INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2017; **27**:2773–2797
Published online 14 November 2016 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.3710

Active disturbance rejection control approach to output-feedback stabilization of lower triangular nonlinear systems with stochastic uncertainty

Bao-Zhu Guo[1,2] and Ze-Hao Wu[1,*,†]

[1] *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China*
[2] *School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg 2050, South Africa*

SUMMARY

In this paper, we apply the active disturbance rejection control approach to output-feedback stabilization for uncertain lower triangular nonlinear systems with stochastic inverse dynamics and stochastic disturbance. We first design an extended state observer (ESO) to estimate both unmeasured states and stochastic total disturbance that includes unknown system dynamics, unknown stochastic inverse dynamics, external stochastic disturbance, and uncertainty caused by the deviation of control parameter from its nominal value. The stochastic total disturbance is then compensated in the feedback loop. The constant gain and the time-varying gain are used in ESO design separately. The mean square practical stability for the closed-loop system with constant gain ESO and the mean square asymptotic stability with time-varying gain ESO are developed, respectively. Some numerical simulations are presented to demonstrate the effectiveness of the proposed output-feedback control scheme. Copyright © 2016 John Wiley & Sons, Ltd.

Received 13 November 2015; Revised 4 July 2016; Accepted 17 October 2016

KEY WORDS: lower triangular nonlinear systems; extended state observer; stochastic disturbance; stabilization

1. INTRODUCTION

The active disturbance rejection control (ADRC), as an unconventional design strategy, was first proposed by Han in his pioneer work [1]. It is now acknowledged to be an effective control strategy in dealing with the total disturbance that can be the coupling between the external disturbance, unknown system dynamics, and the superadded unknown part of control input. The key idea of ADRC is that the total disturbance is considered as an extended state and is estimated, in real time, through extended state observer (ESO). The total disturbance is then compensated in the feedback loop by its estimation. This estimation/cancelation nature of ADRC makes it capable of eliminating the uncertainty before it causes negative effect to control plant and the control energy can therefore be saved significantly in engineering applications.

In the last several years, some progresses have been made in building the theoretical foundation of ADRC, see, for instance, [2–12]. The convergence of linear ESO, which is proposed in [3] in terms of bandwidth, is discussed in [12]. The linear ADRC has been addressed for different systems like those for control and disturbance unmatched systems [8], lower triangular systems [10], and the system without known nominal control parameter [7]. In particular, linear ADRC with adaptive

*Correspondence to: Ze-Hao Wu, Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China.
†E-mail: zehaowu@amss.ac.cn

Copyright © 2016 John Wiley & Sons, Ltd.

gain ESO is investigated in [9, 11] and extended state filter is addressed for filtering problem of general discrete nonlinear uncertain systems in [2]. The convergence of nonlinear ADRC for SISO systems is proved firstly in [4] and extended subsequently to MIMO systems in [5] and then to lower triangular systems in [6].

These literatures address ADRC for deterministic systems, and very little is known for the stochastic counterpart. In contrast to deterministic cases, the main technical obstacle in stochastic systems is that the Itô differentiation involves not only the gradient but also the Hessian term of the Lyapunov function. As breakthrough in stochastic nonlinear control area, a recursive backstepping control design approach is presented to solve stabilization for strict-feedback stochastic systems driven by white noise based on a risk-sensitive cost criterion in [13]. It is recognized that output-feedback control is more difficult and challenging than full state feedback. In recent years, output-feedback design for stochastic nonlinear systems driven by white noise has been an active area of research [14–19]. By using a quartic Lyapunov function, the paper [20] presents a backstepping design to achieve a first result on global output-feedback stabilization for stochastic nonlinear systems driven by white noise. Several output-feedback control designs are available for stochastic nonlinear systems driven by white noise with unmeasured states, such as tracking control [16] and decentralized control [18].

However, in these works, the system functions are supposed to be known or the system uncertainties are linearly parameterized with respect to known nonlinear functions. To overcome this obstacle, an adaptive neural network backstepping output-feedback control approach is investigated for uncertain stochastic nonlinear systems driven by white noise, where the uncertain nonlinear terms are allowed to be functions of the output [17] or even related with all states variables [14]. However, all these output-feedback controllers are constructed recursively in framework of conventional backstepping design technique, which inevitably leads to the problem of 'explosion of complexity' caused by repeated differentiations of virtual controllers [21], which makes the complexity of controller grow dramatically as the order of system increases. By combining dynamic surface control technique [21], a simplified adaptive fuzzy backstepping output-feedback control approach is developed in [22] to overcome 'explosion of complexity' with the unknown nonlinear functions being approximated by fuzzy logic systems, guaranteeing that all signals of the closed-loop system are semi-globally uniformly ultimately bounded in mean square topology.

On the other hand, very few results are available on output-feedback stabilization for nonlinear systems with both uncertain nonlinear system functions and stochastic non-white disturbance. In this paper, we consider output-feedback stabilization for uncertain lower triangular nonlinear systems with bounded exogenous stochastic disturbance that satisfies an uncertain Itô-type stochastic differential equation. A typical example of such kind of exogenous disturbance is the 'colored noise' whose fundamental noise sources through various feedback mechanisms may be regarded as white so that it can be produced by passing the white noise through a filter, described by an Itô-type stochastic differential equation, see, for instance, [23, 24]. Actually, 'colored noise' exists in many practical systems such as physical model systems [25, 26] and chemical model systems [27, 28]. In addition, we also consider the effect of inverse dynamics that is disturbed by both non-white external stochastic disturbance and white noise. Precisely, the system that we consider is an uncertain lower triangular SISO nonlinear system with stochastic inverse dynamics and stochastic disturbance described by

$$\begin{cases} dx_1(t) = [x_2(t) + h_1(x_1(t))]dt, \\ dx_2(t) = [x_3(t) + h_2(x_1(t), x_2(t))]dt, \\ \qquad \vdots \\ dx_n(t) = [f(t, x(t), \zeta(t), w(t)) + h_n(x(t)) + bu(t)]\,dt, \\ d\zeta(t) = f_1(t, x(t), \zeta(t), w(t))dt + f_2(t, x(t), \zeta(t), w(t))dB_1(t), \\ y(t) = x_1(t), \end{cases} \tag{1.1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^\top \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$ are the state, control (input), and output (measurement) of system, respectively. $\zeta(t) \in \mathbb{R}^m$ denotes the state of stochastic inverse dynamics. The functions $h_i : \mathbb{R}^i \to \mathbb{R}, i = 1, 2, \cdots, n$ are known, whereas those

Copyright © 2016 John Wiley & Sons, Ltd.

$f : [0, \infty) \times \mathbb{R}^{n+m+s} \to \mathbb{R}$, $f_1 : [0, \infty) \times \mathbb{R}^{n+m+s} \to \mathbb{R}^m$, and $f_2 : [0, \infty) \times \mathbb{R}^{n+m+s} \to \mathbb{R}^{m\times p}$ are unknown but measurable. The constant $b \neq 0$ is the control coefficient that is not exactly known yet has a nominal value b_0 that is sufficiently closed to b; $\{B_1(t)\}_{t\geqslant 0}$ is a p-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with Ω being a sample space, $\mathcal{F}$ a σ-field, $\{\mathcal{F}_t\}_{t\geqslant 0}$ a filtration, and P the probability measure. The $w(t) \in \mathbb{R}^s$ is used to describe the external stochastic disturbance that is assumed to satisfy the following uncertain stochastic differential equation:

$$dw(t) = \phi\,(t, w(t))\,dt + \psi\,(t, w(t))\,dB_2(t),\ w(0) = w_0, \tag{1.2}$$

where $\{B_2(t)\}_{t\geqslant 0}$ is a q-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ as well and is mutually independent with $\{B_1(t)\}_{t\geqslant 0}$. The functions $\phi : [0, \infty) \times \mathbb{R}^s \to \mathbb{R}^s$, $\psi : [0, \infty) \times \mathbb{R}^s \to \mathbb{R}^{s\times q}$ are unknown measurable functions.

The difference of the x-subsystem of (1.1) with Itô-type stochastic systems studied in [14, 15] is that in the x-subsystem, $w(t)$ is considered completely as an unknown external stochastic disturbance without any statistic characteristic. The x-subsystem can be regarded as a class of stochastic systems driven by colored noise because colored noise is a typical example of $w(t)$, not as a state as in [14, 15] where the stochastic systems are driven by white noise although the x-subsystem is also equivalent to an uncertain Itô-type stochastic system specified in (2.25) after combination with stochastic inverse dynamics and (1.2) together.

It should be noted that the system (1.1) covers some special systems studied in literature such as the deterministic lower triangular SISO nonlinear systems investigated in [29–32] when $f_2(\cdot) = w(\cdot) \equiv 0$; the uncertain deterministic lower triangular SISO nonlinear systems with deterministic disturbance investigated via ADRC approach in [6] when $f_2(\cdot) = \psi(\cdot) \equiv 0$ and $\phi(\cdot)$ is independent of w; the uncertain nonminimum phase lower triangular SISO nonlinear systems where the inverse dynamics equations are disturbed by white noise when $w(\cdot) \equiv 0$; the uncertain lower triangular SISO nonlinear systems with stochastic disturbance when $\zeta(\cdot) \equiv 0$, and hence, system considered in [33] is a special case of (1.1) with $h_i(\cdot) \equiv 0$ and $b = b_0 = 1$. Actually, it is indicated in [34] that any uniform observable SISO nonlinear system can be transformed into the lower triangular form.

Define the stochastic total disturbance as

$$x_{n+1}(t) \triangleq f(t, x(t), \zeta(t), w(t)) + (b - b_0)u(t), \tag{1.3}$$

which contains unknown system dynamics, unknown stochastic inverse dynamics, external stochastic disturbance, and uncertainty caused by the deviation of control parameter b from its nominal value b_0.

The main contributions of this paper are the total disturbance is estimated by ESO and an ESO-based output-feedback control is designed to stabilize the x-subsystem of (1.1), avoiding 'explosion of complexity' inherent in existing output-feedback control methods. It is noted that most available output-feedback controls are guaranteeing global asymptotic stability in probability with assumption of noise vector field being vanishing at the origin based on stochastic LaSalle theorem [20] or the noise-to-state (or input-to-state) stability in probability [15, 16] otherwise. In this paper, however, we address mean square asymptotic stability with nonvanishing non-white noise vector field by designing a time-varying gain ESO-based output-feedback control.

We proceed as follows. In the next section, Section 2, we design a constant high-gain ESO and an ESO-based feedback control for the x-subsystem of (1.1). The mean square practical stability for the closed-loop of x-subsystem of (1.1) is proved. In Section 3, we propose a time-varying gain ESO and an ESO-based feedback control for x-subsystem of (1.1). The mean square asymptotic stability is developed. Finally, in Section 4, we present some numerical simulations for illustration of the convergence and the peaking value reduction.

The following notations are used throughout the paper. The $\mathbb{R}^n$ represents the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ stands for the space of real $n \times m$-matrices. The $C(\mathbb{R}^n; \mathbb{R})$ and $C^1(\mathbb{R}^n; \mathbb{R})$ denote, respectively, spaces of all continuous and continuous differentiable functions defined on $\mathbb{R}^n$. For a given vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm and $x^\top$ denotes its transpose. For a square matrix X, we use $\mathrm{Tr}(X)$ to denote its trace. $(a^{(ij)})_{m\times n}$ denotes an $m \times n$ matrix with entries $a^{(ij)}$. In addition, $f_1 = (f_1^{(i1)})_{m\times 1}$, $f_2 = (f_2^{(ij)})_{m\times p}$, $\phi = (\phi^{(i1)})_{s\times 1}$, $\psi = (\psi^{(ij)})_{s\times q}$, $x(t) = (x_1(t), \cdots, x_n(t))^\top$, $\hat{x}(t) = (\hat{x}_1(t), \cdots, \hat{x}_n(t))^\top$, $\theta(t) = (\theta_1(t), \cdots, \theta_n(t))^\top$.

Copyright © 2016 John Wiley & Sons, Ltd.

2. ACTIVE DISTURBANCE REJECTION CONTROL WITH CONSTANT GAIN EXTENDED STATE OBSERVER

Although linear ESO takes its advantage of simple turning parameter, it also brings the peaking value problem, slow convergence, and many other problems contrast to fast tracking and small peaking value indicated numerically in [35] by nonlinear ESO. By taking these points into account, we introduce the nonlinear ESO proposed in [4, 6] with constant high-gain tuning parameter for system (1.1) as follows:

$$\begin{cases} d\hat{x}_1(t) = \left[\hat{x}_2(t) + \varepsilon^{n-1} g_1\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right) + h_1(\hat{x}_1(t))\right]dt, \\ d\hat{x}_2(t) = \left[\hat{x}_3(t) + \varepsilon^{n-2} g_2\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right) + h_2(\hat{x}_1(t), \hat{x}_2(t))\right]dt, \\ \quad\vdots \\ d\hat{x}_n(t) = \left[\hat{x}_{n+1}(t) + g_n\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right) + h_n(\hat{x}(t)) + b_0 u(t)\right]dt, \\ d\hat{x}_{n+1}(t) = \frac{1}{\varepsilon} g_{n+1}\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right)dt, \end{cases} \tag{2.1}$$

where $g_i \in C(\mathbb{R};\mathbb{R})$, $i = 1, 2, \cdots, n+1$ are designed functions to be specified later and $\varepsilon > 0$ is the tuning parameter. The main idea of ESO is to choose some appropriate $g_i(\cdot)$'s so that when ε is small enough, the $\hat{x}_i(t)$ approaches $x_i(t)$ for all $i = 1, 2, \cdots, n+1$ and sufficiently large t, where $x_{n+1}(t)$ is the stochastic total disturbance defined by (1.3). Here and throughout the paper, we always drop ε for the solution of (2.1) by abuse of notation without confusion.

The ESO (2.1)-based output-feedback control is designed as

$$u(t) = \frac{1}{b_0}\left[\tau v(\tau^{n-1}\hat{x}_1(t), \tau^{n-2}\hat{x}_2(t), \cdots, \hat{x}_n(t)) - \hat{x}_{n+1}(t)\right], \tag{2.2}$$

where $\tau \geqslant 1$ is a constant, $\hat{x}_{n+1}(t)$ is used to compensate (cancel) the total disturbance $x_{n+1}(t)$, and $v : \mathbb{R}^n \to \mathbb{R}$ is to be specified later.

The Assumption (A1) is a prior assumption about the functions $h_i(\cdot)$, $f(\cdot)$, $f_1(\cdot)$, $f_2(\cdot)$, $\phi(\cdot)$, and $\psi(\cdot)$.

Assumption (A1)
$f(\cdot)$ is twice continuously differentiable with respect to its arguments, and there exist (known) constants $C_i > 0$ ($i = 0, \cdots, 4$) and a nonnegative function $\varsigma \in C(\mathbb{R}^s;\mathbb{R})$ such that for all $t \geqslant 0$, $x = (x_1, \cdots, x_n)^\top \in \mathbb{R}^n$, $\zeta = (\zeta_1, \cdots, \zeta_m)^\top \in \mathbb{R}^m$, $w = (w_1, \cdots, w_s)^\top \in \mathbb{R}^s$,

$$\begin{aligned} &|h_i(x_1, \cdots, x_i) - h_i(\hat{x}_1, \cdots, \hat{x}_i)| \leqslant C_0\|((x - \hat{x}_1), \cdots, (x_i - \hat{x}_i))\|, \\ &h_i(0, \cdots, 0) = 0, (i = 1, 2, \cdots, n); \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\left|\frac{\partial f(t,x,\zeta,w)}{\partial t}\right| + \sum_{i=1}^m \left|f_1^{(i1)}(t,x,\zeta,w)\right| \\ &\leqslant C_1 + C_2\|x\| + \varsigma(w); \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\sum_{i=1}^n \left|\frac{\partial f(t,x,\zeta,w)}{\partial x_i}\right| + \sum_{i=1}^m \left|\frac{\partial f(t,x,\zeta,w)}{\partial \zeta_i}\right| + \sum_{i,j=1}^m \left|\frac{\partial f(t,x,\zeta,w)}{\partial \zeta_i \partial \zeta_j}\right| \\ &+ \sum_{i=1}^s \left|\frac{\partial f(t,x,\zeta,w)}{\partial w_i}\right| + \sum_{i,j=1}^s \left|\frac{\partial^2 f(t,x,\zeta,w)}{\partial w_i \partial w_j}\right| \\ &+ \sum_{j=1}^p \sum_{i=1}^m \left|f_2^{(ij)}(t,x,\zeta,w)\right| \leqslant C_3 + \varsigma(w); \end{aligned} \tag{2.5}$$

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

$$\sum_{i=1}^{s} |\phi^{(i1)}(t,w)| + \sum_{j=1}^{q} \sum_{i=1}^{s} |\psi^{(ij)}(t,w)| \leqslant C_4 + \varsigma(w). \tag{2.6}$$

Remark 2.1

Because the stochastic total disturbance is regarded as an extended state variable of system (1.1) to be estimated by ESO, its 'variation' certainly needs to limited. The conditions (2.4), (2.5), and (2.6) in Assumption (A1) are essentially about the Itô differential (or 'variation') of stochastic total disturbance, where the 'variation' satisfies linear growth of x and nonlinear growth of w because the estimation is considered in mean square sense.

The following Assumption (A2) is a prior assumption about $v(\cdot)$ chosen in (2.2).

Assumption (A2)

$v(y)$ is continuously differentiable and Lipschitz continuous with Lipschitz constant L_0, $v(0) = 0$. There exist constants $\lambda_{1i} (i = 1, 2, 3, 4)$ and continuously differentiable function $V_1 : \mathbb{R}^n \to \mathbb{R}$ that is positive definite and radially unbounded such that

$$\begin{cases} \lambda_{11}\|y\|^2 \leqslant V_1(y) \leqslant \lambda_{12}\|y\|^2, \lambda_{13}\|y\|^2 \leqslant W_1(y) \leqslant \lambda_{14}\|y\|^2, \\ \sum\limits_{i=1}^{n-1} y_{i+1} \frac{\partial V_1(y)}{\partial y_i} + v(y) \frac{\partial V_1(y)}{\partial y_n} \leqslant -W_1(y), \\ \left| \frac{\partial V_1(y)}{\partial y_i} \right| \leqslant \alpha \|y\|, i = 1, 2, \cdots, n, \\ \forall \ y = (y_1, y_2, \cdots, y_n)^\top \in \mathbb{R}^n, \end{cases} \tag{2.7}$$

for some nonnegative continuous function $W_1 : \mathbb{R}^n \to \mathbb{R}$ and constant $\alpha > 0$.

In Assumption (A2), the continuous differentiability and Lipschitz continuity of $v(y)$ imply that

$$\left| \frac{\partial v(y)}{\partial y_i} \right| \leqslant L_0, \forall \ y \in \mathbb{R}^n, i = 1, 2, \cdots, n. \tag{2.8}$$

Remark 2.2

Essentially, the Assumption (A2) is to ensure that $v : \mathbb{R}^n \to \mathbb{R}$ is chosen so that the following system is globally asymptotically stable:

$$\dot{y}(t) = (y_2(t), \cdots, y_n(t), v(y_1(t), \cdots, y_n(t)))^\top. \tag{2.9}$$

The following Assumption (A3) is on the designed functions $g_i(\cdot)'s$ in ESO (2.1) and the unknown control parameter b.

Assumption (A3)

$|g_i(r)| \leqslant a_i |r|$ for some positive constants a_i for all $i = 1, 2, \cdots, n+1$. There exist constants $\lambda_{2i} (i = 1, 2, 3, 4)$ and twice continuously differentiable function $V_2 : \mathbb{R}^{n+1} \to \mathbb{R}$ that is positive definite and radially unbounded such that

$$\begin{cases} \lambda_{21}\|y\|^2 \leqslant V_2(y) \leqslant \lambda_{22}\|y\|^2, \lambda_{23}\|y\|^2 \leqslant W_2(y) \leqslant \lambda_{24}\|y\|^2, \\ \sum\limits_{i=1}^{n} \frac{\partial V_2(y)}{\partial y_i}(y_{i+1} - g_i(y_1)) - \frac{\partial V_2(y)}{\partial y_{n+1}} g_{n+1}(y_1) \leqslant -W_2(y), \\ \left| \frac{\partial V_2(y)}{\partial y_i} \right| \leqslant \beta \|y\|, \left| \frac{\partial^2 V_2(y)}{\partial y_{n+1}^2} \right| \leqslant \gamma, i = 1, 2, \cdots, n+1, \\ \forall \ y = (y_1, y_2, \cdots, y_{n+1})^\top \in \mathbb{R}^{n+1}, \end{cases} \tag{2.10}$$

for some nonnegative continuous function $W_2 : \mathbb{R}^{n+1} \to \mathbb{R}$ and constants $\beta, \gamma > 0$. Moreover, the parameter b satisfies $|b - b_0| < \frac{\lambda_{23}|b_0|}{\beta a_{n+1}}$.

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

Remark 2.3
Essentially, the Lyapunov functions $V_2(y)$ and $W_2(y)$ are used mainly to make the following $g_i(\cdot)$ involving system

$$\dot{y}(t) = (y_2(t) - g_1(y_1(t)), \cdots, y_{n+1}(t) - g_n(y_1(t)), -g_{n+1}(y_1(t)))^\top, \tag{2.11}$$

be globally asymptotically stable.

Theorem 2.1
Let $\tau > \max\{1, \frac{n\alpha C_0}{\lambda_{13}}\}$. Suppose that $\sup_{t\geqslant 0}\|w(t)\| \leqslant B$ almost surely for some constant $B > 0$. Then under Assumptions (A1)–(A3), the closed-loop of x-subsystem of (1.1), (2.1), and (2.2) has the following mean square practical convergence: there are a constant $\varepsilon^* > 0$ (specified by (2.31) later) and an ε-dependent constant $t_\varepsilon^* > 0$ with $\varepsilon \in (0, \varepsilon^*)$ such that for any initial values $x(0) \in \mathbb{R}^n$, $(\hat{x}(0), \hat{x}_{n+1}(0)) \in \mathbb{R}^{n+1}$, $\zeta(0) \in \mathbb{R}^m$,

$$\mathbb{E}[x_i(t) - \hat{x}_i(t)]^2 \leqslant \Gamma\varepsilon^{2n+3-2i}, \ \forall t \geqslant t_\varepsilon^*, \ i = 1, 2, \cdots, n+1,$$

and

$$\mathbb{E}\sum_{i=1}^{n} x_i^2(t) \leqslant \Gamma\varepsilon, \ \forall t \geqslant t_\varepsilon^*,$$

where $\Gamma > 0$ is an ε-independent constant.

Proof
Set

$$\begin{cases} \tilde{x}_i(t) = x_i(t) - \hat{x}_i(t), \ \eta_i(t) = \frac{\tilde{x}_i(\varepsilon t)}{\varepsilon^{n+1-i}}, \ i = 1, 2, \cdots, n+1, \\ e_i(t) = \tau^{n-i} x_i(\varepsilon t), \\ \Delta_i(t) = h_i(x_1(t), \cdots, x_i(t)) - h_i(\hat{x}_1(t), \cdots, \hat{x}_i(t)), \\ D_n(t) = \tau v(\tau^{n-1}\hat{x}_1(t), \cdots, \hat{x}_n(t)) - \tau v(\tau^{n-1}x_1(t), \cdots, x_n(t)), \\ \eta(t) = (\eta_1(t), \cdots, \eta_{n+1}(t))^\top, \\ e(t) = (e_1(t), \cdots, e_n(t))^\top. \end{cases} \tag{2.12}$$

Let $\varepsilon > 0$ be chosen so that $\varepsilon < \varepsilon_0 \triangleq \min\{\frac{1}{\tau}, 1\}$. Then by Assumption (A1), we can obtain

$$\begin{aligned} |\Delta_i(\varepsilon t)|^2 &\leqslant C_0^2\left[(x_1(\varepsilon t) - \hat{x}_1(\varepsilon t))^2 + \cdots + (x_i(\varepsilon t) - \hat{x}_i(\varepsilon t))^2\right] \\ &= C_0^2\left[\varepsilon^{2n}|\eta_1(t)|^2 + \cdots + \varepsilon^{2(n+1-i)}|\eta_i(t)|^2\right] \\ &\leqslant C_0^2\varepsilon^{2(n+1-i)}\|\eta(t)\|^2, \ i = 1, 2, \cdots, n, \end{aligned} \tag{2.13}$$

and hence,

$$|h_i(\hat{x}_1(\varepsilon t), \cdots, \hat{x}_i(\varepsilon t))| \leqslant C_0\varepsilon^{n+1-i}\|\eta(t)\| + C_0\|(x_1(\varepsilon t), \cdots, x_i(\varepsilon t))\|, \ i = 1, 2, \cdots, n. \tag{2.14}$$

In addition, it follows from Assumption (A2) that

$$\begin{aligned} |D_n(\varepsilon t)|^2 &\leqslant L_0^2\left[\frac{1}{\varepsilon^{2n}}(x_1(\varepsilon t) - \hat{x}_1(\varepsilon t))^2 + \cdots + \frac{1}{\varepsilon^2}(x_n(\varepsilon t) - \hat{x}_n(\varepsilon t))^2\right] \\ &= L_0^2\left[|\eta_1(t)|^2 + \cdots + |\eta_n(t)|^2\right] \leqslant L_0^2\|\eta(t)\|^2, \end{aligned} \tag{2.15}$$

and so

$$\begin{aligned} |\tau v(\tau^{n-1}\hat{x}_1(\varepsilon t), \cdots, \hat{x}_n(\varepsilon t))| &\leqslant L_0\|\eta(t)\| + |\tau v(\tau^{n-1}x_1(\varepsilon t), \cdots, x_n(\varepsilon t))| \\ &\leqslant L_0\|\eta(t)\| + \tau L_0\|e(t)\|. \end{aligned} \tag{2.16}$$

Copyright © 2016 John Wiley & Sons, Ltd.

In terms of the Itô's formula, it is obtained that

$$
\begin{aligned}
& df(t,x(t),\zeta(t),w(t))\big|_{\text{along (1.1)}} \\
&= \left\{ \frac{\partial}{\partial t} f(t,x(t),\zeta(t),w(t)) + \sum_{i=1}^{n-1} [x_{i+1}(t) + h_i(x_1(t),\cdots,x_i(t))] \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial x_i} \right. \\
&\quad + [f(t,x(t),\zeta(t),w(t)) + bu(t) + h_n(x(t))] \cdot \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial x_n} \\
&\quad + \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial \zeta} f_1(t,x(t),\zeta(t),w(t)) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left\{ f_2^{\top}(t,x(t),\zeta(t),w(t)) \frac{\partial^2 f(t,x(t),\zeta(t),w(t))}{\partial \zeta^2} f_2(t,x(t),\zeta(t),w(t)) \right\} \\
&\quad + \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial w} \phi(t,w(t)) \\
&\quad \left. + \frac{1}{2}\mathrm{Tr}\left\{ \psi^{\top}(t,w(t)) \frac{\partial^2 f(t,x(t),\zeta(t),w(t))}{\partial w^2} \psi(t,w(t)) \right\} \right\} dt \\
&\quad + \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial \zeta} (f_2(t,x(t),\zeta(t),w(t)) dB_1(t)) \\
&\quad + \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial w} (\psi(t,w(t)) dB_2(t)) \\
&\triangleq \Lambda_1(t)dt + \Lambda_2(t)dB_1(t) + \Lambda_3(t)dB_2(t),
\end{aligned}
\tag{2.17}
$$

where we set

$$
\begin{aligned}
\Lambda_2(t) &= (\Lambda_2^j(t))_{1\times p},\ \Lambda_2^j(t) = \sum_{i=1}^{m} \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial \zeta_i} f_2^{(ij)}(t,x(t),\zeta(t),w(t)); \\
\Lambda_3(t) &= (\Lambda_3^j(t))_{1\times q},\ \Lambda_3^j(t) = \sum_{i=1}^{s} \frac{\partial f(t,x(t),\zeta(t),w(t))}{\partial w_i} \psi^{(ij)}(t,w(t)).
\end{aligned}
\tag{2.18}
$$

A direct computation shows that

$$
\begin{aligned}
|f(\varepsilon t,x(\varepsilon t),\zeta(\varepsilon t),w(t)) + bu(\varepsilon t)| &= |x_{n+1}(\varepsilon t) - \hat{x}_{n+1}(\varepsilon t) + \tau v(\tau^{n-1}\hat{x}_1(\varepsilon t),\cdots,\hat{x}_n(\varepsilon t))| \\
&\leqslant |\eta_{n+1}(t)| + L_0\|\eta(t)\| + \tau L_0\|e(t)\|.
\end{aligned}
\tag{2.19}
$$

Thus, it follows from Assumption (A1) that there exist constants $C_5, C_6, C_7, C_8 > 0$ such that

$$
|\Lambda_1(\varepsilon t)| \leqslant C_5 + C_6\|\eta(t)\| + C_7\|e(t)\|,\ \|\Lambda_2(\varepsilon t)\|^2 + \|\Lambda_3(\varepsilon t)\|^2 \leqslant C_8. \tag{2.20}
$$

Finding the derivative of $u(t)$ along the solution of (2.1) to obtain

$$
\begin{aligned}
& \frac{du(t)}{dt}\bigg|_{\text{along (2.1)}} \\
&= \frac{1}{b_0} \left\{ \sum_{i=1}^{n-1} \tau^{n+1-i} \left(\hat{x}_{i+1}(t) + \varepsilon^{n-i} g_i\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n} \right) + h_i(\hat{x}_1(t),\cdots,\hat{x}_i(t)) \right) \right. \\
&\qquad \times \frac{\partial v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))}{\partial(\tau^{n-i}\hat{x}_i)} + \tau\left(\hat{x}_{n+1}(t) + g_n\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n} \right) + b_0u(t) + h_n(\hat{x}(t)) \right) \\
&\qquad \left. \cdot \frac{\partial v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))}{\partial \hat{x}_n} - \frac{1}{\varepsilon} g_{n+1}\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^n} \right) \right\} \triangleq \Lambda_4(t).
\end{aligned}
\tag{2.21}
$$

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

So, it follows from Assumption (A3), (2.8), (2.14), and (2.16) that

$$
\begin{aligned}
\Lambda_4(\varepsilon t)| \leqslant\ & \frac{L_0}{|b_0|}\left\{\sum_{i=1}^{n-1}\tau^{n+1-i}\varepsilon^{n-i}|\eta_{i+1}(t)| + \tau^2\sum_{i=2}^{n}|e_i(t)| + \sum_{i=1}^{n-1}(\ a_i\tau^{n+1-i}\varepsilon^{n-i}|\eta_1(t)|\right.\\
& + \tau^{n+1-i}\varepsilon^{n+1-i}C_0\|\eta(t)\| + \tau^{n+1-i}C_0\|(x_1(\varepsilon t),\cdots,x_i(\varepsilon t))\|\ )\\
& \left. +\tau\left(a_n|\eta_1(t)|+|\tau v(\tau^{n-1}\hat{x}_1(\varepsilon t),\cdots,\hat{x}_n(\varepsilon t))|+\varepsilon C_0\|\eta(t)\| + C_0\|x(\varepsilon t)\|\right)\right\} + \frac{a_{n+1}}{|b_0|\varepsilon}|\eta_1(t)|\\
\leqslant\ & \frac{L_0}{|b_0|}\left\{\sum_{i=1}^{n-1}\tau^{n+1-i}\varepsilon^{n-i}|\eta_{i+1}(t)| + \tau^2\sum_{i=2}^{n}|e_i(t)| + \sum_{i=1}^{n-1}(\ a_i\tau^{n+1-i}\varepsilon^{n-i}|\eta_1(t)|\right.\\
& + \tau^{n+1-i}\varepsilon^{n+1-i}C_0\|\eta(t)\| + \tau C_0\|e(t)\|\ )\\
& \left. +\tau\ (\ a_n|\eta_1(t)| + L_0\|\eta(t)\| + \tau L_0\|e(t)\| + \varepsilon C_0\|\eta(t)\| + C_0\|e(t)\|)\right\} + \frac{a_{n+1}}{|b_0|\varepsilon}|\eta_1(t)|\\
\leqslant\ & \frac{L_0}{|b_0|}\left\{(n-1)\tau^n\varepsilon_0 + (n-1)\tau^n\varepsilon_0\max_{1\leqslant i\leqslant n-1}\{a_i\} + (n-1)\tau^n\varepsilon_0^2C_0 + \tau a_n + \tau L_0 + \tau\varepsilon_0 C_0\right\}\|\eta(t)\|\\
& + \frac{L_0}{|b_0|}\left\{(n-1)\tau^2 + (n-1)\tau C_0 + \tau^2 L_0 + \tau C_0\right\}\|e(t)\| + \frac{a_{n+1}}{|b_0|\varepsilon}\|\eta(t)\|\\
=\ & C_9\|\eta(t)\| + C_{10}\|e(t)\| + \frac{a_{n+1}}{|b_0|\varepsilon}\|\eta(t)\|,
\end{aligned}
\tag{2.22}
$$

where we set

$$
\begin{aligned}
C_9 &= \frac{L_0}{|b_0|}\left\{(n-1)\tau^n\varepsilon_0 + (n-1)\tau^n\varepsilon_0\max_{1\leqslant i\leqslant n-1}\{a_i\} + (n-1)\tau^n\varepsilon_0^2C_0 + \tau a_n + \tau L_0 + \tau\varepsilon_0 C_0\right\},\\
C_{10} &= \frac{L_0}{|b_0|}\left\{(n-1)\tau^2 + (n-1)\tau C_0 + \tau^2 L_0 + \tau C_0\right\}.
\end{aligned}
\tag{2.23}
$$

Let

$$
\Theta_1(t) = \Lambda_1(t) + (b-b_0)\Lambda_4(t),\ \Theta_2(t) = \Lambda_2(t), \Theta_3(t) = \Lambda_3(t). \tag{2.24}
$$

Then, the x-subsystem of (1.1) can be written as

$$
\begin{cases}
dx_1(t) = [x_2(t) + h_1(x_1(t))]\,dt,\\
dx_2(t) = [x_3(t) + h_2(x_1(t), x_2(t))]\,dt,\\
\qquad\vdots\\
dx_n(t) = [x_{n+1}(t) + b_0 u(t) + h_n(x(t))]\,dt\\
dx_{n+1}(t) = \Theta_1(t)dt + \Theta_2(t)dB_1(t) + \Theta_3(t)dB_2(t).
\end{cases}
\tag{2.25}
$$

Notice that for any $\varepsilon > 0$, $\hat{B}_1(t) = \frac{1}{\sqrt{\varepsilon}}B_1(\varepsilon t)$ and $\hat{B}_2(t) = \frac{1}{\sqrt{\varepsilon}}B_2(\varepsilon t)$ are also two mutually independent standard Brownian motions taking values in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. A direct computation shows that the closed-loop of x-subsystem of (1.1), (2.1), and (2.2) is equivalent to

$$
\begin{cases}
de_1(t) = \varepsilon\left[\tau e_2(t) + \tau^{n-1}h_1\left(\frac{e_1(t)}{\tau^{n-1}}\right)\right]dt,\\
de_2(t) = \varepsilon\left[\tau e_3(t) + \tau^{n-2}h_2\left(\frac{e_1(t)}{\tau^{n-1}}, \frac{e_2(t)}{\tau^{n-2}}\right)\right]dt,\\
\qquad\vdots\\
de_n(t) = \varepsilon\left[\tau v(e_1(t),\cdots,e_n(t)) + D_n(\varepsilon t) + \eta_{n+1}(t) + h_n\left(\frac{e_1(t)}{\tau^{n-1}},\cdots,e_n(t)\right)\right]dt,\\
d\eta_1(t) = \left[\eta_2(t) - g_1(\eta_1(t)) + \frac{1}{\varepsilon^{n-1}}\Delta_1(\varepsilon t)\right]dt,\\
d\eta_2(t) = \left[\eta_3(t) - g_2(\eta_1(t)) + \frac{1}{\varepsilon^{n-2}}\Delta_2(\varepsilon t)\right]dt,\\
\qquad\vdots\\
d\eta_n(t) = [\eta_{n+1}(t) - g_n(\eta_1(t)) + \Delta_n(\varepsilon t)]dt,\\
d\eta_{n+1}(t) = [\varepsilon\Theta_1(\varepsilon t) - g_{n+1}(\eta_1(t))]dt + \sqrt{\varepsilon}\Theta_2(\varepsilon t)d\hat{B}_1(t) + \sqrt{\varepsilon}\Theta_3(\varepsilon t)d\hat{B}_2(t).
\end{cases}
\tag{2.26}
$$

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

Consider the positive definite function $V : \mathbb{R}^{2n+1} \to \mathbb{R}$ given by

$$V(e, \eta) = V(e_1, \cdots, e_n, \eta_1, \cdots, \eta_{n+1}) = V_1(e_1, \cdots, e_n) + V_2(\eta_1, \cdots, \eta_{n+1}). \tag{2.27}$$

Apply Itô's formula to $V(e(t), \eta(t))$ with respect to t along the solutions $(e(t), \eta(t))$ of system (2.26) to obtain

$$\begin{aligned}
dV(e(t), \eta(t)) = & \left[\tau\varepsilon \sum_{i=1}^{n-1} \frac{\partial V_1(e(t))}{\partial e_i} e_{i+1}(t) + \tau\varepsilon \frac{\partial V_1(e(t))}{\partial e_n} v(e_1(t), \cdots, e_n(t)) \right. \\
& \left. + \sum_{i=1}^{n} \varepsilon\tau^{n-i} \frac{\partial V_1(e(t))}{\partial e_i} h_i \left(\frac{e_1(t)}{\tau^{n-1}}, \cdots, \frac{e_i(t)}{\tau^{n-i}} \right) + \varepsilon \frac{\partial V_1(e(t))}{\partial e_n} D_n(\varepsilon t) + \varepsilon \frac{\partial V_1(e(t))}{\partial e_n} \eta_{n+1}(t) \right] dt \\
& + \left[\sum_{i=1}^{n} \frac{\partial V_2(\eta(t))}{\partial \eta_i} (\eta_{i+1}(t) - g_i(\eta_1(t))) - \frac{\partial V_2(\eta(t))}{\partial \eta_{n+1}} g_{n+1}(\eta_1(t)) \right] dt \\
& + \sum_{i=1}^{n} \frac{1}{\varepsilon^{n-i}} \frac{\partial V_2(\eta(t))}{\partial \eta_i} \Delta_i(\varepsilon t) dt + \varepsilon \frac{\partial V_2(\eta(t))}{\partial \eta_{n+1}} \Theta_1(\varepsilon t) dt + \frac{1}{2} \varepsilon \frac{\partial^2 V_2(\eta(t))}{\partial \eta_{n+1}^2} \|\Theta_2(\varepsilon t)\|^2 dt \\
& + \frac{1}{2} \varepsilon \frac{\partial^2 V_2(\eta(t))}{\partial \eta_{n+1}^2} \|\Theta_3(\varepsilon t)\|^2 dt + \sqrt{\varepsilon} \frac{\partial V_2(\eta(t))}{\partial \eta_{n+1}} \Theta_2(\varepsilon t) d\hat{B}_1(t) + \sqrt{\varepsilon} \frac{\partial V_2(\eta(t))}{\partial \eta_{n+1}} \Theta_3(\varepsilon t) d\hat{B}_2(t).
\end{aligned} \tag{2.28}$$

By (2.20), there exists constant $C_8 > 0$ such that $\mathbb{E}\|\Theta_2(\varepsilon t)\|^2 \leqslant C_8$, $\mathbb{E}\|\Theta_3(\varepsilon t)\|^2 \leqslant C_8$ for all $t \geqslant 0$. Because $\tau > \max\{1, \frac{n\alpha C_0}{\lambda_{13}}\}$, $|b - b_0| < \frac{\lambda_{23}|b_0|}{\beta a_{n+1}}$, we then have

$$\xi_0 \triangleq \lambda_{13}\tau - n\alpha C_0 > 0, \quad \xi_1 \triangleq \lambda_{23} - \frac{\beta a_{n+1}|b - b_0|}{|b_0|} > 0. \tag{2.29}$$

We also notice that there exist $\mu > 0$ and $\varepsilon_1 > 0$ such that

$$\begin{aligned}
& \xi_0 - 2\mu > 0, \\
& \xi_2 \triangleq \xi_1 - \varepsilon_1 \left(n\beta C_0 + \frac{1}{2} + \beta C_6 + \frac{\beta^2 C_7^2}{2\mu} + \beta C_9 |b - b_0| + \frac{\beta^2 C_{10}^2 |b - b_0|^2}{2\mu} \right) > 0, \\
& \xi_3 \triangleq \xi_2 - \left(\frac{\varepsilon_1 \alpha^2 L_0^2}{2\mu} + \frac{\varepsilon_1 \alpha^2}{2\mu} \right) > 0.
\end{aligned} \tag{2.30}$$

Now, we suppose that

$$0 < \varepsilon < \varepsilon^* \triangleq \min \left\{ 1, \varepsilon_0, \varepsilon_1, \frac{\xi_3}{\xi_0 - 2\mu} \right\}. \tag{2.31}$$

It follows from (2.13), (2.15), (2.20), (2.22), (2.28), Assumptions (A2) and (A3) that

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

$$
\begin{aligned}
\frac{d\mathbb{E}V(e(t),\eta(t))}{dt} &\leqslant -\tau\varepsilon\mathbb{E}W_1(e(t)) + n\alpha\varepsilon C_0\mathbb{E}\|e(t)\|^2 + \alpha\varepsilon L_0\mathbb{E}(\|e(t)\|\cdot\|\eta(t)\|) \\
&+ \alpha\varepsilon\mathbb{E}(\|e(t)\|\cdot\|\eta(t)\|) - \mathbb{E}W_2(\eta(t)) + n\beta\varepsilon C_0\mathbb{E}\|\eta(t)\|^2 \\
&+ \beta\varepsilon\mathbb{E}\left\{\|\eta(t)\|\cdot\left(C_5 + C_6\|\eta(t)\| + C_7\|e(t)\|\right.\right. \\
&\left.\left. +|b-b_0|(C_9\|\eta(t)\| + C_{10}\|e(t)\| + \frac{a_{n+1}}{|b_0|\varepsilon}\|\eta(t)\|)\right)\right\} + \gamma\varepsilon C_8 \\
&\leqslant -\xi_0\varepsilon\mathbb{E}\|e(t)\|^2 + \frac{\mu\varepsilon}{2}\mathbb{E}\|e(t)\|^2 + \frac{\varepsilon\alpha^2L_0^2}{2\mu}\mathbb{E}\|\eta(t)\|^2 \\
&+ \frac{\mu\varepsilon}{2}\mathbb{E}\|e(t)\|^2 + \frac{\varepsilon\alpha^2}{2\mu}\mathbb{E}\|\eta(t)\|^2 - \lambda_{23}\mathbb{E}\|\eta(t)\|^2 + n\beta\varepsilon C_0\mathbb{E}\|\eta(t)\|^2 \\
&+ \frac{\varepsilon}{2}\mathbb{E}\|\eta(t)\|^2 + \frac{\varepsilon\beta^2C_5^2}{2} + \varepsilon\beta C_6\mathbb{E}\|\eta(t)\|^2 + \frac{\mu\varepsilon}{2}\mathbb{E}\|e(t)\|^2 \\
&+ \frac{\varepsilon\beta^2C_7^2}{2\mu}\mathbb{E}\|\eta(t)\|^2 + \varepsilon\beta C_9|b-b_0|\mathbb{E}\|\eta(t)\|^2 + \frac{\mu\varepsilon}{2}\mathbb{E}\|e(t)\|^2 + \frac{\varepsilon\beta^2C_{10}^2|b-b_0|^2}{2\mu}\mathbb{E}\|\eta(t)\|^2 \\
&+ \frac{\beta a_{n+1}|b-b_0|}{|b_0|}\mathbb{E}\|\eta(t)\|^2 + \gamma\varepsilon C_8 \\
&\leqslant -(\xi_0-2\mu)\varepsilon\mathbb{E}\|e(t)\|^2 - \xi_3\mathbb{E}\|\eta(t)\|^2 + \frac{\varepsilon\beta^2C_5^2}{2} + \gamma\varepsilon C_8 \\
&\leqslant -\frac{(\xi_0-2\mu)\varepsilon}{\max\{\lambda_{12},\lambda_{22}\}}\mathbb{E}V(e(t),\eta(t)) + \frac{\varepsilon\beta^2C_5^2}{2} + \gamma\varepsilon C_8.
\end{aligned} \tag{2.32}
$$

Hence, for any $\varepsilon \in (0,\varepsilon^*)$, there exists $t_\varepsilon \triangleq \frac{1}{\varepsilon^\varrho}$, $\varrho > 1$ such that for all $t \geqslant t_\varepsilon$,

$$
\begin{aligned}
&\mathbb{E}V(e(t),\eta(t)) \\
&\leqslant e^{-\frac{(\xi_0-2\mu)\varepsilon t}{\max\{\lambda_{12},\lambda_{22}\}}}\mathbb{E}V(e(0),\eta(0)) + \left(\frac{\varepsilon\beta^2C_5^2}{2} + \gamma\varepsilon C_8\right)\int_0^t e^{-\frac{(\xi_0-2\mu)\varepsilon}{\max\{\lambda_{12},\lambda_{22}\}}(t-s)}ds \\
&\leqslant e^{-\frac{(\xi_0-2\mu)}{\max\{\lambda_{12},\lambda_{22}\}\varepsilon^{\varrho-1}}}\mathbb{E}V(e(0),\eta(0)) + \frac{(\beta^2C_5^2 + 2\gamma C_8)\max\{\lambda_{12},\lambda_{22}\}}{(\xi_0-2\mu)} \\
&\leqslant \Gamma_1,
\end{aligned} \tag{2.33}
$$

for some ε-independent constant $\Gamma_1 > 0$. By a similar technique used in (2.32), it follows from (2.33) that for all $t \geqslant t_\varepsilon$,

$$
\begin{aligned}
\frac{d\mathbb{E}V_2(\eta(t))}{dt} &\leqslant -\xi_2\mathbb{E}\|\eta(t)\|^2 + \mu\varepsilon\mathbb{E}\|e(t)\|^2 + \frac{\varepsilon\beta^2C_5^2}{2} + \gamma\varepsilon C_8 \\
&\leqslant -\frac{\xi_2}{\lambda_{22}}\mathbb{E}V_2(\eta(t)) + \frac{\mu\varepsilon\Gamma_1}{\lambda_{11}} + \frac{\varepsilon\beta^2C_5^2}{2} + \gamma\varepsilon C_8,
\end{aligned} \tag{2.34}
$$

and hence,

$$
\mathbb{E}V_2\left(\eta\left(\frac{t}{\varepsilon}\right)\right) \leqslant e^{-\frac{\xi_2}{\lambda_{22}}(\frac{t}{\varepsilon}-t_\varepsilon)}\mathbb{E}V_2(\eta(t_\varepsilon)) + \left(\frac{\mu\varepsilon\Gamma_1}{\lambda_{11}} + \frac{\varepsilon\beta^2C_5^2}{2} + \gamma\varepsilon C_8\right)\int_{t_\varepsilon}^{\frac{t}{\varepsilon}} e^{-\frac{\xi_2}{\lambda_{22}}(\frac{t}{\varepsilon}-s)}ds. \tag{2.35}
$$

Because it follows from (2.33) that the first term of the right-hand side of (2.35) is bounded by $e^{-\frac{\xi_2 t_\varepsilon}{\lambda_{22}}(\frac{1}{\varepsilon^*}-1)}$ multiplied by an ε-independent constant, and the second term is bounded by ε multiplied by an ε-independent constant, there exists an ε-independent constant $\Gamma_2 > 0$ such that for all $t \geqslant t_\varepsilon$,

$$
\mathbb{E}V_2\left(\eta\left(\frac{t}{\varepsilon}\right)\right) \leqslant \Gamma_2\varepsilon, \tag{2.36}
$$

Copyright © 2016 John Wiley & Sons, Ltd.

and so

$$\mathbb{E}\left\|\eta\left(\frac{t}{\varepsilon}\right)\right\|^2 \leqslant \frac{\mathbb{E}V_2\left(\eta\left(\frac{t}{\varepsilon}\right)\right)}{\lambda_{21}} \leqslant \frac{\Gamma_2\varepsilon}{\lambda_{21}}. \tag{2.37}$$

Thus, for any $i = 1, 2, \cdots, n+1$ and all $t \geqslant t_\varepsilon$,

$$\mathbb{E}\tilde{x}_i^2(t) = \varepsilon^{2n+2-2i}\mathbb{E}\left|\eta_i\left(\frac{t}{\varepsilon}\right)\right|^2 \leqslant \varepsilon^{2n+2-2i}\mathbb{E}\left\|\eta\left(\frac{t}{\varepsilon}\right)\right\|^2 \leqslant \frac{\Gamma_2}{\lambda_{21}}\varepsilon^{2n+3-2i}. \tag{2.38}$$

This completes the proof of the first part.

Set $\theta_i(t) = \tau^{n-i}x_i(t), i = 1, 2, \cdots, n$. Then the x-subsystem of (1.1) is equivalent to

$$\begin{cases} d\theta_1(t) = \left[\tau\theta_2(t) + \tau^{n-1}h_1\left(\frac{\theta_1(t)}{\tau^{n-1}}\right)\right]dt, \\ d\theta_2(t) = \left[\tau\theta_3(t) + \tau^{n-2}h_2\left(\frac{\theta_1(t)}{\tau^{n-1}}, \frac{\theta_2(t)}{\tau^{n-2}}\right)\right]dt, \\ \quad\vdots \\ d\theta_n(t) = \left[\tau v(\theta_1(t), \cdots, \theta_n(t)) + D_n(t) + \eta_{n+1}\left(\frac{t}{\varepsilon}\right) + h_n\left(\frac{\theta_1(t)}{\tau^{n-1}}, \cdots, \theta_n(t)\right)\right]dt. \end{cases} \tag{2.39}$$

Hence, for any $\varepsilon \in (0, \varepsilon^*)$ and all $t \geqslant t_\varepsilon$, it follows from Assumption (A2), (2.15), and 2.37 that

$$\begin{aligned} \frac{d\mathbb{E}V_1(\theta(t))}{dt} &\leqslant -\tau\mathbb{E}W_1(\theta(t)) + \mathbb{E}\left(\frac{\partial V_1(\theta(t))}{\partial\theta_n}|D_n(t)|\right) \\ &\quad + \mathbb{E}\sum_{i=1}^{n}\tau^{n-i}\frac{\partial V_1(\theta(t))}{\partial\theta_i}h_i\left(\frac{\theta_1(t)}{\tau^{n-1}}, \cdots, \frac{\theta_i(t)}{\tau^{n-i}}\right) + \mathbb{E}\left(\frac{\partial V_1(\theta(t))}{\partial\theta_n}\eta_{n+1}\left(\frac{t}{\varepsilon}\right)\right) \\ &\leqslant -\lambda_{13}\tau\mathbb{E}\|\theta(t)\|^2 + \alpha L_0\mathbb{E}\left(\|\theta(t)\| \cdot \left\|\eta\left(\frac{t}{\varepsilon}\right)\right\|\right) \\ &\quad + n\alpha C_0\mathbb{E}\|\theta(t)\|^2 + \alpha\mathbb{E}\left(\|\theta(t)\| \cdot \left|\eta_{n+1}\left(\frac{t}{\varepsilon}\right)\right|\right) \\ &\leqslant -(\lambda_{13}\tau - n\alpha C_0)\mathbb{E}\|\theta(t)\|^2 + \mu\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2 L_0^2}{4\mu}\mathbb{E}\left\|\eta\left(\frac{t}{\varepsilon}\right)\right\|^2 \\ &\quad + \mu\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2}{4\mu}\mathbb{E}\left|\eta_{n+1}\left(\frac{t}{\varepsilon}\right)\right|^2 \\ &\leqslant -(\xi_0 - 2\mu)\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2(L_0^2+1)\Gamma_2}{4\mu\lambda_{21}}\varepsilon. \end{aligned} \tag{2.40}$$

We then have, for any $\varepsilon \in (0, \varepsilon^*)$ and all $t \geqslant t_\varepsilon$ that

$$\begin{aligned} \mathbb{E}\|\theta(t)\|^2 \leqslant \frac{1}{\lambda_{11}}e^{-\frac{(\xi_0-2\mu)}{\lambda_{12}}(t-t_\varepsilon)}\mathbb{E}\|V_1(\theta(t_\varepsilon))\| \\ + \frac{\alpha^2(L_0^2+1)\Gamma_2}{4\mu\lambda_{21}\lambda_{11}}\varepsilon\int_{t_\varepsilon}^{t}e^{-\frac{(\xi_0-2\mu)}{\lambda_{12}}(t-s)}ds. \end{aligned} \tag{2.41}$$

Because by (2.33), the first term of the right-hand side of (2.41) tends to zero as t goes to infinity, and the second term is bounded by ε multiplied by an ε-independent constant, it follows that there exist $t_\varepsilon^* > t_\varepsilon$ and $\Gamma \geqslant \frac{\Gamma_2}{\lambda_{21}} > 0$ such that

$$\mathbb{E}\sum_{i=1}^{n}x_i^2(t) \leqslant \mathbb{E}\|\theta(t)\|^2 \leqslant \Gamma\varepsilon, \forall\ t \geqslant t_\varepsilon^*.$$

This completes the proof of the theorem. □

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

The simplest example of constant gain ADRC satisfying conditions of Theorem 2.1 is the linear one, that is, $g_i(\cdot), i = 1, \cdots, n+1$ in ESO (2.1) and $v(\cdot)$ in feedback control (2.2) are linear functions. Let

$$g_i(r) = k_i r, \quad v(y_1, \cdots, y_n) = c_1 y_1 + \cdots + c_n y_n. \tag{2.42}$$

Define the matrices E and F as follows:

$$E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ddots & 1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{pmatrix}_{n\times n}, \quad F = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_n & 0 & 0 & \ddots & 1 \\ -k_{n+1} & 0 & 0 & \cdots & \end{pmatrix}_{(n+1)\times(n+1)}. \tag{2.43}$$

Let $\lambda_{max}(H)$ be the maximal eigenvalue of matrix H that is the unique positive definite matrix solution of the Lyapunov equation $HE + E^\top H = -I_{n\times n}$ for n-dimensional identity matrix $I_{n\times n}$. In addition, let $\lambda_{max}(Q)$ be the maximal eigenvalue of matrix Q that is the unique positive definite matrix solution of the Lyapunov equation $QF + F^\top Q = -I_{(n+1)\times(n+1)}$ for $(n+1)$-dimensional identity matrix $I_{(n+1)\times(n+1)}$.

Corollary 2.1
Let $\tau > \max\{1, 2n\lambda_{max}(H)C_0\}$. Suppose that $\sup_{t\geq 0}\|w(t)\| \leqslant B$ almost surely for some constant $B > 0$, the matrices E and F are Hurwitz, and $|b - b_0| < \frac{|b_0|}{2k_{n+1}\lambda_{max}(Q)}$. Then under Assumption (A1), the closed-loop of x-subsystem of (1.1),(2.1), and (2.2) has the following mean square practical convergence: There are a constant $\varepsilon^* > 0$ and an ε-dependent constant $t_\varepsilon^* > 0$ with $\varepsilon \in (0, \varepsilon^*)$ such that for any initial values $x(0) \in \mathbb{R}^n$, $(\hat{x}(0), \hat{x}_{n+1}(0)) \in \mathbb{R}^{n+1}$, $\zeta(0) \in \mathbb{R}^m$,

$$\mathbb{E}[x_i(t) - \hat{x}_i(t)]^2 \leqslant \Gamma\varepsilon^{2n+3-2i}, \forall\, t \geqslant t_\varepsilon^*, i = 1, 2, \cdots, n+1,$$

and

$$\mathbb{E}\sum_{i=1}^n x_i^2(t) \leqslant \Gamma\varepsilon, \forall\, t \geqslant t_\varepsilon^*,$$

where $\Gamma > 0$ is an ε-independent constant.

Proof
Define the Lyapunov functions $V_1, W_1 : \mathbb{R}^n \to \mathbb{R}$ by $V_1(\eta) = \eta^\top H\eta$, $W_1(\eta) = \eta^\top\eta$ for $\eta \in \mathbb{R}^n$ and the Lyapunov functions $V_2, W_2 : \mathbb{R}^{n+1} \to \mathbb{R}$ by $V_2(\eta) = \eta^\top Q\eta$, $W_2(\eta) = \eta^\top\eta$ for $\eta \in \mathbb{R}^{n+1}$. Then it is easy to verify that all conditions of Assumptions (A2) and (A3) are satisfied. The results then follow directly from Theorem 2.1. □

Remark 2.4
When $\zeta(\cdot) \equiv 0$, $h_i(\cdot) \equiv 0, i = 1, 2, \cdots, n$, $b = b_0 = 1$, system (1.1) is of the form:

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = x_3(t)dt, \\ \quad\vdots \\ dx_n(t) = [f(t, x_1(t), \cdots, x_n(t), w(t)) + u(t)]\,dt, \\ y(t) = x_1(t), \end{cases} \tag{2.44}$$

In this case, we can easily see that the parameter τ in (2.2) can be chosen as $\tau = 1$. We thus conclude the results of [33] by Theorem 2.1 and Corollary 2.1.

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

Remark 2.5
The traditional ADRC approach needs to have a nominal value of unknown control coefficient b, which is specified in Assumption (A3) in this paper. However, an adaptive projected gradient method is developed in [7] to estimate b without a priori estimate. Therefore, it would be interesting and challenging to generalize some adaptive way like the algorithm in [7] to the stochastic counterpart of systems considered in this paper.

To end this section, we indicate the relation of stability of closed-loop system claimed by Theorem 2.1 and Corollary 2.1, with the total disturbance assumptions (2.4–2.6), which is reflected mainly in constant Γ in mean square convergence. Because this relation is complicated to our nonlinear system, we use an example to explain this point. Consider the following first-order system:

$$\begin{cases} \dot{x}(t) = w(t) + u(t), \\ y(t) = x(t), \end{cases}$$

where $w(t)$ is the deterministic external disturbance that satisfies: $\sup_{t\geqslant 0} |\dot{w}(t)| \leqslant M$ with $M > 0$ being the bound of the variation of external disturbance. We design the following linear ESO:

$$\begin{cases} \dot{\hat{x}}(t) = \hat{w}(t) + \frac{1}{\varepsilon}\left(y(t) - \hat{x}(t)\right) + u(t), \\ \dot{\hat{w}}(t) = \frac{1}{\varepsilon^2}\left(y(t) - \hat{x}_1(t)\right), \end{cases}$$

and ESO-based output-feedback control: $u(t) = -\hat{x}(t) - \hat{w}(t)$.

Let $\tilde{x}_1(t) = x(t) - \hat{x}(t)$ and $\tilde{x}_2(t) = w(t) - \hat{w}(t)$. Then a direct computation shows that the closed-loop system is equivalent to

$$\begin{cases} \dot{x}(t) = -x(t) + \tilde{x}_1(t) + \tilde{x}_2(t), \\ \dot{\tilde{x}}_1(t) = -\frac{1}{\varepsilon}\tilde{x}_1(t) + \tilde{x}_2(t), \\ \dot{\tilde{x}}_2(t) = -\frac{1}{\varepsilon^2}\tilde{x}_1(t) + \dot{w}(t). \end{cases} \tag{2.45}$$

The solution of (2.45) is found explicitly as

$$\begin{cases} \tilde{x}_1(t) = \tilde{x}_1(0)\left(e^{-\frac{t}{2\varepsilon}}\cos\frac{\sqrt{3}t}{2\varepsilon} - \frac{\sqrt{3}}{3}e^{-\frac{t}{2\varepsilon}}\sin\frac{\sqrt{3}t}{2\varepsilon}\right) + \tilde{x}_2(0)\left(\frac{2\sqrt{3}}{3}\varepsilon e^{-\frac{t}{2\varepsilon}}\sin\frac{\sqrt{3}t}{2\varepsilon}\right) \\ \qquad + \int_0^t \left(\frac{2\sqrt{3}}{3}\varepsilon e^{-\frac{t-s}{2\varepsilon}}\sin\frac{\sqrt{3}(t-s)}{2\varepsilon}\right)\dot{w}(s)ds, \\ \tilde{x}_2(t) = \tilde{x}_1(0)\left(-\frac{2\sqrt{3}}{3\varepsilon}e^{-\frac{t}{2\varepsilon}}\sin\frac{\sqrt{3}t}{2\varepsilon}\right) + \tilde{x}_2(0)\left(e^{-\frac{t}{2\varepsilon}}\cos\frac{\sqrt{3}t}{2\varepsilon} + \frac{\sqrt{3}}{3}e^{-\frac{t}{2\varepsilon}}\sin\frac{\sqrt{3}t}{2\varepsilon}\right) \\ \qquad + \int_0^t \left(e^{-\frac{t-s}{2\varepsilon}}\cos\frac{\sqrt{3}(t-s)}{2\varepsilon} + \frac{\sqrt{3}}{3}e^{-\frac{t-s}{2\varepsilon}}\sin\frac{\sqrt{3}(t-s)}{2\varepsilon}\right)\dot{w}(s)ds. \end{cases} \tag{2.46}$$

Thus, for any constant $a > 0$,

$$\begin{aligned} |\tilde{x}_1(t)| &\leqslant \left(\left(1 + \frac{\sqrt{3}}{3}\right)e^{-\frac{t}{2\varepsilon}}|\tilde{x}_1(0)| + \frac{2\sqrt{3}\varepsilon}{3}e^{-\frac{t}{2\varepsilon}}|\tilde{x}_2(0)|\right) + \frac{8\sqrt{3}}{3}M\varepsilon^2 \\ &\leqslant \varepsilon^2\Gamma \text{ uniformly in } t \in [a, +\infty), \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} |\tilde{x}_2(t)| &\leqslant \frac{2\sqrt{3}}{3\varepsilon}e^{-\frac{t}{2\varepsilon}}|\tilde{x}_1(0)| + \left(1 + \frac{\sqrt{3}}{3}\right)e^{-\frac{t}{2\varepsilon}}|\tilde{x}_2(0)| + 4\left(1 + \frac{\sqrt{3}}{3}\right)M\varepsilon \\ &\leqslant \varepsilon\Gamma \text{ uniformly in } t \in [a, +\infty), \end{aligned} \tag{2.48}$$

where we notice that $\Gamma > 0$ is an ε-independent constant and is directly proportional to M. Therefore, the tracking effect would become better as the variation of external disturbance becomes smaller, and it becomes worse otherwise.

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

In addition, it follows from (2.45) that the closed-loop signal $x(t)$ is given by

$$x(t) = e^{-(t-a)}x(a) + \int_a^t e^{-(t-s)}\left[\tilde{x}_1(s) + \tilde{x}_2(s)\right] ds.$$

Thus, it follows from (2.47) and (2.48) that

$$|x(t)| \leqslant e^{-(t-a)}|x(a)| + 4\varepsilon\Gamma, \ \forall t \geqslant a,$$

where $\Gamma > 0$ is specified in (2.47) and (2.48), and thus, we can see that the stabilization effect become better as the variation of external disturbance becomes smaller, and it becomes worse otherwise.

3. ACTIVE DISTURBANCE REJECTION CONTROL WITH TIME-VARYING GAIN EXTENDED STATE OBSERVER

In this section, we propose a time-varying gain ESO for (1.1) as follows:

$$\begin{cases} d\hat{x}_1(t) = \left[\hat{x}_2(t) + \frac{1}{r^{n-1}(t)} g_1\left(r^n(t)(y(t) - \hat{x}_1(t))\right) + h_1\left(\hat{x}_1(t)\right)\right] dt, \\ d\hat{x}_2(t) = \left[\hat{x}_3(t) + \frac{1}{r^{n-2}(t)} g_2\left(r^n(t)(y(t) - \hat{x}_1(t))\right) + h_2(\hat{x}_1(t), \hat{x}_2(t))\right] dt, \\ \quad \vdots \\ d\hat{x}_n(t) = \left[\hat{x}_{n+1}(t) + g_n\left((r^n(t)(y(t) - \hat{x}_1(t))\right) + h_n(\hat{x}(t)) + b_0 u(t)\right] dt, \\ d\hat{x}_{n+1}(t) = r(t) g_{n+1}\left(r^n(t)(y(t) - \hat{x}_1(t))\right) dt, \end{cases} \tag{3.1}$$

where $g_i \in C(\mathbb{R};\mathbb{R})$ are designed functions satisfying Assumption (A3), and $r \in C([0,\infty);(0,\infty))$ is the gain function to be required to satisfy the following Assumption (A4).

Assumption (A4)
$r \in C^1([0,\infty),(0,\infty))$, $r(t) > 0, \dot{r}(t) > k > 0$, and $|\frac{\dot{r}(t)}{r(t)}| \leqslant K$ for all $t \geqslant 0$, where $k > 0$ and $K > 0$ are constants.

Theorem 3.1
Let $\tau > \max\{1, \frac{n\alpha C_0}{\lambda_{13}}\}$. Suppose that $\sup_{t\geqslant 0}\|w(t)\| \leqslant B$ almost surely for some constant $B > 0$. Then under Assumptions (A1)–(A4), for any initial values $x(0) \in \mathbb{R}^n$, $(\hat{x}(0), \hat{x}_{n+1}(0)) \in \mathbb{R}^{n+1}$, $\zeta(0) \in \mathbb{R}^m$, the closed-loop of x-subsystem of (1.1),(3.1), and (2.2) is asymptotically mean square stable in the sense that

$$\lim_{t\to\infty} \mathbb{E}\sum_{i=1}^{n+1}[x_i(t) - \hat{x}_i(t)]^2 = 0, \quad \lim_{t\to\infty} \mathbb{E}\sum_{i=1}^{n} x_i^2(t) = 0.$$

Proof
Set

$$\begin{cases} \theta_i(t) = \tau^{n-i}x_i(t), \ i = 1,2,\cdots,n, \\ \eta_i(t) = r^{n+1-i}(t)(x_i(t) - \hat{x}_i(t)), \ i = 1,2,\cdots,n+1. \end{cases} \tag{3.2}$$

By Assumption (A4), there exists $t_1 > 0$ such that $r(t) > \max\{1,\tau\}$ for all $t \geqslant t_1$. Similar to the computations in the proof of Theorem 2.1, for all $t \geqslant t_1$, by Assumption (A1), we can obtain

$$\begin{aligned} |\Delta_i(t)|^2 &\leqslant C_0^2\left[(x_1(t) - \hat{x}_1(t))^2 + \cdots + (x_i(t) - \hat{x}_i(t))^2\right] \\ &= C_0^2\left[\frac{1}{r^{2n}(t)}|\eta_1(t)|^2 + \cdots + \frac{1}{r^{2(n+1-i)}(t)}|\eta_i(t)|^2\right] \\ &\leqslant \frac{C_0^2}{r^{2(n+1-i)}(t)}\|\eta(t)\|^2, i = 1,2,\cdots,n, \end{aligned} \tag{3.3}$$

and so

$$|h_i(\hat{x}_1(t),\cdots,\hat{x}_i(t))| \leqslant \frac{C_0}{r^{n+1-i}(t)}\|\eta(t)\| + C_0\|(x_1(t),\cdots,x_i(t))\|, i = 1,2,\cdots,n, \tag{3.4}$$

where $\Delta_i(t), i = 1,2,\cdots,n$ are as that defined in (2.12). Moreover, it follows from Assumption (A2) that

$$|D_n(t)| \leqslant L_0\|\eta(t)\|, \tag{3.5}$$

and

$$\begin{aligned}|\tau v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))| &\leqslant |D_n(t)| + |\tau v(\tau^{n-1}x_1(t),\cdots,x_n(t))|\\ &\leqslant L_0\|\eta(t)\| + \tau L_0\|\theta(t)\|,\end{aligned} \tag{3.6}$$

$$\begin{aligned}|f(t,x(t),\zeta(t),w(t)) + bu(t)| &= |x_{n+1}(t) - \hat{x}_{n+1}(t) + \tau v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))|\\ &\leqslant |\eta_{n+1}(t)| + L_0\|\eta(t)\| + \tau L_0\|\theta(t)\|,\end{aligned} \tag{3.7}$$

where $D_n(t)$ is defined in (2.12). It then follows from Assumption (A1) and (2.17) that there exist constants $C_5^*, C_6^*, C_7^*, C_8^* > 0$ such that

$$|\Lambda_1(t)| \leqslant C_5^* + C_6^*\|\eta(t)\| + C_7^*\|\theta(t)\|, \quad \|\Lambda_2(t)\|^2 + \|\Lambda_3(t)\|^2 \leqslant C_8^*, \tag{3.8}$$

where $\Lambda_i(t), i = 1,,2,3$ are in (2.17). Finding the derivative of $u(t)$ along the solution of (3.1) to obtain

$$\begin{aligned}&\frac{du(t)}{dt}\Big|_{\text{along (3.1)}}\\ &= \frac{1}{b_0}\left\{\sum_{i=1}^{n-1}\tau^{n+1-i}\left(\hat{x}_{i+1}(t) + \frac{1}{r^{n-i}(t)}g_i(\eta_1(t)) + h_i(\hat{x}_1(t),\cdots,\hat{x}_i(t))\right)\frac{\partial v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))}{\partial(\tau^{n-i}\hat{x}_i)}\right.\\ &\quad + \tau\left(\hat{x}_{n+1}(t) + g_n(\eta_1(t)) + b_0u(t) + h_n(\hat{x}(t))\right)\cdot\frac{\partial v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))}{\partial\hat{x}_n}\\ &\quad \left. - r(t)g_{n+1}(\eta_1(t))\right\} \triangleq \Lambda_4^*(t).\end{aligned} \tag{3.9}$$

By Assumption (A3), (2.8), (3.4), and (3.6), it follows that

$$\begin{aligned}|\Lambda_4^*(t)| &\leqslant \frac{L_0}{|b_0|}\left\{\sum_{i=1}^{n-1}\frac{\tau^{n+1-i}}{r^{n-i}(t)}|\eta_{i+1}(t)| + \tau^2\sum_{i=2}^{n}|\theta_i(t)| + \sum_{i=1}^{n-1}\left(\frac{a_i\tau^{n+1-i}}{r^{n-i}(t)}|\eta_1(t)|\right.\right.\\ &\quad \left. + \frac{C_0\tau^{n+1-i}}{r^{n+1-i}(t)}\|\eta(t)\| + C_0\tau^{n+1-i}\|(x_1(t),\cdots,x_i(t))\|\right)\\ &\quad \left. +\tau\left(a_n|\eta_1(t)| + |\tau v(\tau^{n-1}\hat{x}_1(t),\cdots,\hat{x}_n(t))| + \frac{C_0}{r(t)}\|\eta(t)\| + C_0\|x(t)\|\right)\right\} + \frac{a_{n+1}}{|b_0|}r(t)|\eta_1(t)|\\ &\leqslant \frac{L_0}{|b_0|}\left\{\sum_{i=1}^{n-1}\frac{\tau^{n+1-i}}{r^{n-i}(t)}|\eta_{i+1}(t)| + \tau^2\sum_{i=2}^{n}|\theta_i(t)| + \sum_{i=1}^{n-1}\left(\frac{a_i\tau^{n+1-i}}{r^{n-i}(t)}|\eta_1(t)|\right.\right.\\ &\quad \left. + \frac{C_0\tau^{n+1-i}}{r^{n+1-i}(t)}\|\eta(t)\| + \tau C_0\|\theta(t)\|\right)\\ &\quad \left. +\tau\left(a_n|\eta_1(t)| + L_0\|\eta(t)\| + \tau L_0\|\theta(t)\| + \frac{C_0}{r(t)}\|\eta(t)\| + C_0\|\theta(t)\|\right)\right\} + \frac{a_{n+1}}{|b_0|}r(t)|\eta_1(t)|\\ &\leqslant \frac{L_0}{|b_0|}\left\{\frac{(n-1)\tau^n}{r(t_1)} + \frac{(n-1)\tau^n}{r(t_1)}\max_{1\leqslant i\leqslant n-1}\{a_i\} + \frac{(n-1)\tau^nC_0}{r^2(t_1)} + \tau a_n + \tau L_0 + \frac{\tau C_0}{r(t_1)}\right\}\|\eta(t)\|\\ &\quad + \frac{L_0}{|b_0|}\left((n-1)\tau^2 + (n-1)\tau C_0 + \tau^2L_0 + \tau C_0\right)\|\theta(t)\| + \frac{a_{n+1}}{|b_0|}r(t)\|\eta(t)\|\\ &= C_9^*\|\eta(t)\| + C_{10}^*\|\theta(t)\| + \frac{a_{n+1}}{|b_0|}r(t)\|\eta(t)\|, \forall\ t \geqslant t_1,\end{aligned} \tag{3.10}$$

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

where we set

$$
\begin{aligned}
C_9^* &\triangleq \frac{L_0}{|b_0|}\left\{\frac{(n-1)\tau^n}{r(t_1)} + \frac{(n-1)\tau^n}{r(t_1)}\max_{1\leqslant i\leqslant n-1}\{a_i\} + \frac{(n-1)\tau^n C_0}{r^2(t_1)} + \tau a_n + \tau L_0 + \frac{\tau C_0}{r(t_1)}\right\},\\
C_{10}^* &\triangleq \frac{L_0}{|b_0|}\left((n-1)\tau^2 + (n-1)\tau C_0 + \tau^2 L_0 + \tau C_0\right).
\end{aligned}
\tag{3.11}
$$

Let

$$
\Theta_1(t) = \Lambda_1(t) + (b-b_0)\Lambda_4^*(t),\;\; \Theta_2(t) = \Lambda_2(t),\; \Theta_3(t) = \Lambda_3(t). \tag{3.12}
$$

Then a direct computation shows that the closed-loop of x-subsystem of (1.1), (2.2), and (3.1) is equivalent to

$$
\begin{cases}
d\theta_1(t) = \left[\tau\theta_2(t) + \tau^{n-1}h_1\left(\frac{\theta_1(t)}{\tau^{n-1}}\right)\right]dt,\\
d\theta_2(t) = \left[\tau\theta_3(t) + \tau^{n-2}h_2\left(\frac{\theta_1(t)}{\tau^{n-1}},\frac{\theta_2(t)}{\tau^{n-2}}\right)\right]dt,\\
\quad\vdots\\
d\theta_n(t) = \left[\tau v(\theta_1(t),\cdots,\theta_n(t)) + D_n(t) + \eta_{n+1}(t) + h_n\left(\frac{\theta_1(t)}{\tau^{n-1}},\cdots,\theta_n(t)\right)\right]dt,\\
d\eta_1(t) = \left[r(t)(\eta_2(t) - g_1(\eta_1(t))) + \frac{n\dot r(t)}{r(t)}\eta_1(t) + r^n(t)\Delta_1(t)\right]dt,\\
d\eta_2(t) = \left[r(t)(\eta_3(t) - g_2(\eta_1(t))) + \frac{(n-1)\dot r(t)}{r(t)}\eta_2(t) + r^{n-1}(t)\Delta_2(t)\right]dt,\\
\quad\vdots\\
d\eta_n(t) = \left[r(t)(\eta_{n+1}(t) - g_n(\eta_1(t))) + \frac{\dot r(t)}{r(t)}\eta_n(t) + r(t)\Delta_n(t)\right]dt,\\
d\eta_{n+1}(t) = [\Theta_1(t) - r(t)g_{n+1}(\eta_1(t))]dt + \Theta_2(t)dB_1(t) + \Theta_3(t)dB_2(t).
\end{cases}
\tag{3.13}
$$

Consider the positive definite function $V:\mathbb{R}^{2n+1}\to\mathbb{R}$ given by

$$
V(\theta,\eta) = V(\theta_1,\cdots,\theta_n,\eta_1,\cdots,\eta_{n+1}) = V_1(\theta_1,\cdots,\theta_n) + V_2(\eta_1,\cdots,\eta_{n+1}). \tag{3.14}
$$

Apply Itô's formula to $V(\theta(t),\eta(t))$ with respect to t along the solutions $(\theta(t),\eta(t))$ of system (3.13) to obtain

$$
\begin{aligned}
dV(\theta(t),\eta(t)) = &\left[\tau\sum_{i=1}^{n-1}\frac{\partial V_1(\theta(t))}{\partial\theta_i}\theta_{i+1}(t) + \tau\frac{\partial V_1(\theta(t))}{\partial\theta_n}v(\theta_1(t),\cdots,\theta_n(t))\right.\\
&+\sum_{i=1}^{n}\tau^{n-i}\frac{\partial V_1(\theta(t))}{\partial\theta_i}h_i\left(\frac{\theta_1(t)}{\tau^{n-1}},\cdots,\frac{\theta_i(t)}{\tau^{n-i}}\right)\\
&\left.+\frac{\partial V_1(\theta(t))}{\partial\theta_n}D_n(t) + \frac{\partial V_1(\theta(t))}{\partial\theta_n}\eta_{n+1}(t)\right]dt\\
&+r(t)\left[\sum_{i=1}^{n}\frac{\partial V_2(\eta(t))}{\partial\eta_i}(\eta_{i+1}(t) - g_i(\eta_1(t))) - \frac{\partial V_2(\eta(t))}{\partial\eta_{n+1}}g_{n+1}(\eta_1(t))\right]dt\\
&+\sum_{i=1}^{n}r^{n+1-i}(t)\frac{\partial V_2(\eta(t))}{\partial\eta_i}\Delta_i(t)dt + \frac{\dot r(t)}{r(t)}\sum_{i=1}^{n}(n+1-i)\frac{\partial V_2(\eta(t))}{\partial\eta_i}\eta_i(t)\\
&+\frac{\partial V_2(\eta(t))}{\partial\eta_{n+1}}\Theta_1(t)dt + \frac{1}{2}\frac{\partial^2 V_2(\eta(t))}{\partial\eta_{n+1}^2}\|\Theta_2(t)\|^2dt\\
&+\frac{1}{2}\frac{\partial^2 V_2(\eta(t))}{\partial\eta_{n+1}^2}\|\Theta_3(t)\|^2dt + \frac{\partial V_2(\eta(t))}{\partial\eta_{n+1}}\Theta_2(t)dB_1(t) + \frac{\partial V_2(\eta(t))}{\partial\eta_{n+1}}\Theta_3(t)dB_2(t).
\end{aligned}
\tag{3.15}
$$

Copyright © 2016 John Wiley & Sons, Ltd.

By Assumption (A4), there exist $\mu > 0$, $t_2 > t_1 > 0$ such that

$$\begin{aligned}&\xi_0 - 2\mu > 0,\\ &\frac{\xi_1 r(t_2)}{2} - \left(\frac{\alpha^2 L_0^2}{2\mu} + \frac{\alpha^2}{2\mu} + n\beta C_0 + Kn^2\beta + 1 + \beta C_6^* \right.\\ &\qquad \left. + \frac{\beta^2 C_7^{*2}}{2\mu} + \beta C_9^* |b - b_0| + \frac{\beta^2 C_{10}^{*2} |b - b_0|^2}{2\mu}\right) > 0,\end{aligned} \tag{3.16}$$

where ξ_0 and ξ_1 are given by (2.29). It follows from (3.3), (3.5), (3.8), (3.10), and Assumptions (A2) and (A3) that

$$\begin{aligned}\frac{d\mathbb{E}V(\theta(t),\eta(t))}{dt} &\leqslant -\tau\mathbb{E}W_1(\theta(t)) + n\alpha C_0\mathbb{E}\|\theta(t)\|^2 + \alpha L_0\mathbb{E}(\|\theta(t)\|\cdot\|\eta(t)\|)\\ &+ \alpha\mathbb{E}(\|\theta(t)\|\cdot\|\eta(t)\|) - r(t)\mathbb{E}W_2(\eta(t)) + n\beta C_0\mathbb{E}\|\eta(t)\|^2\\ &+ n^2\beta\frac{\dot r(t)}{r(t)}\mathbb{E}\|\eta(t)\|^2 + \beta\mathbb{E}\left\{\|\eta(t)\|\cdot\left(C_5^* + C_6^*\|\eta(t)\| + C_7^*\|\theta(t)\|\right.\right.\\ &\left.\left.+|b-b_0|(C_9^*\|\eta(t)\| + C_{10}^*\|\theta(t)\| + \frac{a_{n+1}}{|b_0|}r(t)\|\eta(t)\|)\right)\right\} + \gamma C_8^*\\ &\leqslant -\xi_0\mathbb{E}\|\theta(t)\|^2 + \frac{\mu}{2}\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2L_0^2}{2\mu}\mathbb{E}\|\eta(t)\|^2\\ &+ \frac{\mu}{2}\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2}{2\mu}\mathbb{E}\|\eta(t)\|^2 - \lambda_{23}r(t)\mathbb{E}\|\eta(t)\|^2 + n\beta C_0\mathbb{E}\|\eta(t)\|^2\\ &+ Kn^2\beta\mathbb{E}\|\eta(t)\|^2 + \mathbb{E}\|\eta(t)\|^2 + \frac{\beta^2C_5^{*2}}{4} + \beta C_6^*\mathbb{E}\|\eta(t)\|^2 + \frac{\mu}{2}\mathbb{E}\|\theta(t)\|^2 + \frac{\beta^2C_7^{*2}}{2\mu}\mathbb{E}\|\eta(t)\|^2\\ &+ \beta C_9^*|b-b_0|\mathbb{E}\|\eta(t)\|^2 + \frac{\mu}{2}\mathbb{E}\|\theta(t)\|^2 + \frac{\beta^2C_{10}^{*2}|b-b_0|^2}{2\mu}\mathbb{E}\|\eta(t)\|^2\\ &+ \frac{\beta a_{n+1}|b-b_0|}{|b_0|}r(t)\mathbb{E}\|\eta(t)\|^2 + \gamma C_8^*,\\ &\leqslant -(\xi_0 - 2\mu)\mathbb{E}\|\theta(t)\|^2 - \frac{\xi_1 r(t)}{2}\mathbb{E}\|\eta(t)\|^2 + \frac{\beta^2C_5^{*2}}{4} + \gamma C_8^*,\ \forall\ t \geqslant t_2.\end{aligned} \tag{3.17}$$

From Assumption (A4), for any $\delta > 0$, there exists a positive constant $t_3 > t_2$ such that $r(t) > \frac{\beta^2 C_5^{*2} + 4\gamma C_8^* + 4\xi_1}{2\xi_1\delta}$ for all $t \geqslant t_3$. This together with (3.17) yields that if $\mathbb{E}\|\eta(t)\|^2 > \delta$, then

$$\frac{d\mathbb{E}V(\theta(t),\eta(t))}{dt} \leqslant -\xi_1 < 0. \tag{3.18}$$

Therefore, there exists $t_4 > t_3$ such that $\mathbb{E}\|\eta(t)\|^2 \leqslant \delta$ for all $t \geqslant t_4$, and hence,

$$\lim_{t\to\infty}\mathbb{E}\|\eta(t)\|^2 = 0. \tag{3.19}$$

This shows that

$$\lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n+1}[x_i(t) - \hat x_i(t)]^2 = \lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n+1}\left|\frac{\eta_i(t)}{r^{n+1-i}(t)}\right|^2 = 0, \tag{3.20}$$

the first part of the theorem.

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

By Assumption (A2) and (3.5), we find the derivative of $V_1(\theta(t))$ with respect to t along the solution $\theta(t)$ of system (3.13) to obtain

$$\begin{aligned}
\frac{d\mathbb{E}V_1(\theta(t))}{dt} &\leqslant -\tau\mathbb{E}W_1\theta(t)) + \mathbb{E}\left(\frac{\partial V_1(\theta(t))}{\partial\theta_n}|D_n(t)|\right) \\
&+ \mathbb{E}\sum_{i=1}^{n}\tau^{n-i}\frac{\partial V_1(\theta(t))}{\partial\theta_i}h_i\left(\frac{\theta_1(t)}{\tau^{n-1}},\cdots,\frac{\theta_i(t)}{\tau^{n-i}}\right) + \mathbb{E}\left(\frac{\partial V_1(\theta(t))}{\partial\theta_n}\eta_{n+1}(t)\right) \\
&\leqslant -\lambda_{13}\tau\mathbb{E}\|\theta(t)\|^2 + \alpha L_0\mathbb{E}(\|\theta(t)\|\cdot\|\eta(t)\|) + n\alpha C_0\mathbb{E}\|\theta(t)\|^2 + \alpha\mathbb{E}(\|\theta(t)\|\cdot|\eta_{n+1}(t)|) \\
&\leqslant -(\lambda_{13}\tau - n\alpha C_0)\mathbb{E}\|\theta(t)\|^2 + \mu\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2L_0^2}{4\mu}\mathbb{E}\|\eta(t)\|^2 + \mu\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2}{4\mu}\mathbb{E}\eta_{n+1}^2(t) \\
&\leqslant -(\xi_0 - 2\mu)\mathbb{E}\|\theta(t)\|^2 + \frac{\alpha^2(L_0^2+1)}{4\mu}\mathbb{E}\|\eta(t)\|^2.
\end{aligned} \tag{3.21}$$

Because $\lim_{t\to\infty}\mathbb{E}\|\eta(t)\|^2 = 0$, for any $\delta > 0$, there exists a positive constant $t_3^* > t_2$ such that $\mathbb{E}\|\eta(t)\|^2 < \frac{2\mu(\xi_0-2\mu)\delta}{\alpha^2(L_0^2+1)}$ for all $t \geqslant t_3^*$. It follows from (3.21) that if $\mathbb{E}\|(\theta(t))\|^2 > \delta$, then

$$\frac{d\mathbb{E}V_1(\theta(t))}{dt} \leqslant -\frac{(\xi_0 - 2\mu)\delta}{2} < 0. \tag{3.22}$$

Therefore, there exists $t_4^* > t_3^*$ such that $\mathbb{E}\|\theta(t)\|^2 \leqslant \delta$ for all $t \geqslant t_4^*$. This shows that

$$\lim_{t\to\infty}\mathbb{E}\|\theta(t)\|^2 = 0, \tag{3.23}$$

and thus,

$$\lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n}x_i^2(t) \leqslant \lim_{t\to\infty}\mathbb{E}\|\theta(t)\|^2 = 0. \tag{3.24}$$

This completes the proof of the theorem. □

Similarly, the simplest example of time-varying gain ADRC satisfying conditions of Theorem 3.1 is the linear one, that is, $g_i(\cdot), i = 1,\cdots,n+1$ in ESO (3.1) and $v(\cdot)$ in feedback control (2.2) are linear functions as defined in (2.42). Similar to the proof of Corollary 2.1, we have Corollary 3.1.

Corollary 3.1
Let $\tau > \max\{1, 2n\lambda_{max}(H)C_0\}$. Suppose that $\sup_{t\geqslant 0}\|w(t)\| \leqslant B$ almost surely for some constant $B > 0$, the matrices E and F are Hurwitz, and $|b - b_0| < \frac{|b_0|}{2k_{n+1}\lambda_{max}(Q)}$. Then under Assumptions (A1) and (A4), for any initial values $x(0) \in \mathbb{R}^n$, $(\hat{x}(0), \hat{x}_{n+1}(0)) \in \mathbb{R}^{n+1}$, $\zeta(0) \in \mathbb{R}^m$, the closed-loop of x-subsystem of (1.1), (3.1), and (2.2) is asymptotically mean square stable in the sense that

$$\lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n+1}[x_i(t) - \hat{x}_i(t)]^2 = 0, \quad \lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n}x_i^2(t) = 0.$$

Remark 3.1
As indicated in [6], the time-varying gain ESO degrades the ability of ESO to filter high-frequency noise, while the constant gain ESO does not. In practical applications, we can use time-varying gain $r(t)$ as follows: (i) given a small initial value $r(0) > 0$; (ii) from the constant high gain, we obtain the convergent high-gain value $\frac{1}{\varepsilon}$ $(0 < \varepsilon < 1)$ that can also be obtained by trial-and-error experiment for practical systems; (iii) the gain function is initialed from the small value $r(0) > 0$ and then increases continuously to a large constant high gain $\frac{1}{\varepsilon}$. Specially, $r(t)$ can be chosen as

$$r(t) = \begin{cases} e^{at}, & 0 \leqslant t \leqslant -\frac{1}{a}\ln\varepsilon, \\ \frac{1}{\varepsilon}, & t \geqslant -\frac{1}{a}\ln\varepsilon, \end{cases} \tag{3.25}$$

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

where $a > 0$ is used to control the convergent speed and the peaking value. The larger a is, the faster convergence but larger peaking; while the smaller a is, the lower convergence speed and smaller peaking. The mean square practical ability of the closed-loop system of x-subsystem of (1.1), (2.2), and (3.1) with time-varying gain $r(t)$ given by (3.25) can also be achieved since the ESO (3.1) is reduced to ESO (2.1) when $t \geqslant -\frac{1}{a}\ln\varepsilon$.

4. NUMERICAL SIMULATIONS

In this section, we present an example to illustrate the effectiveness of the proposed ADRC approach. Consider the following uncertain lower triangular system with stochastic inverse dynamic and exogenous stochastic disturbance:

$$\begin{cases} dx_1(t) = [x_2(t) + \sin(x_1(t))]dt, \\ dx_2(t) = [\alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3\cos(\zeta(t)) + \cos(\alpha_4 t + \alpha_5 B_2(t)) + u(t) + \sin(x_2(t))]dt, \\ d\zeta(t) = \alpha_6 \sin(\zeta(t)) \cdot x_2(t)dt + \alpha_7\cos(\zeta(t)) \cdot \cos(\alpha_4 t + \alpha_5 B_2(t))\, dB_1(t), \\ y(t) = x_1(t), \end{cases} \tag{4.1}$$

where α_i $(i = 1, 2, \cdots, 7)$ are unknown parameters satisfying $|\alpha_i| \leqslant M$ $(i = 1, 2, \cdots, 7)$ for any given (known) constant $M > 0$. The $w(t) \triangleq \cos(\alpha_4 t + \alpha_5 B_2(t))$ is a bounded non-white noise appeared often in many practical dynamical systems like the motion of oscillators [25], where α_4 and α_5^2 are constants representing the central frequency and strength of frequency disturbance, respectively. In this case, $n = 2, m = 1, s = 1, p = q = 1, b = b_0 = 1$. It is easy to check that all the uncertainties in (4.1) satisfy Assumption (A1). So we can design a constant gain ESO (4.2) for system (4.1) as follows:

$$\begin{cases} d\hat{x}_1(t) = \left[\hat{x}_2(t) + \frac{6}{\varepsilon}(y(t) - \hat{x}_1(t)) + \varepsilon\Psi\left(\frac{y(t)-\hat{x}_1(t)}{\varepsilon^2}\right) + \sin(\hat{x}_1(t))\right]dt, \\ d\hat{x}_2(t) = \left[\hat{x}_3(t) + \frac{12}{\varepsilon^2}(y(t) - \hat{x}_1(t))dt + u(t) + \sin(\hat{x}_2(t))\right]dt, \\ d\hat{x}_3(t) = \frac{8}{\varepsilon^3}(y(t) - \hat{x}_1(t))dt, \end{cases} \tag{4.2}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\Psi(s) = \begin{cases} -\frac{1}{\pi}, & s \in (-\infty, -1], \\ \frac{1}{\pi}\sin\frac{\pi s}{2}, & s \in (-1, 1), \\ \frac{1}{\pi}, & s \in [1, +\infty). \end{cases} \tag{4.3}$$

First, we notice that the corresponding matrix in (2.43) for the linear part of (4.2) is

$$F = \begin{pmatrix} -6 & 1 & 0 \\ -12 & 0 & 1 \\ -8 & 0 & 0 \end{pmatrix}, \tag{4.4}$$

which has eigenvalues equal to -2 and hence is Hurwitz. In this case, $g_i(\cdot), i = 1, 2, 3$ in (2.1) can be specified as

$$g_1(y_1) = 6y_1 + \Psi(y_1),\ g_2(y_1) = 12y_1,\ g_3(y_1) = 8y_1. \tag{4.5}$$

The Lyapunov function $V_2 : \mathbb{R}^3 \to \mathbb{R}$ for this case is given by

$$V_2(y) = y^\top Q y + \int_0^{y_1} \Psi(s)ds, \forall\ y = (y_1, y_2, y_3)^\top \in \mathbb{R}^3, \tag{4.6}$$

where

$$Q = \begin{pmatrix} \frac{67}{32} & -\frac{1}{2} & -\frac{97}{128} \\ -\frac{1}{2} & \frac{97}{128} & -\frac{1}{2} \\ -\frac{97}{128} & -\frac{1}{2} & \frac{643}{512} \end{pmatrix} \tag{4.7}$$

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

is the positive definite solution of the Lyapunov equation $QF + F^\top Q = -I_{3\times 3}$ for F given by (4.4). A direct computation shows that

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{\partial V_2(y)}{\partial y_i}(y_{i+1} - g_i(y_1)) - \frac{\partial V_2(y)}{\partial y_3} g_3(y_1) \\
&= -y_1^2 - y_2^2 - y_3^2 - \left(\frac{67}{16}y_1 - y_2 - \frac{97}{64}y_3 + \Psi(y_1)\right)\cdot \Psi(y_1) + (y_2 - 6y_1)\cdot\Psi(y_1) \\
&\leqslant -\frac{63}{256}y_1^2 - \frac{1}{3}y_2^2 - \frac{159}{256}y_3^2 \triangleq -W_2(y),\ \forall\ y = (y_1, y_2, y_3)^\top \in \mathbb{R}^3.
\end{aligned} \tag{4.8}$$

So all conditions of Assumption (A3) are satisfied. Choose $v : \mathbb{R}^2 \to \mathbb{R}$ in (2.42) as follows:

$$v(\hat{x}_1, \hat{x}_2) = -2\hat{x}_1 - 3\hat{x}_2 \tag{4.9}$$

with the corresponding matrix in (2.43)

$$E = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}. \tag{4.10}$$

being Hurwitz and

$$H = \begin{pmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{4.11}$$

is the positive definite solution of the Lyapunov equation $HE + E^\top H = -I_{2\times 2}$. A simple computation shows that the maximal eigenvalue of matrix H is $\lambda_{\max\{H\}} = \frac{3+\sqrt{5}}{4} < \frac{3}{2}$. We also notice that $C_0 = 1, \lambda_{13} = 1, n = 2, \alpha < 3$, and thus, we can choose the parameter τ for this case in (2.2) as $\tau = 6$. It follows from Assumption (A2) and Theorem 2.1 that (4.2) serves as a well-defined nonlinear constant gain ESO for system (4.1) under the ESO (4.2)-based feedback control designed as

$$u(t) = -72\hat{x}_1(t) - 18\hat{x}_2(t) - \hat{x}_3(t). \tag{4.12}$$

The Milstein approximation method [36] is used to discretize systems (4.1) and (4.2). Figures 1–4 display the numerical results for (4.1) and (4.2) where we take

$$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = \frac{1}{3}, \alpha_5 = \frac{1}{3}, \alpha_6 = \frac{1}{2}, \alpha_7 = \frac{1}{2}. \tag{4.13}$$

The initial values are

$$x_1(0) = 1, x_2(0) = -1, \zeta(0) = 0, \hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 0, \tag{4.14}$$

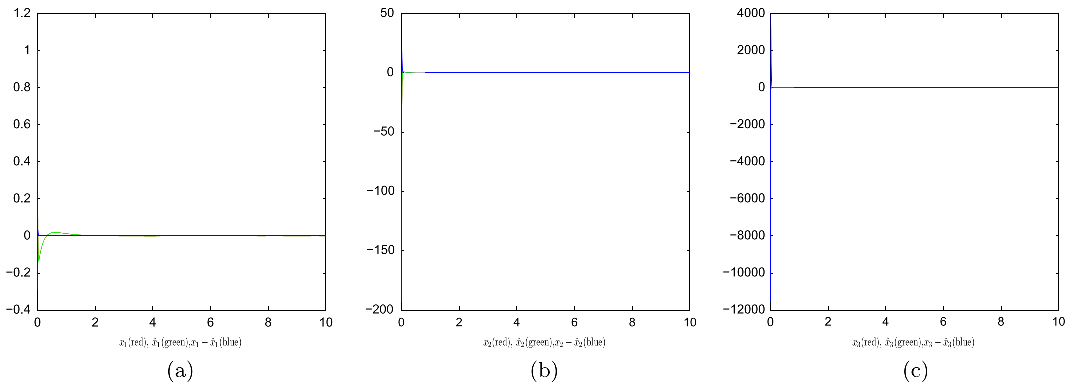
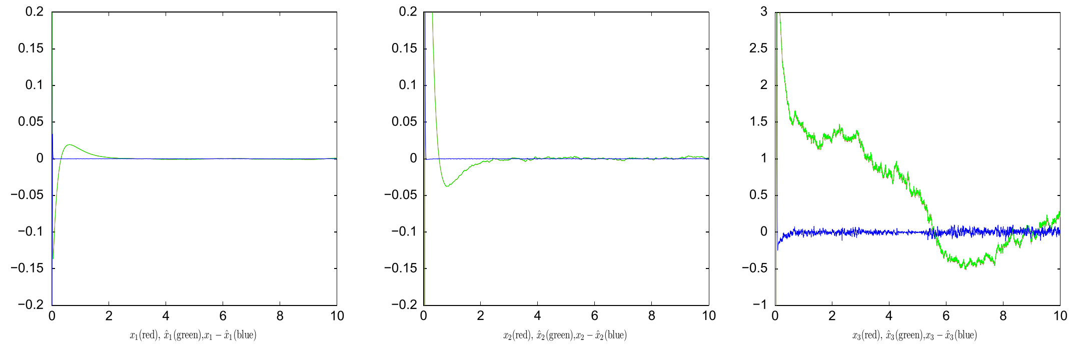


Figure 1. The closed-loop state $(x_1(t), x_2(t))$, stochastic total disturbance $x_3(t)$, and their estimates $(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t))$ under nonlinear constant gain ESO (4.2)-based feedback control (4.12) with $\varepsilon = 0.01$. [Colour figure can be viewed at wileyonlinelibrary.com]

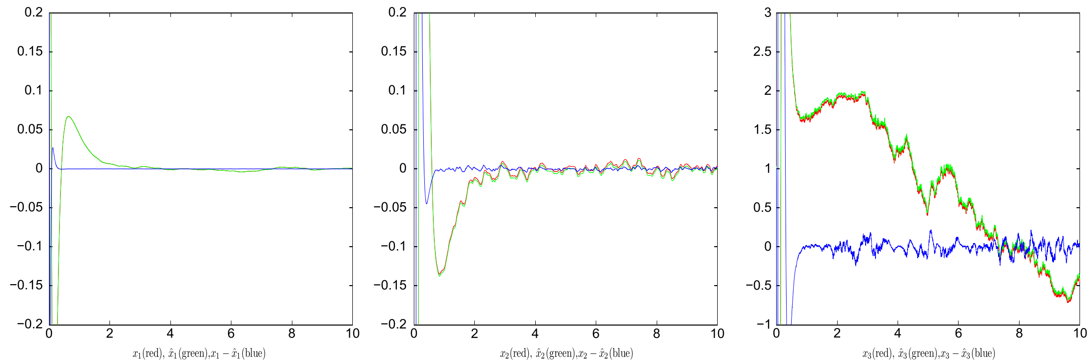
Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc



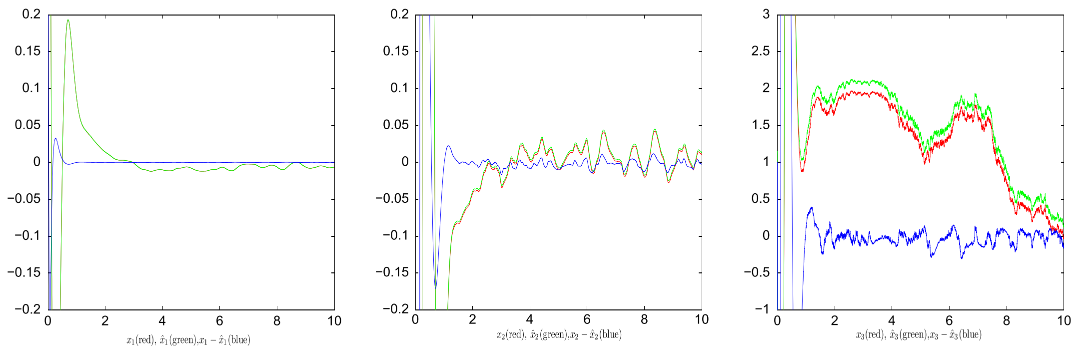
(a) Magnification of (1(a)) (b) Magnification of (1(b)) (c) Magnification of (1(c))

Figure 2. The closed-loop state $(x_1(t), x_2(t))$, stochastic total disturbance $x_3(t)$, and their estimates $(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t))$ under nonlinear constant gain ESO (4.2)-based feedback control (4.12) with $\varepsilon = 0.01$. (a) Magnification of (1(a)), (b) magnification of (1(b)), and (c) magnification of (1(c)). [Colour figure can be viewed at wileyonlinelibrary.com]



(a) (b) (c)

Figure 3. The closed-loop state $(x_1(t), x_2(t))$, stochastic total disturbance $x_3(t)$, and their estimates $(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t))$ under nonlinear constant gain ESO (4.2)-based feedback control (4.12) with $\varepsilon = 0.05$. [Colour figure can be viewed at wileyonlinelibrary.com]



(a) (b) (c)

Figure 4. The closed-loop state $(x_1(t), x_2(t))$, stochastic total disturbance $x_3(t)$, and their estimates $(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t))$ under nonlinear constant gain ESO (4.2)-based feedback control (4.12) with $\varepsilon = 0.1$. [Colour figure can be viewed at wileyonlinelibrary.com]

and time discrete step is taken as

$$\Delta t = 0.001. \tag{4.15}$$

Theoretically, we can conclude from Theorem 2.1 that under the ESO (4.2)-based output-feedback control (4.12), the estimation errors for $x_1(t)$, $x_2(t)$, $x_3(t)$ are bounded by $O(\varepsilon^5)$, $O(\varepsilon^3)$, $O(\varepsilon)$ in practical mean square sense, respectively. In addition , the states $x_1(t)$ and $x_2(t)$ are

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

bounded by $O(\varepsilon)$ in practical mean square sense. In Figures 1 and 2, the tuning parameter is $\varepsilon = 0.01$. The local amplification of Figure 1 is plotted in Figure 2. It is seen from Figure 2 that the nonlinear constant gain ESO (4.2) is very effective in tracking system (4.1) not only for the state $(x_1(t), x_2(t))$ but also for the extended state (stochastic total disturbance) $x_3(t)$ defined by

$$x_3(t) = x_1(t) + 2x_2(t) + \cos(\zeta(t)) + \cos\left(\frac{1}{3}t + \frac{1}{3}B_2(t)\right). \tag{4.16}$$

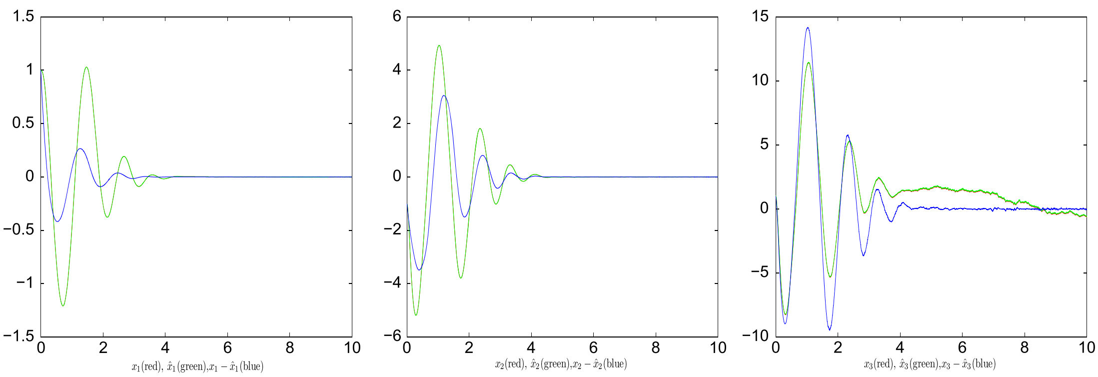
It is observed from Figure 2 that the estimation effect for $x_1(t)$ is the best, $x_2(t)$ the second , and $x_3(t)$ the last, which are coincident with the theoretical estimations. Moreover, it is seen from Figure 2(a) and (b) that stabilization for each trajectory of $x_1(t)$ and $x_2(t)$ is very satisfactory. To validate further the theoretical convergence in Theorem 2.1, Figures 3 and 4 are plotted in comparison with Figure 2, where the tuning parameters are chosen as $\varepsilon = 0.05$ and $\varepsilon = 0.1$, respectively. On one hand, it is seen that the effects of estimation and stabilization in Figure 2 are the best, Figures 3 the second, and Figure 4 the worst because of the increase in tuning parameter ε, which is also consistent with the theoretical estimation. On the other hand, it is observed from Figures 3 and 4 that the estimation for states $x_1(t)$ and $x_2(t)$ still maintains good performances although the tuning parameter ε is increased from 0.01 to 0.05 and 0.1. However, the estimation effect for stochastic total disturbance $x_3(t)$ becomes much worse when the tuning parameter ε is increased from 0.01 to 0.05 and 0.1. These are exactly consistent with the theoretical estimation that the estimation errors for $x_1(t)$ and $x_2(t)$ are bounded by $O(\varepsilon^5)$ and $O(\varepsilon^3)$ in practical mean square sense, respectively, but the estimation error for stochastic total disturbance $x_3(t)$ is only bounded by $O(\varepsilon)$.

The main problem for constant high-gain ESO, likewise many other high-gain designs, is the peaking value problem near the initial stage caused by different initial values of system (4.1) and ESO (4.2) ([6]). The large peaking values of $\hat{x}_2(t)$ and $\hat{x}_3(t)$ are observed near the initial stage because of the high gain $\frac{1}{\varepsilon} = 100$: The absolute peaking value of $\hat{x}_2(t)$ is near 200 and that of $\hat{x}_3(t)$ is even greater than 10^4 in Figure 1(b) and (c), respectively.

Now, we apply the following time-varying gain ESO (4.17) to system (4.1), which comes from (3.1) with nonlinear functions $g_i(\cdot), i = 1, 2, 3$ as that in (4.5):

$$\begin{cases} d\hat{x}_1(t) = \left[\hat{x}_2(t) + 6r(t)(y(t) - \hat{x}_1(t)) + \frac{1}{r(t)}\Psi\left(r^2(t)(y(t) - \hat{x}_1(t)\right) + \sin(\hat{x}_1(t))\right]dt, \\ d\hat{x}_2(t) = \left[\hat{x}_3(t) + 12r^2(t)(y(t) - \hat{x}_1(t))dt + u(t) + \sin(\hat{x}_2(t))\right]dt, \\ d\hat{x}_3(t) = 8r^3(t)(y(t) - \hat{x}_1(t))dt, \end{cases} \tag{4.17}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is given by (4.3). In what follows, we use the time-varying gain $r(t) = e^{0.5t}$ for the numerical simulation. It is observed from Figure 5 that the estimation of $(x_1(t), x_2(t), x_3(t))$



(a) (b) (c)

Figure 5. The closed-loop state $(x_1(t), x_2(t))$, stochastic total disturbance $x_3(t)$, and their estimates $(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t))$ under the nonlinear time-varying gain ESO (4.17)-based feedback control (4.12) with time-varying gain $r(t) = e^{0.5t}$. [Colour figure can be viewed at wileyonlinelibrary.com]

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

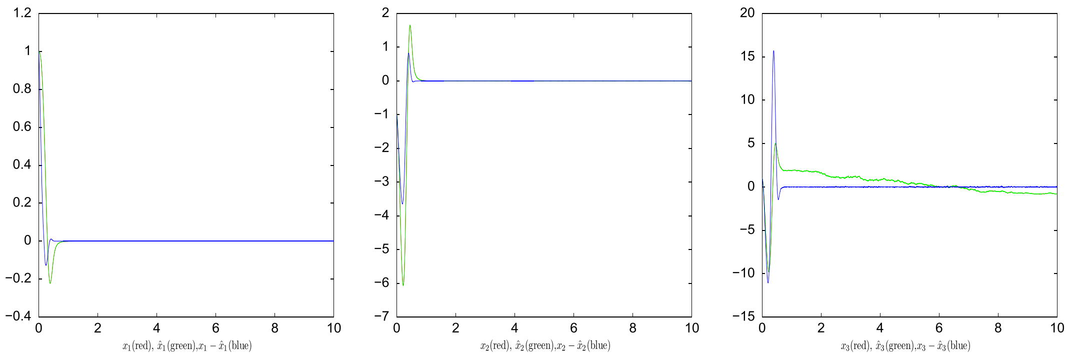


Figure 6. The closed-loop state $(x_1(t), x_2(t))$, stochastic total disturbance $x_3(t)$, and their estimates $(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t))$ under the nonlinear time-varying gain ESO (4.17)-based feedback control (4.12) with time-varying gain $r(t)$ given by (4.18) and $\varepsilon = 0.01$. [Colour figure can be viewed at wileyonlinelibrary.com]

and the stabilization for state $(x_1(t), x_2(t)$ are also very satisfactory after a short time. In addition, there are no peaking values near the initial stages for $\hat{x}_2(t)$ and $\hat{x}_3(t)$.

In general, the large gain value needs small integration step. Thus, as recommended in [6] and Remark (3.1), in practice, the time-varying gain should be small value in the beginning and gradually increases to a large constant high gain for which we choose as

$$r(t) = \begin{cases} e^{6t}, & 0 \leqslant t \leqslant \ln 100/6, \\ \frac{1}{\varepsilon} = 100, & t \geqslant \ln 100/6. \end{cases} \tag{4.18}$$

The numerical results for (4.1) with time-varying gain ESO (4.17) and time-varying gain $r(t)$ given by (4.18) are plotted in Figure 6 with the same initial values and time discrete step as that in Figures 1–5. Figure 6 shows that the nonlinear time-varying gain ESO (4.17) tracks the state $(x_1(t), x_2(t))$ of system (4.1) and stochastic total disturbance $x_3(t)$ defined in (4.16) well. In addition, Figure 6(a) and (b) show that the stabilization under time-varying gain ESO (4.17)-based feedback control (4.12) is also very satisfactory. More importantly, the absolute peaking value near the initial stage of $\hat{x}_2(t)$ is around one (near 200 by constant high gain) and that of $\hat{x}_3(t)$ is less than 5 (over 10^4 by constant high gain). This shows that the time-varying gain method reduces dramatically the peaking value caused by the constant high gain. Finally, the effects of estimation and stabilization in Figure 6 are satisfactorily after a shorter time than Figure 5, which is because the gain value of the former is larger than the latter in the beginning.

5. CONCLUDING REMARKS

In this paper, we apply ADRC approach to output-feedback stabilization for a class of lower triangular nonlinear systems with large stochastic uncertainty in the control channel. Both constant gain ESO and time-varying gain ESO are designed to estimate, in real time, both the unmeasured states and the stochastic total disturbance that includes unknown system dynamics, unknown stochastic inverse dynamics, external stochastic disturbance, and uncertainty caused by the deviation of control parameter from its nominal value. The stochastic total disturbance is then compensated in the feedback loop. An ESO-based output-feedback control is designed analogously as for the system without disturbance. It is shown that the resulting closed-loop of x-subsystem is practically mean square stable with constant gain ESO and asymptotically mean square stable with time-varying gain ESO, respectively. The numerical results validate the efficiency of both design methods. By combination of the time-varying gain in the initial stage and the constant high gain, the peaking value reduction near the initial stage is also addressed through numerical simulations.

Finally, we indicate a potential application of the ADRC approach to more complicated systems like models in [21, 22] with mismatched unknown nonlinear system uncertainty and stochastic

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc

disturbance. That is, the nonlinear system dynamics $h_i(\cdot)$ $(i = 1, 2, \cdots, n)$ in system (1.1) are also unknown and the x-subsystem could be modeled by Itô-type stochastic differential equations. In this case, the diffusion term in the x-subsystem would bring essential difficulty for ESO to estimate the stochastic total disturbance because high-gain ESO is sensitive to white noise in x-subsystem. Mathematically, a feasible way like reforming ESO should be excavated to tackle the Hessian term brought by Itô differential to estimate stochastic total disturbance.

ACKNOWLEDGEMENTS

This work was carried out with the support of the National Natural Science Foundation of China and the National Research Foundation of South Africa.

REFERENCES

1. Han JQ. From PID to active disturbance rejection control. *IEEE Transactions on Industrial Electronics* 2009; **56**: 900–906.
2. Bai WY, Xue WC, Huang Y, Fang HT. Extended state filter design for general nonlinear uncertain systems. *SICE Annual Conference*, Hangzhou, China, 2015; 712–717.
3. Gao ZQ. Scaling and bandwidth-parameterization based controller tuning. *American Control Conference*, Denver, Colorado, 2003; 4989–4996.
4. Guo BZ, Zhao ZL. On the convergence of extended state observer for nonlinear systems with uncertainty. *Systems & Control Letters* 2011; **60**:420–430.
5. Guo BZ, Zhao ZL. On convergence of the nonlinear active disturbance rejection control for MIMO systems. *SIAM Journal on Control and Optimization* 2013; **51**:1727–1757.
6. Zhao ZL, Guo BZ. Active disturbance rejection control approach to stabilization of lower triangular systems with uncertainty. *International Journal of Robust and Nonlinear Control* 2016; **26**:2314–2337.
7. Jiang TT, Huang CD, Guo L. Control of uncertain nonlinear systems based on observers and estimators. *Automatica* 2015; **59**:35–47.
8. Li SH, Yang J, Chen WH, Chen XS. Generalized extended state observer based control for systems with mismatched uncertainties. *IEEE Transactions on Industrial Electronics* 2012; **59**:4792–4802.
9. Pu ZQ, Yuan RY, Yi JQ, Tan XM. A class of adaptive extended state observers for nonlinear disturbed systems. *IEEE Transactions on Industrial Electronics* 2015; **62**:5858–5869.
10. Xue WC, Huang Y. On performance analysis of ADRC for a class of MIMO lower-triangular nonlinear uncertain system. *ISA Transactions* 2014; **53**:955–962.
11. Xue WC, Bai WY, Yang S, Song K, Huang Y, Xie H. ADRC with adaptive extended state observer and its application to air-fuel ratio control in gasoline engines. *IEEE Transactions on Industrial Electronics* 2015; **62**:5847–5857.
12. Zheng Q, Gao L, Gao ZQ. On stability analysis of active disturbance rejection control for nonlinear time-varying plants with unknown dynamics. *IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007; 3501–3506.
13. Pan Z, Basar T. Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion. *SIAM Journal on Control and Optimization* 1999; **37**:957–995.
14. Chen WS, Jiao LC, Li J, Li RH. Adaptive NN backstepping output-feedback control for stochastic nonlinear strict-feedback systems with time-varying delays. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* 2010; **40**:939–950.
15. Deng H, Krstić M. Output-feedback stabilization of stochastic nonlinear systems driven by noise of unknown covariance. *Systems & Control Letters* 2000; **39**:173–182.
16. Ji HB, Xi HS. Adaptive output-feedback tracking of stochastic nonlinear systems. *IEEE Transactions on Automatic Control* 2006; **51**:355–360.
17. Li J, Chen WS, Li JM, Fang YQ. Adaptive NN output-feedback stabilization for a class of stochastic nonlinear strict-feedback systems. *ISA Transactions* 2009; **48**:468–475.
18. Liu SJ, Zhang JF, Jiang ZP. Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems. *Automatica* 2007; **43**:238–251.
19. Pan ZG, Liu YG, Shi SJ. Output feedback stabilization for stochastic nonlinear systems in observer canonical form with stable zero-dynamics. *Science in China (Series F)* 2001; **44**:292–308.
20. Deng H, Krstić M. Output-feedback stochastic nonlinear stabilization. *IEEE Transactions on Automatic Control* 1999; **44**:328–333.
21. Wang D, Huang J. Neural network-based adaptive dynamic surface control for a class of uncertain nonlinear systems in strict-feedback form. *IEEE Transactions on Neural Networks* 2005; **16**:195–202.
22. Tong SC, Li Y, Li YM, Liu YJ. Observer-based adaptive fuzzy backstepping control for a class of stochastic nonlinear strict-feedback systems. *IEEE Transactions On Systrems, Man, And Cybernetics -Part B: Cybernetics* 2011; **41**: 1693–1704.

Copyright © 2016 John Wiley & Sons, Ltd.

23. Faetti S, Grigolini P. Unitary point of view on the puzzling problem of nonlinear systems driven by colored noise. *Physical Review A* 1987; **36**:441–444.
24. Jia Y, Zheng XP, Hu XM, Li JR. Effects of colored noise on stochastic resonance in a bistable system subject to multiplicative and additive noise. *Physical Review E* 2001; **63**:031107.
25. Huang ZL, Zhu WQ, Ni YQ, Ko JM. Stochastic averaging of strongly non-linear oscillators under bounded noise excitation. *Journal of Sound Vibration* 2002; **254**:245–267.
26. Misono M, Kohmoto T, Fukuda Y, Kunitomo M. Stochastic resonance in an optical bistable system driven by colored noise. *Optics Communications* 1998; **152**:255–258.
27. Sako O. Dynamical behavior of fluctuations in a non-linear reactor system perturbed by non-white reactivity noise. *Proceedings of the Faculty of Engineering of Tokai University*, Vol. 11, Tokyo Metropolitan, 1985; 23–34.
28. Zhong S, Xin HW. Effects of colored noise on internal stochastic resonance in a chemical model system. *Chemical Physics Letters* 2001; **333**:133–138.
29. Andrieu V, Praly L, Astolfi A. Asymptotic tracking of a reference trajectory by output-feedback for a class of nonlinear systems. *Systems & Control Letters* 2009; **45**:412–418.
30. Andrieu V, Praly L, Astolfi A. High gain observers with updated gain and homogeneous correction terms. *Automatica* 2009; **58**:652–663.
31. Praly L, Jiang ZP. Linear output feedback with dynamic high gain for nonlinear systems. *Systems & Control Letters* 2004; **53**:107–116.
32. Shen Y, Xia X. Semi-global finite-time observers for nonlinear systems. *Automatica* 2008; **44**:3152–3156.
33. Guo BZ, Wu ZH, Zhou HC. Active disturbance rejection control approach to output-feedback stabilization of a class of uncertain nonlinear systems subject to stochastic disturbance. *IEEE Transactions on Automatic Control* 2016; **61**:1613–1618.
34. Gauthier JP, Hammouri H, Othman S. A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on Automatic Control* 1992; **37**:875–880.
35. Han JQ. *Active Disturbance Rejection Control: A Technique for Estimating and Compensating the Uncertainties*. National Defence Industry Press: Beijing, 2008. (in Chinese).
36. Higham DJ. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review* 2001; **43**:525–546.

Copyright © 2016 John Wiley & Sons, Ltd.

DOI: 10.1002/rnc