_0^t e^{-rC_1(t-s)} \sum_{j=1}^{n} \frac{\partial \eta_j^\top(s) Q_j \eta_j(s)}{\partial \eta_{j2}} \psi_j^*(s)dB(s). \quad (46)$$

Since $\eta_j(t)$ $(j = 1, 2, \ldots, n)$ satisfy the Itô-type stochastic differential equations (10), it is also easy to conclude that for all $t \geq 0$, $\int_0^t e^{-rC_1(t-s)} \sum_{j=1}^{n} [(\partial \eta_j^\top(s) Q_j \eta_j(s))/(\partial \eta_{j2})]\psi_j^*(s)dB(s)$ is a martingale (not just a local martingale). By taking expectation on both sides of (46), there holds

$$\mathbb{E}V_n(t) \leq e^{-rC_1 t}\mathbb{E}V_n(0) + \frac{M_n}{rC_1}. \quad (47)$$

It can be easily concluded that

$$\mathbb{E}V_n(0) \leq \frac{1}{2}\sum_{j=1}^{n} \mathbb{E}|x_j(0) - x_j^*(0)|^2 + \sum_{j=1}^{n} \lambda_{\max}(Q_j) \left[r^2|x_j(0) - \hat{x}_j(0)|^2 + \mathbb{E}|w_j(0) - \hat{w}_j(0)|^2\right] \leq C_2 r^2 \quad (48)$$

for some positive constant

$$C_2 \triangleq \frac{1}{2}\sum_{j=1}^{n} \mathbb{E}|x_j(0) - x_j^*(0)|^2 + \sum_{j=1}^{n} \lambda_{\max}(Q_j) \times \left[|x_j(0) - \hat{x}_j(0)|^2 + \mathbb{E}|w_j(0) - \hat{w}_j(0)|^2\right] \quad (49)$$

independent of $r$. Hence, for any $T > 0$ and $t \in [T, \infty)$, it has

$$e^{-rC_1 t}\mathbb{E}V_n(0) \leq e^{-rC_1 T}C_2 r^2 \leq \frac{C_3}{r} \quad (50)$$

where $C_3 \triangleq \sup_{s\in(r^*,\infty)} e^{-sC_1 T}C_2 s^3$ is a positive constant independent of $r$. Let $C_4 = \max\{C_3, (M_n)/(C_1)\}$. By (47) and (50), it follows that for any $T > 0$:

$$\mathbb{E}V_n(t) \leq \frac{C_4}{r} \quad \text{uniformly in } t \in [T, \infty). \quad (51)$$

Therefore, for any $T > 0$

$$\mathbb{E}|w_i(t) - \hat{w}_i(t)|^2 = \mathbb{E}|\eta_{i2}(t)|^2 \leq \mathbb{E}\|\eta_i(t)\|^2 \leq \frac{C_4}{\min_{1\leq i\leq n} \lambda_{\min}(Q_i) r} \leq \frac{C}{r} \quad (52)$$

$$\mathbb{E}|y(t) - v(t)|^2 = \mathbb{E}|\varrho_1(t)|^2 \leq 2\mathbb{E}V_n(t) = \frac{2C_4}{r} \leq \frac{C}{r} \quad (53)$$

both uniformly in $t \in [T, \infty)$, where

$$C \triangleq \max\left\{2C_4, \frac{C_4}{\min_{1\leq i\leq n} \lambda_{\min}(Q_i)}\right\}. \quad (54)$$

This completes the proof of assertions 1) and 2). It remains to prove assertion 3). Indeed, by (51), it follows that for any $T > 0$, there holds:

$$\mathbb{E}\|\varrho(t)\|^2 \leq 2\mathbb{E}V_n(t) \leq \frac{C}{r} \text{ uniformly in } \quad t \in [T, \infty). \quad (55)$$

By Chebyshev's inequality (see [37, p. 5]), for any positive constant $\lambda > 0$ and any $t \in [T, \infty)$

$$P\{\|\varrho(t)\| \geq \lambda\} \leq \frac{1}{\lambda^2}\mathbb{E}\|\varrho(t)\|^2 \leq \frac{C}{\lambda^2 r} \quad (56)$$

which means that

$$P\{|\varrho_i(t)| < \lambda\} \geq 1 - \frac{C}{\lambda^2 r}, \quad i = 1, 2, \ldots, n. \quad (57)$$

Similarly, it follows from (52), the boundedness of $w_i(t)$ $(i = 1, 2, \ldots, n)$ in Assumption (A1), the boundedness of $v_i(t)$ $(i = 2, \ldots, n)$ in Assumption (A2), and Chebyshev's inequality again, we can easily conclude that

$$P\{|\hat{w}_i(t)| < \lambda r\} \geq 1 - \frac{1}{\lambda^2 r^2}\left(\frac{2C}{r} + 2\beta_i^2\right), \quad i = 1, \ldots, n \quad (58)$$

$$P\{|v_i(t)| < \lambda r\} \geq 1 - \frac{M^2}{\lambda^2 r^2}, \quad i = 2, \ldots, n. \quad (59)$$

We only give a detailed proof of (58) with (59) being similarly concluded. Actually for a random variable $X$, Chebyshev's inequality (see [37, p. 5]) is

$$P\{\omega{:}|X(\omega)| \geq c\} \leq c^{-p}\mathbb{E}|X|^p$$

whenever $c > 0, p > 0, X \in L^p$. Since

$$P\{|\hat{w}_i(t)| \geq \lambda r\} \leq \frac{1}{\lambda^2 r^2}\mathbb{E}|\hat{w}_i(t)|^2 \leq \frac{1}{\lambda^2 r^2}[2\mathbb{E}|w_i(t) - \hat{w}_i(t)|^2 + 2\mathbb{E}|w_i(t)|^2] \leq \frac{1}{\lambda^2 r^2}\left(\frac{2C}{r} + 2\beta_i^2\right).$$

Equation (58) can then be concluded from

$$P\{|\hat{w}_i(t)| < \lambda r\} = 1 - P\{|\hat{w}_i(t)| \geq \lambda r\} \geq 1 - \frac{1}{\lambda^2 r^2}\left(\frac{2C}{r} + 2\beta_i^2\right).$$

By (53) and the boundedness of the reference signal $v_1(t)$ in Assumption (A2), we have

$$\mathbb{E}|x_1(t)|^2 = \mathbb{E}|y(t)|^2 \leq \frac{2C}{r} + 2M^2. \quad (60)$$

Again, by Chebyshev's inequality further, for any positive constant $\lambda > 0$ and any $t \in [T, \infty)$

$$P\{|x_1(t)| < N_1(r)\} \geq 1 - \gamma_1(r) \quad (61)$$

where we denote

$$N_1(r) = \lambda r, \gamma_1(r) = \frac{1}{\lambda^2 r^2}\left(\frac{2C}{r} + 2M^2\right). \quad (62)$$

It can be seen that $\gamma_1$ is a monotonically decreasing function satisfying $\lim_{r\to\infty} \gamma_1(r) = 0$.

The practical boundedness in the probability of $x_i(t)$ $(i = 2, \ldots, n)$ can be proved successively. This is because $x_i(t) = \varrho_i(t) + x_i^*(t)$ $(i = 2, \ldots, n)$ so that the practical boundedness in the probability of $x_i(t)$ can be concluded by the practical boundedness in the probability of $\varrho_i(t)$ obtained in (57) and the practical boundedness in the probability of $x_i^*(t)$, which are proven step by step as well as they are designed.

We first show in detail the practical boundedness in the probability of the virtual ADRC controller $x_2^*(t)$ designed in (14). From (14), it is seen that $x_2^*(t)$ is a combination of $\varrho_1(t)$, $h_1(x_1(t))$, $\hat{w}_1(t)$, and $\nu_2(t)$ by the scalar multiplication operation and the addition and subtraction operations. The key step to obtain the practical boundedness in the probability of $x_2^*(t)$ is to have the same property of $\varrho_1(t)$, $h_1(x_1(t))$, $\hat{w}_1(t)$, and $\nu_2(t)$. By (57)

$$P\left\{\left|\left(r+\frac{1}{2\varepsilon_1}\right)\varrho_1(t)\right| < \lambda\left(r+\frac{1}{2\varepsilon_1}\right)\right\} \geq 1-\frac{C}{\lambda^2 r}. \tag{63}$$

Denote the maximal value of the continuous function $h_1$ on the compact set $[0, N_1(r)]$ as $\alpha_1(r)$, that is, $\alpha_1(r) \triangleq \max_{s\in[0,N_1(r)]} |h_1(s)|$.

By (61) and the continuity of the function $h_1$, we also have

$$P\{|h_1(x_1(t))| \leq \alpha_1(r)\} \geq 1-\gamma_1(r). \tag{64}$$

Then

$$\begin{aligned}
&P\left\{\left|\left(r+\frac{1}{2\varepsilon_1}\right)\varrho_1(t)| + |h_1(x_1(t))\right| < \lambda\left(r+\frac{1}{2\varepsilon_1}\right)+\alpha_1(r)\right\}\\
&\geq P\left(\left\{\left|\left(r+\frac{1}{2\varepsilon_1}\right)\varrho_1(t)\right| < \lambda\left(r+\frac{1}{2\varepsilon_1}\right)\right\}\right.\\
&\qquad\left.\cap\{|h_1(x_1(t))| \leq \alpha_1(r)\}\right)\\
&\geq P\left\{\left|\left(r+\frac{1}{2\varepsilon_1}\right)\varrho_1(t)\right| < \lambda\left(r+\frac{1}{2\varepsilon_1}\right)\right\}\\
&\quad - P(\{|h_1(x_1(t))| \leq \alpha_1(r)\}^c)\\
&\geq 1-\frac{C}{\lambda^2 r}-\gamma_1(r).
\end{aligned} \tag{65}$$

Similarly, by (58) and (59), we have

$$\begin{aligned}
&P\{|\hat{w}_1(t)| + |\nu_2(t)| < 2\lambda r\}\\
&\quad\geq P(\{|\hat{w}_1(t)| < \lambda r\} \cap \{|\nu_2(t)| < \lambda r\})\\
&\quad\geq P\{|\hat{w}_1(t)| < \lambda r\} - P(\{|\nu_2(t)| < \lambda r\}^c)\\
&\quad\geq 1-\frac{1}{\lambda^2 r^2}\left(\frac{2C}{r}+2\beta_1^2\right)-\frac{M^2}{\lambda^2 r^2}.
\end{aligned} \tag{66}$$

Therefore, by the virtual ADRC controller $x_2^*(t)$ designed in (14) and (65)–(66), similar to the procedures in (65) or (66), we conclude that for any $t \in [T,\infty)$, there holds

$$\begin{aligned}
&P\left\{|x_2^*(t)| < \frac{\lambda}{2\varepsilon_1}+3\lambda r+\alpha_1(r)\right\}\\
&\quad\geq 1-\left[\frac{C}{\lambda^2 r}+\gamma_1(r)+\frac{1}{\lambda^2 r^2}\left(\frac{2C}{r}+2\beta_1^2\right)+\frac{M^2}{\lambda^2 r^2}\right].
\end{aligned} \tag{67}$$

Therefore, it follows from (57) and (67) that:

$$\begin{aligned}
&P\{|x_2(t)| = |\varrho_2(t)+x_2^*(t)| < N_2(r)\}\\
&\geq P\left(\{|\varrho_2(t)| < \lambda\} \cap \left\{|x_2^*(t)| < \frac{\lambda}{2\varepsilon_1}+3\lambda r+\alpha_1(r)\right\}\right)\\
&\geq P\{|\varrho_2(t)| < \lambda\} - P\left(\left\{|x_2^*(t)| < \frac{\lambda}{2\varepsilon_1}+3\lambda r+\alpha_1(r)\right\}^c\right)\\
&\geq 1-\gamma_2(r)
\end{aligned} \tag{68}$$

where we denoted

$$\begin{aligned}
N_2(r) &= \lambda+\frac{\lambda}{2\varepsilon_1}+3\lambda r+\alpha_1(r)\\
\gamma_2(r) &= \gamma_1(r)+\frac{2C}{\lambda^2 r}+\frac{1}{\lambda^2 r^2}\left(\frac{2C}{r}+2\beta_1^2+M^2\right)
\end{aligned} \tag{69}$$

and it can be seen that $\gamma_2$ is a monotonically decreasing function satisfying $\lim_{r\to\infty}\gamma_2(r) = 0$.

It should be emphasized again that, for all $i = 3,\ldots,n$, the virtual ADRC control $x_i^*(t)$ is recursively designed by

$$\begin{aligned}
x_i^* &= -\Lambda_{(i-1)r}\varrho_{i-1}-\varrho_{i-2}-h_{i-1}(\bar{x}_{i-1})-\hat{w}_{i-1}\\
&\quad+\sum_{j=1}^{i-2}\frac{\partial x_{i-1}^*}{\partial x_j}(x_{j+1}+h_j(\bar{x}_j)+\hat{w}_j)+\sum_{j=1}^{i-1}\frac{\partial x_{i-1}^*}{\partial \nu_j}\nu_{j+1}
\end{aligned} \tag{70}$$

which are a combination of $\varrho_j(t)$ $(j = i-2, i-1)$, $\hat{w}_j(t)$ $(j = 1,2,\ldots,i-1)$, $\nu_j(t)$ $(j = 2,\ldots,i)$, and $g_{i-1}(\bar{x}_{i-1}(t))$ for some continuous functions $g_{i-1}$ guaranteed by the smooth assumption on $h_i$ with the scalar multiplication operation, the addition and subtraction operations, and the multiplication. First, we notice that the boundedness in the probability of $\varrho_j(t)$ $(j = i-2, i-1)$, $\hat{w}_j(t)$ $(j = 1,2,\ldots,i-1)$, and $\nu_j(t)$ $(j = 2,\ldots,i)$ are always true as presented in (57), (58), and (59), respectively. Thus, if the practical boundedness in the probability of $\bar{x}_{i-1}(t)$ has been concluded, similar to the above derivations (61) and (64), the practical boundedness in the probability of $g_{i-1}(\bar{x}_{i-1}(t))$ can be concluded. Similar to the derivations of (63), (65), and (66), we can conclude that the scalar multiplication operation and the addition and subtraction operation among $\varrho_j(t)$ $(j = i-2, i-1)$, $\hat{w}_j(t)$ $(j = 1,2,\ldots,i-1)$, $\nu_j(t)$ $(j = 2,\ldots,i)$, and $g_{i-1}(\bar{x}_{i-1}(t))$ will keep the practical boundedness in probability. Next, we will prove that the case by the multiplication operation is still true. That is, for any two stochastic processes $q_1(t)$ and $q_2(t)$ which are practical boundedness in probability, $q_1(t)q_2(t)$ is still practical boundedness in probability. Actually, suppose that

$$P\{|q_1(t)| < \alpha_1^*(r)\} \geq 1-\gamma_1^*(r) \tag{71}$$

$$P\{|q_2(t)| < \alpha_2^*(r)\} \geq 1-\gamma_2^*(r) \tag{72}$$

where $\gamma_j^*$ $(j = 1, 2)$ are monotonically decreasing non-negative functions satisfying $\lim_{r\to\infty}\gamma_j^*(r) = 0$ $(j = 1, 2)$ and $\alpha_j^*$ $(j = 1, 2)$ are non-negative functions. Then

$$\begin{aligned}
&P\{|q_1(t)\cdot q_2(t)| < \alpha_1^*(r)\cdot\alpha_2^*(r)\}\\
&\quad\geq P(\{|q_1(t)| < \alpha_1^*(r)\} \cap \{|q_2(t)| < \alpha_2^*(r)\})\\
&\quad\geq P\{|q_1(t)| < \alpha_1^*(r)\} - P(\{q_2(t)| < \alpha_2^*(r)\}^c)\\
&\quad\geq 1-\gamma_1^*(r)-\gamma_2^*(r)
\end{aligned} \tag{73}$$

which concludes the practical boundedness in the probability of $q_1(t)q_2(t)$. In a word, the practical boundedness in the probability of $x_i^*(t)$ can be obtained and then the practical boundedness in the probability of $x_i(t)$ could also be concluded for $i = 3,\ldots,n$. Altogether, we could conclude recursively that there exist two non-negative functions $N_i$ and $\gamma_i$, such that for any $t \in [T,\infty)$, it holds

$$P\{|x_i(t)| < N_i(r)\} \geq 1-\gamma_i(r),\quad i = 3,\ldots,n \tag{74}$$

where $\gamma_i$ are monotonically decreasing functions satisfying $\lim_{r\to\infty}\gamma_i(r) = 0$. These procedures are as clear as stated above yet are tedious and we omit some trivial details. This completes the proof of Theorem 1. ■

*Remark 3:* The parameters in the ADRC controller (37) are the gain parameter $r$, $\varepsilon_j$ $(j = 1, 2, \ldots, n)$, and $a_{i2}$ $(i = 1, 2, \ldots, n-1)$. To guarantee the convergence of the resulting closed loop of system (1) under control (37) shown by Theorem 1, they are chosen by the principle of the following.

1) $r > r^* \triangleq \max\{(\varepsilon^*)/(1-2\varepsilon^*), 1\}$, where $\varepsilon_j$ $(j = 1, 2, \ldots, n)$ are some small positive parameters satisfying $\varepsilon^* \triangleq \sum_{j=1}^{n}\varepsilon_j < (1)/(2)$.
2) $a_{i2}$ $(i = 1, 2, \ldots, n-1)$ are chosen to make the matrices defined in (8) be Hurwitz.

In addition, it is shown from (39) and (40) in Theorem 1 that the higher the estimation and tracking accuracy is required, the larger the gain parameter $r$ needs to be tuned. That is, large gain parameter $r$ can make satisfactory estimation and tracking accuracy. Nevertheless, the peaking values of the ADRC controller (37) resulted from a large gain parameter should be also taken into consideration.

*Remark 4:* It is especially pointed out that the convergence of the output tracking error is addressed by the "practically mean square" sense in Theorem 1, where the "practicability" is demonstrated by the fact that the transient tracking performance is guaranteed by tuning the gain parameter $r$. In addition, it should also be noted that the boundedness of the closed-loop states is addressed by the "practically bounded in probability" sense. The practicability could be embodied by the fact that the probability of the boundedness of the closed-loop states is regulated by the gain parameter $r$, which can be as large as possible by regulating the $\gamma_i(r)$ but the bounds $N_i(r)$ may amplify at the same time. This is coincident to the reality.

*Remark 5:* By tuning the gain parameter $r$, (40) means that the tracking error can be regulated to be arbitrarily small. It is also noticed that (40) indicates the transient tracking performance but not only the steady tracking one given in the existing literature. Actually, the steady tracking performance can be obtained directly by (47) in the above proof that

$$\limsup_{t\to\infty}\mathbb{E}|y(t)-v(t)|^2 \le \frac{2M_n}{rC_1} \tag{75}$$

where $(2M_n)/(C_1) \le C$. This means a reasonable fact that the steady tracking performance could be more satisfactory than the transient tracking one. It is also worth mentioning that both the estimation performance and the tracking performance are related to the parameter $C$ which has a positive correlation with the initial estimation/tracking errors $|x_j(0) - \hat{x}_j(0)|$, $\mathbb{E}|w_j(0) - \hat{w}_j(0)|^2$, and $\mathbb{E}|x_j(0) - x_j^*(0)|^2$. The key point to optimize this parameter is to minimize these initial estimation/tracking errors. However, an accurate evaluation of the parameter $C$ would be very sophisticated because there are many implicit and explicit parameters involved.

## IV. Numerical Simulations

### A. Numerical Example

In this section, some numerical simulations are presented to demonstrate the effectiveness of the proposed backstepping ADRC approach. Consider the following second-order nonlinear systems with mismatched bounded stochastic disturbances:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + h_1(x_1(t)) + w_1(t) \\ \dot{x}_2(t) = h_2(x_1(t), x_2(t)) + w_2(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{76}$$

where $h_i$ $(i = 1, 2)$ and $w_i(t)$ $(i = 1, 2)$ are known system functions and unknown bounded stochastic disturbances, respectively. In particular, $w_1(t) \triangleq c_1\cos(c_2 t + c_3 B_1(t))$ and $w_2(t) \triangleq c_4\sin(c_5 t + c_6 B_2(t))$ in system (76) are bounded stochastic noises existing in many practical dynamical systems such as the motion of oscillators [38], [39], where $c_i$ $(i = 1, 2, \ldots, 6)$ are unknown parameters with a known bound, and $B(t) = (B_1(t), B_2(t))^\top$ is a two-dimensional standard Brownian motion. Let $v(t) = \sin(2t+1)$ be the reference signal. Now, we design a backstepping ADRC controller to guarantee that the output $y(t)$ of system (76) tracks the reference signal $\sin(2t + 1)$ in practically mean square with good transient performance and robustness and the states are practically bounded in probability. Two second-order ESOs for system (76) are designed as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = x_2(t) + h_1(x_1(t)) + \hat{w}_1(t) + 2r(x_1(t) - \hat{x}_1(t)) \\ \dot{\hat{w}}_1(t) = 3r^2(x_1(t) - \hat{x}_1(t)) \end{cases} \tag{77}$$

$$\begin{cases} \dot{\hat{x}}_2(t) = h_2(\bar{x}_2(t)) + \hat{w}_2(t) + u(t) + 2r(x_2(t) - \hat{x}_2(t)) \\ \dot{\hat{w}}_2(t) = 3r^2(x_2(t) - \hat{x}_2(t)) \end{cases} \tag{78}$$

where the matrices in (8) are specified as

$$A_1 = A_2 = \begin{pmatrix} -2 & 1 \\ -3 & 0 \end{pmatrix} \tag{79}$$

which is Hurwitz for their eigenvalues are $-1 \pm \sqrt{2}i$. To illustrate the controller design more clearly, the known nonlinear functions $h_i$ $(i = 1, 2)$ in the following numerical simulations are specified as:

$$h_1(x_1) = x_1^2 + x_1^3 + e^{x_1},\ \ h_2(x_1, x_2) = x_1^3 + x_2 + x_2^4 + e^{x_1^2 + x_2}. \tag{80}$$

In this case, the actual backstepping ADRC controller $u$ is designed as follows:

$$\begin{aligned} u = &-\Lambda_{2r}(x_2 - x_2^*) - (x_1 - \sin(2t+1)) \\ &- (x_1^3 + x_2 + x_2^4 + e^{x_1^2+x_2}) - \hat{w}_2 \\ &- \left(r + \frac{1}{2\varepsilon_1} + 2x_1 + 3x_1^2 + e^{x_1}\right) \\ &\times (x_2 + x_1^2 + x_1^3 + e^{x_1} + \hat{w}_1) \\ &+ 2\left(r + \frac{1}{2\varepsilon_1}\right)\cos(2t+1) - 4\sin(2t+1) \end{aligned} \tag{81}$$

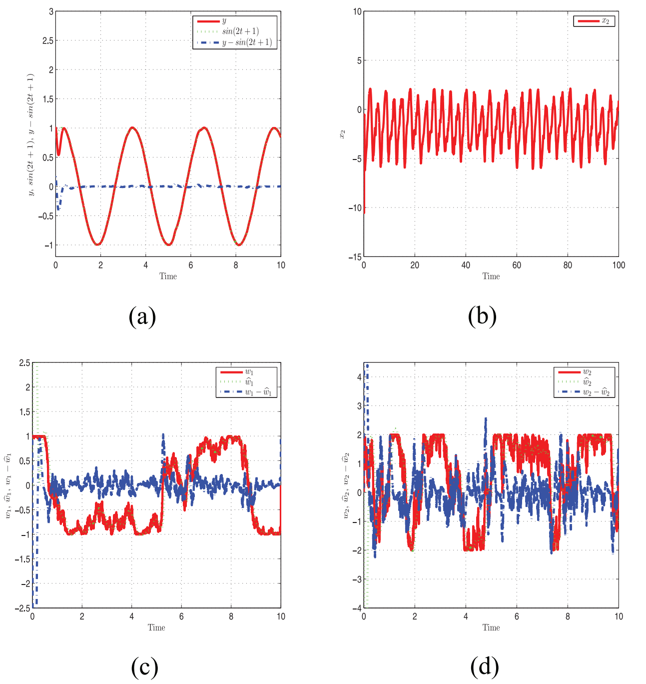


Fig. 1. Output tracking, the boundedness of the state $x_2(t)$, and the estimation of bounded stochastic noises of the closed-loop system (76) under the backstepping ADRC controller (81) with $r = 10$ and uncertain parameters given in (85).

where

$$\Lambda_{2r} = \frac{1}{2\varepsilon_2} + \frac{1}{4\varepsilon_2}\left[r + \frac{1}{2\varepsilon_1} + 2x_1 + 3x_1^2 + e^{x_1}\right]^2 + \left(1 + \frac{9}{4\varepsilon_2}\right)r \tag{82}$$

and $x_2^*$ is a virtual control defined by

$$x_2^* = -\left(r + \frac{1}{2\varepsilon_1}\right)(x_1 - \sin(2t+1)) - x_1^2 - x_1^3 - e^{x_1} - \hat{w}_1 + 2\cos(2t+1). \tag{83}$$

The parameters $r$ and $\varepsilon_i$ $(i = 1, 2)$ are chosen as $r = 10$ and $\varepsilon_1 = \varepsilon_2 = 0.2$. Figs. 1 and 2 show the numerical results for (76)–(81) where we take the initial values as

$$x_1(0) = 1, x_2(0) = -1$$
$$\hat{x}_1(0) = \hat{x}_2(0) = \hat{w}_1(0) = \hat{w}_2(0) = 0. \tag{84}$$

In Fig. 1, the uncertain parameters $c_i$ $(i = 1, 2, \ldots, 6)$ are specified as

$$c_1 = c_2 = c_3 = 1, \; c_4 = c_5 = c_6 = 2. \tag{85}$$

It is seen from Fig. 1(c) and (d) that the estimation effects of the ESOs (77) and (78) for bounded stochastic disturbances $w_1(t)$ and $w_2(t)$ are very satisfactory. It is also observed from Fig. 1(a) that the output $y(t)$ of the closed loop (76) under control (81) is very effective in tracking the reference signal $\sin(2t+1)$, and from Fig. 1(b), we can see that the absolute value of the closed-loop state $x_2(t)$ is within 10 in a long time.

Fig. 2(a) and (b) presents the numerical results for (76)–(81) with uncertain parameters $c_i$ $(i = 1, 2, \ldots, 6)$ chosen as

$$c_1 = c_2 = c_3 = 2, \; c_4 = 3, c_5 = c_6 = 4 \tag{86}$$

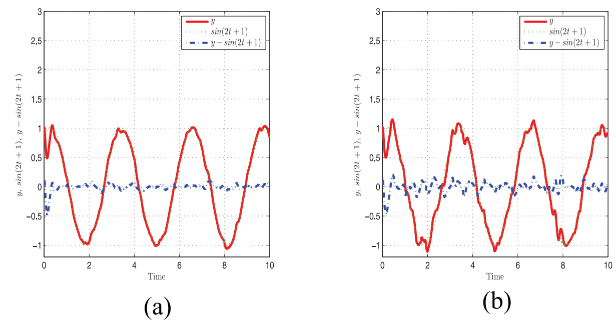


Fig. 2. Output tracking of the closed-loop system (76) under the backstepping ADRC controller (81) with $r = 10$ and uncertain parameters given in (86) and (87).

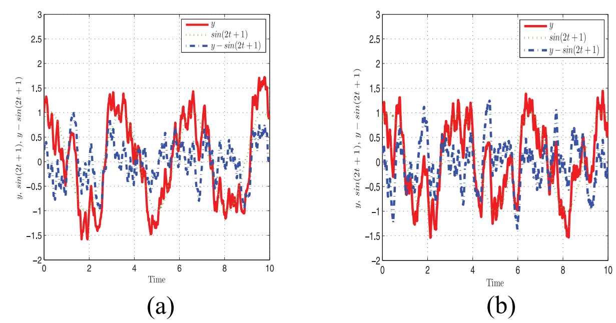


Fig. 3. Output tracking of the closed-loop system (76) under the backstepping ADRC controller (81) with $r = 3$ and uncertain parameters given as $c_i = 8$ $(i = 1, \ldots, 6)$ and $c_i = 10$ $(i = 1, \ldots, 6)$.

and

$$c_1 = c_2 = c_3 = 3, \; c_4 = 4, c_5 = c_6 = 5 \tag{87}$$

respectively.

It should be noticed that the uncertain parameters $c_i$ $(i = 1, 2, \ldots, 6)$ in (86) and (87) are larger than the ones in (85), which means that the intensity of bounded stochastic disturbances becomes stronger. However, we observe clearly from Fig. 2(a) and (b) that the output $y(t)$ of the closed loop under the backstepping ADRC controller (81) still tracks the reference signal $\sin(2t+1)$ with very satisfactory effects, which validates that the proposed backstepping ADRC controller has a good robustness.

To further verify the effectiveness of the proposed backstepping ADRC approach, the gain parameter is chosen to be smaller as $r = 3$ while the uncertain parameters are increased as $c_i = 8$ $(i = 1, \ldots, 6)$ and $c_i = 10$ $(i = 1, \ldots, 6)$. It can be seen that the tracking effects are comparatively ineffective in Fig. 3 compared with Figs. 1 and 2. This shows that control effects can be significantly improved by an appropriate gain parameter.

### B. Application to Practical Example

A simple one-link robot manipulator driven directly by a current-controlled permanent magnet dc motor with a servo electrical driver is considered in this section which has been addressed by the active disturbance rejection adaptive control in [40]. The kinematic equation of the inertia load is described

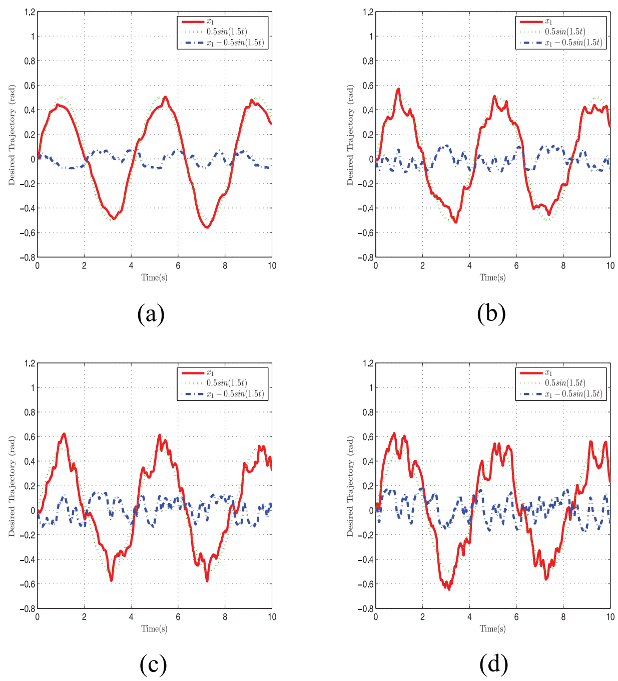



Fig. 4. Output tracking of the closed-loop system (89) under the backstepping ADRC controller (95) with $r = 10$ and uncertain parameters given in (90)–(93), respectively.

by [40]

$$J\ddot{q}(t) = K_\theta \theta(t) - [\alpha_1 \dot{q}(t) + \alpha_2 \tanh(\beta_1 \dot{q}(t)) + \alpha_3(\tanh(\beta_2 \dot{q}(t)) - \tanh(\beta_3 \dot{q}(t))] - T_d(t)$$
$$L\frac{d\theta(t)}{dt} = K_u u(t) - R\theta(t) - K_e \dot{q}(t) \tag{88}$$

where $q(t)$ and $J$ represent, respectively, the angular displacement and the inertia load, $K_\theta$ denotes the torque constant with respect to the unit of current, $\theta(t)$ is the control current, $T_d(t)$ is the time-varying external disturbance, the term $\alpha_1\dot{q}(t)+\alpha_2\tanh(\beta_1\dot{q}(t))+\alpha_3(\tanh(\beta_2\dot{q}(t))-\tanh(\beta_3\dot{q}(t)))$ represents the nonlinear friction behaviors originally modeled in [41] with $\alpha_i$ $(i = 1, 2, 3)$ being different friction levels and $\beta_i$ $(i = 1, 2, 3)$ being various shape coefficients, $L$ is the armature inductance of the motor, $K_u$ is the electrical gain, $R$ is the armature resistance of the motor, $u(t)$ is the control input voltage, and $K_e$ is the electromotive force coefficient, see [40] for more detailed explanations of the physical meanings.

Set the state variable $x(t) = (x_1(t), x_2(t), x_3(t))^\top = (q(t), \dot{q}(t), (K_\theta)/(J)\theta(t))^\top$. System (88) can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) - \dfrac{1}{J}[\alpha_1 x_2(t) + \alpha_2 \tanh(\beta_1 x_2(t)) + \alpha_3(\tanh(\beta_2 x_2(t)) - \tanh(\beta_3 x_2(t)))] - \dfrac{T_d(t)}{J} \\ \dot{x}_3(t) = \dfrac{K_\theta K_u}{JL}u(t) - \dfrac{R}{L}x_3(t) - \dfrac{K_\theta K_e}{JL}x_2(t). \end{cases} \tag{89}$$

The desired motion trajectory of the angular displacement in the simulation is given as $v(t) = 0.5\sin(1.5t)$rad. Compared with the time-varying external disturbance $T_d(t) = 0.15\sin(t)$ in [40], we consider it to be more general bounded stochastic noise case: $T_d(t) = c_1 \sin(c_2 t + c_3 B_1(t))$ with unknown parameters $c_i$ $(i = 1, 2, 3)$, where the uncertain parameters $c_i$ $(i = 1, 2, 3)$ in Fig. 4(a)–(d) are chosen as

$$c_1 = c_2 = c_3 = 2 \tag{90}$$
$$c_1 = c_2 = c_3 = 3 \tag{91}$$
$$c_1 = c_2 = c_3 = 4 \tag{92}$$

and

$$c_1 = c_2 = c_3 = 5 \tag{93}$$

respectively. Set $w(t) = -[(T_d(t))/(J)$. A second-order ESO is designed to estimate the mismatched stochastic disturbance $w(t)$ as follows:

$$\begin{cases} \dot{\hat{x}}_2(t) = x_3(t) - \dfrac{1}{J}[\alpha_1 x_2(t) + \alpha_2 \tanh(\beta_1 x_2(t)) + \alpha_3(\tanh(\beta_2 x_2(t)) - \tanh(\beta_3 x_2(t)))] + \hat{w}(t) + 2r(x_2(t) - \hat{x}_2(t)) \\ \dot{\hat{w}}(t) = 3r^2(x_2(t) - \hat{x}_2(t)). \end{cases} \tag{94}$$

The actual backstepping ADRC controller $u$ is designed as follows:

$$\begin{aligned} u = \frac{JL}{K_\theta K_u}\Bigg\{ &- \Lambda_{3r}(x_3 - x_3^*) - (x_2 - x_2^*) + \left(\frac{R}{L}x_3 + \frac{K_\theta K_e}{JL}x_2\right) - \left[1 + \Lambda_{2r}\left(r + \frac{1}{2\varepsilon_1}\right)\right]x_2 \\ &+ \Bigg[-\Lambda_{2r} + \frac{\alpha_1}{J} + \frac{\beta_1\alpha_2}{J}(1 - \tanh^2(\beta_1 x_2)) + \frac{\alpha_3}{J}(\beta_2(1 - \tanh^2(\beta_2 x_2)) - \beta_3(1 - \tanh^2(\beta_3 x_3)))\Bigg] \\ &\times \left\{x_3 - \frac{1}{J}[\alpha_1 x_2 + \alpha_2 \tanh(\beta_1 x_2) + \alpha_3(\tanh(\beta_2 x_2) - \tanh(\beta_3 x_2))] + \hat{w}\right\} \\ &+ 0.75\cos(1.5t) + \Lambda_{2r}\left(r + \frac{1}{2\varepsilon_1} + 1\right) - \frac{9}{8}\left(r + \frac{1}{2\varepsilon_1}\right)\sin(1.5t) - \frac{27}{16}\cos(1.5t)\Bigg\} \end{aligned} \tag{95}$$

where

$$\Lambda_{2r} = \frac{1}{2\varepsilon_2} + \frac{1}{4\varepsilon_2}\left(r + \frac{1}{2\varepsilon_1}\right)^2 + r \tag{96}$$

$$\begin{aligned} \Lambda_{3r} = \frac{1}{2\varepsilon_3} + \frac{1}{4\varepsilon_3}\Bigg\{&\left(\Lambda_{2r}\left(r + \frac{1}{2\varepsilon_1}\right) + 1\right)^2 \\ &+ \Bigg[-\Lambda_{2r} + \frac{\alpha_1}{J} + \frac{\beta_1\alpha_2}{J}(1 - \tanh^2(\beta_1 x_2)) \\ &+ \frac{\alpha_3}{J}(\beta_2(1 - \tanh^2(\beta_2 x_2)) \end{aligned}$$

$$- \beta_3(1-\tanh^2(\beta_3 x_2))) - \left(r+\frac{1}{2\varepsilon_1}\right)\Big]^2\Bigg\} + \left(1+\frac{9}{4\varepsilon_3}\right)r \quad (97)$$

$$x_2^* = -\left(r+\frac{1}{2\varepsilon_1}\right)(x_1 - 0.5\sin(1.5t)) + 0.75\cos(1.5t) \quad (98)$$

$$\begin{aligned} x_3^* = & -\Lambda_{2r}(x_2 - x_2^*) - (x_1 - 0.5\sin(1.5t)) \\ & + \frac{1}{J}[\alpha_1 x_2 + \alpha_2\tanh(\beta_1 x_2) + \alpha_3(\tanh(\beta_2 x_2) \\ & - \tanh(\beta_3 x_2))] - \hat{w} \\ & - \left(r+\frac{1}{2\varepsilon_1}\right)x_2 + 0.75\left(r+\frac{1}{2\varepsilon_1}\right)\cos(1.5t) \\ & - \frac{9}{8}\sin(1.5t). \end{aligned} \quad (99)$$

The related physical parameters of the controlled system in Fig. 3 are chosen as the same as those in [40]

$$\begin{aligned} & J = 0.01, L = 0.05, R = 2.5, K_\theta = 1.75, K_u = 2 \\ & K_e = 1, \alpha_1 = 0.1, \alpha_2 = 0.05, \alpha_3 = 1.025 \\ & \beta_1 = 700, \beta_2 = 15, \beta_3 = 1.5. \end{aligned} \quad (100)$$

In Fig. 4, the parameters $r$ and $\varepsilon_i$ $(i = 1, 2, 3)$ are chosen as $r = 10$ and $\varepsilon_i = 0.15$ $(i = 1, 2, 3)$, and the initial values are $x_1(0) = x_2(0) = x_3(0) = \hat{x}_2(0) = \hat{w}(0) = 0$.

It can be seen from Fig. 4 that the tracking effect of the angular displacement state variable $x_1(t)$ of the closed loop of (89) under the control (95) to the desired motion trajectory $\nu(t) = 0.5\sin(1.5t)$ rad is very satisfactory under four different noise intensities. The good tracking performance is maintained even the uncertain parameters $c_i$ $(i = 1, 2, 3)$ are increased from $c_i = 2$ to $c_i = 3$, $c_i = 4$, and $c_i = 5$, which demonstrate the good robustness of the designed backstepping ADRC controller.

## V. Conclusion

In this article, an output tracking problem of a class of lower triangular nonlinear systems subject to mismatched bounded stochastic disturbances of unknown statistic characteristics and nonvanishing at the origin is addressed by combining the ADRC approach with the backstepping control strategy. The mismatched bounded stochastic disturbance in each channel is, in real time, estimated by a second-order ESO, and then an ESOs-based backstepping ADRC controller is constructed to obtain desired closed-loop performances. It is shown that the output of the resulting closed loop tracks a time-varying reference signal in practically mean-square sense with not only the steady performance but also the more significant transient one. In addition, some novel techniques in the theoretical proofs are adopted to obtain that the closed-loop states are practically bounded in probability, which is first defined in this article, where the theoretical results and robustness of the proposed controller are confirmed by some numerical simulations.

In addition, we point out some potential interesting problems that could be further considered. First, it is worth noting that the existing ESO designs cannot be used for the real-time estimation of external disturbance or uncertainty of the uncertain Itô-type stochastic systems because of the Hessian term caused by the Itô differentiation. It would be therefore interesting to develop the ADRC approach for the uncertain Itô-type stochastic systems. Second, from the available design ideology and theoretical proofs of ADRC, it seems that the boundedness is a key factor to determine whether the stochastic noises can be estimated and rejected by the ADRC approach. For this reason, it may be feasible that the common colored noise which is bounded in the mean square could also be coped with by ADRC. Finally, the backstepping design technique is adopted in the ADRC controller to overcome the obstacle that the mismatched stochastic disturbances cannot be refined into the control input channel by the frequently used state transformation. However, the backstepping ADRC controller design will become more complex with the increase of the system order. This problem known as the "explosion of complexity" is often inevitable for the backstepping design caused by repeated differentiation of virtual controllers. Thus, it would be interesting to improve the available controller design to surmount the problem of the explosion of complexity.

## References


[1] L. Guo and S. Cao, *Anti-Disturbance Control for Systems With Multiple Disturbances*. Boca Raton, FL, USA: CRC Press, 2013.

[2] P. S. Maybeck, *Stochastic Models, Estimation, and Control*. London, U.K.: Academic, 1982.

[3] K.J. Åström, *Introduction to Stochastic Control Theory*. New York, NY, USA: Academic, 1970.

[4] H. Deng and M. Krstić, "Stochastic nonlinear stabilization-I: A backstepping design," *Syst. Control Lett.*, vol. 32, no. 3, pp. 143–150, 1997.

[5] Z. G. Pan and T. Basar, "Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion," *SIAM J. Control Optim.*, vol. 37, no. 3, pp. 957–995, 1999.

[6] H. Deng, M. Krstić, and R. J. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Trans. Autom. Control*, vol. 46, no. 8, pp. 1237–1253, Aug. 2001.

[7] X. Y. Zhao and F. Q. Deng, "Divided state feedback control of stochastic systems," *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1870–1885, Jul. 2015.

[8] H. F. Min, S. Y. Xu, B. Y. Zhang, and Q. Ma, "Output-feedback control for stochastic nonlinear systems subject to input saturation and time-varying delay," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 359–364, Jan. 2019.

[9] S. Y. Xu and J. Lam, *Robust Control and Filtering of Singular Systems*. Berlin, Germany: Springer, 2006.

[10] B. S. Chen and W. H. Zhang, "Stochastic $H_2/H_\infty$ control with state-dependent noise," *IEEE Trans. Autom. Control*, vol. 49, no. 1, pp. 45–57, Jan. 2004.

[11] D. Wang, C. X. Mu, H. B. He, and D. R. Liu, "Event-driven adaptive robust control of nonlinear systems with uncertainties through NDP strategy," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1358–1370, Jul. 2017.

[12] Z. J. Zhao, X. Y. He, Z. G. Ren, and G. L. Wen, "Boundary adaptive robust control of a flexible riser system with input nonlinearities," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 10, pp. 1971–1980, Oct. 2019.

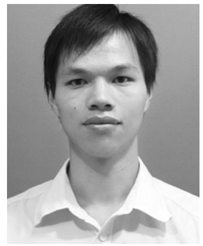
[13] S. H. Li, J. Yang, W. H. Chen, and X. S. Chen, *Disturbance Observer-Based Control: Methods and Applications*. Boca Raton, FL, USA: CRC Press, 2016.

[14] H. B. Sun, S. H. Li, J. Yang, and W. X. Zheng, "Global output regulation for strict-feedback nonlinear systems with mismatched nonvanishing disturbances," *Int. J. Robust Nonlinear Control*, vol. 25, no. 15, pp. 2631–2645, 2015.

[15] Z. J. Zhao, X. Y. He, and C. K. Ahn, "Boundary disturbance observer-based control of a vibrating single-link flexible manipulator," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, May 7, 2019, doi: 10.1109/TSMC.2019.2912900.

[16] J. Q. Han, "From PID to active disturbance rejection control," *IEEE Trans. Ind. Electron.*, vol. 56, no. 3, pp. 900–906, Mar. 2009.

[17] Q. Zheng and Z. Q. Gao, "An energy saving, factory-validated disturbance decoupling control design for extrusion processes," in *Proc. 10th World Congr. Intell. Control Autom.*, Beijing, China, 2012, pp. 2891–2896.
[18] B. S. Sun and Z. Q. Gao, "A DSP-based active disturbance rejection control design for a 1-kW H-bridge DC-DC power converter," *IEEE Trans. Ind. Electron.*, vol. 52, no. 5, pp. 1271–1277, Oct. 2005.
[19] Y. Q. Xia and M. Y. Fu, *Compound Control Methodology for Flight Vehicles*. Berlin, Germany: Springer-Verlag, 2013.
[20] H. Sira-Ramírez, J. Linares-Flores, C. García-Rodríguez, and M. A. Contreras-Ordaz, "On the control of the permanent magnet synchronous motor: An active disturbance rejection control approach," *IEEE Trans. Control Syst. Technol.*, vol. 22, no. 5, pp. 2056–2063, Sep. 2014.
[21] W. C. Xue, W. Y. Bai, S. Yang, K. Song, Y. Huang, and H. Xie, "ADRC with adaptive extended state observer and its application to air-fuel ratio control in gasoline engines," *IEEE Trans. Ind. Electron.*, vol. 62, no. 9, pp. 5847–5857, Sep. 2015.
[22] L. Sun , D. H. Li, K. T. Hu, K. Y. Lee, and F. P. Pan, "On tuning and practical implementation of active disturbance rejection controller: A case study from a regenerative heater in a 1000 MW power plant," *Ind. Eng. Chem. Res.*, vol. 55, no. 23, pp. 6686–6695, 2016.
[23] B.-Z. Guo and Z.-L. Zhao, "On convergence of the nonlinear active disturbance rejection control for MIMO Systems," *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 1727–1757, 2013.
[24] T. T. Jiang, C. D. Huang, and L. Guo, "Control of uncertain nonlinear systems based on observers and estimators," *Automatica*, vol. 59, pp. 35–47, Sep. 2015.
[25] Z. Q. Pu, R. Y. Yuan, J. Q. Yi, and X. M. Tan, "A class of adaptive extended state observers for nonlinear disturbed systems," *IEEE Trans. Ind. Electron.*, vol. 62, no. 9, pp. 5858–5869, Sep. 2015.
[26] B. Z. Guo and Z. L. Zhao, *Active Disturbance Rejection Control for Nonlinear Systems: An Introduction*. New York, NY, USA: Wiley, 2016.
[27] J. Li, Y. Q. Xia, X. H. Qi, and Z. Q. Gao, "On the necessity, scheme, and basis of the linear-nonlinear switching in active disturbance rejection control," *IEEE Trans. Ind. Electron.*, vol. 64, no. 2, pp. 1425–1435, Feb. 2017.
[28] H. C. Zhou and H. Feng, "Disturbance estimator based output feedback exponential stabilization for Euler-Bernoulli beam equation with boundary control," *Automatica*, vol. 91, pp. 79–88, May 2018.
[29] C. Peng, J. C. Fang, and X. B. Xu, "Mismatched disturbance rejection control for voltage-controlled active magnetic bearing via state-space disturbance observer," *IEEE Trans. Power Electron.*, vol. 30, no. 5, pp. 2753–2762, May 2015.
[30] S. H. Li, J. Yang, W. H. Chen, and X. Chen, "Generalized extended state observer based control for systems with mismatched uncertainties," *IEEE Trans. Ind. Electron.*, vol. 59, no. 12, pp. 4792–4802, Dec. 2012.
[31] B.-Z. Guo and Z.-H. Wu, "Output tracking for a class of nonlinear systems with mismatched uncertainties by active disturbance rejection control," *Syst. Control Lett.*, vol. 100, pp. 21–31, Feb. 2017.
[32] Z.-L. Zhao and B.-Z. Guo, "A novel extended state observer for output tracking of MIMO systems with mismatched uncertainty," *IEEE Trans. Autom. Control*, vol. 63, no. 1, pp. 211–218, Jan. 2018.
[33] Y. Yang, J. Tan, and D. Yue, "Prescribed performance tracking control of a class of uncertain pure-feedback nonlinear systems with input saturation," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 5, pp. 1733–1745, May 2020.
[34] B.-Z. Guo, Z.-H. Wu, and H.-C. Zhou, "Active disturbance rejection control approach to output-feedback stabilization of a class of uncertain nonlinear systems subject to stochastic disturbance," *IEEE Trans. Autom. Control*, vol. 61, no. 6, pp. 1613–1618, Jun. 2016.
[35] B. Z. Guo and Z. H. Wu, "Active disturbance rejection control approach to output-feedback stabilization of lower triangular nonlinear systems with stochastic uncertainty," *Int. J. Robust Nonlinear Control*, vol. 27, no. 16, pp. 2773–2797, 2017.
[36] W. Xue, X. Zhang, L. Sun, and H. Fang, "Extended state filter based disturbance and uncertainty mitigation for nonlinear uncertain systems with application to fuel cell temperature control," *IEEE Trans. Ind. Electron.*, vol. 67, no. 12, pp. 10682–10692, Dec. 2020, doi: 10.1109/TIE.2019.2962426.
[37] X. R. Mao, *Stochastic Differential Equations and Applications*, Oxford, U.K.: Horwood, 2007.
[38] Z. L. Huang, W. Q. Zhu, Y. Q. Ni, and J. M. Ko, "Stochastic averaging of strongly non-linear oscillators under bounded noise excitation," *J. Sound Vib.*, vol. 254, no. 2, pp. 245–267, 2002.
[39] F. Hu, L. C. Chen, and W. Q. Zhu, "Stationary response of strongly nonlinear oscillator with fractional derivative damping under bounded noise excitation," *Int. J. Nonlinear Mech.*, vol. 47, no. 10, pp. 1081–1087, 2012.
[40] J. Y. Yao and W. X. Deng, "Active disturbance rejection adaptive control of uncertain nonlinear systems: Theory and application," *Nonlinear Dyn.*, vol. 89, no. 3, pp. 1611–1624, 2017.
[41] C. Makkar, W. E. Dixon, W. G. Sawyer, and G. Hu, "A new continuously differentiable friction model for control systems design," in *Proc. Int. Conf. Adv. Intell. Mechatronics*, 2005, pp. 600–605.



**Ze-Hao Wu** (Member, IEEE) received the B.S. degree in mathematics from Guangdong Polytechnic Normal University, Guangzhou, China, in 2011, the M.S. degree in mathematics from Xiamen University, Xiamen, China, in 2014, and the Ph.D. degree in operational research and cybernetics from the Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, China, in 2017.

He is currently an Associate Professor with Foshan University, Foshan, China. His research interests include estimation and control of stochastic systems and active disturbance rejection control.

**Feiqi Deng** (Senior Member, IEEE) was born in 1962. He received the Ph.D. degree in control theory and control engineering from the South China University of Technology, Guangzhou, China, in 1997.

Since 1999, he has been a Professor with the South China University of Technology, where he is the Director of the Systems Engineering Institute. He has published over 300 journal papers on the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, *Automatica*, *SIAM Journal of Control and Optimization*, *International Journal of Systems Science*, and *Systems and Control Letters*. His current research interests include stability, stabilization, and robust control theory of complex systems, including time-delay systems, nonlinear systems, and stochastic systems.

Dr. Deng is currently serving as the Chair for the IEEE CSS Guangzhou Chapter and the IEEE SMC Guangzhou Chapter, the Vice Editor-in-Chief for the *Journal of South China University of Technology*, and an Editorial Board Member of the journals, such as IEEE ACCESS, *Control Theory and Applications*, *Systems and Control*, *Journal of Systems Engineering and Electronics*, and *Journal of Systems Engineering*. He is currently a Technical Committee Member on Control Theory and the Chinese Association of Automation.

**Bao-Zhu Guo** (Senior Member, IEEE) received the Ph.D. degree in applied mathematics from the Chinese University of Hong Kong, Hong Kong, in 1991.

From 1985 to 1987, he was a Research Assistant with the Beijing Institute of Information and Control, Beijing, China. From 1993 to 2000, he was with the Beijing Institute of Technology, first as an Associate Professor from 1993 to 1998 and subsequently a Professor from 1998 to 2000. Since 2000, he has been with the Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, where he is a Research Professor of Mathematical System Theory. He is the author or coauthor of four books, including *Stability and Stabilization of Infinite Dimensional Systems With Applications* (Springer-Verlag, 1999), *Control of Wave and Beam PDEs: The Riesz Basis Approach* (Springer-Verlag, 2019), and *Active Disturbance Rejection Control for Nonlinear Systems: An Introduction* (Wiley, 2016). His research interests include the theory of control of infinite-dimensional systems and active disturbance rejection control.

Dr. Guo is the recipient of the One Hundred Talent Program from the Chinese Academy of Sciences in 1999 and the National Science Fund for Distinguished Young Scholars in 2003.

**Chufen Wu** received the B.S. degree in pure mathematics and the Ph.D. degree in pure mathematics from the South China Normal University, Guangzhou, China, in 2005 and 2010, respectively.

She is currently a Professor with Foshan University, Foshan, China. Her research interests include nonlinear analysis, applied dynamical systems, mathematical biology, and control theory.

**Qiaomin Xiang** received the B.S. and M.S. degrees in mathematics from Nanchang University, Nanchang, China, in 2012 and 2015, respectively, and the Ph.D. degree in mathematics from the South China University of Technology, Guangzhou, China, in 2018.

She is currently an Associate Professor with Foshan University, Foshan, China. Her research interests include dynamical systems and control theory.