Output tracking for a class of nonlinear systems with mismatched uncertainties by active disturbance rejection control

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**ABSTRACT**

In this paper, we apply active disturbance rejection control, an emerging control technology, to achieve practical output tracking for a class of nonlinear systems in the presence of vast matched and mismatched uncertainties including unknown internal system dynamic uncertainty, external disturbance, and uncertainty caused by the deviation of control parameter from its nominal value. The total disturbance influencing the performance of controlled output is refined first and then estimated by an extended state observer (ESO). Under the assumption that the inverse dynamics of the uncertain systems are bounded-input-bounded-state stable, a constant high gain ESO based output feedback is constructed to guarantee that the state is bounded and the output tracks practically a given reference signal. A time-varying gain ESO is also discussed to reduce the peaking value near the initial stages of ESO caused by constant high gain. Numerical simulations are presented to demonstrate the effectiveness of the proposed output-feedback control scheme.

1. Introduction

Dealing with uncertainty is a key issue in modern control theory since the inception of the modern control theory in the later years of 1950s, seeded in [1] where it is stated that the control operation “must not be influenced by internal and external disturbances” [1, p. 228]. Many methods have been developed since 1970s to cope with uncertainty like robust control [2], high-gain control [3], internal model principle [4–6], adaptive control [7], among them, the robust control is a remarkable paradigm shift in modern control theory [8]. However, most of these control methods are based on the worst case scenario, which makes the controller designed rather conservative. Very different strategy is the estimation/cancellation strategy which can be found in adaptive control and internal model principle for dealing with almost known uncertainty.

The idea of estimation/cancellation strategy is carried forward by known as active disturbance rejection control (ADRC) to this day, proposed by Han [9] in later 1980s. ADRC lumps vast uncertainty into “total disturbance” which may include the coupling between unknown system dynamics, external disturbance, the superadded unknown part of control input, or even if whatever the part of hardly to be dealt with by practitioner. This spans significantly the concept of “disturbance”. The key idea of ADRC is that the “total disturbance”, as a signal of time, no matter it is state-dependent or free, time invariant or variant, linear or nonlinear, is reflected entirely in the observable measured output and can hence be estimated. The estimation of total disturbance as well as state is realized through a device called extended state observer (ESO). The “total disturbance” is then compensated in the feedback loop by its estimate. This estimation/cancellation nature of ADRC makes it capable of eliminating the uncertainties before it causes negative effect to control plant.

In the last few years, some progresses have been made leading to theoretical foundation of ADRC in [10–21], among many others. The convergence of linear ESO, which is proposed in [22] in terms of bandwidth, is discussed in [17, 21]. Linear ADRC has been addressed for different systems like those for control and disturbance unmatched systems [14], lower triangular systems [16], and the system without known nominal control parameter [13]. In addition, linear ADRC with adaptive gain ESO is investigated in [15]. The convergence of nonlinear ADRC for SISO systems is proved firstly in [10] and extended secondly to MIMO system in [11], and then to lower triangular system in [19, 20], and to system with stochastic disturbance in [12]. The convergence of nonlinear ADRC with time-varying gain ESO is discussed in [18].

On the other hand, most of aforementioned literatures mainly address ADRC for essential-integral-chain systems with matched...
uncertainties, and very little attention is paid to systems with uncertainties that are not in the control channel. Actually, systems with non-integral chain form and mismatched uncertainties are more general and widely exist in practical engineering systems. For example, in flight control systems, the lumped disturbance torques caused by un-modeled dynamics, external winds, parameter perturbations, etc., always influence the states directly but not through the input channels [23]. To this end, a generalized ESO based control approach was proposed for general systems with mismatched uncertainties and non-integral chain form in [14], whose feasibility and validity are mainly demonstrated by numerical and application design examples. The stability analysis in [14] is addressed under strong conditions that the mismatched uncertainties are bounded, independent of states, and have constant values in steady state. In addition, [16] addresses ADRC to achieve desired performance for a class of MIMO lower-triangular nonlinear systems with vast mismatched uncertainties by state feedback.

In this paper, we address ADRC approach to output tracking for lower triangular nonlinear systems with more general mismatched uncertainties without restrictive conditions like that in [14], and output feedback control instead of state feedback like that in [16] is concerned.

The remainder of the paper is organized as follows. In the next section, Section 2, the total disturbance that affects the output of the system is first determined. We then design a constant high gain ESO to estimate the total disturbance in real time, and finally a constant gain ESO based output feedback control is designed. It is shown that the output feedback control law can guarantee the boundedness of the state of the closed-loop and the output tracks practically a given reference signal. In Section 3, a time-varying gain ESO is briefly discussed to reduce the peaking value near the initial stage of ESO caused by constant high gain. Finally, in Section 4, we present some numerical simulations for illustration of the performance of closed-loop and the peaking value reduction.

2. ADRC with constant gain ESO

In this paper, we consider output tracking problem for a class of uncertain nonlinear systems in lower triangular form described as follows:

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), \zeta(t), u(t)), \\
&= f(x_1, x_2, \ldots, x_n, t, \zeta(t), u(t)) \\
&= f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \zeta(t), u(t)), \\
&= \phi(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \zeta(t), u(t)) \\
&= \phi_0(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \zeta(t), u(t)).
\end{align*}
\] (2.1)

where \(x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n\) and \(x(t) \in \mathbb{R}^m\) are system states with \(\zeta(t)\) the zero dynamics, \(y(t) \in \mathbb{R}\) the measured output, \(u(t) \in \mathbb{R}\) the control input, \(\bar{w}(t) \in \mathbb{R}\) the unknown exogenous signal or external disturbance, and \(b(\cdot) : [0, +\infty) \times \mathbb{R} \to \mathbb{R}\) the control coefficient which is not exactly known yet has a nominal value \(b_0(t)\) sufficiently close to \(b(\cdot)\). The functions \(h_i(\cdot) : \mathbb{R}^{n+i+m+1} \to \mathbb{R} (i = 1, 2, \ldots, n - 1), f(\cdot) : [0, +\infty) \times \mathbb{R}^{n+m+1} \to \mathbb{R}, \) and \(\phi(\cdot) : \mathbb{R}^{n+2} \to \mathbb{R}\) are generally unknown. So system (2.1) allows nonlinear uncertainties in all channels, not only in the control channel as considered in existing literature. As indicated in [9], the key point in application of ADRC is how to reformation the problem by lumping various known and unknown quantities that affect the system performance into “total disturbance”. This is a crucial step in transforming a complex control problem into a simple one. A natural requirement is that the total disturbance can be identified from the measured output. The idea of addressing ADRC for deterministic systems with mismatched uncertainties is originated from [24] where no theoretical proof is given. Motivated by [24], we set:

\[
\begin{align*}
\bar{x}_i(t) &= x_i(t), \\
\dot{x}_i(t) &= x_i(t) + h_i(x_i(t), \zeta(t), u(t)), \\
\dot{\bar{x}}(t) &= \dot{x}(t) + \sum_{j=1}^{n-1} h_{i-j}(x_i(t), \ldots, x_{i-j}(t), \zeta(t), u(t)), \\
\end{align*}
\] (2.2)

where \(h_{i-j}(\cdot)\) represents the \((j - 1)\)-th derivative of \(h_i(\cdot)\) with respect to time variable \(t\). A straightforward computation shows that for all \(i \geq 3\),

\[
\begin{align*}
\dot{\bar{x}}_i(t) &= f_i(x_i(t), \ldots, x_{i-j}(t), \zeta(t), u(t)) \\
&= f_{i-j}(x_i(t), \ldots, x_{i-j}(t), \zeta(t), u(t)) \\
&= f_{i-j}(\bar{x}_i(t), \ldots, \bar{x}_{i-j}(t), \zeta(t), u(t)).
\end{align*}
\] (2.3)

for some continuous function \(f_{i-j}(\cdot)\) when \(h_i(\cdot) \in C^{n+1-1}(\mathbb{R}^{n+i+m+1}; \mathbb{R}), f_{i-j}(\cdot) \in C^{n-1}(\mathbb{R}^{n+i+m+2}; \mathbb{R})\), and \(u(\cdot)\) is \(n\)-th continuously differentiable with respect to time variable \(t\) supposed in Assumption (A1) later. Equivalently, there are continuous functions \(\phi_i(\cdot) (i = 1, 2, \ldots, n - 1)\) such that

\[
\begin{align*}
(x_i(t) = \bar{x}_i(t), \\
\dot{x}_i(t) = \dot{x}_i(t), \\
\dot{\bar{x}}_i(t) = \dot{\bar{x}}_i(t), \\
\dot{\bar{x}}_i(t) = \dot{\bar{x}}_i(t).
\end{align*}
\] (2.4)

Under the new state variable \(\bar{x}(t) = (\bar{x}_1(t), \ldots, \bar{x}_n(t))\), the \(n\)-subsystem of (2.1) is transformed into an essentially integral-chain system with control matched total disturbance as follows:

\[
\begin{align*}
\dot{\bar{x}}_1(t) &= \bar{x}_1(t), \\
\dot{\bar{x}}_2(t) &= \bar{x}_2(t), \\
&\vdots \\
\dot{\bar{x}}_n(t) &= \bar{x}_n(t) + b_0(t)u(t), \\
y(t) &= x_1(t),
\end{align*}
\] (2.5)

where the “total disturbance” \(x_{n+1}(t)\) is given by

\[
\begin{align*}
x_{n+1}(t) &= f(t, x(t), \zeta(t), u(t)) + (b(t, w(t)) - b_0(t))u(t) \\
&= \sum_{j=0}^{n-1} h_{n-j}(x_1(t), \ldots, x_{n-j}(t), \zeta(t), u(t)).
\end{align*}
\] (2.6)

Our control objective is to design an output feedback control so that for all initial states in given compact set, the state \((x(t), \zeta(t))\) is bounded and the output \(y(t)\) tracks practically a given, bounded, reference signal \(r(t)\) whose derivatives \(\dot{r}(t), \ddot{r}(t), \ldots, r^{(n+1)}(t)\) are supposed to be bounded. Let

\[
(r_1(t), r_2(t), \ldots, r_{n+1}(t)) = (r(t), \dot{r}(t), \ldots, r^{(n+1)}(t)).
\] (2.7)

The key step of ADRC is to design an extended state observer (ESO) for \(n\)-subsystem of (2.1) to estimate the total disturbance, which can be reduced to design ESO for system (2.5). This is because these two systems have the same controlled output and there exist continuous invertible transformations between \(x\)-variable and \(\bar{x}\)-variable as shown in (2.2) and (2.4). The simplest ESO is linear one which takes advantage of simple tuning parameter but it may bring the peaking value problem, slow convergence, and many other problems contrast to fast tracking and small peaking value indicated numerically in [25] by nonlinear ESO. By taking these points into account, we introduce a nonlinear ESO [10,19,20] with constant high gain tuning parameter for system
\[ (2.5) \text{ as follows:} \]
\[
\begin{align*}
\dot{x}_1(t) &= \dot{x}_2(t) + e^{\gamma_1} g_1(n_1(t)), \\
\dot{x}_2(t) &= \dot{x}_3(t) + e^{\gamma_2} g_2(n_1(t)), \\
&\vdots \\
\dot{x}_n(t) &= \dot{x}_{n+1}(t) + g_1(n_1(t)) + b_0 u(t), \\
\dot{x}_{n+1}(t) &= \frac{1}{\epsilon} g_{n+1}(n_1(t)), \quad n_1(t) = \frac{y(t) - \hat{x}_n(t)}{e^n},
\end{align*}
\]

where \( g_i \in C(\mathbb{R}; \mathbb{R}), i = 1, 2, \ldots, n + 1 \) are functions to be specified later and \( \epsilon > 0 \) is the parametrization. The main idea of ESO is to choose some appropriate \( g_i(\cdot) \)'s so that when \( \epsilon \) is small enough, the \( \dot{x}_i(t) \) approaches \( \dot{x}_n(t) \) for each \( i = 1, 2, \ldots, n + 1 \) and sufficiently large \( t \), where \( \dot{x}_{n+1}(t) \) is the total disturbance defined by (2.6). Here and throughout the paper, we always drop \( \epsilon \) for the solution of (2.8) by abuse of notation without confusion.

The ESO (2.8) based output feedback is designed as
\[
\begin{align*}
u(t) &= \frac{1}{b_0(t)} \left( \rho \left( \text{sat}_{\rho_0}(\dot{x}_1(t) - r_1(t)), \ldots, \text{sat}_{\rho_0}(\dot{x}_n(t) - r_n(t)) \right) \
&\quad - \text{sat}_{\rho_0}(\dot{x}_{n+1}(t) + r_{n+1}(t)) \right), \quad (2.9)
\end{align*}
\]

where \( \dot{x}_{n+1}(t) \) is used to compensate (cancel) the total disturbance \( x_{n+1}(t) \), and \( \rho(\dot{x}_1(t) - r_1(t)), \ldots, \rho(\dot{x}_n(t) - r_n(t)) + r_{n+1}(t) \) is to guarantee the output tracking and \( \dot{x}_n(t) - r_n(t) \) is bounded by using saturated functions \( \text{sat}_{\rho_0}(\cdot) \) respectively to limit the peaking value in control signal. The continuous differentiable saturation odd functions \( \text{sat}_{\rho_0} : \mathbb{R} \rightarrow \mathbb{R} \) are defined by (the counterpart for \( t \in (-\infty, 0) \) is obtained by symmetry)
\[
\text{sat}_{\rho_0}(x) = \begin{cases} z, & 0 \leq z \leq Q_0, \\
-\frac{1}{2} z^2 + (Q_0 + 1)z - \frac{1}{2} Q_0^2, & Q_0 < z \leq Q + 1, \\
Q_0 + \frac{1}{2} z, & z > Q + 1,
\end{cases} \tag{2.10}
\]

where \( Q_i \leq i \leq n \) are constants depending on the bound of initial values, the reference signal \( r(t) \), external disturbance \( u(t) \), and their derivatives up to \( n \) and \( Q_{n+1} \) is a constant depending not only on this bound, but also known of matched uncertainties \( h_i(\cdot) \) \( i = 1, 2, \ldots, n - 1 \) and their derivatives up to \( n \) on some compact set. It is easy to compute that |sat\(_{\rho_0}(z)| \leq 1.

Since estimation/cancellation strategy is adopted in real time, the control signal in ADRC avoids being unnecessarily large, which implies that ADRC would spend less energy in control to cancel the effect of total disturbance.

To obtain convergence for closed-loop of system (2.1) under ESO (2.8) based output feedback control (2.9), we need the following assumptions.

The Assumption (A1) is a prior assumption about the functions \( h_i(\cdot), f_i(\cdot), b_i(\cdot), b_0(\cdot) \).

**Assumption (A1)**, \( h_i(\cdot) \in C^{n_i-1}([0, m+1]; R), f_i(\cdot) \in C([0, \infty) \times R^{n_i+1}; R), b_i(\cdot) \in C([0, \infty) \times R^m; R); u(\cdot) \) is \( n \)-th continuously differentiable with respect to time variable \( t \), and there exist known non-negative functions \( \sigma_1 \in C([0, \infty); [0, \infty]) \) and \( \sigma_2 \in C([0, \infty); [0, \infty]) \) such that for all \( t \geq 0, x \in R^n, \xi \in R^n, w \in R^n,\]
\[
\max(||(t, x, \xi, w)||, ||(V(t, x, \xi, w)||) \leq \sigma_1(x, \xi, w), \max(||(t, u, w)||, ||(b(t, u, w)||, ||b_0(u)||) \leq \sigma_2(w).
\]

In succeeding Assumption (A2), we assume that the initial values of system (2.1) lie in a compact set and the reference signal \( r(t) \), external disturbance \( u(t) \), and their derivatives up to \( n + 1 \) and \( n \) respectively are bounded. These conditions are needed to construct a saturated feedback control to avoid the peaking value of feedback control caused by the high gain in ESO.

**Assumption (A2)**, there exist positive constants \( C_i \) and \( C_j \) such that \( ||(x(0), \xi(0)) \| \leq C_i \) and \( ||(r_1(t), r_2(t), \ldots, r_{n+1}(t), \tilde{r}_n(t)) \| \leq C_j \) for all \( t \geq 0 \). In addition, \( v(t) \in B, w(t) \in B \) for all \( t \geq 0 \) and \( i = 1, 2, \ldots, n \), where \( B = [-B, B] \subset R, B > 0 \).

The following Assumption (A3) is a prior assumption about \( \rho(\cdot) \) chosen in (2.9).

**Assumption (A3)**, \( \rho(z) \) is continuously differentiable. There exists a continuously differentiable function \( V_1 : R^n \rightarrow R^n \) which is positive definite and radially unbounded such that
\[
\begin{align*}
\sum_{i=1}^{n-1} \frac{\partial V_1(z)}{\partial z_i} + \rho(z) \frac{\partial V_1(z)}{\partial z_n} &\leq -c_1 V_1(z), \forall z = (z_1, z_2, \ldots, z_n) \in R^n, \quad (2.12)
\end{align*}
\]
for some positive constant \( c_1 > 0 \).

**Remark 2.1.** Essentially speaking, the Assumption (A3) is to ensure that \( \rho : R^n \rightarrow R \) is chosen so that the following system is globally asymptotically stable:
\[
\begin{align*}
\dot{z}(t) &= (z_2(t), \ldots, z_n(t), \rho(z_1(t), \ldots, z_n(t))), \quad (2.13)
\end{align*}
\]
with \( z(t) = (z_1(t), z_2(t), \ldots, z_n(t)) \).

The following Assumption (A4) is on the designed functions \( g_i(\cdot) \) in ESO (2.8).\n
**Assumption (A4)**, \( |g_i(z)| \leq k_i |z| \) for some positive constants \( k_i \) for all \( i = 1, 2, \ldots, n + 1 \). There exists a continuously differentiable function \( V_2 : R^{n+1} \rightarrow R^n \) which is positive definite and radially unbounded such that
\[
\begin{align*}
\sum_{i=1}^{n+1} \frac{\partial V_2(z)}{\partial z_i} - c_2 |g_i(z)| &\leq c_2 |g_i(z)|, \quad \frac{\partial V_2(z)}{\partial z_{n+1}} \leq c_4 V_2(z), 0 < \theta < 1, \quad (2.14)
\end{align*}
\]
for some positive constants \( c_1 > 0 \), \( i = 2, 3, 4, 5 \).

**Remark 2.2.** Assumption (A4) is made essentially to guarantee that the following system
\[
\begin{align*}
\dot{z}(t) &= (z_2(t) - g_1(z_1(t)), \ldots, z_{n+1}(t)) \\
&\quad - g_0(z_1(t)), - g_{n+1}(z_1(t)) \quad (2.15)
\end{align*}
\]
is globally asymptotically stable, where \( z(t) = (z_1(t), z_2(t), \ldots, z_{n+1}(t)) \).

By (2.2) and Assumption (A2), we can conclude that \( ||(x(0)) \| \leq C_3 \) for some positive constant \( C_3 \).

Define two compact subsets of \( R^n \) as follows:
\[
\begin{align*}
\Theta_1 \triangleq \{ v \in R^n : V_1(v) \leq \max_{z \in R^n, \|z\| \leq C_3} V_1(z) + 1 \}, \\
\Theta_2 \triangleq \{ v \in R^n : V_1(v) \leq \max_{z \in R^n, \|z\| \leq C_3} V_1(z) \}.
\end{align*}
\]

The positive constants \( C_1 \) used in saturation functions are chosen so that
\[
\begin{align*}
Q_i \geq \sup_{v \in \Theta_1} ||v||, \quad i = 1, 2, \ldots, n, \\
Q_{n+1} = 2N_1 + 2N_2 + N_3 + C_2 + \frac{1}{2} \tag{2.17}
\end{align*}
\]
We define a compact set as follows:

\[ \Theta_3 = \{(v_1, \ldots, v_n) \in \mathbb{R}^n : |v_i| \leq Q_3, \ i = 1, 2, \ldots, n\}. \]  

(2.18)

Let

\[ \Theta_j^i = \{v^i \in (v_1, \ldots, v_i) \in \mathbb{R}^i : v^i = (v_1, \ldots, v_n) \in \Theta_j\}, \ j = 1, 2, 3, 4. \]  

(2.19)

Assumption (A5) There exists a continuously differentiable function \( v_0(\xi) : \mathbb{R}^m \rightarrow \mathbb{R} \) which is positive definite and radially unbounded such that for all \( x_t, \zeta \in \mathbb{R}^m \) and \( w, \varepsilon \in \mathbb{R} \),

\[ \frac{\partial v_0(x_t, \zeta, w)}{\partial \zeta} f_0(x_1, \zeta, w) \leq 0, \ \forall \| \zeta \| \geq \alpha(|x_t|, |w|), \]  

(2.20)

where \( \alpha(\cdot) \) is a known class \( \mathcal{K} \) function [26].

Assumption (A5) ensures that the system \( \dot{\xi}(t) = f_0(x_1(t), \xi(t), w(t)) \) with input \( (x_1(t), w(t)) \) is bounded-input-bounded-state stable.

Under Assumption (A5), we can conclude that there exist a constant \( C_0 \geq 0 \) and a known class \( \mathcal{K}_\infty \) function \( \beta(\cdot) \) such that

\[ C \triangleq \{x \in \mathbb{R}^m : \|x\| \leq \max_{x \in \Theta_3} \beta(|x_t|, |w|) + C_0 \}. \]  

(2.21)

is a positively invariant set of \( \dot{\xi}(t) = f_0(x_1(t), \xi(t), w(t)) \) for all \( x_t(0) \in C_0 \) and \( w(t) \in B \).

Suppose that \( M_i \ (i = 1, 2, \ldots, n) \) are positive constants such that

\[ M_i = Q_1 + C_2, \]

\[ M_i \geq \sup_{x \in C_0, \xi : x, \ldots, x^{(i-2)}} |\phi_i(\xi, \zeta, w)|, \]  

(2.22)

Assumption (A6) \[ \inf_{x \in C_0} |b_0(t)| \geq \alpha_0 > 0, \]

\[ \beta_0 \triangleq \sup_{(t, w) \in \mathbb{R}^0} |b(t, w) - b_0(t)| \]

\[ < \min \left\{ \frac{1}{2} \alpha_0, \frac{\alpha_0 C_2}{C_3 \kappa_n} \right\}. \]  

(2.24)

Theorem 2.1. Suppose that \( \zeta(0) \in C \). Then under Assumptions (A1)–(A6), the closed-loop system composed of (2.1), (2.8), and (2.9) has the following convergence.

(i) The closed-loop state \( (x(t), \xi(t)) \) is bounded: \( \|x(t)\| \leq \Gamma_t \), \( \|\xi(t)\| \leq \Gamma_t \) for all \( t \geq 0 \), where \( \Gamma_t \) is an \( \varepsilon \)-independent positive constant;

(ii) The output \( y(t) \) of system (2.1) tracks practically the reference signal \( r(t) \) in the sense that: For any \( \sigma > 0 \), there exists a constant \( \varepsilon^n > 0 \) such that for any \( \varepsilon \in \varepsilon^n \),

\[ |y(t) - r(t)| \leq \sigma \text{ uniformly in } t \in [t_e, \infty), \]

where \( t_e > 0 \) is an \( \varepsilon \)-dependent constant. In particular,

\[ \lim_{t \rightarrow \infty} |y(t) - r(t)| \leq \sigma. \]

Proof. Set

\[ \eta(0) = \frac{\tilde{x}_1(t) - \tilde{x}_1(t)}{e^{n+1-i}} \quad (i = 1, 2, \ldots, n+1), \]

\[ \eta(t) = \eta(t), \eta_n(t), \eta_{n+1}(t), \]

\[ e(t) = \dot{\eta}(t), e(t) = \dot{e}(t), \quad \Delta(t) = \rho \left( \Delta ight. \]  

(2.25)

\[ \Delta(t) = \rho \left( \Delta \right. \]  

which satisfy

\[ \dot{e}_1(t) = e_2(t), \]

\[ e_2(t) = e_3(t), \]

\[ \ddot{e}_n(t) = \rho(e(t)) + \Delta(t) + \dot{\xi}_n+1(t) - \frac{1}{e} \]  

(2.26)

\[ \dot{\eta}(t) = \frac{1}{e} \rho(\eta(t)) - g_1(\eta_1(t)), \]

\[ \ddot{\eta}_n(t) = \frac{1}{e} \rho(\eta_n(t)) - g_n(\eta_n(t)), \]

\[ \dot{\eta}_n(t) = \frac{1}{e} \rho(\eta_n(t)) - g_n(\eta_n(t)). \]

The proof is split into three steps.

Step 1: There exists \( \epsilon_2 > 0 \) such that \( |e(t) : t \in [0, \infty) \subset \Theta_1 \) for all \( \epsilon \in (0, \epsilon_2) \). This concludes that there exists an \( \epsilon \)-independent constant \( \Gamma > 0 \) such that \( \|x(t)\| \leq \Gamma, \|\xi(t)\| \leq \Gamma \) for all \( t \geq 0 \).

Now we prove the claim of Step 1. Since \( e(0) \in \Theta_2 \) is an interior point of \( \Theta_2 \), \( e(t) \) would lie in \( \Theta_2 \) within a short time from \( t = 0 \) by its continuity in \( t \). Before \( e(t) \) escaping from \( \Theta_2, e(t) \in \Theta_2 \subset \Theta_1 \).

By (2.18), \( \dot{\xi}(t) \in \Theta_2 \). Since \( w(t) \in B, \zeta(0) \in C, \) and \( \dot{e}(t) \) in Assumption (A5) is bounded-input-bounded-state stable, if \( x_1(t) \in \Theta_1^i \), then \( \xi(t) \in C \). It follows from (2.4), (2.22), and (2.23) that \( x(t) \in \Theta_4 \).

Let

\[ N_1 = \sup_{(x, t, \xi, w) : t \in [0, \infty) \times \Theta_2 \times \Theta_2 \times \Theta_2} |\|f(t, x, \xi, w)\|, \|\nabla f(t, x, \xi, w)\|, |b(t, w)|, |\nabla b(t, w)|, |\dot{b}_0(t)|\|, \]  

(2.27)

\[ N_2 = \sum_{i=1}^{n-1} \sup_{(x, t, \xi, w) : t \in [0, \infty) \times \Theta_2 \times \Theta_2 \times \Theta_2 \times \Theta_2} \]

\[ + \sup_{i=1}^{n-1} \left[ |h_i(x, \zeta, w)| + |h_i(x, \zeta, w)| \right], \]  

(2.28)

\[ N_3 = \sup_{|x_1| < \frac{1}{2}} |\rho(z_1, \ldots, z_n)|, |\nabla \rho(z_1, \ldots, z_n)|. \]  

(2.29)

where \( \zeta = (x_1, \ldots, x_n) \). The “e-part” of (2.26), if \( x \in \Theta_4 \), then

\[ |e_1(t)| \leq |e_1(0)| + |e_2(0)|t + \cdots + \frac{1}{(n-1)!} |e_1(0)| t^{n-1} + \]

\[ + \frac{1}{m} \left[ N_3 + N_1 + \frac{\rho_0}{\alpha_0} \left( N_3 + Q_n+1 + \frac{1}{2} + C_2 \right) \right], \]  

(2.30)

\[ + N_2 + Q_{n+1} + \frac{1}{2} t^n, \]

\[ |e_{n-1}(t)| \leq |e_{n-1}(0)| + |e_n(0)| t + \frac{1}{2} \left[ N_3 + N_1 + \frac{\rho_0}{\alpha_0} \left( N_3 + Q_n+1 + \frac{1}{2} + C_2 \right) \right], \]  

\[ + N_2 + Q_{n+1} + \frac{1}{2} t^n, \]

\[ |e_n(t)| \leq |e_n(0)| + \left[ N_3 + N_1 + \frac{\rho_0}{\alpha_0} \left( N_3 + Q_n+1 + \frac{1}{2} + C_2 \right) \right], \]  

\[ + N_2 + Q_{n+1} + \frac{1}{2} t^n. \]  

By the “e-part” of (2.26), if \( x \in \Theta_4 \), then
It is observed that all terms on the right hand side of (2.30) are \( \varepsilon \)-independent. Since \( e(0) \) is an interior point of \( \Theta_2 \), there exists an \( \varepsilon \)-independent constant \( T > 0 \) such that \( e(t) \in \Theta_2 \) for all \( t \in [0, T] \).

We suppose that the conclusion of Step 1 is false and obtain a contradiction. Actually, by continuity of \( e(t) \) in \( t \), there exist \( \varepsilon \)-dependent constants \( t_1 \) and \( t_2 \) satisfying \( t_2 > t_1 \geq T \) such that

\[
e(t_1) \in \partial \Theta_2, \quad e(t_2) \in \partial \Theta_1, \quad \{e(t) : t \in [t_1, t_2] \} \subset \Theta_1 - \Theta_2,
\]

\[
\{e(t) : t \in [0, t_2] \} \subset \Theta_1.
\]

(2.31)

In this case, we can also conclude from (2.4), (2.22) and (2.23) that

\[
\{x(t) : t \in [0, t_2] \} \subset \Theta_4.
\]

(2.32)

Finding the derivative of the total disturbance \( \hat{x}_{n+1}(t) \) with respect to \( t \) gives

\[
\dot{\hat{x}}_{n+1}(t) = \frac{d}{dt} \left[ f(t, x(t), \zeta(t), u(t)) + (b(t, w(t)) - b_0(u(t))\right] + \sum_{j=1}^{n} h_{n-j}^i(x_j(t), \ldots, x_{n-j}(t), \zeta(t), u(t)).
\]

(2.33)

A direct computation shows that

\[
\frac{df(t, x(t), \zeta(t), u(t))}{dt} = \frac{df(t, x(t), \zeta(t), u(t))}{dt} + \sum_{i=1}^{n} \left[ \frac{\partial f(t, x(t), \zeta(t), u(t))}{\partial x_i} \right] + \left[ \frac{\partial f(t, x(t), \zeta(t), u(t))}{\partial \zeta} \right] \cdot f_0(x_1(t), \zeta(t), u(t)) + \left[ \frac{\partial f(t, x(t), \zeta(t), u(t))}{\partial u} \right] \cdot \dot{w}(t).
\]

(2.34)

By (2.27), (2.28), and (2.29), it follows that

\[
\left| \frac{df(t, x(t), \zeta(t), u(t))}{dt} \right| \leq N_1 \left[ 1 + (n - 1)(N_4 + N_2) + \frac{N_4 + Q_{n+1} + \frac{1}{2} C_2}{\alpha_0} \right],
\]

(2.35)

\( \forall t \in [0, t_2] \).

Similarly,

\[
\left| \frac{db(t, w(t))}{dt} - b_0(t) \right| \leq N_1 (B + 2), \quad \forall t \in [0, t_2].
\]

(2.36)

Finding the derivative of \( u(t) \) along the solution of (2.8) to obtain

\[
\frac{du(t)}{dt} \bigg|_{t_1} = \frac{1}{b_0(t)} \left( \sum_{i=1}^{n} \left( \hat{\phi}_{i+1}(t) + e_i \hat{g}(\eta_i(t)) - r_{i+1}(t) \right) \right) + \hat{\rho}_{n+1} \left( \hat{\phi}_{n+1}(t) - r_{n+1}(t) \right)
\]

\[
\hat{\rho}(\hat{\phi}_{n+1}(t) - r_{n+1}(t), \ldots, \hat{\phi}(t) - r_{n+1}(t))
\]

\[
\hat{\rho}(\hat{\phi}_{n+1}(t) - r_{n+1}(t), \ldots, \hat{\phi}(t) - r_{n+1}(t))
\]

\[
\frac{d}{dt} \left( \hat{\phi}(t) - r_{n+1}(t), \ldots, \hat{\phi}(t) - r_{n+1}(t) \right)
\]

(2.37)

where

\[
\rho_{n+1}(\hat{\phi}(t) - r_{n+1}(t), \ldots, \hat{\phi}(t) - r_{n+1}(t))
\]

denotes the \( i \)th partial derivative of \( \rho(\cdot) \) at \( sat_{n+1}(\hat{\phi}(t) - r_{n+1}(t), \ldots, \hat{\phi}(t) - r_{n+1}(t)) \). We then deduce from (2.24), (2.27), (2.29), and assumption for \( g_i(\cdot) \) that

\[
\frac{d}{dt} \left| u(t) \right| \bigg|_{t_1} \leq \frac{N_3}{\alpha_0} \left( \sum_{i=1}^{n} e_i \eta_i(t) + \sum_{i=2}^{n} (|e(t)| + |r(t)|) \right) + \sum_{i=1}^{n} k_i e_i \eta_i(t) + (n - 1)C_2
\]

(2.38)

\[
|\eta_{n+1}(t) + |x_{n+1}(t) + k_0|\eta_1(t)| + N_4 + Q_{n+1} + \frac{1}{2} C_2 \right],
\]

\( \forall t \in [0, t_2] \).

By (2.6), (2.24), (2.27), (2.28), and (2.29), it follows that

\[
|\hat{x}_{n+1}(t)| \leq N_1 + \frac{C_0 \epsilon}{\alpha_0} \left( N_1 + N_4 + Q_{n+1} + \frac{1}{2} C_2 \right), \quad \forall t \in [0, t_2].
\]

(2.39)

Thus, it follows from (2.24), (2.31), (2.35), (2.36), (2.38), and (2.39) that there exist \( \varepsilon \)-independent constants \( D_1, D_2 > 0 \) such that

\[
|\hat{x}_{n+1}(t)| \leq D_1 + D_2 \| \eta(t) \| + \frac{C_0 \epsilon}{\alpha_0} \| \eta_1(t) \|, \quad \forall t \in [0, t_2].
\]

(2.40)

By (2.24), we can define

\[
\xi_0 = C_2 - \frac{C_0 \epsilon}{\alpha_0} - 2 \epsilon - 0 > 0.
\]

(2.41)

We also notice that there exists \( \epsilon_0 > 0 \) such that

\[
c_1 D_2 - \frac{\xi_0}{2 \epsilon_0} > 0.
\]

(2.42)

Suppose that \( 0 < \varepsilon < \epsilon_0 \). It follows from (2.40) and Assumption (A4) that

\[
\frac{dV_2(\eta(t))}{dt} = \frac{1}{\varepsilon} \left( \sum_{i=1}^{n} \frac{\partial V_2(\eta(t))}{\partial \eta_i}(\eta_i(t) - g_i(\eta_i(t))) \right)
\]

\[
= \frac{\partial V_2(\eta(t))}{\partial \eta_i}(\eta_i(t) - g_i(\eta_i(t)))
\]

\[
+ \frac{\partial V_2(\eta(t))}{\partial \eta_{n+1}}(\eta_{n+1}(t) - g_{n+1}(\eta_{n+1}(t)))
\]

\[
+ \frac{\partial V_2(\eta(t))}{\partial \eta_i}(\eta_i(t) + c_0 \hat{\phi}(\eta_i(t)) - r_{n+1}(t))
\]

\[
\leq \frac{-\xi_0}{2 \epsilon} V_2(\eta(t)) + c_4 D_1 V_2(\eta(t)), \quad \forall t \in [t_1, t_2].
\]

(2.43)

Thus,

\[
\frac{d}{dt} \left( V_2(\eta(t)) \right) \leq \frac{1 + \theta - \xi_0}{2 \epsilon} V_2(\eta(t)) + (1 - \theta) c_4 D_1,
\]

(2.44)
and so
\[
V^1_{2}(\eta(t)) \leq e^{-\frac{(1-\eta_0)t}{\xi_0}}V^1_{2}(\eta(0)) + (1-\eta)\xi D_1 \int_0^t e^{-\frac{(1-\eta_0)(t-s)}{\xi_0}} ds 
\]
\[
\leq e^{-\frac{(1-\eta_0)t}{\xi_0}}V^1_{2}(\eta(0)) + \frac{4\xi_0 c_1 D_1}{\xi_0}, \forall t \in [t_1, t_2].
\]
(2.45)

By Assumption (A4),
\[
e^{-\frac{(1-\eta_0)t}{\xi_0}}V^1_{2}(\eta(0)) \leq e^{-\frac{(1-\eta_0)t}{\xi_0}} \left( \sum_{i=1}^{n+1} \left( \frac{|\hat{\chi}(0) - \hat{\chi}(\eta)|}{\xi_0^{i-1}} \right) \right)^{1-\theta} \rightarrow 0 \text{ as } \epsilon \rightarrow 0,
\]
(2.46)

and hence
\[
V^1_{2}(\eta(t)) \rightarrow 0 \text{ as } \epsilon \rightarrow 0, \forall t \in [t_1, t_2].
\]
(2.47)

Let
\[
\hat{\xi} = \min \left\{ \frac{1}{2}, \frac{c_1 \min_{\epsilon \in \epsilon_i}[V_1(\epsilon)]}{N_2(2n N_3 + 3)} \right\}
\]
(2.48)

Since \(V_2(\cdot)\) is continuous, positive definite, and radially unbounded, it follows from Lemma 4.3 of [26, p. 145] that there exists continuous class \(K_{\infty}\)-function \(\kappa : [0, \infty) \rightarrow [0, \infty)\) such that \(V_2(\eta) \geq \kappa(\|\eta\|)\) for all \(\eta \in \mathbb{R}^{n+1}\). By (2.47), there exists \(\epsilon_1 \leq \epsilon_2\) such that \(V_2(\eta(t)) \leq \kappa(\hat{\xi})\) uniformly in \(t \in [t_1, t_2]\) for all \(\epsilon \in (0, \epsilon_1]\). That is, \(\|\eta(t)\| \leq \hat{\xi}\) uniformly in \(t \in [t_1, t_2]\) for all \(\epsilon \in (0, \epsilon_1]\). Suppose that \(0 < \epsilon < \epsilon_2 \leq \hat{\epsilon}(\epsilon_1, \epsilon_2)\). Then
\[
|\hat{\chi}(t) - \hat{\xi}(t)| = \left\| \hat{\chi}(t) - \hat{\xi}(t), \ldots, \hat{\chi}(t) - \hat{\xi}(t) \right\| \leq \|\eta(t)\| \leq \hat{\xi} \leq \frac{1}{2} \text{ for all } t \in [t_1, t_2], i = 1, \ldots, n + 1.
\]
(2.49)

We also conclude from (2.17) and (2.31) that \(|e_i(t)| \leq Q_i (i = 1, 2, \ldots, n)\) for all \(t \in [t_1, t_2]\) and from (2.17), (2.24), and (2.39), \(|\hat{\chi}_{n+1}(t)| \leq Q_{n+1}\) for all \(t \in [t_1, t_2]\). These together with (2.49) yield
\[
|\hat{\chi}_i(t) - r_i(t)| = |\hat{\xi}(t) - \hat{\xi}(t) + e_i(t)| \leq Q_i + \frac{1}{2}, \forall t \in [t_1, t_2], i = 1, 2, \ldots, n.
\]
(2.50)

If \(|\hat{\chi}_i(t) - r_i(t)| \leq Q_i\), then
\[
|\hat{\chi}_i(t) - r_i(t) - \text{sat}_{Q_i}(\hat{\chi}_i(t) - r_i(t))| = 0, i = 1, 2, \ldots, n.
\]
(2.51)

If \(|\hat{\chi}_i(t) - r_i(t)| > Q_i\) and \(\hat{\chi}_i(t) - r_i(t) > 0\), we have \(\hat{\chi}_i(t) - r_i(t) > Q_i\)

\[
|\hat{\chi}_i(t) - r_i(t) - Q_i| = \hat{\chi}_i(t) - r_i(t) - Q_i \leq \hat{\xi}(t) - r_i(t) - e_i(t) \leq \|\eta(t)\| \leq \hat{\xi}, \forall t \in [t_1, t_2].
\]
(2.52)

This, together with (2.10), gives
\[
|\hat{\chi}_i(t) - r_i(t) - \text{sat}_{Q_i}(\hat{\chi}_i(t) - r_i(t))| \leq \hat{\chi}_i(t) - r_i(t) + \frac{1}{2} \hat{\xi}(t) - r_i(t) \leq \frac{1}{2} Q_i^2.
\]
(2.53)

Similarly, we can also conclude (2.53) when \(\hat{\chi}_i(t) - r_i(t) > Q_i\), and \(\hat{\chi}_i(t) - r_i(t) < 0\) because \(\text{sat}_{Q_i}(\cdot)\) is an odd function. Therefore
\[
|\hat{\chi}_i(t) - r_i(t) - \text{sat}_{Q_i}(\hat{\chi}_i(t) - r_i(t))| \leq \xi (i = 1, 2, \ldots, n) \text{ for all } t \in [t_1, t_2].\]
(2.54)

For any \(\epsilon > 0\), we have \(|\hat{\chi}_i(t) - r_i(t)| \leq Q_i (i = 1, 2, \ldots, n)\) and \(|\hat{\chi}_{n+1}(t)| \leq Q_{n+1}\) for all \(t \in [0, \infty)\). Since \(V_2(\cdot)\) is continuous, positive definite, and radially unbounded, it follows from Lemma 4.3 of [26, p. 145] that there exists continuous class \(K_{\infty}\)-function \(\chi : [0, \infty) \rightarrow [0, \infty)\)
From (2.67), we have
\[ \frac{dv}{dt}(t) = -c_1 \chi(\sigma) + \frac{2c_1 \chi(\sigma)}{3N_k}, \quad \forall t \in [a, \infty). \] (2.62)

So when \(|\|e(t)\|| \geq \sigma|\), by (2.61) and (2.62),
\[ \frac{dv}{dt}(t) \leq -c_1 \chi(\sigma) + \frac{2c_1 \chi(\sigma)}{3} = -\frac{c_1}{3} \chi(\sigma) < 0, \quad \forall t \in [a, \infty). \] (2.63)

Hence there exists an \(\varepsilon\)-dependent constant \(t_\varepsilon > 0\) such that
\[ \|e(t)\| \leq \sigma \quad \text{for all } t \in [t_\varepsilon, \infty). \] (2.64)

Therefore,
\[ |y(t) - r(t)| \leq \|e(t)\| \leq \sigma \quad \text{uniformly in } t \in [t_\varepsilon, \infty). \] (2.65)

This completes the proof of the theorem.  

The simplest example of constant gain ADRC satisfying conditions of Theorem 2.1 is the linear one, i.e., \(g_c(\cdot) \equiv 1\), \(i = 1, \ldots, n + 1\) in ESO (2.8) and \(\rho(\cdot)\) in feedback control (2.9) are linear functions. Let
\[ g_c(z_1) = k_c z_1, \quad \rho(z_1, \ldots, z_n) = a_1 z_1 + \cdots + a_n z_n. \] (2.66)

Define the matrices \(E\) and \(F\) as follows:
\[ E = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_{n-1} & a_n \\
\end{pmatrix}_{n \times n}, \quad F = \begin{pmatrix}
-k_1 & 1 & 0 & \cdots & 0 \\
-k_2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-k_n & 0 & 0 & \cdots & 1 \\
\end{pmatrix}_{(n+1) \times (n+1)}. \] (2.67)

Let \(\lambda_{\max}(H)\) be the maximal eigenvalue of matrix \(H\) that is the unique positive definite solution matrix of the Lyapunov equation \(HE + E^T H = -I_{n \times n}\) for \(n\)-dimensional identity matrix \(I_{n \times n}\). In addition, let \(\lambda_{\max}(Q)\) and \(\lambda_{\min}(Q)\) be the maximal and minimal eigenvalues of matrix \(Q\) that is the unique positive definite matrix solution of the Lyapunov equation \(QF + F^T Q = -I_{(n+1) \times (n+1)}\) for \((n+1)\)-dimensional identity matrix \(I_{(n+1) \times (n+1)}\).

**Corollary 2.1.** Suppose that \(\chi(\cdot) \in C\) and the matrices \(E\) and \(F\) in (2.67) are Hurwitz. Then under Assumptions (A1)–(A2) and (A5–A6) with \(c_2 = \frac{1}{\lambda_{\max}(Q)}\) and \(c_3 = \frac{2\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\), the closed-loop system composed of (2.1), (2.8), and (2.9) has the following convergence:

(i) The state \(x(t)\), \(\xi(t)\) is bounded: \(\|x(t)\| \leq \Gamma\), \(\|\xi(t)\| \leq \Gamma\) for all \(t \geq 0\), where \(\Gamma\) is an \(\varepsilon\)-dependent positive constant;

(ii) The output \(y(t)\) of system (2.1) tracks practically the reference signal \(r(t)\) in the sense that: For any \(\sigma > 0\), there exists a constant \(\varepsilon^* > 0\) such that for any \(\varepsilon \in (0, \varepsilon^*)\),
\[ |y(t) - r(t)| \leq \sigma \quad \text{uniformly in } t \in [t_\varepsilon, \infty), \] where \(t_\varepsilon > 0\) is an \(\varepsilon\)-dependent constant. In particular,
\[ \lim_{t \to +\infty} |y(t) - r(t)| \leq \sigma. \]

**Proof.** Define the Lyapunov functions \(V_1 : \mathbb{R}^n \to \mathbb{R}\) by \(V_1(z) = z^T H z\) for \(z \in \mathbb{R}^n\) and \(V_2 : \mathbb{R}^{n+1} \to \mathbb{R}\) by \(V_2(z) = z^T Q z\) for \(z \in \mathbb{R}^{n+1}\). Then it is easy to verify that all conditions of Assumptions (A3)–(A4) are satisfied, where the parameters in Assumptions (A3–A4) are specified as \(c_1 = \frac{1}{\lambda_{\max}(H)}\), \(c_2 = \frac{1}{\lambda_{\max}(Q)}\), \(c_3 = \frac{2\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\), \(c_4 = \frac{2\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\), \(\theta = \frac{1}{2}\), \(\gamma_2 = \frac{1}{2}\), \(c_5 = \lambda_{\max}(Q)\). The results then follow directly from Theorem 2.1.

The uncertainties in system (2.1) seem too complicated and the results of Theorem 2.1 may be overwhelmed by this complexity. At the end of this section, we consider a special case where the mismatched uncertainties in system (2.1) are only external disturbances, that is, system (2.1) is of the form:

\[ \dot{x}_1(t) = x_2(t) + u(t), \]
\[ \dot{x}_2(t) = x_3(t) + u(t), \]
\[ \vdots \]
\[ \dot{x}_n(t) = x_{n+1}(t), \]
\[ \dot{x}_{n+1}(t) = f(t, x(t), \xi(t), w(t)) + b(t, w(t))u(t), \]
\[ \gamma(t) = x_\sigma(t), \]
\[ \zeta(t) = f_\sigma(x(t), \xi(t), w(t)), \]
\[ \zeta(t) = \sigma(t), \]
where we used \(u(t)\), \(i = 1, 2, \ldots, n - 1\) to represent mismatched external disturbances in different channels, and \(w(t) = (w_1(t), w_2(t), \ldots, w_n(t))\). In this case, we notice from (2.2) that
\[ x_1(t) = x_2(t), x_2(t) = x_3(t) + \sum_{i=1}^{n-1} w_{i-1}(t) (i = 2, 3, \ldots, n) \]
and \(w_{n-1}(t) = \sum_{i=1}^{n-1} w_{n-1}(t) (i = 2, \ldots, n)\) from the conclusion of Step 3 in Theorem 2.1, for any \(\sigma > 0\), there exists an \(\varepsilon^* > 0\) such that for any \(\varepsilon \in (0, \varepsilon^*)\),
\[ |y(t) - r(t)| \leq \sigma, \]
\[ |x_i(t) - r_{i-1}(t)| \leq \sigma + \sup_{t \geq 0} \sum_{i=1}^{n-1} |w_{i-1}(t)|, \]
\[ V(t) \leq t, \quad \forall t \in [t_\varepsilon, \infty), \quad i = 2, 3, \ldots, n, \]
where \(t_\varepsilon > 0\) is an \(\varepsilon\)-dependent constant. We notice from (2.69) that the performance of tracking of \(x_i(t)\) to \(r_{i-1}(t)\) for each \(i = 2, 3, \ldots, n\) is closely related to intensity of external disturbances and their variations (derivatives), i.e., the effects of the tracking would become more satisfactory as the intensity becomes smaller, and becomes worse otherwise.

In particular, when system (2.1) has no mismatched uncertainties, i.e., \(w_i(\cdot) \equiv 0, i = 1, 2, \ldots, n - 1\), then by (2.69), for any \(\sigma > 0\), there exists an \(\varepsilon^* > 0\) such that for any \(\varepsilon \in (0, \varepsilon^*)\),
\[ |x_i(t) - r_{i-1}(t)| \leq \sigma \quad \text{for all } t \in [t_\varepsilon, \infty), \]
\[ i = 1, 2, \ldots, n - 1 \]
and \(t_\varepsilon > 0\) is an \(\varepsilon\)-dependent constant. In addition, we see also from (2.70) that this general formulation covers not only the special output regulation problem, but also the output feedback stabilization by setting \(r(t) \equiv 0\). That is, when \(w_i(\cdot) \equiv 0, i = 1, 2, \ldots, n - 1\) and \(r(t) \equiv 0\), then for any \(\sigma > 0\), there exists an \(\varepsilon^* > 0\) such that for any \(\varepsilon \in (0, \varepsilon^*)\),
\[ |x_i(t)| \leq \sigma \quad \text{for all } t \in [t_\varepsilon, \infty), \]
\[ i = 1, 2, \ldots, n, \]
where \(t_\varepsilon > 0\) is an \(\varepsilon\)-dependent constant.

### 3. ADRC with time-varying gain ESO

In the last section, the constant high gain ESO (2.8) is designed to estimate total disturbance of system (2.1) and the corresponding
ESO based output feedback control guarantees that the output of closed-loop of (2.1) tracks practically reference signal and the closed-loop state $(x(t), \zeta(t))$ is bounded. The merit of constant high gain lies in its fast convergence and filter function for high frequency noise [20]. However, the main problem for constant high gain ESO, likewise many other high gain designs, is the peaking value problem near the initial stage of ESO caused by different initial values of system (2.1) and ESO [18–20]. To solve this problem, a time-varying gain ESO is proposed in [18], where the time-varying gain increases slowly from a small initial value to reach its maximal value. The peaking value reduction with time-varying gain ESO is illustrated through numerical simulations in Section 4. Precisely, motivated by [18–20], a time-varying gain ESO for (2.1) is designed as follows:

\[
\dot{\hat{x}}_1(t) = \hat{x}_2(t) + \frac{1}{\vartheta(t)}g_1(\vartheta(t)y(t) - \hat{x}_1(t)), \\
\dot{\hat{x}}_2(t) = \hat{x}_3(t) + \frac{1}{\vartheta(t)}g_2(\vartheta(t)y(t) - \hat{x}_1(t)), \\
\vdots \\
\dot{\hat{x}}_{n-1}(t) = \hat{x}_n(t) + \frac{1}{\vartheta(t)}g_{n-1}(\vartheta(t)y(t) - \hat{x}_1(t)), \\
\dot{\hat{x}}_n(t) = \vartheta(t)\hat{x}_n(t) + \frac{1}{\vartheta(t)}g_n(\vartheta(t)y(t) - \hat{x}_1(t)),
\]

(3.1)

where \(g_i \in C(\mathbb{R}; \mathbb{R})\) are designed functions satisfying Assumption (A4) and \(\vartheta(t)\) is chosen as

\[
\vartheta(t) = \begin{cases} 
\frac{e^a t}{\epsilon}, & 0 \leq t \leq -\frac{1}{a} \ln \epsilon, \\
\frac{1}{\epsilon}, & t \geq -\frac{1}{a} \ln \epsilon,
\end{cases}
\]

(3.2)

where \(a > 0\) is used to control the convergent speed and the peaking value. The larger \(a\) is, the faster convergence but larger peaking; while the smaller \(a\) is, the lower convergence speed and smaller peaking.

We can easily generalize the results claimed by Theorem 2.1 and Corollary 2.1 to the closed-loop system of (2.1) under time-varying gain ESO (3.1) based output feedback control (2.9) since the ESO (3.1) is reduced to ESO (2.8) when \(t \geq -\frac{1}{a} \ln \epsilon\), which is summarized in the succeeding Theorem 3.1 and Corollary 3.1.

**Theorem 3.1.** Suppose that \(\zeta(0) \in C\). Then under Assumption (A1)–(A6), the closed-loop system composed of (2.1), (3.1) and (2.9) has the following convergence:

(i) The closed-loop state \((x(t), \zeta(t))\) is bounded: \(\|x(t)\| \leq \Gamma\), \(\|\zeta(t)\| \leq \Gamma\) for all \(t \geq 0\), where \(\Gamma\) is an \(\epsilon\)-dependent positive constant;

(ii) The output \(y(t)\) of system (2.1) tracks practically the reference signal \(r(t)\) in the sense that: For any \(\sigma > 0\), there exists a constant \(\epsilon^* > 0\) such that for any \(\epsilon \in (0, \epsilon^*)\),

\[
|y(t) - r(t)| \leq \sigma \text{ uniformly in } t \in [t_\epsilon, \infty),
\]

where \(t_\epsilon > 0\) is an \(\epsilon\)-dependent constant. In particular,

\[
\lim_{t \to +\infty} |y(t) - r(t)| \leq \sigma.
\]

**Corollary 3.1.** Suppose that \(\zeta(0) \in C\) and the matrices \(E\) and \(F\) in (2.67) are Hurwitz. Then Assumptions (A1)–(A2) and (A5–A6) with \(c_2 = \frac{1}{\max_{\lambda \in \sigma(E)}}, c_3 = \frac{2}{\max_{\lambda \in \sigma(E)}}, \) and \(k_{n-1}\) specified in (2.66), the closed-loop system composed of (2.1), (3.1) and (2.9) has the following convergence:

(i) The state \((x(t), \zeta(t))\) is bounded: \(\|x(t)\| \leq \Gamma\), \(\|\zeta(t)\| \leq \Gamma\) for all \(t \geq 0\), where \(\Gamma\) is an \(\epsilon\)-dependent positive constant;

(ii) The output \(y(t)\) of system (2.1) tracks practically the reference signal \(r(t)\) in the sense that: For any \(\sigma > 0\), there exists a constant \(\epsilon^* > 0\) such that for any \(\epsilon \in (0, \epsilon^*)\),

\[
|y(t) - r(t)| \leq \sigma \text{ uniformly in } t \in [t_\epsilon, \infty),
\]

where \(t_\epsilon > 0\) is an \(\epsilon\)-dependent constant. In particular,

\[
\lim_{t \to +\infty} |y(t) - r(t)| \leq \sigma.
\]

4. Numerical simulations

In this section, we present several numerical simulations to illustrate the effectiveness of the proposed ADRC approach. Consider the following lower triangular system with mismatched uncertainties:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) + h_1(x_1(t), \zeta(t), w(t)), \\
\dot{x}_2(t) &= f(t, x_1(t), x_3(t), \zeta(t), w(t)) + b(t, w(t))u(t), \\
\zeta(t) &= f_0(x_1(t), \zeta(t), w(t)), \\
y(t) &= x_1(t),
\end{align*}
\]

(4.1)

where \(h_1(\cdot): \mathbb{R}^3 \to \mathbb{R}, f(\cdot): [0, \infty) \times \mathbb{R}^4 \to \mathbb{R}, \) and \(f_0(\cdot): \mathbb{R}^3 \to \mathbb{R}\) are unknown nonlinear functions, \(b(\cdot): [0, \infty) \times \mathbb{R} \to \mathbb{R}\) is a partially unknown input gain function with its constant nominal value \(b_0 \in \mathbb{R}\), and \(u(t)\) is the external disturbance. Let \(r(t) = \sin(t + 1)\). Now we design an ESO based output feedback so that the output \(y(t)\) of (4.1) tracks practically the reference signal \(r(t)\) and keep the state \((x_1(t), x_2(t), \zeta(t))\) being bounded.

Similarly to [10,11,18,19], we design a nonlinear ESO as follows:

\[
\begin{align*}
\dot{x}_1(t) &= \hat{x}_2(t) + \frac{6}{\epsilon} (y(t) - \hat{x}_1(t)) + \epsilon \Psi \left(\frac{y(t) - \hat{x}_1(t)}{\epsilon^2}\right) \\
\dot{x}_2(t) &= \hat{x}_3(t) + \frac{1}{\epsilon} (y(t) - \hat{x}_1(t)) + b_0u(t), \\
\dot{x}_3(t) &= \frac{6}{\epsilon^3} (y(t) - \hat{x}_1(t)),
\end{align*}
\]

(4.2)

where \(\Psi: \mathbb{R} \to \mathbb{R}\) is defined as

\[
\Psi(s) = \begin{cases} 
\frac{1}{4\pi} s, & s \in (-\pi/2, 0), \\
\frac{1}{4\pi} s, & s \in [0, \pi/2], \\
\frac{1}{4\pi} s, & s \in [\pi/2, +\infty).
\end{cases}
\]

(4.3)

We notice that the corresponding matrix in (2.67) for the linear part of (4.2)

\[
F = \begin{pmatrix} -6 & 1 & 0 \\
-11 & 0 & 1 \\
-6 & 0 & 0 \end{pmatrix}
\]

(4.4)

is Hurwitz with eigenvalues \(-1, -2, -3\). The \(g_i(\cdot)\) in (2.8) can be specified as

\[
g_1(y_1) = 6y_1 + \Psi(y_1), \quad g_2(y_1) = 11y_1, \quad g_3(y_1) = 6y_1.
\]

(4.5)

The nonlinear function \(g_i(\cdot)\) is constructed by linear function perturbed by a Lipschitz continuous nonlinear function with small Lipschitz constant.

A linear ESO (4.2) based output feedback control is designed as follows:

\[
u(t) = \frac{1}{b_0}\left[-2\text{sat}_3(\hat{x}_1(t) - \sin(t + 1)) - 3\text{sat}_3(\hat{x}_2(t) - \cos(t + 1)) - \text{sat}_3(\hat{x}_3(t) - \sin(t + 1))\right],
\]

(4.6)

with the corresponding matrix in (2.67)

\[
E = \begin{pmatrix} 0 & 1 & 0 \\
-2 & -3 \end{pmatrix}
\]

(4.7)
The input gain function \( b(x) \) given by

\[
b(x) = x_1^2 + \cos x_1 + x_3^2 + w^2,
\]

and the external disturbance \( w(t) \) as follows:

\[
\begin{align*}
h_1(x_1, \zeta, w) &= x_1^2 + \cos x_1 + x_3^2 + w^2, \\
f(t, x_1, x_2, \zeta, w) &= t^2e^{-t} + x_1 + x_2 + \zeta^2 + w, \\
f_0(x_1, \zeta, w) &= -[x_1^2 + w^2]\zeta, \\
w(t) &= \cos(3t + 1).
\end{align*}
\]

The input gain function \( b(\cdot) \) and its constant nominal value \( b_0 \) are given by

\[
b(t, w) = 3 + \frac{1}{2}\sin(t + x_1 + 2x_2 + 3w), \quad b_0 = 3.
\]

In addition, the initial values are

\[
x_1(0) = x_2(0) = \zeta(0) = \frac{1}{2}, \quad \hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 0,
\]

and the discrete step and the gain constant \( \varepsilon \) are taken as \( \Delta t = 0.001 \) and \( \varepsilon = 0.01 \), respectively.

In Section 2, we indicate that the ESO (4.2) is designed to estimate, in real time, \( \hat{x}_i(t) \) \((i = 1, 2, 3)\) given by

\[
\begin{align*}
\hat{x}_1(t) &= x_1(t), \\
\hat{x}_2(t) &= x_2(t) + h_1(x_1(t), \zeta(t), w(t)), \\
\hat{x}_3(t) &= f(t, x_1(t), x_2(t), \zeta(t), w(t)) + [bf(t, w(t)) - b_0]u(t) \\
&\quad + [2x_1(t) - \sin(x_1(t))] \cdot [x_2(t) + h_1(x_1(t), \zeta(t), w(t))] \\
&\quad + 3\zeta^2(t) \cdot f_0(x_1(t), \zeta(t), w(t)) + 2w(t)u(t),
\end{align*}
\]

where \( \hat{x}_3(t) \) is the actual total disturbance that affects the performance of the controlled output \( y(t) \).

It is observed from Fig. 1 that the estimation effects of the constant gain ESO (4.2) for \( \hat{x}_1(t) \) \(,\hat{x}_2(t) \) and the total disturbance \( \hat{x}_3(t) \) defined by (4.11) are very satisfactory. It is also seen from Fig. 2 that the output of closed-loop system (4.1) under constant gain ESO (4.2) based feedback control (4.6) is very effective in tracking the reference signal \( \sin(t + 1) \). The closed-loop state \( (x_2(t), \zeta(t)) \) have small bound over a long period of time, and \( \zeta(t) \) even converges to zero after a short time. In addition, we can see from Fig. 3(c) that the saturations of \( \dot{\hat{x}}_1(t) - \sin(t + 1), \dot{\hat{x}}_2(t) - \cos(t + 1), \) and \( \dot{\hat{x}}_3(t) \) make the control value less than 8. However, the large peaking values of \( \hat{x}_3(t) \) and \( x_1(t) \) are observed near the initial stage because of the high gain \( \frac{1}{\varepsilon} = 100 \). The absolute peaking value of \( \hat{x}_3(t) \) is near 100 and that of \( \hat{x}_3(t) \) is near 5000 in Figs. 3(a) and 3(b), respectively.

To avoid the peaking phenomenon near the initial stage for \( \hat{x}_3(t) \), \( \hat{x}_3(t) \) of closed-loop system (4.1) under time-varying gain ESO (4.12) based feedback control (4.6) to system (4.1), which comes from (3.1) with nonlinear functions \( g_i(\cdot) \) \((i = 1, 2, 3)\) as that in (4.5):

\[
\begin{align*}
\dot{\hat{x}}_1(t) &= \hat{x}_2(t) + 6\theta(t)y(t) - \hat{x}_1(t) + \hat{\varphi}(\theta(t)y(t) - \hat{x}_1(t)), \\
\dot{\hat{x}}_2(t) &= \hat{x}_3(t) + 11\theta^2(t)y(t) - \hat{x}_3(t) + b_0u(t), \\
\dot{\hat{x}}_3(t) &= 6\theta^3(t)y(t) - \hat{x}_3(t),
\end{align*}
\]

where \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is given by (4.3) and the time-varying gain \( \theta(t) \) is given by

\[
\begin{align*}
\theta(t) &= \begin{cases}
e^{-\varepsilon t}, & 0 \leq t \leq \frac{1}{5} \ln 100, \\
\frac{1}{\varepsilon} = 100, & t \geq \frac{1}{5} \ln 100.
\end{cases}
\end{align*}
\]

The numerical results for (4.1) with time-varying gain ESO (4.12) and time-varying gain \( \theta(t) \) given by (4.13) are plotted in Figs. 4 and 5 with the same initial values and time discrete step as that in Figs. 1–3. Fig. 4 shows that the time-varying gain ESO (4.12) tracks the \( \hat{x}_1(t) \), \( \hat{x}_2(t) \) and the total disturbance \( \hat{x}_3(t) \) of (4.11) well. In addition, Fig. 5 shows that the output of the closed-loop system (4.1) under time-varying gain ESO (4.12) based feedback control (4.6) is also very effective in tracking the reference signal \( \sin(t + 1) \). The closed-loop state \( (x_2(t), \zeta(t)) \) has small bound over a long period of time, and \( \zeta(t) \) even converges to zero after a short time. More importantly, Figs. 4(b) and 4(c) show that the absolute peaking value near the initial stage of \( \hat{x}_3(t) \) is less than 2 (near 100 by constant high gain) and that of \( \hat{x}_3(t) \) is less than 4.

---

**Fig. 1.** The \( \hat{x}_1(t), \hat{x}_2(t), \) total disturbance \( \hat{x}_3(t) \), and their estimates \( \hat{\hat{x}}_1(t), \hat{\hat{x}}_2(t), \hat{\hat{x}}_3(t) \) with constant gain (4.2).

**Fig. 2.** The output tracking and boundedness of state \( (x_1(t), x_2(t), \zeta(t)) \) of closed-loop system (4.1) under constant gain ESO (4.2) based feedback control (4.6).
Finally, to validate the ADRC capacity for external disturbance attenuation indicated in (2.69), we consider a special case where mismatched uncertainties in system (4.1) are only external disturbances, that is, \(h_1(x_1(t), \zeta(t), w(t)) = w(t)\). Fig. 6 shows that when the external disturbance is small: \(w(t) = \frac{1}{100} \cos(3t + 1)\), the states \(x_2(t)\) of system (4.1) under constant gain ESO (4.2) and time-varying gain ESO (4.12) based feedback control (4.6) tracks \(f(t) = \cos(t + 1)\) very well, which are plotted in Fig. 6(a) and 6(b), respectively.

5. Concluding remarks

In this paper, we apply the active disturbance rejection control (ADRC) approach to output tracking for a class of lower triangular systems with vast matched and mismatched uncertainties including unknown internal uncertainty, external disturbance, and uncertainty caused by the deviation of control parameter from its nominal value. The total disturbance is first determined by a state variable transformation. An extended state observer (ESO) with constant high gain is then designed to estimate in real time the total disturbance. An ESO based output feedback is then designed to guarantee that all signals in the closed-loop are bounded and the tracking error is in an arbitrarily given area. To avoid the peaking phenomenon that occurs near the initial stages of ESO caused by constant high gain, a time-varying gain ESO is addressed and the corresponding closed-loop performance and output tracking are guaranteed by the time-varying gain ESO based output feedback control. The simulation examples illustrate that good tracking performance and peaking value reduction can be achieved by the proposed approach.
Fig. 6. The tracking of state $x_2(t)$ of closed-loop system (4.1) under constant gain ESO (4.2) and time-varying gain ESO (4.12) based feedback control (4.6) to $\dot{r}(t) = \cos(t+1)$, where $h_t(x_1(t), \zeta(t), w(t)) = w(t) = \frac{1}{10}\cos(3t+1)$.

References