Active disturbance rejection control to MIMO nonlinear systems with stochastic uncertainties: approximate decoupling and output-feedback stabilisation

Ze-Hao Wu $^a$ and Bao-Zhu Guo $^{a,b,c}$

$^a$School of Mathematics and Big Data, Foshan University, Foshan, People's Republic of China; $^b$Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing, People's Republic of China; $^c$School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa

ABSTRACT
In this paper, we apply the active disturbance rejection control, an emerging control technology, to output-feedback stabilisation for a class of uncertain multi-input multi-output nonlinear systems with vast stochastic uncertainties. Two types of extended state observers (ESO) are designed to estimate both unmeasured states and stochastic total disturbance which includes unknown system dynamics, unknown stochastic inverse dynamics, external stochastic disturbance without requiring the statistical characteristics, uncertain nonlinear interactions between subsystems, and uncertainties caused by the deviation of control parameters from their nominal values. The estimations decouple approximately the system after cancelling stochastic total disturbance in the feedback loop. As a result, we are able to design an ESO-based stabilising output-feedback and prove the practical mean square stability for the closed-loop system with constant gain ESO and the asymptotic mean square stability with time-varying gain ESO, respectively. Some numerical simulations are presented to demonstrate the effectiveness of the proposed output-feedback control scheme.

1. Introduction
Disturbance rejection is a different paradigm in control theory since the inception of the modern control theory in the late years of 1950s, seeded in Tsien (1954) where it was stated that the control operation ‘must not be influenced by internal and external disturbances’ (Tsien, 1954, p. 228). Many available modern robust control methods focus, however, on the worst case scenario which makes the controller design rather conservative. Motivated highly from PID model free nature, Han (2009) proposed an almost model free control technology known as active disturbance rejection control (ADRC) to this day in the late 1980s. ADRC adopts estimation/cancellation strategy in dealing with ‘total disturbance’ which can contain the coupling between the unknown system dynamics, external disturbance, and the superadded unknown part of control input, or even if whatever the part of hardly to be dealt with by practitioners. This spans significantly the concept of ‘disturbance’. The central idea of ADRC is that the ‘total disturbance’, as a signal of time no matter it is state-dependent or free, time invariant or variant, linear or nonlinear, is reflected entirely in the observable measured output and can hence be estimated. Han himself developed a scheme to estimate the ‘total disturbance’ from the measured output, which is now known as extended state observer (ESO) to be used for estimation of not only the state but also the total disturbance. Once the total disturbance is estimated, it is then cancelled (compensated) in the feedback loop by ESO-based output-feedback control. Although we could find estimation/cancellation strategy sporadically in other control methods like the adaptive control where most often, the internal unknown parameters are estimated/cancelled and the internal model principle where a class of external disturbances from a dynamic exosystem can be estimated/cancelled, ESO is a systematic scheme to estimate in large scale ‘total disturbance’. In the past two decades, the effectiveness of ADRC control strategy has been demonstrated in many engineering applications like control of synchronous motors (Sira-Ramírez, Linares-Flores, García-Rodríguez, & Contreras-Ordaz, 2014, DC–DC power converter Sun & Gao, 2005, control system in super-conducting RF cavities Vincent et al., 2011, flight vehicles control Xia & Fu, 2013 and Gasoline Engines Xue et al., 2015), to name just a few. On the other hand, many theoretical issues including convergence of ESO and applicability of ADRC in stabilisation and output regulation for uncertain nonlinear systems have recently been developed, see Guo and Zhao (2013), Guo and Zhao (2016), Huang and Xue (2014), Jiang, Huang, and Guo (2015), Li, Yang, Chen, and Chen (2012), Pu, Yuan, Yi, and Tan (2015), Shao and Gao (2017), Xue and Huang (2014), Zheng, Gao, and Gao (2007). Although great progress has been achieved, it should be noted however that very few literatures address ADRC for stochastic systems. In contrast to deterministic cases, a fundamental technical obstacle in the stochastic Lyapunov analysis is that the Itô differentiation involves not
only the gradient but also the Hessian of the Lyapunov function. We could find some breakthrough efforts in stabilisation for stochastic nonlinear systems driven by white noise like the generalisation of Sontag’s stabilisation formula to the stochastic counterpart (Florchinger, 1993), recursive backstepping control design to stochastic strict-feedback systems driven by white noise based on a risk-sensitive cost criterion in Pan and Basar (1999), and the results extending inverse optimal stabilisation for systems with deterministic uncertainties to the stochastic cases (Deng & Krstić, 1999). The robust mixed \( H_2/H_\infty \) globally linearised filter design problem is investigated for a general nonlinear stochastic time-varying delay system with external disturbance in Mao, Deng, and Wan (2016) and the divided state feedback control of stochastic systems is addressed in Zhao and Deng (2015). It is recognised that output-feedback control is more difficult and challenging than full state-feedback. In recent years, output-feedback design for stochastic nonlinear systems driven by white noise has become an intensive investigated research topic. Examples can be found in quartic Lyapunov function approach to overcome the obstacle caused by the Itô differentiation to obtain the first result on output-feedback stabilisation for stochastic output-feedback nonlinear systems driven by white noise (Deng & Krstić, 1999) and then for the systems driven by white noise whose covariance is with unknown bound (Deng & Krstić, 2000). A series of continuous progresses are made such as decentralised robust adaptive output-feedback stabilisation for large-scale stochastic nonlinear systems driven by white noise (Liu, Zhang, & Jiang, 2007) and output-feedback stabilisation for a class of stochastic nonminimum-phase nonlinear systems driven by white noise (Liu, Ji, & Zhang, 2008), to name just a few. The stochastic \( H_\infty \) (or mixed \( H_2/H_\infty \)) control or filtering for uncertain stochastic nonlinear systems where the system state is corrupted not only by the white noise but also by the exogenous disturbance signal have been investigated in Niu, Ho, and Wang (2008), Zhang, Chen, and Tseng (2005) and the references therein. The adaptive neural network (Chen, Jiao, Li, & Li, 2010) based output-feedback control combining with the backstepping technique has been developed to deal with stochastic nonlinear systems driven by white noise, where the uncertain nonlinear terms are allowed to be functions of all states variables (Chen et al., 2010), and the adaptive fuzzy backstepping output-feedback control approach is developed in Tong, Li, Li, and Liu (2011) to overcome ‘explosion of complexity’ with unknown nonlinear system functions being approximated by fuzzy logic systems guaranteeing that all signals of the closed-loop system are semi-globally uniformly ultimately bounded (SGUUUB) in mean square topology.

In spite of huge works on output-feedback stabilisation for the stochastic nonlinear systems driven by the white noise, the output-feedback stabilisation for uncertain stochastic nonlinear systems driven by external bounded stochastic noise without known statistical characteristics receives little attention. Actually, non-white bounded stochastic noise exists widely in practical systems, see, for instance, Huang, Zhu, Ni, and Ko (2002), Hu, Chen, and Zhu (2012) and Huang and Zhu (2004). In addition, it is often difficult to require knowledge of the statistical characteristics of the noise so that it may not be described by the Wiener process with some known statistical characteristics. On the other hand, many available output-feedback controls for stochastic systems driven by the white noise are constructed recursively in the framework of conventional backstepping design technique that has the conspicuous drawbacks from two respects. Firstly, it requires very much the knowledge of the model or the system uncertainties; Secondly, it inevitably leads to the problem of ‘explosion of complexity’ caused by the repeated differentiations of virtual controls, which makes the complexity of control grow dramatically as the order of system increases. Thirdly, most of the robust control approaches to deal with stochastic uncertainties such as the stochastic \( H_\infty \) and mixed \( H_2/H_\infty \) focus on the worst-case scenario which makes the controller design conservative. However, ADRC is an estimation/cancellation strategy where the disturbance is estimated in real time through ESO and cancelled in the feedback loop, which reduces significantly the control energy in practice.

Based on ADRC’s nature of almost free of mathematical models, simple control structure, and powerful decoupling function, we address, in this paper, ADRC approach to output-feedback stabilisation for a class of multi-input multi-output (MIMO) nonlinear systems with vast stochastic uncertainties including unknown system dynamics, unknown stochastic inverse dynamics, external stochastic disturbance without known statistical characteristics, uncertain nonlinear interactions between subsystems, and uncertainties caused by the partially unknown control parameters, where the inverse dynamics equations are disturbed by both the external stochastic disturbance and white noise. Precisely, the system that we consider in this paper is the following partial exact feedback linearisable MIMO system (Guo & Zhao, 2013; Isidori, 1995) with vast stochastic uncertainties:

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= A_{ni}x_i(t)dt + B_{ni} \\
&+ f_i(t,x(t),\zeta(t),w(t)) + \sum_{l=1}^{m} c_{il}u_l(t) \, dt, \\
\frac{dx(t)}{dt} &= h_1(t,x(t),\zeta(t),w(t))dt \\
&+ h_2(t,x(t),\zeta(t),w(t))dw_1(t), \\
y_i(t) &= C_{ni}x_i(t), \quad i = 1, 2, \ldots, m,
\end{align*}
\]

where \( x(t) = (x_1(t), \ldots, x_m(t))^T \in \mathbb{R}^n \) (\( n = n_1 + \cdots + n_m \)), \( u(t) = (u_1(t), \ldots, u_m(t))^T \in \mathbb{R}^m \), and \( y(t) = (y_1(t), \ldots, y_m(t))^T \in \mathbb{R}^m \) are the state, control (input), and output (measurement) of the system, respectively; \( \zeta(t) \in \mathbb{R}^d \) denotes the state of stochastic inverse dynamics; The functions \( f_i : [0, \infty) \times \mathbb{R}^{n+i+1} \to \mathbb{R} \), \( h_1 : [0, \infty) \times \mathbb{R}^{n+i+1} \to \mathbb{R}^s \), and \( h_2 : [0, \infty) \times \mathbb{R}^{n+i+1} \to \mathbb{R}^{sp} \) are unknown; The constants \( c_{il} \) (\( i = 1, 2, \ldots, m \)) are the control coefficients which are not exactly known yet have nominal values \( c_{il}^* \) that are sufficiently close to \( c_{il}^* \) (\( W_i(t) \)) is a \( p \)-dimensional standard Wiener process defined on a complete probability space \((\Omega, \mathcal{F}, [\mathcal{F}_t]_{t \geq 0}, P)\) with \( \Omega \) being a sample space, \( \mathcal{F} \) a \( \sigma \)-field, \( [\mathcal{F}_t]_{t \geq 0} \) a filtration, and \( P \) the probability measure; The \( W(t) \) is \( \psi(t, W_2(t)) \in \mathbb{R} \) for some bounded (possibly) unknown function \( \psi(\cdot) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) is the external stochastic disturbance, where \( \{W_2(t)\}_{t \geq 0} \) is a \( q \)-dimensional standard Wiener process defined on \((\Omega, \mathcal{F}, [\mathcal{F}_t]_{t \geq 0}, P)\) as well and is mutually
independent with \([W_i(t)]_{t \geq 0}\); In addition,

\[
A_n = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix}_{n_i \times n_i}, \quad B_n = (0, \ldots, 0, 1)_{n_i \times 1},
\]

\[
C_n = (1, 0, \ldots, 0)_{1 \times n_i}.
\]

It should be pointed out that the external stochastic disturbance \(w(t)\) is quite general. First, the statistical characteristics of \(w(t)\) is not necessarily to be known due to unknown of the function \(\psi(\cdot)\). Second, the \(w(t)\) covers the disturbance without stochastic properties investigated via ADRC in aforementioned papers when \(\psi(\cdot)\) is the function of time \(t\) only: \(w(t) \triangleq \psi(t)\). In this case, system (1) is reduced to the class of MIMO nonlinear systems composed of coupled essentially integral-chain subsystems with controls matched uncertainties considered in Guo and Zhao (2013) and Guo and Zhao (2016) when inverse dynamics are no stochastic properties, i.e. \(h_2(\cdot) \equiv 0\). In addition, for the stochastic case, the bounded stochastic noise considered in Guo and Zhao (2013), Guo and Zhao (2016), and Guo, Wu, and Zhou (2016) in many practical systems are also covered. Finally, system (1) covers special SISO nonlinear systems with stochastic uncertainties considered in Guo et al. (2016) as a special case of \(m = 1\), \(\zeta(\cdot) \equiv 0\), and \(c_1 = 1\).

For each \(1 \leq i \leq m\), we define the stochastic total disturbance as

\[
x_{i(n_i+1)}(t) \triangleq f_i(t, x(t), \xi(t), w(t)) + \sum_{l=1}^{m} (c_{il} - c_{il}^*)u_l(t),
\]

which contains unknown system dynamics, unknown stochastic inverse dynamics, external stochastic disturbance, uncertain nonlinear interactions between subsystems, and uncertainties caused by the deviation of control parameters from their nominal values. So the stochastic uncertainties in the considered system (1) are stated. Next, we propose a time-varying gain ESO and an ESO-based feedback control for \(x\)-subsystem of (1). The asymptotic mean square stability is stated. And last, both constant high gain and time-varying gain ESO, as well as the corresponding ESO-based output-feedback controls are applied to \(x\)-subsystem of (1) where the stochastic total disturbance includes only unknown stochastic inverse dynamics, external stochastic disturbance, and uncertainties caused by the deviation of control parameters from their nominal values. The proofs of the main results are presented in Section 3. Finally, in Section 4, we present some numerical simulations for illustration of estimation effects of ESO, stabilisation effects of ADRC, and peaking value reduction by the proposed method.

The following notations are used throughout the paper. The \(\mathbb{R}^n\) represents the \(n\)-dimensional Euclidean space and \(\mathbb{R}^{n \times m}\) stands for the space of real \(n \times m\)-matrices; For a vector or matrix \(X, X^\top\) denotes its transpose; For a square matrix \(X, \text{Tr}(X)\) denotes its trace; \(I_{n \times n}\) denotes the \(n \times n\) unit matrix; \(\lambda_{\min}(X)\) and \(\lambda_{\max}(X)\) represent the minimal and maximal eigenvalues of the symmetric real matrix \(X\); respectively; \(|X|\) denotes the Euclidean norm of the vector \(X\), and the corresponding induced norm when \(X\) is a matrix; The \((a^{(ij)})_{m \times n}\) denotes an \(m \times n\) matrix with entries \(a^{(ij)}\); For a differentiable function \(f : \mathbb{R}^n \to \mathbb{R}\), \(\frac{df}{dz}\) is a matrix valued function \(\frac{df}{dz} : \mathbb{R}^n \to \mathbb{R}^{m \times s}\), \(i = 1, 2, \ldots, n\) for \(z = (z_1, \ldots, z_n)^\top\); For a twice differentiable function \(f : \mathbb{R}^n \to \mathbb{R}\), \(\frac{df}{dz} \triangleq \left(\frac{df}{dz_j}\right)_{n \times s}\), \(j, i = 1, 2, \ldots, n\) for \(z = (z_1, \ldots, z_n)^\top\); For a matrix valued function \(f : \mathbb{R}^n \to \mathbb{R}^{m \times s}\), \(\frac{df}{dz} \triangleq \left(\frac{df}{dz_j}\right)_{m \times s}\), \(j, i = 1, 2, \ldots, m\), \(m, n, s \geq 1\), \(s \geq 1\); In addition, for all \(1 \leq i \leq m\),

\[
\eta_i = (\eta_{1i}, \ldots, \eta_{mi})^\top, \quad \eta = (\eta_1^\top, \ldots, \eta_m^\top)^\top,
\]

\[
x_i = (x_{1i}, \ldots, x_{mi})^\top,
\]

\[
\bar{x}_i = (x_{1i}, \ldots, x_{(n_i+1)i})^\top,
\]

\[
\kappa = (\kappa_1^\top, \ldots, \kappa_m^\top)^\top,
\]

\[
\alpha = (\alpha_{1i}, \ldots, \alpha_{mi}),
\]

\[
\zeta = (\zeta_1, \ldots, \zeta_i)^\top,
\]

\[
\phi = (\phi_1, \ldots, \phi_q)^\top.
\]

\[
(4)
\]

2. Main results

2.1 ADRC with constant gain ESO

Although the linear ESO takes its advantage of simple parameter turning, it also brings the peaking value problem, slow convergence, and many other problems contrast to fast tracking and small peaking value indicated numerically in Han (2008) by nonlinear ESO. By taking these points into account and for generality as well, we introduce more general one-parameter tuning ESO proposed in Guo and Zhao (2013) for \(x\)-subsystem of (1) as follows:

\[
\begin{align*}
\dot{\hat{x}}_{i1}(t) &= \hat{x}_{i2}(t) + e^{n_i-1}g_{i1}(\eta_{i1}(t)), \\
\dot{\hat{x}}_{i2}(t) &= \hat{x}_{i3}(t) + e^{n_i-2}g_{i2}(\eta_{i1}(t)), \\
& \vdots \\
\dot{\hat{x}}_{im}(t) &= \hat{x}_{i(n_i+1)}(t) + g_{im}(\eta_{i1}(t)) + \sum_{l=1}^{m} c_{il}^*u_l(t), \\
\dot{\hat{x}}_{i(n_i+1)}(t) &= \frac{1}{\varepsilon}g_{i(n_i+1)}(\eta_{i1}(t)), \quad i = 1, 2, \ldots, m,
\end{align*}
\]
where $\eta_i(t) = \frac{\gamma_i(t)}{\epsilon^n}$ with the output $\gamma_i(t)$ injected from system (1). $g_i \in C(\mathbb{R}; \mathbb{R}) (i = 1, 2, \ldots, n_i + 1)$ are designed functions which could be linear or nonlinear to be specified later, $\epsilon > 0$ is the tuning parameter, and $\epsilon^n_i$ is the nominal value of $\epsilon_i^n$ indicated after (1). The ESO (4) covers the one-parameter tuning linear ESO proposed in Gao (2003) in terms of bandwidth as its special case by choosing $g_i$ to be linear functions. The main idea of ESO (4) is estimating total disturbance. Therefore, we need to choose appropriate $g_i(\cdot)$'s so that the $\tilde{x}_i(t)$ approaches $x_i(t)$ by tuning the parameter $\epsilon$. In particular, $\tilde{x}_i(n_i+1)(t)$ is an approximation of the stochastic total disturbance $x_i(n_i+1)(t)$ defined by (3). There is only one parameter $\epsilon$ in ESO (4) to be tuned on the basis of the estimation accuracy and the variation of the stochastic total disturbance. In general, the higher the estimation accuracy is required and the faster the stochastic total disturbance changes, the smaller the parameter $\epsilon$ needs to be tuned.

**Remark 2.1:** It should be noted that the actual estimation error used in ESO (4) is $\gamma_i - \tilde{x}_i$, not $\frac{\gamma_i}{\epsilon^n}$. In ESO (4), $\epsilon^{n_i -} g_i \left(\frac{\gamma_i}{\epsilon^n}\right) (j = 1, 2, \ldots, n_i + 1)$ behave as the nonlinear gain functions with the estimation error $\gamma_i - \tilde{x}_i$ injected. The reason for designing this class of nonlinear ESO (with linear ESO as its special case) is that a sufficient condition for the choice of the nonlinear functions $g_i(\cdot) (j = 1, 2, \ldots, n_i + 1)$ could be given as that in Assumption A.4 so that ESO (4) has only one parameter $\epsilon$ to be tuned.

Here and throughout the paper, we always drop $\epsilon$ for the solution of (4) by abuse of notation without confusion.

The ESO (4) based output-feedback control is designed as

$$
u_i(t) = \sum_{j=1}^{m} c_{ij}^g(v_l(\tilde{x}_1(t)), \tilde{x}_2(t), \ldots, \tilde{x}_{m}(t)) - \tilde{x}_i(n_i+1)(t),$$

$$1 \leq i \leq m,$$  \hspace{1cm} (5)

where $c_{ij}^g$ will be specified in (15) later, and $v_l : \mathbb{R}^m \to \mathbb{R}$ could be linear or nonlinear, and is chosen so that the following system (the nominal part of $x$-subsystem of (1) without stochastic uncertainties and coupling interactions between subsystems) is exponentially stable for all $1 \leq i \leq m$:

$$\dot{x}_i(t) = (x_i(2)(t), x_i(3)(t), \ldots, v_i(x_i(t)))^\top,$$

$$x_i = (x_{i1}, \ldots, x_{in_i})^\top \in \mathbb{R}^{n_i},$$  \hspace{1cm} (6)

which is essentially guaranteed by Assumption A.3.

To obtain stability for the closed-loop of $x$-subsystem of (1) under ESO (4) based output-feedback control (5) including ESO's estimation of unmeasured states and stochastic total disturbance, we need the following assumptions.

Assumption A.1 is about the unknown function $\psi(\cdot)$ that defines the external stochastic disturbance.

**Assumption A.1:** The $\psi(t, \vartheta) : [0, \infty) \times \mathbb{R}^q \to \mathbb{R}$ is twice continuously differentiable with respect to their arguments and there exists a (known) constant $D_1 > 0$ such that for all $\vartheta \in \mathbb{R}^q$,

$$|\psi(t, \vartheta)| + \left|\frac{\partial \psi(t, \vartheta)}{\partial t}\right| + \left|\frac{\partial \psi(t, \vartheta)}{\partial \vartheta}\right| + \sum_{i=1}^{q} \left|\frac{\partial^2 \psi(t, \vartheta)}{\partial \vartheta_i^2}\right| \leq D_1.$$  \hspace{1cm} (7)

Assumption A.2 is a prior assumption about the unknown functions $f_i(\cdot), h_1(\cdot)$, and $h_2(\cdot)$ in system (1).

**Assumption A.2:** The $f_i(\cdot) (i = 1, \ldots, m)$ are twice continuously differentiable with respect to their arguments and $h_i(\cdot) (i = 1, 2)$ are locally Lipschitz continuous in $(x, \xi, w) \in \mathbb{R}^n \times \mathbb{R}^r$ uniformly in $t \in [0, \infty)$. There exist (known) constants $D_i > 0 (i = 2, 3, 4)$ and a non-negative continuous function $\varsigma \in C(\mathbb{R}; \mathbb{R})$ such that for all $t \geq 0, x \in \mathbb{R}^n, \xi \in \mathbb{R}^r, w \in \mathbb{R}$,

$$\sum_{i=1}^{m} \left|\frac{\partial f_i(t, x, \xi, w)}{\partial t}\right| + \left|\frac{\partial h_1(t, x, \xi, w)}{\partial x}\right| \leq D_2$$

$$+ D_3 \|x\| + \varsigma(w), \hspace{1cm} i = 1, 2, \ldots, m;$$  \hspace{1cm} (8)

$$\sum_{i=1}^{m} \left|\frac{\partial f_i(t, x, \xi, w)}{\partial x}\right| + \left|\frac{\partial h_1(t, x, \xi, w)}{\partial \xi}\right| + \sum_{j=1}^{s} \left|\frac{\partial^2 f_i(t, x, \xi, w)}{\partial \xi_j^2}\right|$$

$$+ \sum_{i=1}^{m} \left|\frac{\partial f_i(t, x, \xi, w)}{\partial w}\right| + \left|\frac{\partial h_2(t, x, \xi, w)}{\partial w^2}\right|$$

$$+ \sum_{j=1}^{p} \sum_{i=1}^{s} \left|\frac{\partial f_i(t, x, \xi, w)}{\partial w}\right| \leq D_4 + \varsigma(w), \hspace{1cm} i = 1, 2, \ldots, m.$$  \hspace{1cm} (9)

**Remark 2.2:** Since the external stochastic disturbance is coupled with unmodelled dynamics as part of the stochastic total disturbance estimated by ESO, it is reasonable to assume that both the external stochastic disturbance and its ‘variation’ are bounded as stated in Assumption A.1.

**Remark 2.3:** Since the stochastic total disturbance is regarded as an extended state variable of the $x$-subsystem of (1) to be estimated by ESO, its ‘variation’ certainly needs to be limited or can be ‘absorbed’ by decaying parts in the closed-loop. The conditions of Assumption A.1 and (8), (9) in Assumption A.2 are essentially about the L0 differential (or ‘variation’) of $f_i(\cdot) (i = 1, 2, \ldots, m)$ computed as that in (2). This is to ensure that its drift term is growing linearly with respect to $x$ and $\eta$, and its diffusion term is bounded almost surely as concluded in (34) since $f_i(\cdot)$ is a part of the stochastic total disturbance.

Assumption A.3 is a prior assumption about the functions $v_i(\cdot)$ in (5).

**Assumption A.3:** For each $1 \leq i \leq m, v_i(\cdot)$ in (5) is continuously differentiable and globally Lipschitz continuous with Lipschitz constant $L_i, v_i(0) = 0$. There exist constants $\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \lambda_{i4}, \alpha_i,$
and continuously differentiable functions \(V_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}\) which are positive definite and radially unbounded such that
\[
\lambda_1 \|z\|^2 \leq V_1(z) \leq \lambda_2 \|z\|^2, \quad \lambda_3 \|z\|^2 \leq U_1(z) \leq \lambda_4 \|z\|^2,
\]
for some \(n_1 \in \mathbb{N}\) and constants \(\lambda_i > 0\).

In Assumption A.3, the continuously differentiability and globally Lipschitz continuity of \(v_i(z)\) imply that
\[
\left| \frac{\partial v_i(z)}{\partial z_j} \right| \leq L_i, \quad \forall z = (z_1, z_2, \ldots, z_m)^T \in \mathbb{R}^m,
\]
for some non-negative continuous functions \(L_i : \mathbb{R}^m \rightarrow \mathbb{R}\) and constants \(\alpha_i > 0\).

A simple feedback control (5) satisfying conditions of Assumption A.3 is certainly the linear one. Let
\[
v_i(z_1, \ldots, z_m) = c_{i1}z_1 + \cdots + c_{in}z_n, \quad 1 \leq i \leq m. \tag{12}
\]
In this case, the system (6) is linear: \(\dot{z}_i(t) = E_i z_i(t)\), where \(z_i = (z_{i1}, \ldots, z_{im})^T\) and
\[
E_i = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
c_{i1} & c_{i2} & \cdots & c_{i(n-1)} & c_{in}
\end{pmatrix}_{n \times n_i}, \tag{13}
\]
Thus, (9) in Assumption A.3 is to make the parameters \(c_{ij} (j = 1, 2, \ldots, n_i, i = 1, 2, \ldots, m)\) be chosen so that the matrices \(E_i\) are Hurwitz, which is verified in the proof of Corollary 2.1.

**Remark 2.4:** The stability of the closed-loop system cannot be concluded from this Assumption A.3 directly. For instance, there exists a positive correlation between \(\sigma(I)\) and \(n_{lid(n+1)}\) (t) in (39) (the terms influencing the stability of the closed-loop system (39)) and the estimation errors of ESO for the unmeasured states and stochastic total disturbance. However, the estimation error analysis of ESO with respect to the type error equation (\(\eta\)-subsystem of (39)) is never trivial in the stochastic Lyapunov analysis due to the complicated stochastic total disturbance including external stochastic disturbance, unknown state-dependent dynamics, unknown stochastic inverse dynamics, coupling interactions between subsystems, and uncertainties caused by the deviation of control parameters from their nominal values.

Assumption A.4 is on the designed functions \(g_{ij}()\) in ESO (4) and the prior estimates \(\hat{c}_{ij} (i, l = 1, 2, \ldots, m)\) for the unknown control parameters \(c_{il} (i, l = 1, 2, \ldots, m)\) in system (1).

**Assumption A.4:** For each \(1 \leq i \leq m\), \(|g_{ij}(z)| \leq k_{ij}|z|\) for all \(z \in \mathbb{R}\) and \(1 \leq j \leq n_i + 1\). There exist constants \(\lambda_{i1}^*, \lambda_{i2}^*, \lambda_{i3}^*, \lambda_{i4}^*, \beta_i\), \(F_i\) are Hurwitz, which is verified in the proof of Corollary 2.1.

Thus, (14) in Assumption A.4 is to make the parameters \(k_{ij} (j = 1, 2, \ldots, n_i + 1, i = 1, 2, \ldots, m)\) be chosen so that the matrices \(F_i\) are Hurwitz, which is verified in the proof of Corollary 2.1.
where the parameters $\beta_i (i = 1, 2, \ldots, m)$, $\tilde{c}^{+}_{id} (l = 1, 2, \ldots, m)$, and $\lambda_0^m (i = 1, 2, \ldots, m)$ are specified in (13), (14), (18), and (13), respectively.

In particular, if we only design linear ESO, as stated in Corollary 2.1, the priori information on $\tilde{c}^{+}_{id} (i = 1, 2, \ldots, m)$ can be expressed more concisely as follows:

$$
\sum_{i, l, d=1}^{m} 2\lambda_{\max}(Q_i) |(\tilde{c}^{+}_{id} - c^{+}_{id})| < 1.
$$

We take a simple first-order SISO system as an example to look at this priori information assumption. Now, $n = m = 1$ in system (1), and we design the following second-order linear ESO:

$$
\begin{align*}
\dot{x}_{11}(t) &= \dot{x}_{12}(t) + \frac{1}{\varepsilon}(y_1(t) - \dot{x}_{11}(t)) + c^{+}_{11} u_{11}(t), \\
\dot{x}_{12}(t) &= \frac{1}{\varepsilon}(y_1(t) - \dot{x}_{11}(t)),
\end{align*}
$$

where the matrix $F_1$ in (18) and $Q_1$ satisfying $Q_1 F_1 + F_1^T Q_1 = -I_{2 \times 2}$ can be easily obtained as follows:

$$
F_1 = \begin{pmatrix}
-1 & 1 \\
-1 & 0
\end{pmatrix}_{2 \times 2},
$$

$$
Q_1 = \begin{pmatrix}
1 & -1/2 \\
-1/2 & 1
\end{pmatrix}_{2 \times 2}.
$$

A direct computation shows that the prior estimate $c^{+}_{11}$ for unknown control parameter $c_{11}$ should satisfy

$$
\frac{|c_{11} - c^{+}_{11}|}{|c^{+}_{11}|} < \frac{2}{5 + \sqrt{5}}.
$$

Remark 2.6: We notice from (5) and (14) that

$$
\sum_{l=1}^{m} c^{+}_{il} u_l(t) = v_l(x_{il}(t), \ldots, x_{im}(t)) - \dot{x}_{i}(n_{i} + 1)(t),
$$

where $-\dot{x}_{i}(n_{i} + 1)(t)$ is used to cancel (decouple or compensate) the stochastic total disturbance $x_{i}(n_{i} + 1)(t)$ defined by (3) for all $1 \leq i \leq m$. Because of the real-time cancellation effect of $-\dot{x}_{i}(n_{i} + 1)(t)$, the $m$ subsystems become approximately independent $m$ linear time invariant systems without any disturbances and interactions between subsystems, which reveals the powerful decoupling function of ADRC.

The main result on practical mean square stability of the closed-loop of $x$-subsystem of (1), (4), and (5), which includes ESO's practical mean square estimation of unmeasured states and stochastic total disturbance, is summarised in the succeeding Theorem 2.1.

**Theorem 2.1:** Under Assumptions A.1–A.4, for any initial values $x(0) \in \mathbb{R}^n$, $\dot{x}(0) \in \mathbb{R}^{n+m}$, $\zeta(0) \in \mathbb{R}^n$, the closed-loop of the $x$-subsystem of (1), (4), and (5) is practically mean square stable in the sense that there are a constant $c^+ > 0$ (specified by (44) later) and an $\varepsilon$-dependent constant $t^*_\varepsilon > 0$ with $\varepsilon \in (0, \varepsilon^*)$ such that for all $t \geq t^*_\varepsilon$,

$$
E|x(t)|^2 \leq \Gamma \varepsilon^{2n+3-2j},
$$

and

$$
E|x(t)|^2 \leq \Gamma \varepsilon, \quad 1 \leq j \leq n + 1, 1 \leq i \leq m,
$$

where $\Gamma > 0$ is an $\varepsilon$-independent constant.

A simple constant gain ADRC satisfying Assumptions A.3–A.4 is the linear one where all $g_i(\cdot)$'s in ESO (4) and $v_i(\cdot)$'s in feedback control (5) are linear functions as defined in (18) and (12), respectively. Let $H_i$ and $Q_1$ be the matrices satisfying the Lyapunov equations $H_i E_i + E_i^\top H_i = -I_{n \times n}$ and $Q_1 F_i + F_i^\top Q_1 = -I_{(n_i + 1) \times (n_i + 1)}$, respectively. By Theorem 2.1, we have the following Corollary 2.1.

**Corollary 2.1:** Suppose that the matrices $E_i$ and $F_i$ are Hurwitz for all $1 \leq i \leq m$, (14) holds and $\sum_{i, l, d=1}^{m} 2\lambda_{\max}(Q_i) |(c_{il} - c^{+}_{il})| < 1$. Then under Assumptions A.1–A.2, for any initial values $x(0) \in \mathbb{R}^n$, $\dot{x}(0) \in \mathbb{R}^{n+m}$, $\zeta(0) \in \mathbb{R}^n$, the closed-loop of $x$-subsystem of (1) under linear ESO (4) based linear output-feedback control (5) is practically mean square stable in the sense that there are a constant $c^+ > 0$ and an $\varepsilon$-dependent constant $t^*_\varepsilon > 0$ with $\varepsilon \in (0, \varepsilon^*)$ such that for all $t \geq t^*_\varepsilon$,

$$
E|x(t)|^2 \leq \Gamma \varepsilon^{2n+3-2j},
$$

and

$$
E|x(t)|^2 \leq \Gamma \varepsilon, \quad 1 \leq j \leq n + 1, 1 \leq i \leq m,
$$

where $\Gamma > 0$ is an $\varepsilon$-independent constant.

### 2.2 ADRC with time-varying gain ESO

In last subsection, the constant high gain ESO (4) is designed to cancel the stochastic total disturbance for system (1) and the corresponding ESO-based output-feedback control stabilises the $x$-subsystem of (1) at the origin. The merit of constant high gain lies in its fast convergence and filter function for high frequency noise (Guo & Zhao, 2016). However, the main problem for constant high gain ESO, likewise many other high gain designs, is the peaking value problem near the initial stage caused by different initial values of $x$-subsystem of (1) and ESO (Guo & Zhao, 2016). In addition, in Section 2, we notice that Theorem 2.1 and Corollary 2.1 give only practical mean square stability for the closed-loop of $x$-subsystem. In this subsection, we present asymptotic mean square stability for the closed-loop system with time-varying gain ESO by properly selecting gain function, where the uniform boundedness of the external stochastic disturbance in previous assumption is also relaxed to
be exponential growth. The peaking value reduction with time-varying gain ESO is illustrated through numerical simulations in Section 4. Precisely, we propose a time-varying gain ESO for \( x \)-subsystem of (1) as follows:

\[
\begin{aligned}
\dot{x}_{11}(t) &= \dot{x}_{12}(t) + \sum_{i=1}^{m} \frac{1}{\rho_{ii}(t)} g_{ii} \left( r^m(t) y_i(t) - \dot{x}_{i1}(t) \right), \\
\dot{x}_{12}(t) &= \dot{x}_{13}(t) + \sum_{i=1}^{m} \frac{1}{\rho_{ii}(t)} g_{ii} \left( r^m(t) y_i(t) - \dot{x}_{i1}(t) \right), \\
&\quad \vdots \\
\dot{x}_{ni}(t) &= \dot{x}_{n(i+1)}(t) + \sum_{i=1}^{m} \frac{1}{\rho_{ii}(t)} g_{ii} \left( r^m(t) y_i(t) - \dot{x}_{i1}(t) \right), \\
\dot{x}_{n(i+1)}(t) &= r(t) \dot{x}_{n(i+1)}(t) + \sum_{i=1}^{m} \frac{1}{\rho_{ii}(t)} g_{ii} \left( r^m(t) y_i(t) - \dot{x}_{i1}(t) \right),
\end{aligned}
\]

\[ (21) \]

where \( g_{ii} \in C([0, \infty); \mathbb{R}) \) \((j = 1, 2, \ldots, n_i + 1)\) could be linear or nonlinear designed functions satisfying Assumption A.4, \( r \in C((0, \infty); (0, \infty)) \) is the time-varying gain function, and \( y_i(t) \) and \( \rho_{ii}(t) \) are the output and nominal value of control coefficients \( c_{ij} \) of system (1).

The Assumption A.1 about the unknown function \( \psi(t, \theta) \) defining the external stochastic disturbance is relaxed (at least theoretically) to the following Assumption A.1*.

**Assumption A.1**: \( \psi(t, \theta) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) is twice continuously differentiable with respect to arguments and there exist constants \( M_1, M_2, b > 0 \) such that for all \( \theta \in \mathbb{R}^d \),

\[
\begin{aligned}
&|\psi(t, \theta)| + \left\| \frac{\partial \psi(t, \theta)}{\partial t} \right\| + \left\| \frac{\partial \psi(t, \theta)}{\partial \theta} \right\| \\
&\quad + \sum_{i=1}^{q} \left| \frac{\partial^2 \psi(t, \theta)}{\partial \theta_i \partial \theta_j} \right| \leq M_1 + M_2 e^{bt}.
\end{aligned}
\]

The asymptotic mean square stability of the closed-loop of \( x \)-subsystem of (1), (5), and (21), which includes ESO's asymptotic mean square estimation of unmeasured states and stochastic total disturbance, is summarised in the succeeding Theorem 2.2.

**Theorem 2.2**: Let \( r(t) = e^{at}, a > 3b \). Then under Assumptions A.1* and A.2–A.4 with \( \varsigma(w) = |w| \) for \( w \in \mathbb{R} \), for any initial values \( x(0) \in \mathbb{R}^n, \dot{x}(0) \in \mathbb{R}^{n+m}, \zeta(0) \in \mathbb{R}^d \), the closed-loop of \( x \)-subsystem of (1), (5), and (21) is asymptotically mean square stable in the sense that

\[
\lim_{t \to \infty} \mathbb{E}[x_{ij}(t) - \dot{x}_{ij}(t)]^2 = 0, \quad 1 \leq j \leq n_i + 1, 1 \leq i \leq m,
\]

and

\[
\lim_{t \to \infty} \mathbb{E}[x_{ij}(t)]^2 = 0, \quad 1 \leq j \leq n_i, 1 \leq i \leq m.
\]

Similarly, a simple time-varying gain ADRC satisfying conditions of Assumptions A.3–A.4 is the linear one where all \( g_{ij}(\cdot) \)'s in ESO (21) and \( v_{ij}(\cdot) \)'s in feedback control (5) are linear functions as defined in (18) and (12), respectively. Similar to Corollary 2.1, we have Corollary 2.2.

**Corollary 2.2**: Suppose that the matrices \( E_{i} \) and \( F_{i} \) are Hurwitz for all \( 1 \leq i \leq m \), (15) holds, and \( \sum_{i=1}^{m} \lambda_{\max}(Q_i)(c_{ii} - c_{ii}^*)^2 \leq (\xi_0^*)^{2}k_{d(n+1)}^{2} \). Let \( r(t) = e^{at}, a > 3b \). Then under Assumptions A.1* and A.2 with \( \varsigma(w) = |w| \) for \( w \in \mathbb{R} \), for any initial values \( x(0) \in \mathbb{R}^n, \dot{x}(0) \in \mathbb{R}^{n+m}, \zeta(0) \in \mathbb{R}^d \), the closed-loop of \( x \)-subsystem of (1) under linear time-varying gain ESO (21) based linear output-feedback (5) is asymptotically mean square stable in the sense that

\[
\lim_{t \to \infty} \mathbb{E}[x_{ij}(t) - \dot{x}_{ij}(t)]^2 = 0, \quad 1 \leq j \leq n_i + 1, 1 \leq i \leq m,
\]

and

\[
\lim_{t \to \infty} \mathbb{E}[x_{ij}(t)]^2 = 0, \quad 1 \leq j \leq n_i, 1 \leq i \leq m.
\]

**Remark 2.7**: If Assumption A.1* in both Theorem 2.2 and Corollary 2.2 is relaxed to Assumption A.1, the time-varying gain function \( r(t) \) can be chosen as \( r(t) = e^{at} \) for any \( a > 0 \).

**Remark 2.8**: As indicated in Guo and Zhao (2016), the time-varying gain ESO degrades the ability of ESO to filter high noise while the constant gain ESO does not. In practical applications, we can use time-varying gain \( r(t) \) as follows: (a) Given a small initial value \( r(0) > 0 \); (b) From the constant high gain, we obtain the convergent high gain value \( \frac{1}{e} (0 < \varepsilon < 1) \) which can also be obtained by trial and error experiment for practical systems; (c) The gain function is initiated from the small value \( r(0) > 0 \) and then increases continuously to a large constant high gain \( \frac{1}{e} \). Specially, \( r(t) \) can be chosen as

\[
r(t) = \begin{cases}
\frac{e^{at}}{1 - \frac{1}{k} \ln \varepsilon}, & 0 \leq t \leq -\frac{1}{a} \ln \varepsilon, \\
\frac{1}{e}, & t \geq -\frac{1}{a} \ln \varepsilon,
\end{cases}
\]

where \( a > 0 \) is used to control the convergent speed and the peaking value. The larger \( a \) is, the faster convergence but larger peaking; while the smaller \( a \) is, the lower convergence speed and smaller peaking. The mean square practical ability of the closed-loop system of \( x \)-subsystem of (1), (5), and (21) with time-varying gain \( r(t) \) given by (23) can also be achieved since the ESO (21) is reduced ESO (4) when \( t \geq -\frac{1}{a} \ln \varepsilon \).

### 2.3 ADRC utilising partial information

In this subsection, a special ADRC output-feedback is constructed in which the system functions are partially known in the sense that for all \( 1 \leq i \leq m \), \( f_i(t, x, \zeta, w) = f_i^x(t, x) + f_i^{**}(\zeta, w) \), where \( f_i^x(t, x) \) is known. In other words, the stochastic uncertainties only come from stochastic inverse dynamics, external stochastic disturbance, and deviation of control parameters from their nominal values. In this case, we try to utilise known information as much as possible. The ESO in this case
can be modified as

\[
\begin{align*}
\dot{x}_{i1}(t) &= \dot{x}_{i2}(t) + \frac{1}{\rho_{n-1}} g_{i1}
(r_{n1}(t)(y_{i1}(t) - \dot{x}_{i1}(t))), \\
\dot{x}_{i2}(t) &= \dot{x}_{i3}(t) + \frac{1}{\rho_{n-2}} g_{i2}
(r_{n2}(t)(y_{i2}(t) - \dot{x}_{i2}(t))), \\
&\vdots \\
\dot{x}_{in_{(n-1)}(t)} &= \dot{x}_{(n-1)}(t) + g_{in}(r_{n1}(t)(y_{i1}(t) - \dot{x}_{i1}(t)))
+ f_{i}^*(t, \dot{x}(t)) + \sum_{l=1}^{m} c_{il} u_l(t), \\
\dot{x}_{in_{(n-1)}}(t) &= r(t) g_{in_{(n-1)}}(r_{n1}(t)(y_{i1}(t) - \dot{x}_{i1}(t))), \\
1 &\leq i \leq m, \\
\end{align*}
\]

(24)

where \( r \in C([0, \infty); (0, \infty)) \) is either the constant high gain: \( r(t) \equiv \frac{1}{\varepsilon} \) given in Section 2.1 or the time-varying gain function given in Section 2.2; The \( \dot{x}_{i1}(t) \) (1 \( \leq j \leq n_i \)) is used to estimate unmeasured state \( x_{i1}(t) \) and \( \dot{x}_{(n-1)}(t) \) is to estimate the stochastic total disturbance \( x_{(n-1)}(t) \) uniformly in \( t \). Similarly, \( u_l(t) \) is used to estimate \( u_l(t) \) and \( \dot{z}_l(t) \) is the output and nominal value of the control coefficient of system (1). The ESO (23) based output-feedback control is designed as

\[
u_l(t) = \sum_{i=1}^{m} c_{il}^*(f_{i}^*(t, \dot{x}(t)) + v_l(\dot{x}_{i1}(t), \dot{x}_{i2}(t), \ldots, \dot{x}_{in_{(n-1)}}(t)) - \dot{x}_{in_{(n-1)}}(t))
\]

(25)

where \( v_l : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( c_{il}^* \) are given in Assumption A.3 and (15), respectively. For convergence of ESO (24), we replace Assumption A.2 by the following Assumption A.2*.

**Assumption A.2**: The \( f_{i}^*(\cdot) (i = 1, 2, \ldots, m) \) are twice continuously differentiable with respect to their arguments and \( h_i(\cdot) (i = 1, 2) \) are locally Lipschitz continuous with respect to \((x, w)\) uniformly in \( t \in [0, \infty) \). There exist (known) constants \( L > 0, D_i^* > 0 (i = 1, 2, 3, 4) \) and a non-negative continuous function \( \varepsilon \in C(\mathbb{R}; \mathbb{R}) \) such that for all \( t \geq 0, x \in \mathbb{R}^n, z \in \mathbb{R}^s, \) and \( w \in \mathbb{R}^r \)

\[
\begin{align*}
|f_{i}^*(t, x_{i1}, \ldots, x_{in_{(n-1)}}) - f_{i}^*(t, \dot{x}_{i1}, \ldots, \dot{x}_{in_{(n-1)}})|
&\leq L \|((x_{i1} - \dot{x}_{i1}), \ldots, (x_{in_{(n-1)}}, \dot{x}_{in_{(n-1)}}))\|, \\
n_{i}^*(t, 0, 0, \ldots, 0) &\equiv 0, \quad i = 1, 2, \ldots, m; \\
\|h_i(t, x, z, w)\| &\leq D_i^* + D_i^* \|x\| + \varphi(w); \\
\left|\frac{\partial f_{i}^*(\zeta, w)}{\partial \zeta}\right| &+ \sum_{j=1}^{s} \left|\frac{\partial^2 f_{i}^*(\zeta, w)}{\partial \zeta \partial w}\right| \\
&+ \sum_{j=1}^{p} \sum_{j=1}^{s} \left|\frac{\partial f_{i}^*(t, x, \zeta, w)}{\partial w}\right|
&\leq D_i^* + \varphi(w), \quad i = 1, 2, \ldots, m.
\end{align*}
\]

(26)

**Corollary 2.3**: Suppose that all the matrices \( E_i \) and \( F_i \) are Hurwitz for all \( 1 \leq i \leq m \), (15) holds and \( \sum_{l=1}^{m} \lambda_{\max}(Q_l) (c_{il}^* - c_{il}) c_{il}^* k_{d(n-1)}(n_{(n-1)}) < 1 \). For any initial values \( x(0) \in \mathbb{R}^n, \dot{x}(0) \in \mathbb{R}^{n+m}, \zeta(0) \in \mathbb{R}^s \),

\[
\begin{align*}
(i) &\text{ under Assumptions A.1, A.2*, and A.3–A.4, the closed-loop of } x\text{-subsystem of (1), (24), and (25) with constant high gain } r(t) \equiv \frac{1}{\varepsilon} \text{ is practically mean square stable in the sense that there are a constant } \varepsilon^* > 0 \text{ and an } \varepsilon\text{-dependent constant } t_\varepsilon^* > 0 \text{ with } \varepsilon \in (0, \varepsilon^*) \text{ such that for all } t \geq t_\varepsilon^*, \\
\text{ }E|\dot{x}_{ij}(t) - \dot{x}_{ij}(t)|^2 &\leq \Gamma_2 2\varepsilon^n + 3 - 2\varepsilon, \\
1 &\leq j \leq n_i, 1 \leq i \leq m,
\end{align*}
\]

and

\[
\text{ }E|x_{ij}(t)|^2 \leq \Gamma_\varepsilon, \quad 1 \leq j \leq n_i, 1 \leq i \leq m,
\]

where \( \Gamma_\varepsilon > 0 \) is an \( \varepsilon \)-independent constant.

(ii) Under Assumptions A.1*–A.2*, the closed-loop of \( x\)-subsystem of (1), (24), and (25) with time-varying gain \( r(t) = e^{at} (a > 3b) \) is asymptotically mean square stable in the sense that

\[
\begin{align*}
\lim_{t \rightarrow \infty} E|\dot{x}_{ij}(t) - \dot{x}_{ij}(t)|^2 &\equiv 0, \quad 1 \leq j \leq n_i, 1 \leq i \leq m, \\
\text{and}
\end{align*}
\]

\[
\lim_{t \rightarrow \infty} E|x_{ij}(t)|^2 = 0, \quad 1 \leq j \leq n_i, 1 \leq i \leq m.
\]
3. Proof of main results

Proof of Theorem 2.1: Set

\[ \eta_j(t) = \frac{x_j(t) - \tilde{x}_j(t)}{\varepsilon^{n_i+1-j}}, \quad 1 \leq j \leq n_i + 1, \ 1 \leq i \leq m, \quad (29) \]

\[ \Phi_i(x_i) = (x_{i2}, x_{i3}, \ldots, v_1(\tilde{x}_{i1}), \ldots, \tilde{x}_{in_i}) + x_{i(n_i+1)} - \tilde{x}_{i(n_i+1)}, \quad (30) \]

\[ \Phi(x) = (\Phi_1(x_1)^T, \Phi_2(x_2)^T, \ldots, \Phi_m(x_m)^T)^T. \quad (31) \]

By virtue of Itô’s formula, it is obtained that

\[
\frac{d f_i(t, x(t), \zeta(t), w(t))}{\partial t} |_{t \to \infty} = \left\{ \begin{array}{l}
\frac{\partial f_i(t, x(t), \zeta(t), w(t))}{\partial x}^T \Phi(x(t)) \\
+ \left( \frac{\partial f_i(t, x(t), \zeta(t), w(t))}{\partial \zeta} \right)^T h_1(t, x(t), \zeta(t), w(t)) \\
+ \frac{1}{2} \text{Tr} \left[ \frac{\partial^2 f_i(t, x(t), \zeta(t), w(t))}{\partial \zeta^2} \right] h_2(t, x(t), \zeta(t), w(t)) \\
+ \frac{1}{2} \sum_{l=1}^{q} \left( \frac{\partial \psi(t, W_2(t))}{\partial \zeta} \right)^T \frac{\partial^2 \psi(t, W_2(t))}{\partial \zeta^2} \\
+ \frac{1}{2} \sum_{l=1}^{q} \left( \frac{\partial \psi(t, W_2(t))}{\partial \zeta} \right)^T \frac{\partial^2 \psi(t, W_2(t))}{\partial \zeta^2} \\
+ \frac{1}{2} \sum_{l=1}^{q} \left( \frac{\partial \psi(t, W_2(t))}{\partial \zeta} \right)^T \frac{\partial^2 \psi(t, W_2(t))}{\partial \zeta^2} \\
+ \frac{1}{2} \sum_{l=1}^{q} \left( \frac{\partial \psi(t, W_2(t))}{\partial \zeta} \right)^T \frac{\partial^2 \psi(t, W_2(t))}{\partial \zeta^2} \\
\end{array} \right. \\
\vdots \\
\frac{d f_i(t, x(t), \zeta(t), w(t))}{\partial t} \Big|_{t \to \infty} = \Lambda_{i1}(t) dt + \Lambda_{i2}(t) dw_1(t) + \Lambda_{i3}(t) dw_2(t), \quad (32) \]

where we set

\[
\begin{align*}
\Lambda_{i1}(t) &= (\Lambda_{i1,l}(t))_{1 \times p}, \\
\Lambda_{i2,l}(t) &= \sum_{j=1}^{p} \frac{\partial f_i(t, x(t), \zeta(t), w(t))}{\partial \zeta} h_2^{(j)}(t, x(t), \zeta(t), w(t)), \\
\Lambda_{i3}(t) &= (\Lambda_{i3,l}(t))_{1 \times q}, \\
\Lambda_{i3,l}(t) &= \frac{\partial f_i(t, x(t), \zeta(t), w(t))}{\partial w} \frac{\partial \psi(t, W_2(t))}{\partial \zeta} \\
\end{align*} \quad (33) \]

Thus, it follows from Assumptions A.1–A.2, (29), \( v_i(0) = 0 \), and the globally Lipschitz continuity of \( v_i(\cdot) \) in Assumption A.3 that there exist \( \varepsilon \)-independent positive constants \( \xi_{i1}, \xi_{i2}, \xi_{i3}, \xi_{i4} \) such that

\[
|\Lambda_{i1}(t)| \leq \xi_{i1} + \xi_{i2} \| \eta(t) \| + \xi_{i3} \| x(t) \|, \quad \Lambda_{i2}(t) \leq \xi_{i4} \| \eta(t) \|, \quad (34) \]

In addition,

\[
\frac{d}{dt} \left[ \sum_{l=1}^{m} (c_{il} - c_{il}^*) \hat{z}^*(l) \right] = \frac{d}{dt} \left[ \sum_{l=1}^{m} (c_{il} - c_{il}^*) \hat{z}^*(l) \right] \\
+ \frac{v_i(\tilde{x}_{d1}(t), \ldots, \tilde{x}_{d2n}(t))}{\partial \tilde{x}_{d1}} + \frac{v_i(\tilde{x}_{d1}(t), \ldots, \tilde{x}_{d2n}(t))}{\partial \tilde{x}_{d2}} \\
+ \varepsilon^n \sum_{l=1}^{m} (c_{il} - c_{il}^*) \hat{z}^*(l) \\
+ \sum_{l=1}^{m} \xi_{i4} \| \eta(t) \|, \quad (35) \]

Suppose that \( 0 < \varepsilon < 1 \). It follows from \( v_i(0) = 0 \), the globally Lipschitz continuity of \( v_i(\cdot) \), (10) in Assumption A.3, \( |g_{ij}(z)| \leq k_{ij} |z| \) for all \( z \in \mathbb{R} \) in Assumption A.4, and (20) that there exist \( \varepsilon \)-independent positive constants \( \xi_{i5}, \xi_{i6}, \xi_{i7} \) such that

\[
|\Lambda_{i4}(t)| \leq \xi_{i5} \| \eta(t) \| + \xi_{i6} \| x(t) \| + \frac{\xi_{i7}}{\varepsilon} \| \eta(t) \|, \quad (36) \]

where

\[
\xi_{i} = \sum_{l=1}^{m} |(c_{il} - c_{il}^*) \hat{z}^*(l)|. \quad (37) \]

Then, the \( x \)-subsystem of (1) can be written as

\[
\left\{ \begin{array}{l}
\frac{d \tilde{x}_i(t)}{dt} = \Lambda_{i1}(t) \tilde{x}_i(t) + \left( \begin{array}{c}
B_{i1} \\
0
\end{array} \right) \sum_{l=1}^{m} c_{il} u_l(t) \\
+ B_{i1} \Lambda_{i2}(t) dw_1(t) + B_{i1} \Lambda_{i3}(t) dw_2(t), \\
y_i(t) = C_{i1} \tilde{x}_i(t), \quad i = 1, 2, \ldots, m.
\end{array} \right. \quad (38) \]
By (4), (20), and (38), it follows that the closed-loop of x-subsystem of (1), (4), and (5) is equivalent to

\[
\begin{align*}
d\tilde{x}_i(t) &= A_n x_i(t) dt + B_n [v_i(x_i(t), \ldots, x_m(t)) \\
&+ \varphi_i(t) + \eta_i(n+1)(t)] dt, \\
d\eta_i(t) &= \frac{1}{e} A_{n+1} \eta_i(t) dt - \frac{1}{e} \left( \sum_{i=1}^{m} g_i(\eta_i(t)) \right) dt \\
&+ B_{n+1} (A_i(t) + \Lambda_{i4}(t)) dt \\
&+ B_{n+1} \Lambda_{i2}(t) dW_1(t) + B_{n+1} \Lambda_{i3}(t) dW_2(t),
\end{align*}
\]

where \(i \leq m\).

By the globally Lipschitz continuity of \(v_i(\cdot)\) with Lipschitz constant \(L_i\) in Assumption A.3, a straightforward computation shows that

\[
|\varphi_i(t)|^2 \leq L_i^2 \left[ (x_i(t) - \tilde{x}_i(t))^2 + \cdots + (x_m(t) - \tilde{x}_m(t))^2 \right] \\
= L_i^2 \left[ e^{2n} |\eta_i(t)|^2 + \cdots + e^2 |\eta_m(t)|^2 \right] \\
\leq L_i^2 e^{2n} |\eta_i(t)|^2, \quad i = 1, 2, \ldots, m.
\]

We proceed the proof in three steps as follows.

**Step 1**: The solution \((x(t), \eta(t))\) of system (39) is practically mean square bounded. Define a positive definite function \(V : \mathbb{R}^{2n+\cdots+2m+m} \rightarrow \mathbb{R}\) by

\[
V(x, \eta) = V_1(x_1, \ldots, x_m) + V_2(\eta_1, \ldots, \eta_m)
\]

Applying Itô’s formula to \(V(x(t), \eta(t))\) with respect to \(t\) along the solution \((x(t), \eta(t))\) of system (39) to obtain

\[
\begin{align*}
dV(x(t), \eta(t)) &= \sum_{i=1}^{m} \left\{ \left( \sum_{j=1}^{n-1} \frac{\partial V_1(x_i(t))}{\partial x_{ij}} \cdot \dot{x}_{ij}(t) \right) \\
&+ \frac{1}{e} \left( \sum_{j=1}^{n} \frac{\partial V_2(\eta_i(t))}{\partial \eta_{ij}} \left( \eta_{i(j+1)}(t) - g_i(\eta_i(t)) \right) \right) dt \\
&- \frac{1}{e} \frac{\partial V_2(\eta_i(t))}{\partial \eta_i(n+1)} (\eta_i(n+1)(t)) \right\} dt \\
&+ \frac{1}{2} \frac{\partial^2 V_2(\eta_i(t))}{\partial \eta_i(n+1)} \left[ \sum_{j=1}^{n} \frac{\partial V_2(\eta_i(t))}{\partial \eta_{ij}} \right] dt \\
&\leq \frac{1}{2} \frac{\partial^2 V_2(\eta_i(t))}{\partial \eta_i(n+1)} \left[ \sum_{j=1}^{n} \frac{\partial V_2(\eta_i(t))}{\partial \eta_{ij}} \right] dt \\
&\leq -\sum_{i=1}^{m} \left\{ -\lambda_{i3} E||x_i(t)||^2 + \frac{\beta_{i2} x_i(t)^2}{2} + \frac{\lambda_{i3}}{e} E||\eta_i(t)||^2 \right\}
\]

where \(\lambda_{i3}\) is given in (16). Now we suppose that

\[
0 < \varepsilon < \varepsilon^* \leq \min \left\{ 1, \varepsilon_1, \frac{\theta_1}{2 \theta_2} \right\}.
\]

Set

\[
\lambda = \max \left\{ \max_{1 \leq i \leq m} \{ \lambda_{i2} \}, \max_{1 \leq i \leq m} \{ \lambda_{i3} \} \right\}.
\]

It follows from (34), (36), (40), (43), Assumptions A.3–A.4, and Young’s inequality that

\[
\begin{align*}
&\frac{d}{dt} \mathbb{E} V(x(t), \eta(t)) \\
&\leq \sum_{i=1}^{m} \left\{ -\varepsilon U_1(x_i(t)) + \varepsilon \alpha_i L_i \mathbb{E}(||x_i(t)|| \cdot ||\eta_i(t)||) \\
&+ \alpha_i \mathbb{E}(||x_i(t)|| \cdot ||\eta_i(n+1)(t)||) \\
&- \frac{1}{e} \mathbb{E} U_2(\eta_i(t)) + \beta_i \mathbb{E}(||\eta_i|| \cdot (\xi_i + \xi_{d2} ||\eta_i|| + \xi_{d3} ||x_i|| + \xi_{d6} ||\eta_i||) + \gamma_{i5} \xi_{i4}) \\
&\leq \sum_{i=1}^{m} \left\{ -\lambda_{i3} E||x_i(t)||^2 + \frac{\beta_{i2} x_i(t)^2}{2} + \frac{\lambda_{i3}}{e} E||\eta_i(t)||^2 \right\}
\end{align*}
\]

We also notice that there exist \(\mu > 0\) and \(\varepsilon_1 > 0\) such that

\[
\begin{align*}
\theta_2 &\triangleq \min_{1 \leq i \leq m} \{ \lambda_{i3} \} - \mu - \frac{\varepsilon_1}{2} > 0, \\
\theta_1 &\leq \frac{\varepsilon_1}{2} \frac{3}{\mu} \max_{1 \leq i \leq m} \{ \alpha_i \} + \frac{1}{2} + \sum_{i=1}^{m} \beta_{i2} x_i(t)^2 \\
&+ \frac{3m}{4 \mu} \max_{1 \leq i \leq m} \{ \beta_i \} \xi_{i2} + \sum_{i=1}^{m} \beta_{i2} x_i(t)^2 + \frac{3m}{4 \mu} \max_{1 \leq i \leq m} \{ \beta_{i2} x_i(t)^2 \} > 0,
\end{align*}
\]

where \(\theta_1\) is given in (16). It now follows that

\[
0 < \varepsilon < \varepsilon^* \leq \min \left\{ 1, \varepsilon_1, \frac{\theta_1}{2 \theta_2} \right\}.
\]

Set

\[
\lambda = \max \left\{ \max_{1 \leq i \leq m} \{ \lambda_{i2} \}, \max_{1 \leq i \leq m} \{ \lambda_{i3} \} \right\}.
\]

It follows from (34), (36), (40), (43), Assumptions A.3–A.4, and Young’s inequality that

\[
\begin{align*}
&\frac{d}{dt} \mathbb{E} V(x(t), \eta(t)) \\
&\leq \sum_{i=1}^{m} \left\{ -\lambda_{i3} E||x_i(t)||^2 + \frac{\beta_{i2} x_i(t)^2}{2} + \frac{\lambda_{i3}}{e} E||\eta_i(t)||^2 \right\}
\end{align*}
\]

where \(\theta_1\) and \(\theta_2\) are defined in (16) and (43), respectively. It is easy to check that \(\mathbb{E} V(x(0), \eta(0))\) is bounded by a polynomial of
Thus for all $t \geq t_e + 1, 1 \leq j \leq n_i + 1, 1 \leq i \leq m$,
\[
\mathbb{E}|x_{ij}(t) - \hat{x}_{ij}(t)|^2 = e^{2n_i+2-2j}\mathbb{E}|\eta_j(t)|^2 \\
\leq e^{2n_i+2-2j}\mathbb{E}||\eta_j(t)||^2 \\
\leq \frac{\Gamma_4}{\min_{1 \leq i \leq m, \lambda_{ji}^*}} e^{2n_i+2-2j}.
\] (51)

**Step 3**: Practical mean square stability of the closed-loop of $x$-subsystem of (1), (4), and (5). For any $\varepsilon \in (0, \varepsilon^*)$ and all $t \geq t_e + 1$, it follows from Assumption A.3, (45), and (50) that
\[
\frac{d}{dt}\mathbb{E}V_2(x(t)) \leq -\frac{\theta_1}{2\varepsilon} \mathbb{E}\|\eta(t)\|^2 + \frac{2\mu}{3} \mathbb{E}\|x(t)\|^2 \\
+ \frac{\theta_3}{2\varepsilon} \max_{1 \leq i \leq m, \lambda_{ji}^*} \mathbb{E}V_2(\eta(t)) \\
+ \frac{2\mu\Gamma_1}{3 \min_{1 \leq i \leq m, \lambda_{ji}}^*} + \frac{1}{2} \sum_{i=1}^m (\beta_i^2 \xi_i^2 + 2\gamma_i \xi_i a_i).
\] (47)

Set
\[
\Gamma_2 = \frac{\theta_1}{2\varepsilon} \max_{1 \leq i \leq m, \lambda_{ji}^*}, \\
\Gamma_3 = \frac{2\mu\Gamma_1}{3 \min_{1 \leq i \leq m, \lambda_{ji}}^*} + \frac{1}{2} \sum_{i=1}^m (\beta_i^2 \xi_i^2 + 2\gamma_i \xi_i a_i).
\]

Then
\[
\mathbb{E}V_2(\eta(t)) \leq e^{-\frac{\Gamma_2}{2\varepsilon}(t-t_e)} \mathbb{E}V_2(\eta(t_e)) + \Gamma_3 \int_{t_e}^t e^{-\frac{\Gamma_3}{2\varepsilon}(t-s)} ds.
\] (48)

For all $t \geq t_e + 1$, since from (46), the first term of the right-hand side of (48) is bounded by $e^{-\frac{\Gamma_2}{2\varepsilon}}$ multiplied by an $\varepsilon$-independent constant, and the second term is bounded by $\varepsilon$ multiplied by an $\varepsilon$-independent constant, there exists an $\varepsilon$-independent constant $\Gamma_4 > 0$ such that for all $t \geq t_e + 1$,
\[
\mathbb{E}V_2(\eta(t)) \leq \varepsilon \Gamma_4,
\] (49)

and so
\[
\mathbb{E}\|\eta_i(t)\|^2 \leq \frac{\mathbb{E}V_2(\eta_i(t))}{\min_{1 \leq i \leq m, \lambda_{ji}^*}} \leq \frac{\varepsilon \Gamma_4}{\min_{1 \leq i \leq m, \lambda_{ji}^*}}.
\] (50)

This completes the proof of the theorem.

**Proof of Corollary 2.1**: For $1 \leq i \leq m$, we define the Lyapunov functions $V_{ii}, U_{ii} : \mathbb{R}^{n_i} \to \mathbb{R}$ by $V_{ii}(z) = z^\top H_i z, U_{ii}(z) = z^\top \Gamma_i z$ and...
\[ z^T z \text{ for } z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n \text{ and the Lyapunov functions } V_{12}, U_{12} : \mathbb{R}^{n+1} \to \mathbb{R} \text{ by } V_{12}(z) = z^T Q z, \text{ } U_{12}(z) = z^T z \\
\text{for } z = (z_1, z_2, \ldots, z_{n+1})^T \in \mathbb{R}^{n+1}. \text{ Then}
\]
\[
\lambda_{\min}(H_i) \| z \|^2 \leq V_{1i}(z) \leq \lambda_{\max}(H_i) \| z \|^2,
\]
\[
\sum_{i=1}^{n+1} z_{j+1} \frac{\partial V_{1i}(z)}{\partial z_j} + (c_{i1} z_1 + \cdots + c_{in} z_n) \frac{\partial V_{1i}(z)}{\partial z_n} = -\| z \|^2 = -U_{1i}(z),
\]
\[
\frac{\partial V_{1i}(z)}{\partial z_n} \leq \left\| \frac{\partial V_{1i}(z)}{\partial z} \right\| = 2\| z \| Q_i \| z \| = 2\lambda_{\max}(H_i) \| z \| \text{, } \forall z
\]
\[
(2z^T H_i \| z \|) \leq 2\| H_i \| \| z \|^2.
\]
\[\text{and}
\]
\[
\lambda_{\min}(Q_i) \| z \|^2 \leq V_{1i}(z) \leq \lambda_{\max}(Q_i) \| z \|^2,
\]
\[
\sum_{i=1}^{n+1} z_{j+1} \frac{\partial V_{1i}(z)}{\partial z_j} + (c_{i1} z_1 + \cdots + c_{in} z_n) \frac{\partial V_{1i}(z)}{\partial z_n} = -\| z \|^2 = -V_{1i}(z),
\]
\[
\frac{\partial V_{1i}(z)}{\partial z_n} \leq \left\| \frac{\partial V_{1i}(z)}{\partial z} \right\| = 2\| z \| Q_i \| z \| = 2\lambda_{\max}(Q_i) \| z \| \text{, } \forall z
\]
\[
(2z^T Q_i \| z \|) \leq 2\| Q_i \| \| z \|^2.
\]

All conditions of Assumptions A.3–A.4 are therefore satisfied. The results then follow directly from Theorem 2.1.

**Proof of Theorem 2.2:** We first prove the ESO’s asymptotic mean square estimation of unmeasured states of x-subsystem of (1) and stochastic total disturbance (3) in the sense that
\[
\lim_{t \to \infty} \mathbb{E}[\| x_{ij}(t) - \hat{x}_{ij}(t) \|^2] = 0, \quad 1 \leq j \leq n_i + 1, \quad 1 \leq i \leq m.
\]
(56)

To this end, set
\[
\eta_{ij}(t) = e^{(n_i+1-j)a t}(x_{ij}(t) - \hat{x}_{ij}(t)), \quad 1 \leq j \leq n_i + 1, \quad 1 \leq i \leq m.
\]
(57)

Since
\[
\frac{d}{dt} \left| \sum_{i=1}^{m} (c_{ii} - c_{ii}) u_i(t) \right| = \frac{d}{dt} \left| \sum_{l=1}^{m} (c_{il} - c_{il}) \hat{x}_{il}(t) \right|
\]

we can write, based on (32) and (58), the x-subsystem of (1) as
\[
\begin{align*}
\frac{dx_i(t)}{dt} &= A_{x_i} x_i(t) + B_{x_i} \left[ \sum_{i=1}^{m} c_{il} u_i(t) \right] dt \\
&+ B_{x_i} (A_{x_i}(t) + A_{x_i} \left( t \right)) dt \\
&+ B_{x_i} A_{x_i} \left( t \right) dW_1(t) + B_{x_i} A_{x_i} \left( t \right) dW_2(t) \\
n_i(t) &= C_i x_i(t), \quad i = 1, 2, \ldots, m,
\end{align*}
\]
(59)

where \( A_{x_i}(t) \) (\( j = 1, 2, 3 \)) are defined as that in (32). A direct computation shows that the closed-loop of x-subsystem of (1), (5), and (21) is equivalent to
\[
\begin{align*}
\frac{dx_i(t)}{dt} &= A_{x_i} x_i(t) + B_{x_i} \left[ \sum_{i=1}^{m} c_{il} u_i(t) \right] dt \\
&+ B_{x_i} (A_{x_i}(t) + A_{x_i} \left( t \right)) dt \\
&+ B_{x_i} A_{x_i} \left( t \right) dW_1(t) + B_{x_i} A_{x_i} \left( t \right) dW_2(t), \\
\end{align*}
\]

where \( \Lambda_{x_i}(t) \) (\( j = 1, 2, 3 \)) are defined as that in (32). A direct computation shows that the closed-loop of x-subsystem of (1), (5), and (21) is equivalent to
\[
\begin{align*}
\frac{dx_i(t)}{dt} &= A_{x_i} x_i(t) + B_{x_i} \left[ \sum_{i=1}^{m} c_{il} u_i(t) \right] dt \\
&+ B_{x_i} (A_{x_i}(t) + A_{x_i} \left( t \right)) dt \\
&+ B_{x_i} A_{x_i} \left( t \right) dW_1(t) + B_{x_i} A_{x_i} \left( t \right) dW_2(t), \\
\end{align*}
\]

We notice from Assumptions A.1*, A.2, (30), \( v_i(0) = 0 \), and the globally Lipschitz continuity of \( v_i(t) \) in Assumption A.3 that there exist positive constants \( N_{11}, N_{12}, N_{13}, N_{14} \) such that
\[
\| A_{x_i} \| \leq N_{11} e^{3b t + \lambda_{x_i} e^{b t} \| x(t) \|} + N_{13} e^{b t} \| x(t) \|, \\
\| A_{x_i} \|^2 \leq 2N_{14} e^{2b t}.
\]
(61)

It follows from \( v_i(0) = 0 \), the globally Lipschitz continuity of \( v_i(t) \), (11) in Assumption A.3, \( \| g_{ij}(z) \| \leq k_{ij} \| z \| \) for all \( z \in \mathbb{R} \) in Assumption A.4, and (20) that there exist positive constants \( N_{15}, N_{16} \) such that
\[
\| A_{x_i} \| \leq N_{15} \| x(t) \| + N_{16} \| x(t) \| + e^{a t} \| x(t) \|, \\
\| A_{x_i} \|^2 \leq 2N_{14} e^{2b t}.
\]
(62)

where \( \xi_i \) is defined as that in (37). By the globally Lipschitz continuity of \( v_i(t) \) with Lipschitz constant \( L_i \) in Assumption A.3 and
a simple computation, we can obtain that
\[
|\sigma(t)^2| \leq L_1^2 \left[ (x_{i1}(t) - \tilde{x}_{i1}(t))^2 + \cdots + (x_{im}(t) - \tilde{x}_{im}(t))^2 \right]
= L_1^2 \left[ \frac{1}{e^{2\mu t}} |\eta_{i1}(t)|^2 + \cdots + \frac{1}{e^{2\mu t}} |\eta_{im}(t)|^2 \right]
\leq L_1^2 e^{-2\mu t} \|\eta_i(t)\|^2, \quad i = 1, 2, \ldots, m.
\] (63)

Define a positive definite function \( V : \mathbb{R}^{2n_1+\cdots+2n_m+m} \to \mathbb{R} \) by
\[
V(x, \eta) = V_1(x_1, \ldots, x_m) + V_2(\eta_1, \ldots, \eta_m)
= m \left[ V_1(x_i) + V_2(\eta_i) \right].
\] (64)

Apply Ito’s formula to \( V(x(t), \eta(t)) \) with respect to \( t \) along the solutions \( (x(t), \eta(t)) \) of system (60) to obtain
\[
dV(x(t), \eta(t))
= \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n_i-1} \partial V_1(x_{i(j+1)}(t)) + \frac{\partial V_1(x_i(t))}{\partial x_{ni}} \right\} dt
+ e^{\alpha t} \left( \sum_{j=1}^{n_i} \frac{\partial V_2(\eta_{i(j+1)}(t))}{\partial \eta_{i(j+1)}} - g_i(t) \right) dt
+ e^{\alpha t} \left( \sum_{j=1}^{n_i} \frac{\partial V_2(\eta_{i(j)}(t))}{\partial \eta_{i(j)}(t)} - g_i(t) \right) dt
+ \partial V_2(\eta_{i(i)}(t)) \left( \Lambda_{i1}(t) + \Lambda_{i5}(t) \right) dt
+ e^{\alpha t} \left( \sum_{j=1}^{n_i} \frac{\partial V_2(\eta_{i(j)}(t))}{\partial \eta_{i(j)}(t)} - g_i(t) \right) dt
\]
(65)

We also notice that there exist \( \mu > 0 \) and \( t_1 > 0 \) such that for all \( t \geq t_1 \),
\[
\theta^2 \triangleq \min_{1 \leq i \leq m} \{ \lambda_{i3} \} - \mu - \frac{1}{2e^{2\mu t}} > 0,
\]
\[
\theta_1 e^{\alpha t} \triangleq \min_{1 \leq i \leq m} \{ \lambda_{i3} \} - \mu - \frac{3}{4\mu} \max_{1 \leq i \leq m} \beta_i > 0.
\]

\[
\frac{\partial}{\partial \eta_{i(i)}} V_2(\eta_{i(i)}(t)) + \partial V_2(\eta_{i(i)}(t)) \Lambda_{i3}(t) dW_2(t)
= e^{\alpha t} \left( \sum_{j=1}^{n_i} \frac{\partial V_2(\eta_{i(j)}(t))}{\partial \eta_{i(j)}(t)} - g_i(t) \right) \Lambda_{i3}(t) dW_2(t)
\]
(65)

\[
dE \frac{\partial V_2(\eta(t))}{\partial \eta_{i(i)}(t)} + \partial V_2(\eta_{i(i)}(t)) \Lambda_{i3}(t) dW_2(t) \geq 0, \quad \forall t \geq t_1,
\]
(66)

where \( \theta_1 \) is given by (16). By (61), (62), (65), (66), Assumptions A.3–A.4, and Young’s inequality, it follows that for all \( t \geq t_1 \),
\[
\frac{dE \langle x(t), \eta(t) \rangle}{dt} \leq \sum_{i=1}^{m} \left\{ - \mu \|x(t)\|^2 + \frac{\alpha L_i}{e^{at}} E(\|x(t)\|)
\right\}
+ e^{3\theta_1} \sum_{i=1}^{m} \beta_i N_i \sum_{i=1}^{m} \gamma_i N_i \chi(t)^2
\]
\[
= \frac{\mu}{3} \|x(t)\|^2 + \frac{3\alpha^2}{4\mu} E(\|x(t)\|)^2 + \frac{\mu}{3m} \chi(t)^2
\]
\[
+ e^{\delta t} \sum_{i=1}^{m} \gamma_i N_i \chi(t)^2
\]
\[
\leq -\theta^2 \|x(t)\|^2 - \frac{\theta_1 e^{at}}{2} E(\|\eta(t)\|)^2
\]
\[
+ e^{\delta t} \sum_{i=1}^{m} \gamma_i N_i \chi(t)^2
\]
(67)

Since
\[
\lim_{t \to \infty} \left( -\frac{\theta_1 e^{at}}{2} + e^{\delta t} \sum_{i=1}^{m} \gamma_i N_i \chi(t)^2 \right) = -\infty,
\]
(68)

there exists \( t_2 \geq t_1 \) such that for all \( t \geq t_2 \),
\[
-\frac{\theta_1 e^{at}}{2} + e^{\delta t} \sum_{i=1}^{m} \gamma_i N_i \chi(t)^2 \leq -\theta_1 < 0, \quad \forall t \geq t_2.
\]
(69)

This together with (67) yields that if \( E(\|\eta(t)\|) \geq 1 \), then \( E(\|\eta(t)\|) \geq 1 \) and thus
\[
\frac{dE \langle x(t), \eta(t) \rangle}{dt} \leq -\theta_1 < 0, \quad \forall t \geq t_2.
\]
(70)

Therefore, there exists \( t_3 \geq t_2 \) such that \( E(\|\eta(t)\|) \leq 1 \) for all \( t \geq t_3 \). For any \( \delta > 0 \), similarly with (69), there exists \( t_4 \geq t_3 \) such
that for all $t \geq t_4$,
\[-\frac{\delta_1 e^{at}}{2} + e^{3bt} \max_{1 \leq i \leq m} \beta_i N_1 + e^{2bt} \sum_{i=1}^{m} \gamma_i N_i \leq -\theta_1. \quad (71)\]

By (67) and (71), if $\mathbb{E}\|\eta(t)\|^2 > \delta$, then
\[
\frac{d\mathbb{E}V(x(t), \eta(t))}{dt} \leq -\theta_1 < 0, \quad (72)
\]

and hence there exists $t_5 \geq t_4$ such that $\mathbb{E}\|\eta(t)\|^2 \leq \delta$ for all $t \geq t_5$. So we have
\[
\lim_{t \to \infty} \mathbb{E}\|\eta(t)\|^2 = 0. \quad (73)
\]

This shows that for all $1 \leq j \leq n_1 + 1$, $1 \leq i \leq m$,
\[
\lim_{t \to \infty} \mathbb{E}|x_{ij}(t) - \hat{x}_{ij}(t)|^2 = \lim_{t \to \infty} \mathbb{E} \left| \frac{\eta_{ij}(t)}{\sqrt{(n_1 + 1 - j)al}} \right|^2 
\leq \lim_{t \to \infty} \mathbb{E}\|\eta(t)\|^2 = 0, \quad (74)
\]

which is (56). Now we show the asymptotic mean square stability of the closed-loop of $x$-subsystem of (1), (5), and (21) in the sense that
\[
\lim_{t \to \infty} \mathbb{E}|x_{ij}(t)|^2 = 0, \quad 1 \leq j \leq n_1, 1 \leq i \leq m. \quad (75)
\]

By Assumption A.3, we find the derivative of $V_1(x(t))$ with respect to $t$ along the solution $x(t)$ of system (60) to obtain
\[
\frac{d\mathbb{E}V_1(x(t))}{dt} \leq \sum_{i=1}^{m} \left\{ - \mathbb{E}U_{i1}(x_i(t)) + \frac{\alpha_i L_i}{e^{at}} \mathbb{E}(\|x_i(t)\| \cdot \|\eta_i(t)\|) + \alpha_i \mathbb{E}(\|x_i(t)\| \cdot |\eta_i(n_i+1)(t)|) \right\}
\leq \sum_{i=1}^{m} \left\{ - \lambda_i \mathbb{E}(\|x_i(t)\|^2) + \frac{1}{2e^{at}} \mathbb{E}(\|x_i(t)\|^2)
+ \frac{\alpha_i^2 L_i^2}{2e^{at}} \mathbb{E}(\|\eta_i(t)\|^2) + \mu \mathbb{E}(\|x_i(t)\|^2)
+ \frac{\alpha_i^2}{4\mu} \mathbb{E}(\|\eta_i(t)\|^2) \right\}
\leq -\theta_2^* \mathbb{E}(\|x(t)\|^2) + \left( \frac{1}{2e^{at}} \max_{1 \leq i \leq m} \alpha_i^2 L_i^2 + \frac{1}{4\mu} \max_{1 \leq i \leq m} \alpha_i^2 \right) \mathbb{E}(\|\eta(t)\|^2), \quad (76)
\]

where $\theta_2^*$ is defined as in (66). Since $\lim_{t \to \infty} \mathbb{E}\|\eta(t)\|^2 = 0$, for any $\delta > 0$, there exists a positive constant $t_6 \geq t_5$ such that $\mathbb{E}\|\eta(t)\|^2 < \frac{\delta}{2 \mu \max_{1 \leq i \leq m} \alpha_i^2 L_i^2 + \max_{1 \leq i \leq m} \alpha_i^2 e^{at}}$ for all $t \geq t_6$. It follows from (76) that if $\mathbb{E}(\|x(t)\|^2) > \delta$, then
\[
\frac{d\mathbb{E}V_1(x(t))}{dt} \leq -\frac{\theta_2^* \delta}{2} < 0. \quad (77)
\]

Therefore, there exists $t_7 \geq t_6$ such that $\mathbb{E}(\|x(t)\|^2) \leq \delta$ for all $t \geq t_7$. This shows that
\[
\lim_{t \to \infty} \mathbb{E}(\|x(t)\|^2) = 0, \quad (78)
\]

and thus for all $1 \leq j \leq n_i$, $1 \leq i \leq m$,
\[
\lim_{t \to \infty} \mathbb{E}(\|x_{ij}(t)\|^2) = \lim_{t \to \infty} \mathbb{E}(\|x(t)\|^2) = 0. \quad (79)
\]

This gives (75) and thus completes the proof of the theorem. \textbf{■}

\textbf{Proof of Theorem 2.3:} In terms of the Itô’s formula, it is obtained that
\[
\frac{dE_{i}^{\ast}(\xi(t), w(t))}{dt} \bigg|_{\text{along}} = \left\{ \left( \frac{\partial f_{i}^{\ast}(\xi(t), w(t))}{\partial \xi} \right)^{\top} h_{1}(t, x(t), \xi(t), w(t))
+ \frac{1}{2} \text{Tr} \left\{ h_{2}^{\top}(t, x(t), \xi(t), w(t)) \frac{\partial^{2} f_{i}^{\ast}(\xi(t), w(t))}{\partial \xi^{2}} \right\} \right\}
\times h_{2}(t, x(t), \xi(t), w(t)) + \frac{\partial^{2} f_{i}^{\ast}(\xi(t), w(t))}{\partial \psi} \left( \frac{\partial \psi(t, W_{2}(t))}{\partial t} \right)
+ \frac{1}{2} \sum_{j=1}^{q} \frac{\partial \psi(t, W_{2}(t))}{\partial \theta_{j}} \right) \right\}
\leq \frac{1}{2} \sum_{j=1}^{q} \left( \frac{\partial \psi(t, W_{2}(t))}{\partial \theta_{j}} \right)^{2} \right\} \right\}
\leq \left( \frac{\partial \psi(t, W_{2}(t))}{\partial \theta_{j}} \right)^{2} \right\}
\leq \frac{1}{2} \left( \frac{\partial \psi(t, W_{2}(t))}{\partial \theta_{j}} \right)^{2} \left( \frac{\partial \psi(t, W_{2}(t))}{\partial \theta_{j}} \right)^{2} \right\}
\leq \Lambda_{i}^{\ast}(t)dt + \Lambda_{i}^{\ast}(t)dW_{1}(t) + \Lambda_{i}^{\ast}(t)dW_{2}(t), \quad (80)
\]

where we set
\[
\Lambda_{i}^{\ast}(t) = \left( \Lambda_{i}^{\ast}(t) \right)_{1 \times p},
\Lambda_{i}^{\ast}(t) = \sum_{j=1}^{s} \frac{\partial f_{i}^{\ast}(\xi(t), w(t))}{\partial \psi(t, W_{2}(t))} h_{2}^{(ij)}(t, x(t), \xi(t), w(t));
\Lambda_{i}^{\ast}(t) = \left( \Lambda_{i}^{\ast}(t) \right)_{1 \times q},
\Lambda_{i}^{\ast}(t) = \frac{\partial f_{i}^{\ast}(\xi(t), w(t))}{\partial \psi(t, W_{2}(t))} \frac{\partial \psi(t, W_{2}(t))}{\partial \theta_{j}}. \quad (81)
\]
In addition,
\[
\frac{d}{dt}\vert_{\text{along (24)}} \begin{bmatrix} \sum_{l=1}^{m} (c_{il} - c_{il}^*) u_{il}(t) \\ \vdots \end{bmatrix} = \frac{d}{dt}\vert_{\text{along (24)}} \begin{bmatrix} \sum_{l=1}^{m} (c_{il} - c_{il}^*) \hat{c}_{il}^t (v_{ij}(\hat{x})d_1(t), \\ \ldots, \hat{x}_{dn_d}(t)) - \hat{x}_{d(n_d+1)}(t) \end{bmatrix} \\
\begin{bmatrix} \frac{\partial v_{ij}(\hat{x}_d(t), \ldots, \hat{x}_{dn_d}(t))}{\partial \hat{x}_{ij}} \\ \vdots \end{bmatrix} \\
+ \sum_{l=1}^{m} (c_{il} - c_{il}^*) \hat{c}_{il}^t \begin{bmatrix} \hat{x}_{d(n_d+1)}(t) + g_{dn_d}(\eta_d(t)) \\ + f_{g_d}(\hat{x}, \hat{x}_{d}) \end{bmatrix} + \sum_{k=1}^{m} c_{ik}^* u_k(t) \\
+ \sum_{l=1}^{m} (c_{il} - c_{il}^*) \hat{c}_{il}^t \begin{bmatrix} \hat{x}_{d(n_d+1)}(t) + g_{dn_d}(\eta_d(t)) \\ - r(t) g_{d(n_d+1)}(\eta_d(t)) \end{bmatrix} \triangleq \Lambda_{\mu}^2(t). \\
\end{bmatrix}
\]

Then the closed-loop of x-subsystem of (1), (24), and (25) is equivalent to
\[
\begin{cases}
\dot{x}_i(t) = A_{ni} x_i(t) dt + B_{ni} [v_i(x_{i1}(t), \ldots, x_{in_i}(t)) \\
+ \eta_{i(n_i+1)}(t) + \hat{\sigma}_i(t)] dt, \\
\text{let } a(t) = r(t) A_{ni+1} \eta_{i}(t) dt \\
- r(t) \begin{bmatrix} g_{i1}(\eta_{i1}(t)) \\
g_{i(n_i+1)}(\eta_{i1}(t)) \end{bmatrix} dt + \frac{r(t)}{r(t)} \begin{bmatrix} n_1 \eta_{i1}(t) \\
(n_1 - 1) \eta_{i2}(t) \\
\cdots \\
\eta_{in_i}(t) \end{bmatrix} dt + B_{n_i+1} A^*_{ni} (t) + \Lambda_{ni}^2(t) dt \\
+ B_{n_i+1} A^*_{ni} (t) dW_1(t) + B_{n_i+1} A^*_{ni} (t) dW_2(t), \\
1 \leq i \leq m,
\end{cases}
\]
where \( \hat{\sigma}_i(t) = \sigma_{i1}(t) + f^*_i(t, x(t)) - f^*_i(t, \hat{x}(t)) \) and \( \eta_{ij}(t) = r^{(n_j+1-i)}(t)(x_{ij}(t) - \hat{x}_{ij}(t)) \) (1 \( \leq j \leq n_i + 1) ). Notice that for all 1 \( \leq i \leq m \) and all \( t \geq 0 \),
\[
[f^*_i(t, x(t)) - f^*_i(t, \hat{x}(t)) \leq \frac{L}{r(t)} ||\eta(t)||, \\
[f^*_i(t, \hat{x}(t)) \leq \frac{L}{r(t)} ||\eta(t)|| + L||x(t)||, \\
|\hat{\sigma}_i(t)| \leq \frac{1}{r(t)} \left( \max_{1 \leq i \leq m} L_i + L \right) ||\eta(t)||.
\]

Proof of Claim (i). By Assumptions A.1 and A.2*, there exist \( \varepsilon \)-independent positive constants \( \xi_{i1}^*, \xi_{i2}^*, \xi_{i3}^* \) such that
\[
||\Lambda_{\mu}^1(t) || \leq \xi_{i1}^* ||\eta(t)||, \quad ||\Lambda_{\mu}^2(t) ||^2 + ||\Lambda_{\mu}^3(t) ||^2 \leq 2 \xi_{i2}^*, \\
(87)
\]
In addition, it also follows from \( v_i(0) = 0 \), the globally Lipschitz continuity of \( v_i(\cdot) \), in Assumption A.3, \( \| g_{d}(\cdot) \| \leq k_d \| z \| \) for all \( z \in \mathbb{R} \) in Assumption A.4, (20), and (85) that there exist positive constants \( \xi_{i1}^*, \xi_{i2}^*, \xi_{i3}^* \) such that
\[
||\Lambda_{\mu}^1(t) || \leq \xi_{i1}^* ||\eta(t)|| + \xi_{i2}^* ||x(t)|| + \xi_{i3}^* ||\eta(t)||, \\
(88)
\]
where \( \xi_i \) is defined as that in (37). Since \( r(t) \equiv \frac{1}{r(t)} \) for all \( t \geq 0 \), \( r(t) \equiv \frac{1}{r(t)} \) for all \( t \geq 0 \) in system (83). By combining (86), (87) with (88), we can see that the remaining stochastic Lyapunov analysis for the closed-loop system (83) can follow similarly that of Theorem 2.1. The remainder of the proof is omitted.

Proof of Claim (ii). By Assumptions A.1*–A.2*, there exist positive constants \( N_{ni}^*, N_{n_i}^*, N_{n_i}^* \) such that
\[
||\Lambda_{\mu}^1(t) || \leq N_{ni}^* e^{3bt} + N_{n_i}^* e^{bt} ||x(t)||, \\
||\Lambda_{\mu}^2(t) ||^2 + ||\Lambda_{\mu}^3(t) ||^2 \leq 2 N_{ni}^* e^{2bt}. \\
(89)
\]
In addition, it also follows from \( v_i(0) = 0 \), the globally Lipschitz continuity of \( v_i(\cdot) \), in Assumption A.3, \( \| g_{d}(\cdot) \| \leq k_d \| z \| \) for all \( z \in \mathbb{R} \) in Assumption A.4, (20), and (85) that there exist positive constants \( N_{ni}^* \) and \( N_{n_i}^* \) such that
\[
||\Lambda_{\mu}^1(t) || \leq N_{ni}^* ||\eta(t)|| + N_{n_i}^* ||x(t)|| + e^{at} \xi_i ||\eta(t)||, \\
(90)
\]
where \( \xi_i \) is defined as that in (37). Since \( r(t) \equiv e^{at} \) for all \( t \geq 0 \), \( r(t) \equiv a \) for all \( t \geq 0 \) in system (83). By combining (86), (89) with (90), we can see that the remaining stochastic Lyapunov analysis for the closed-loop system (83) can follow similarly that of Theorem 2.2. The remainder of the proof is also omitted. ■

4. Numerical simulations

In this section, we present an example to illustrate the effectiveness of the proposed ADRC approach. Consider the following uncertain MIMO system with stochastic inverse dynamic and exogenous stochastic disturbance:
\[
\begin{align*}
\dot{x}_{11}(t) &= x_{12}(t), \\
\dot{x}_{12}(t) &= [a_1 x_{12}(t) + a_2 x_{21}(t) + a_3 \zeta(t) + a_4 \cos(a_5 t) + a_6 W_2(t)] + u_1(t) + u_2(t), \\
\dot{x}_{21}(t) &= x_{22}(t), \\
\dot{x}_{22}(t) &= [a_7 x_{21}(t) + a_8 x_{12}(t) + a_9 \zeta(t) + a_{10} \cos(a_5 t) + a_6 W_2(t)] + u_1(t) - u_2(t), \\
\zeta(t) &= a_{11} \sin(\zeta(t)) \cdot x_{12}(t) dt + a_{12} \cos(\zeta(t)) \cdot x_{21}(t) dt + a_{13} \cos(\zeta(t)) \cdot \cos(a_5 t) + a_6 W_2(t) dW_1(t), \\
y_1(t) &= x_{11}(t), \\
y_2(t) &= x_{21}(t),
\end{align*}
\]
where \( a_i (i = 1, 2, \ldots, 13) \) are unknown parameters satisfying: \( |a_i| \leq M (i = 1, 2, \ldots, 13) \) for a given (known) constant \( M > 0 \). The \( W(t) \triangleq \cos(a_5 t) + a_6 W_2(t) \) is a bounded non-white
noise appeared often in many practical dynamical systems like the motion of oscillators (Hu et al., 2012; Huang et al., 2002), where $\alpha_5$ and $\alpha_6^2$ are constants representing the central frequency and strength of frequency disturbance, respectively. In this case, $m = 2, n = 2 = n, s = k = 1, p = q = 1, c_{11} = c_{12} = c_{21} = 1, c_{22} = -1$. It is easy to check that all the uncertainties in (91) satisfy Assumptions A.1–A.2. Similar to Guo and Wu (2017), we design a constant gain ESO (92) for system (91) as follows:

$$
\begin{align*}
\dot{x}_{11}(t) &= \dot{x}_{12}(t) + \frac{6}{\varepsilon}(y_1(t) - \dot{x}_{11}(t)) \\
+ &\varepsilon \Psi \left( \frac{y_1(t) - \dot{x}_{11}(t)}{\varepsilon} \right), \\
\dot{x}_{12}(t) &= \dot{x}_{13}(t) + \frac{12}{\varepsilon^2}(y_1(t) - \dot{x}_{11}(t)) + u_1(t) + u_2(t), \\
\dot{x}_{13}(t) &= \frac{8}{\varepsilon}(y_1(t) - \dot{x}_{11}(t)), \\
\dot{x}_{21}(t) &= \dot{x}_{22}(t) + \frac{6}{\varepsilon}(y_2(t) - \dot{x}_{21}(t)), \\
\dot{x}_{22}(t) &= \dot{x}_{23}(t) + \frac{12}{\varepsilon}(y_2(t) - \dot{x}_{21}(t)) + u_1(t) - u_2(t), \\
\dot{x}_{23}(t) &= \frac{8}{\varepsilon}(y_2(t) - \dot{x}_{21}(t))
\end{align*}
$$

(92)

where $\Psi : \mathbb{R} \to \mathbb{R}$ is defined as

$$
\Psi(s) = \begin{cases} 
-\frac{1}{\pi}, & s \in (-\infty, -1), \\
\frac{1}{\pi} \sin \frac{\pi s}{2}, & s \in (-1, 1), \\
\frac{1}{\pi}, & s \in [1, +\infty).
\end{cases}
$$

(93)

First we notice that the corresponding matrices in (19) for the linear part of (92) are

$$
F_1 = F_2 = \begin{pmatrix} -6 & 1 & 0 \\
-12 & 0 & 1 \\
-8 & 0 & 0 \end{pmatrix},
$$

(94)

with eigenvalues identical to -2 and hence are Hurwitz. In this case, $g_6()$ in (4) can be specified as

$$
\begin{align*}
g_{11}(z_1) &= 6 z_1 + \Psi(z_1), & g_{12}(z_1) &= 12 z_1, & g_{13}(z_1) &= 8 z_1; \\
g_{21}(z_1) &= 6 z_1, & g_{22}(z_1) &= 12 z_1, & g_{23}(z_1) &= 8 z_1.
\end{align*}
$$

(95)

The nonlinear function $g_{11}()$ is constructed by linear function perturbed by a Lipschitz continuous nonlinear function with small Lipschitz constant. We notice that the ESO (92) without the nonlinear function $\Psi()$ is reduced to a well-defined linear ESO. The design of ESO (92) with $\Psi()$ aims at showing that Assumption A.4 is 'non-empty' for nonlinear cases. On the other hand, it has been shown in many papers such as Guo and Zhao (2013,2016) that this class of nonlinear ESO is at least as good as linear ESO in tracking states and total disturbance. The Lyapunov functions $V_{12}, V_{22} : \mathbb{R}^3 \to \mathbb{R}$ in Assumption A.4 for this case are given by

$$
V_{12}(z) = z^T Q_{12} z + \int_0^t \Psi(s) ds, \\
V_{22}(z) = z^T Q_{22} z,
$$

\forall z = (z_1, z_2, z_3)^T \in \mathbb{R}^3,
$$
(96)

where

$$
Q_{1} = Q_{2} = \begin{pmatrix} 67 & 1 & -97 \\
32 & 2 & -128 \\
-128 & 2 & 512 \end{pmatrix}
$$

(97)

are the positive definite solutions of the Lyapunov equations of $Q_i F_i + F_i^T Q_i = -I_{3 \times 3}$ for $F_i (i = 1, 2)$ given by (94). Similar to the computations in Guo and Wu (2017), we can prove that all conditions of Assumption A.4 are satisfied. Choose $v_i : \mathbb{R}^2 \to \mathbb{R} (i = 1, 2)$ in (11) as follows:

$$
v_1(\hat{x}_{11}, \hat{x}_{12}) = -2 \hat{x}_{11} - 3 \hat{x}_{12}, \\
v_2(\hat{x}_{21}, \hat{x}_{22}) = -\hat{x}_{21} - 2 \hat{x}_{22}
$$

(98)

with the corresponding matrices in (13)

$$
E_1 = \begin{pmatrix} 0 & 1 \\
-2 & -3 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\
-1 & -2 \end{pmatrix}
$$

(99)

being Hurwitz. It follows from Assumption A.3 and Theorem 2.1 that (92) serves as a well-defined nonlinear constant gain ESO for system (91) under the ESO (92) based feedback control designed as

$$
\begin{align*}
u_1 &= \frac{1}{2}(v_1(\hat{x}_{11}, \hat{x}_{12}) - \hat{x}_{13}) + \frac{1}{2}(v_2(\hat{x}_{21}, \hat{x}_{22}) - \hat{x}_{23}), \\
u_2 &= \frac{1}{2}(v_1(\hat{x}_{11}, \hat{x}_{12}) - \hat{x}_{13}) - \frac{1}{2}(v_2(\hat{x}_{21}, \hat{x}_{22}) - \hat{x}_{23}),
\end{align*}
$$

(100)

where in this case the $\alpha_1^* (i, l = 1, 2)$ in (5) are specified as $\alpha_{11}^* = \alpha_{12}^* = \alpha_{13}^* = \alpha_{21}^* = \alpha_{22}^* = \alpha_{23}^* = -\frac{1}{2}$. The Milstein approximation method (Higham, 2001) is used to discretise systems (91) and (92). Figures 1–4 display the numerical results for (91) and (92) where we take

$$
\begin{align*}
a_1 &= 1, a_2 = 2, a_3 = 1, a_4 = a_5 = a_6 = \frac{1}{3}, \\
a_7 &= a_8 = a_9 = a_{10} = a_{11} = a_{12} = a_{13} = 1, \varepsilon = 0.08.
\end{align*}
$$

(101)

The initial values are

$$
x(0) = (1, -1, -1), \quad \zeta(0) = 0, \quad \tilde{x}(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
$$

(102)

and the time discrete step is taken as

$$
\Delta t = 0.001.
$$

(103)

The whole picture of Figure 1(b,c,e,f) near the initial time are depicted in Figure 2. It is seen from Figure 1 that the nonlinear constant gain ESO (92) is very effective in tracking system (91) not only for the state $x(t) = (x_{11}(t), x_{12}(t), x_{21}(t), x_{22}(t))^T$ but also for the extended state (stochastic total disturbance).
Figure 1. Estimation of unmeasured states and stochastic total disturbance and stabilisation of closed-loop system (91) under nonlinear constant gain ESO (92) based feedback control (100).

Figure 2. Observation of peaking values of $\hat{x}_{12}(t), \hat{x}_{13}(t), \hat{x}_{22}(t)$, and $\hat{x}_{23}(t)$ near the initial time. (a) Whole picture of Figure 1(b) near the initial time (b) Whole picture of Figure 1(c) near the initial time (c) Whole picture of Figure 1(e) near the initial time and (d) Whole picture of Figure 1(f) near the initial time.

$$(x_{13}(t), x_{23}(t))^T$$ defined by

$$x_{13}(t) = x_{12}(t) + 2x_{21}(t) + \xi(t) + \frac{1}{3} \cos \left( \frac{1}{3} t + \frac{1}{3} W_2(t) \right),$$

$$x_{23}(t) = x_{21}(t) + x_{12}(t) + \xi(t) + \cos \left( \frac{1}{3} t + \frac{1}{3} W_2(t) \right).$$

(104)

In addition, it is seen from Figure 1(a,b,d,e) that the stabilisation at the origin for each trajectory of $x(t)$ is very satisfactory. However, the large absolute peaking values of $\hat{x}_{12}(t), \hat{x}_{13}(t), \hat{x}_{22}(t)$, and $\hat{x}_{23}(t)$ are observed near the initial stage in Figure 2 which are near 25, 150, 25, and 150, respectively.

Now we apply the following time-varying gain ESO (105) to system (91), which comes from (21) with nonlinear functions
Figure 3. Estimation of unmeasured states and stochastic total disturbance and stabilisation of closed-loop system (91) under nonlinear time-varying gain ESO (105) based feedback control (100).

\[
\begin{aligned}
\dot{x}_{11}(t) &= \dot{x}_{12}(t) + 6r(t)(y_1(t) - \hat{x}_{11}(t)) \\
&\quad + \frac{1}{r(t)}\Psi(r^2(t)(y_1(t) - \hat{x}_{11}(t))), \\
\dot{x}_{12}(t) &= \dot{x}_{13}(t) + 12r^2(t)(y_1(t) - \hat{x}_{11}(t)) + u_1(t) + u_2(t), \\
\dot{x}_{13}(t) &= 8r^3(t)(y_1(t) - \hat{x}_{11}(t)), \\
\dot{x}_{21}(t) &= \dot{x}_{22}(t) + 6r(t)(y_2(t) - \hat{x}_{21}(t)), \\
\dot{x}_{22}(t) &= \dot{x}_{23}(t) + 12r^2(t)(y_2(t) - \hat{x}_{21}(t)) + u_1(t) - u_2(t), \\
\dot{x}_{23}(t) &= 8r^3(t)(y_2(t) - \hat{x}_{21}(t)),
\end{aligned}
\]

(105)

where \( \Psi : \mathbb{R} \to \mathbb{R} \) is given by (93). As recommended in Guo and Zhao (2016) and Remark 2.8, in practice, the time-varying gain should be small value in the beginning and gradually increases to a large constant gain for which we choose as

\[
r(t) = \begin{cases} 
e^{st}, & 0 \leq t \leq \frac{\ln 12.5}{5}, \\ 1 + \frac{1}{5}, & t \geq \frac{\ln 12.5}{5}. \end{cases}
\]

(106)

Figure 3 shows that the nonlinear time-varying gain ESO (105) tracks the state \( x(t) \) of system (91) and stochastic total disturbance \( (x_{13}(t), x_{23}(t))^\top \) defined in (104) well after a short time. In addition, Figure 3(a,b,d,e) show that the stabilisation at the origin under time-varying gain ESO (105) based feedback control (100) is also very satisfactory. More importantly, the absolute peaking values near the initial stage of \( \hat{x}_{12}(t), \hat{x}_{13}(t), \hat{x}_{22}(t), \hat{x}_{23}(t) \) are all less than 8 in Figure 3. This shows that the time-varying gain method reduces dramatically the peaking value caused by the constant high gain.

Now we show the numerical results in the presence of measurement noise in Figure 4. Suppose that the outputs \( y_i(t) (i = 1, 2) \) are all contaminated by the noise 0.001n(t), where \( n(t) \) is the standard Gaussian noise generated by the Matlab program command ‘randn’. Figure 4 demonstrates that the effects of estimation of unmeasured states and stochastic total disturbance (104) and stabilisation of closed-loop system (91) under nonlinear constant gain ESO (92) based feedback control (100) can still be well maintained in the presence of measurement noise, which means that the nonlinear constant gain ESO (92) based feedback control (100) enjoys the performance of measurement noise tolerance to some extent.
5. Concluding remarks

In this paper, we apply the active disturbance rejection control (ADRC) approach to output-feedback stabilisation for a class of MIMO systems with vast stochastic uncertainties including unknown internal stochastic uncertainties, external stochastic disturbance, and uncertainties caused by the deviation of control parameters from their nominal values. The interactions between various control input channels are also regarded as part of stochastic total disturbance and both constant gain ESO and time-varying gain ESO are designed to estimate stochastic total disturbance for each subsystem so that the stochastic total disturbance is compensated in the ESO-based output-feedback loop and then each subsystem becomes approximately a linear time invariant system, which reveals the powerful decoupling function for MIMO uncertain stochastic systems by ADRC approach. With decoupling, an ESO-based stabilising output-feedback control is designed to stabilise each subsystem at the origin separately. Both the practical mean square stability of the closed-loop system under constant gain ESO-based ADRC and asymptotic mean square stability of the closed-loop system under time-varying gain ESO-based ADRC are developed. In addition, two types of special constant gain ADRC and time-varying ADRC are designed by utilising known information of the MIMO system as much as possible when part of the system dynamics is available. The numerical simulations are presented to demonstrate effectiveness of ESO in estimating unmeasured states and stochastic total disturbance of the systems, efficient stabilisation effects of ADRC, and peaking value reduction achieved by the time-varying gain approach.

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ORCID

Ze-Hao Wu http://orcid.org/0000-0003-4109-1757
Bao-Zhu Guo http://orcid.org/0000-0001-9078-0001
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