Disturbance Observer-Based Boundary Control for an Anti-Stable Stochastic Heat Equation with Unknown Disturbance

Ze-Hao Wu, Hua-Cheng Zhou, Feiqi Deng, and Bao-Zhu Guo

Abstract—In this paper, a novel control strategy namely disturbance observer-based control is first applied to stabilization and disturbance rejection for an anti-stable stochastic heat equation with Neumann boundary actuation and unknown boundary external disturbance generated by an exogenous system. A disturbance observer-based boundary control is designed based on the backstepping approach and estimation/cancellation strategy, where the unknown disturbance is estimated in real time by a disturbance observer and rejected in the closed-loop, while the in-domain multiplicative noise whose intensity is within a known finite interval is attenuated. It is shown that the resulting closed-loop system is exponentially stable in the sense of both mean square and almost surely. A numerical example is demonstrated to validate the effectiveness of the proposed control approach.

Index Terms—Stochastic heat equation; boundary control; disturbance rejection; stabilization; backstepping approach.

I. INTRODUCTION

DISTURBANCES are ubiquitous in many practical control systems, which often cause negative effects on performance of the control plant. For the sake of control precision, many control approaches have been developed since 1970s to cope with disturbances in term of disturbance attenuation or disturbance rejection. The stochastic control and robust control are two of representative disturbance attenuation approaches, where the former is often used for attenuating noises with known statistical characteristics while the latter can deal with more general disturbances. However, most of the robust control approaches are on the worst case scenario, which may lead to control design rather conservative. Based on estimation/cancellation strategy, some novel active anti-disturbance control approaches like disturbance observer based control (DOBC) have been proposed for disturbance rejection for control systems over the past two decades. The core idea of these active anti-disturbance control approaches is that the disturbances affecting system performance can be estimated by a disturbance observer and then be compensated in the closed-loop. Owing to the estimation/cancellation characteristics, the active anti-disturbance control is capable of eliminating the disturbances before negative effects are caused and at the same time, the control energy can be reduced significantly in engineering applications.

The disturbance rejection for distributed parameter systems by active anti-disturbance boundary control approaches has been paid increasing attention in the last two decades, see, for instance [3], [4], [5], [6], [7] and the references therein. Specially, some active anti-disturbance control methods to disturbance rejection for finite-dimensional stochastic systems have been proposed. For example, problems of the composite DOBC and $H_{\infty}$ control for Markovian jump systems and the DOBC for a class of stochastic systems with multiple disturbances have been studied in [10] and [11], respectively; An extended state observer-based output feedback stabilizing control has been designed for a class of stochastic systems subject to bounded stochastic noise [12].

As one of the active anti-disturbance control approaches, the DOBC has been widely applied in engineering applications with good disturbance rejection performance and robustness, see, for instance [13], [14] and the references therein. However, there is still no relevant study from theoretical perspective on DOBC for stochastic distributed parameter systems. In this paper, we demonstrate for the first time, through an anti-stable stochastic heat equation with unknown boundary external disturbance, the DOBC approach to stabilization and disturbance rejection for stochastic distributed parameter systems. The main contributions and novelty of this paper can be summarized as follows. a) From a theoretical perspective, the applicability of the powerful DOBC control technology is first expanded to a class of stochastic distributed parameter systems with unknown boundary external disturbance; b) The unknown boundary external disturbance is rejected completely by virtue of estimation/cancellation strategy of the DOBC approach, while the in-domain multiplicative noise with bounded intensity is attenuated; c) Not only the mean square exponential stability but also the almost surely exponential stability are obtained for the resulting closed-loop system.

We proceed as follows. In the next section, section II, some problem formulation and preliminaries are presented. In section III, both design of the DOBC boundary control and stability of the closed-loop system are discussed and stated. A numerical example is presented in section IV, followed up concluding remarks in section V. The proofs of the main results are arranged in Appendix.

II. PROBLEM FORMULATION AND PRELIMINARIES

We first introduce some notations. The $I_n$ denotes the $n$-dimensional identity matrix and $L^2([0,1])$ is the space of all real-valued functions that are square Lebesgue integrable over $(0,1)$. 

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Let \((\Omega, \mathcal{F}, P)\) be a complete filtered probability space with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) on which a one-dimensional standard Brownian motion \(B(t)\) is defined. A stochastic process \(f(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n\) is called \(\mathcal{F}\)-adapted if for every \(t \geq 0\), the function \(\omega \rightarrow f(t, \omega)\) is \(\mathcal{F}_t\)-measurable. For notational simplicity we use \(f(t)\) to denote a stochastic process \(f(t, \omega)\). Let \(V\) be a Banach space. A sub-\(\sigma\)-algebra \(\mathcal{M}\) of \(\mathcal{F}\), denoted by \(L^2_{\mathcal{M}}(\Omega; N)\), is the set of all \(\mathcal{M}\)-measurable \((N\text{-valued})\) random variables \(f : \Omega \rightarrow N\) such that \(\mathbb{E}[f^2_N] < \infty\). Let \(H^1(0, 1)\) and \(H^2(0, 1)\) be the Sobolev spaces. Set \(L^2_0(0, T; L^2(\Omega; V)) = \{f : (0, T) \times \Omega \rightarrow V(f(\cdot)) \text{ is } \mathcal{F}\text{-adapted and } \int_0^T \mathbb{E}[f^2(t)] dt < \infty\}, C^0_0(0, T; L^2(\Omega; V)) = \{f : [0, T] \times \Omega \rightarrow V(f(\cdot)) \text{ is } \mathcal{F}\text{-adapted and } (\mathbb{E}[f^2(t)])^\frac{1}{2} \text{ is continuous}\}, C^1_0(0, T; L^2(\Omega; V)) = \{f : [0, T] \times \Omega \rightarrow V(f(\cdot)) \text{ is } \mathcal{F}\text{-adapted and } (\mathbb{E}[f^2(t)])^\frac{1}{2} \text{ is continuous}\}, C^2_0(0, T; L^2(\Omega; V)) = \{f : [0, T] \times \Omega \rightarrow V(f(\cdot)) \text{ is } \mathcal{F}\text{-adapted and } (\mathbb{E}[f^2(t)])^\frac{1}{2} \text{ is continuously differentiable}\}.

All the above spaces are endowed with the usual canonical norms.

In this paper, we consider stabilization and disturbance rejection for a one-dimensional anti-stable stochastic heat equation driven by multiplicative white noise with unknown boundary external disturbance as follows:

\[
\begin{align*}
\frac{dy(x, t)}{dt} &= y_{xx}(x, t)dt + a(x)y(x, t)dt + \sigma y(x, t)dB(t), \\
y_0(0, t) &= 0, \quad t \geq 0, \\
y_1(1, t) &= u(t) + w(t), \quad t \geq 0, \\
y(0, 0) &= y_0(x, 0), \quad 0 \leq x \leq 1,
\end{align*}
\]

where \(y(x, t)\) is the system state representing the temperature profile at the spatial position \(x \in [0, 1]\) and the time \(t \in [0, \infty)\), \(a(\cdot) \in L^2(0, 1)\), \(\sigma\) is a constant representing the intensity of the multiplicative white noise with a known upper bound for its absolute value, \(y_0(\cdot) \in L^2_0(\Omega; L^2(0, 1))\) is the initial value, and \(w(t)\) is the boundary control input, \(w(t) \in \mathbb{R}\) is the unknown disturbance which could be the temperature perturbation generated from an exogenous system as follows:

\[
\begin{align*}
\xi(t) &= A\xi(t), \\
w(t) &= C\xi(t),
\end{align*}
\]

where \(\xi(t) \in \mathbb{R}^n\) is an unknown exogenous noise, and \(A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{1 \times n}\) are known matrices. Throughout the paper, the stochastic differentials including \(dy(x, t)\) are with respect to the time \(t\).

The following Assumption (A1) is for estimation of the unknown disturbance \(w(t)\).

**Assumption (A1).** For the exogenous system (2), the pair \((A, C)\) is supposed to be observable.

**Remark 1:** The exogenous system (2) is a typical form for disturbance in output regulation, which covers all finite harmonic disturbance with known frequency but unknown amplitude and phase. The harmonic disturbance can be considered as an approximation of periodic disturbance \([11], [10], [18]\) and has been discussed intensively by means of the DOBC approach (see, e.g., [2, p.47]).

Similar to both mean square exponential stability and almost surely exponential stability of stochastic differential equations (see, e.g.,[22]), we introduce the following stability for system (1).

**Definition 1:** System (1) is said to be mean square exponentially stable if there are positive constants \(M\) and \(\eta\) such that

\[
\mathbb{E}[y(\cdot)]^2_{L^2_0(0, 1)} \leq Me^{-\eta t}, \quad \forall t \geq 0,
\]

for all initial value \(y_0 \in L^2_0(\Omega; L^2(0, 1))\), where \(M\) depends on the initial value \(y_0\).

**Definition 2:** System (1) is said to be almost surely exponentially stable if

\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(\cdot)|^2_{L^2_0(0, 1)} < 0 \quad \text{almost surely},
\]

for all initial value \(y_0 \in L^2_0(\Omega; L^2(0, 1))\).

**Remark 2:** System (1) without boundary control could be neither stable in mean square nor stable in almost surely even if the boundary input is vanishing: \(u(t) = w(t) \equiv 0\). Actually, by taking \(a(x) = 4\pi^2 + 1.005\), \(\sigma = 0.1\) and \(y_0(x) = \cos(2\pi x)\) in (1), \(y(x, t) = \cos(2\pi x) e^{t + 0.1B(t)}\) solves (1). Since \(\lim_{t \to \infty} \frac{B(t)}{t} = 0\) almost surely, it is clear that \(\int_0^1 y^2(x, t)dx \to \infty\) almost surely as \(t \to \infty\). Moreover, \(\mathbb{E}[\int_0^1 y^2(x, t)dx] = \mathbb{E}[\int_0^1 y_0^2(x)dx \cdot e^{2t + 0.2B(t)}] = \mathbb{E}[\int_0^1 y_0^2(x)dx \cdot e^{2t} e^{0.2B(t)}] \to \infty\) as \(t \to \infty\).

In what follows, we consider system (1) in the state space \(L^2_0(\Omega; L^2(0, 1))\) with the usual canonical norm. The control objective is to design a disturbance observer-based boundary control so that the closed-loop system is exponentially stable in both mean square and almost surely.

**III. DOBC BOUNDARY CONTROL DESIGN AND MAIN RESULTS**

The framework of the DBOC boundary control design and theoretical approach can be simply explained in Figure 1. By the backstepping approach, the controlled system is first transformed into an equivalent “good” target system for which we are able to design a feedback stabilizing control without unknown boundary external disturbance being considered. Second, by a test function, a measurement output is introduced which is the solution of an approximately exactly observable Itô-type stochastic system with \(w(t)\) being its external disturbance, and then a finite-dimensional disturbance observer is designed for real-time estimation of the unknown disturbance \(w(t)\). The DOBC boundary control constructed by the feedback stabilizing control and a compensation term by the estimate of the unknown disturbance is finally designed to obtain the resulting closed-loop system.

**Motivated by [19], we introduce an invertible transformation \(\Lambda : y \in L^2_0(\Omega; L^2(0, 1)) \rightarrow \zeta \in L^2_0(\Omega; L^2(0, 1))\) as follows:**

\[
z(x, t) = y(x, t) - \int_0^x k(x, \zeta) \gamma(\zeta, t) d\zeta = \Lambda(y(x, t)),
\]

where \(k(\cdot, \cdot)\) defined on \(G := \{(x, \zeta) \in \mathbb{R}^2 : 0 \leq \zeta \leq x \leq 1\}\) is the kernel function specified in (5) later. Let \(c\) be a positive constant. By Itô’s differential rule, a direct computation shows that

\[
\begin{align*}
dz(x, t) &= -z_{xx}(x, t)dt + cz(x, t)dt - \sigma(z(x, t))dB(t) \\
dz(x, t) &= -\left[k(x, y)(x, t) - k(x, 0)\gamma(0, t)\right]dt - \int_0^x k(\zeta, x) \gamma(\zeta, t) d\zeta dt \\
&\quad - \int_0^x k(\zeta, x) \gamma(\zeta, t) d\zeta dt - \int_0^x k(\zeta, x) \zeta \gamma(\zeta, t) d\zeta dt \\
&\quad - \sigma(x) \int_0^x k(\zeta, x) \gamma(\zeta, t) d\zeta dt \\
&\quad - \left[y_{xx}(x, t) - \frac{dk(x, x)}{dx} \gamma(x, t) - k(x, y_{xx}(x, t))\right]dt \\
&\quad - \left[-k_0(x, x) \gamma(x, t) - \int_0^x k_0(x, \zeta) \gamma(\zeta, t) d\zeta dt \right] \\
&\quad + c[y(x, t) - \int_0^x k(x, \zeta) \gamma(\zeta, t) d\zeta dt - \sigma z(x, t)dB(t)]
\end{align*}
\]
where $\mathbf{Z}(t)$ is the internal state of the disturbance observer (12), $L \in \mathbb{R}^{n \times 1}$ is chosen so that $A + LC$ is Hurwitz by Assumption (A1), and $\hat{\xi}(t)$ and $\hat{w}(t)$ are the estimates of $\xi(t)$ and the unknown disturbance $w(t)$, respectively. Set the observer estimation error as $\eta(t) = \hat{\xi}(t) - \xi(t)$. It is easy to verify that $\eta(t)$ satisfies the following Itô-type stochastic differential equation
\begin{equation}
\eta(t) = (A + LC)\eta(t)dt + \sigma L Z(t) dB(t).
\end{equation}

Remark 4: It is pointed out that the proposed finite-dimensional disturbance observer for real-time estimation of unknown boundary external disturbance of stochastic PDEs is systematic. Actually, let the state of the stochastic PDE also be denoted by $z(x,t)$. The idea is to find an average-type output $\bar{Z}(t) = \int_0^1 h(z(x,t))dx$ where $h(x)$ is a test function governed by the stochastic PDE so that $Z(t)$ satisfies an approximatively exactly observable Itô-type stochastic system with $w(t)$ being the external disturbance determined uniquely by the average-type output $Z(t)$. Specifically, for the case that $z(x,t)$ satisfies (8) with $u(t)$ being designed in (9), it can be obtained that the average-type output $\bar{Z}(t) = \int_0^1 h(z(x,t))dx$ satisfies
\begin{equation}
d\bar{Z}(t) = h(1)\eta(t)dt + \sigma L Z(t) dB(t).
\end{equation}

Hence, to guarantee that $w(t)$ is uniquely determined by the average-type output $Z(t)$, the test function $h(x)$ can be any one that satisfies the condition
\begin{equation}
h'(0) = 0, h(1) \neq 0, h''(x) = v h(x),
\end{equation}
where $v$ is any constant. This is the reason we choose $h(x) = \cos(\pi x)$ for our case. In fact, for any $T > 0$,
\begin{equation}
\bar{Z}(t) \equiv 0, u_0(t) \equiv 0, \forall t \in [0,T] \Rightarrow w(t) \equiv 0, \forall t \in [0,T],
\end{equation}
which means that $w(t)$ is uniquely determined by the average-type output $Z(t)$. This is similar to the exact observability of Itô-type stochastic systems without external disturbance that the state is uniquely determined by output, see for instance [20]. Therefore, a natural way is to use the output $Z(t)$ of system (11) to design a disturbance observer to estimate $w(t)$, and hence the idea of finite-dimensional disturbance observer can be adopted to design the observer (12). In addition, no matter what test function satisfying condition (15) to be chosen, the design framework of the disturbance observer does not change for an approximatively exactly observable linear Itô-type stochastic system (14). The control design also does not change because in the steady state, $\hat{w}(t)$ approaches $w(t)$ in both mean square and almost sure sense.

Since $A + LC$ is Hurwitz, the mean square exponential stability of the error system (13) can be expected if the intensity of the white noise is “small” to some extent, and then the almost surely exponential stability can also be concluded directly due to the common linear growth condition. Thus, we can design the control law $u_0(t) = -\tilde{w}(t)$ in (9) to cancel the unknown disturbance $w(t)$ in real time. We first consider the mean square exponential stability for $(Z(t), \eta(t))$ which is governed by
\begin{equation}
d \left[ \frac{Z(t)}{\eta(t)} \right] \leq \left[ \begin{array}{c} -(c + \pi^2) & C \\ A + LC & \sigma \end{array} \right] \begin{bmatrix} Z(t) \\ \eta(t) \end{bmatrix} dt + \left[ \begin{array}{c} \mathbf{0} \\ \sigma \end{array} \right] \frac{Z(t)}{\eta(t)} dB(t).
\end{equation}
By Assumption (A1), \( \begin{pmatrix} -c + \pi^2 & C \\ 0 & A + LC \end{pmatrix} \) is Hurwitz and hence there exists a unique positive matrix \( Q \) such that
\[
\begin{pmatrix} -c + \pi^2 & C \\ 0 & A + LC \end{pmatrix} Q + Q^\top \begin{pmatrix} -c + \pi^2 & C \\ 0 & A + LC \end{pmatrix} = -I_{(n+1)^2},
\]
where \( Q \in \mathbb{R}^{n+1} \). A direct computation can easily show that \( Q_1 = -\frac{1}{2c+\pi^2}C(A+LC)-(c+\pi^2)I^{-1} \), and \( Q_2 \) satisfies \((A+LC)^\top Q_2 + Q_2A + LC + C^\top Q_2 + Q_3C = -I_n \). It can be further obtained that
\[
\mu_c := \max \left( \left[ \begin{array}{l} 0 \\ Q_2 \end{array} \right] \right) = Q_1 + Q_2L + L^\top Q_3 + L^\top Q_4L,
\]
which is a \( c \)-dependent positive constant because \( Q_2 \) is a positive definite matrix and \( \left[ \begin{array}{l} 0 \\ Q_1 \end{array} \right] \) is a nonzero positive semi-definite one. The mean square and almost surely exponential stability of system (17) can be summarized as the following Lemma 3.2.

**Lemma 3.2:** Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{2c+\pi^2} \) hold. Then, the solution of system (17) satisfies

(i) \( E[|Z(t)|^2 + |\eta(t)|^2] \leq \frac{1}{\mu_c} \max_{Q_2} E[|Z(0)|^2 + |\eta(0)|^2]e^{\mu_c t} \), \( t \geq 0 \);

(ii) \(
\max \{ \limsup_{t \to \infty} \frac{1}{t} \log |Z(t)|, \limsup_{t \to \infty} \frac{1}{t} \log |\eta(t)| \} \leq -\frac{1}{2\mu_c^2} \)
almost surely.

**Proof.** See “Proof of Lemma 3.2” in Appendix A.

The DOBC boundary control is
\[
u(t) = k(1,1) y(t,1) + \int_0^1 k_1(1,\xi) y(\xi, t) d\xi - \hat{w}(t),
\]
under which system (8) becomes
\[
\begin{align*}
\dot{z}_1(t) &= -c z_1(t) + cz_2(t) + \sigma z_2(t) \dot{b}(t), \\
z_1(0) &= 0, \ t \geq 0, \\
z_2(1,t) &= \hat{w}(t) + w(t), \ t \geq 0, \\
z_2(0) &= 0, \ 0 \leq x \leq 1.
\end{align*}
\]

It is noted that in order to design disturbance observer (12) and DOBC boundary control (22), we used actually three measured outputs \( y(t) := \{y(1,t), Z(t), \int_0^1 k_1(1,\xi) y(\xi,t) d\xi\} \), where \( y(1,t) \) is a pointwise measurement output, and \( Z(t) \) defined in (10) and \( \int_0^1 k_1(1,\xi) y(\xi, t) d\xi \) with \( k(\cdot, \cdot) \) specified in (5) are two average-type measurement outputs. Such kinds of measurement outputs can be found in Chapters 4 or 5 of the monograph [15] where they are regarded as an average measurement of the temperature around a certain point. In a recent paper [16], these signals are regarded as the non-local ones. In addition, by choosing appropriately measurement distribution functions, they can be realized by a finite number of sensors in practice, see for instance [6], [17]. Nevertheless, we consider present paper as state feedback. The output feedback stabilization is a separate issue based on state feedback stabilization.

Next, we first show the well-posedness of the closed-loop system (23).

**Lemma 3.3:** Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{2c+\pi^2} \) hold. Then, for any initial value \( z_0 \in \mathbb{L}_2^\infty(\Omega; L^2(0,1)) \), the closed-loop system (23) admits a unique weak solution \( z \in C_y(0, +\infty; L^2(\Omega; L^2(0,1))) \); Moreover, for any \( T > 0 \) there exists a positive constant \( C(T) \) such that
\[
\|z\|_{C_y(0,T; L^2(\Omega; L^2(0,1)))} + \|\hat{w}\|_{L_2^\infty(0,T; L^2(\Omega; L^2(0,1)))} \leq C(T) \left[ |z_0|_{\mathbb{L}_2^\infty(\Omega; L^2(0,1))} + |\hat{w}|_{L_2^\infty(0,T; L^2(\Omega; L^2(0,1)))} \right].
\]

**Proof.** See “Proof of Lemma 3.3” in Appendix B.

The positive constant \( \theta^* \) used hereinbelow is defined as
\[
\theta^* = \begin{cases} \min \{2c - 2 - 3\sigma^2, \frac{1 - \mu_c^2}{\max\{Q_2\}} \} & \text{if } 2c - 2 - 3\sigma^2 \neq \frac{1 - \mu_c^2}{\max\{Q_2\}}, \\
\theta & \text{if } 2c - 2 - 3\sigma^2 = \frac{1 - \mu_c^2}{\max\{Q_2\}}, \end{cases}
\]
where \( \theta \) is a fixed positive constant satisfying \( \theta < 2c - 2 - 3\sigma^2 \).

The mean square exponential stability of the closed-loop system (23) is summarized in the following Lemma 3.4.

**Lemma 3.4:** Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{\sqrt{2c(2c-1)}} \) hold. Then, for any initial value \( z_0 \in L_2^\infty(\Omega; L^2(0,1)) \), the solution of the closed-loop system (23) satisfies
\[
E|z(\cdot,t)|^2_{L_2^\infty(0,1)} \leq \Gamma \exp^{-\theta t}, \quad t \geq 0,
\]
where the positive constant \( \Gamma \) specified in (56) depends on the initial value \( z_0 \) and \( \theta \) for the second case of (25).

**Proof.** See “Proof of Theorem 3.4” in Appendix C.

The resulting closed-loop system comprised of (1), (12) and (22) is then written as
\[
\begin{align*}
\dot{y}(t) &= y_1(x,t)dt + a(x)y_1(x,t)dt + \sigma y_1(x,t)dB(t), \\
y_1(0,t) &= 0, \ t \geq 0, \\
y_1(1,t) &= k(1,1)y_1(1,t) + \int_0^1 k_1(1,\xi)y_1(\xi,t) d\xi + w(t) - \hat{w}(t), \ t \geq 0, \\
z(x,t) &= y(x,t) - \int_0^1 k_1(1,\xi)y_1(\xi,t) d\xi, \ t \geq 0, \\
\hat{w}(t) &= \hat{w}(t) + |AL + (c + \pi^2)\mathbb{Z}(t), t > 0, \\
\hat{z}(t) &= \theta \hat{w}(t) + L\hat{w}(t), \ \hat{w}(t) = C^T \hat{w}(t), \ T > 0.
\end{align*}
\]

The mean square exponential stability of the closed-loop system (27) is summarized in the following Theorem 3.1.

**Theorem 3.1:** Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{\sqrt{2c(2c-1)}} \) hold. Then, for any initial value \( y_0 \in L_2^\infty(\Omega; L^2(0,1)) \), the closed-loop system (27) admits a unique weak solution \( y \in C_y(0, +\infty; L^2(\Omega; L^2(0,1))) \); Moreover, the solution of the closed-loop system (27) satisfies
\[
E|y(\cdot,t)|^2_{L_2^\infty(0,1)} \leq \Gamma \exp^{-\theta^* t}, \quad t \geq 0,
\]
where \( \Gamma^* \) specified in (58) is a positive constant depending on the initial value \( y_0 \).

**Proof.** See “Proof of Theorem 3.1” in Appendix D.

The almost surely exponential stability of the closed-loop system (27) is summarized in the following Theorem 3.2.

**Theorem 3.2:** Suppose that Assumption (A1) and \(|\sigma| < \frac{1}{\sqrt{2c(2c-1)}} \) hold. Then, for any initial value \( y_0 \in L_2^\infty(\Omega; L^2(0,1)) \), the closed-loop system (27) admits a unique weak solution \( y \in C_y(0, +\infty; L^2(\Omega; L^2(0,1))) \); Moreover, the solution of the closed-loop system (27) satisfies
\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(\cdot,t)|_{L_2^\infty(0,1)} \leq -\frac{\theta^*}{2} \quad \text{almost surely}.
\]

**Proof.** See “Proof of Theorem 3.2” in Appendix E.

**Remark 5:** It is noted that system (1) contains two classes of disturbances: one is the in-domain multiplicative noise \( \sigma y(x,t)dB(t) \) and the other is the unknown boundary external disturbance \( w(t) \). The in-domain multiplicative noise whose intensity \( \sigma \) satisfying \(|\sigma| < \frac{1}{\sqrt{2c(2c-1)}} \) is attenuated by the proposed DOBC boundary control (22) in a passive way. However, \( w(t) \) is rejected by an active estimation/cancellation strategy in the feedback loop.
Remark 6: For practical examples with the known matrices $A, L, C$ and the chosen parameter $c$, it is seen from (19) that $\mu_c$ can be computed directly so that the inequality condition $|\sigma| < \min\left\{\frac{1}{\sqrt{n}}, \sqrt{\frac{2(1-c^2)}{2}} \right\}$ in Lemma 3.4, Theorems 3.1 and 3.2 can be checked; And for given matrices $A, L, C$, the relationship of $u_c$ with the chosen parameter $c$ can also be determined, which will provide a guideline on how to determine the parameter $c$ to satisfy the inequality condition. Significantly, we can also see that the maximum tolerance of the in-domain multiplicative noise is $\min\left\{\frac{1}{\sqrt{n}}, \sqrt{\frac{2(1-c^2)}{2}} \right\}$ which depends on $c$ and $L$. This is reasonable because they are used in the design of both the gain $(c + \pi)^2$ of the disturbance observer (12) and the function gain $k(\cdot, \cdot)$ of the DOBC boundary control (22).

Remark 7: From Remark 6, we know that the maximum tolerance of the multiplicative noise depends on the designed $c$ and $L$. For practical example, it is possible to guarantee that the maximum tolerance can be sufficiently large by appropriately designing $c$ and $L$. For example, when $n = 2$, $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $C = [1, 0]$, the matrix $L = [l_1, l_2]^T$ where $l_1 < 0$ and $l_2 < 2$ is chosen so that $A + LC = \begin{bmatrix} h_1 & 2 \\ -h_2 & 0 \end{bmatrix}$ is Hurwitz. Moreover, from (18)-(19), it is seen that $\mu_c$ is calculated to be $\mu_c = \lambda_{\max}\left(\begin{bmatrix} 1 \end{bmatrix}, [0, 0] \right) \frac{1}{\sqrt{2(1-c^2)}} + (c + e)^2 - \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{2(1-c^2)}} l_2 \frac{1}{\sqrt{2(1-c^2)}} \frac{1}{\sqrt{2(1-c^2)}} l_1 \frac{1}{\sqrt{2(1-c^2)}} l_1 \frac{1}{\sqrt{2(1-c^2)}} l_2 > 0$, where $\lambda_1 = (l_1 - c - \pi^2)(c + \pi^2) + l_2 = 0$. Letting $l_2 = l_1$ with $|l_1| < 2$, it is clear that $\mu_c \to 0$ as $(c, l_2) \to (\infty, 0^+)$. Hence, $\min\left\{\frac{1}{\sqrt{n}}, \sqrt{\frac{2(1-c^2)}{2}} \right\} \to \infty$ as $(c, l_2) \to (\infty, 0^+)$, which implies that the maximum tolerance $\min\left\{\frac{1}{\sqrt{n}}, \sqrt{\frac{2(1-c^2)}{2}} \right\}$ can be as large as possible by choosing properly both $c$ and $L$. Certainly, finding $c$ and $L$ to ensure $\min\left\{\frac{1}{\sqrt{n}}, \sqrt{\frac{2(1-c^2)}{2}} \right\}$ to be large enough is generally complicated when $n > 2$.

Remark 8: Compared with the deterministic setting, many new concerns should be addressed for the boundary feedback stabilization of stochastic PDEs subject to boundary external disturbance. On the one hand, the admissibility property that the Ito integral $\int_0^t e^{\delta(t-t')} \mathcal{B}u(s)dB(s) \in L^2_\mathbb{P}(\mathcal{U}, L^2(0, 1))$ for any fixed $t$ in stochastic setting is difficult to be proved, where $e^{\delta t}$ and $\mathcal{B} = \delta(x - 1)$ are the involved semigroup and boundary control operator, respectively. Thus, the proof of Lemma 3.3 is divided into three steps, and some new approaches including stochastic analysis to obtain the martingale property and the approximating method are adopted to prove the well-posedness of the closed-loop system (23). On the other hand, compared with the deterministic situation, it is not only a simple application of the Ito’s differentiation rule in stochastic setting. For example, in the proof of Lemma 3.3, the stopping time technique should be used because no available results to exclude directly the possibility that $E\mathbb{E}[\sigma(\cdot, t) \mathbb{P}_{(t)} | _{\mathbb{U}(0, 1)}] \to \infty$ in finite time so that even $\mathbb{E}[\sigma(\cdot, t) \mathbb{P}_{(t)} | _{\mathbb{U}(0, 1)}] dB(t)$ does not necessarily exist and $\mathbb{E}[\sigma(\cdot, t) \mathbb{P}_{(t)} | _{\mathbb{U}(0, 1)}] dB(t)$ is not necessarily a martingale. In addition, very few references address the almost surely boundary stabilization of stochastic PDEs with or without boundary external disturbance. Thus, in the proof of Theorem 3.2, some stochastic analysis techniques including Chebyshev’s inequality, Burkholder-Davis-Gundy inequality and Borel-Cantelli lemma are used for the proof of almost surely exponential stability of the closed-loop system (27), where Ito’s differentiation rule is just the first step.

IV. NUMERICAL SIMULATIONS

In this section, we present some numerical simulations for the closed-loop system (27) for illustration of the effectiveness of the proposed DOBC control. As pointed out in Remark 2, we take $a(x) = 4\pi^2 + 1.005$, $\sigma = 0.1$, $y_0(x) = \cos(2\pi x)$ in system (1). In this case, system (1) is not stable if there is no boundary control input. This can be seen from Figure 2(a).

The known matrices in the exogenous system (2) are chosen as $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $C = [1, 0]$, and $L$ is taken as $L = [-5, -1]^T$ to guarantee that $A + LC = \begin{bmatrix} -5 & 2 \\ -1 & 0 \end{bmatrix}$ is Hurwitz. The parameter $c$ in system (27) is chosen as $c = 1.02$. Solve (18) to get $Q_c = 0.0134$. 0.0041 0.0041 0.0041 0.1667 0.1667 0.1667 0.1667 0.1667 and $\lambda_{\max}(Q_c) = 0.7172$. Calculate (19) to get $\mu_c = 6.464$. Clearly, the conditions in Lemma 3.2, Theorem 3.1 and Theorem 3.2 are satisfied.

Fig. 2. The state $y(x, t)$ of the open-loop system without boundary control and the state $y(x, t)$ of the closed-loop system (27). (for interpretation of the references to color of the figure’s legend in this section, we refer to the PDF version of this paper).

Fig. 3. The estimation of boundary external disturbance $\hat{w}(t)$, $w(t)$, $\hat{w}(t) - w(t)$, DOBC boundary control $u(t)$ and feedback control $u(t) + \hat{w}(t)$ (for interpretation of the references to color of the figure’s legend in this section, we refer to the PDF version of this paper).

It is seen from Figure 2(b) that the closed-loop state $y(x, t)$ is convergent to zero quite quickly. It is observed from Figure 3(a) that the boundary external disturbance $w(t)$ is effectively estimated by $\hat{w}(t)$. Finally, the effect of both DOBC boundary control $u(t)$ and feedback control $u(t) + \hat{w}(t)$ can be seen from Figure 3(b).

V. CONCLUDING REMARKS

This paper is the first effort to apply the disturbance observer-based control (DOBC) approach to the stabilization and disturbance rejection for the stochastic distributed parameter systems. The considered plant is a one-dimensional anti-stable stochastic heat equation driven by multiplicative white noise subject to boundary external disturbance generated from an exogenous system. A boundary feedback stabilizing control is designed for the system without external boundary disturbance by the backstepping approach. A disturbance observer is designed to estimate in real time the boundary external disturbance,
and then a DOBC boundary control constructed by the boundary feedback stabilizing control and a compensation term by use of the estimate of the boundary external disturbance is designed. It is shown that the resulting closed-loop system is exponentially stable in mean square and almost surely. Some numerical simulations validate the theoretical results and effectiveness of the proposed DOBC approach. A further development of this topic is to address the output feedback boundary control for stochastic PDEs subject to unknown external disturbances only by pointwise measurement.

APPENDIX A: Proof of Lemma 3.2. Let $V(Z, \eta) = \left[ \frac{Z}{\eta} \right]^{T} Q_{e} \left[ \frac{Z}{\eta} \right]$. By (17) and (18), and applying Itô’s formula to $V(Z(t), \eta(t))$ with respect to $t$, we obtain

$$
\begin{align*}
& \mathbb{E} V(Z(t), \eta(t)) = \mathbb{E} V(Z(0), \eta(0)) - \int_{0}^{t} \mathbb{E} \left[ |Z(s)|^2 + |\eta(s)|^2 \right] ds \\
& \quad + \sigma^2 \int_{0}^{t} \mathbb{E} \left[ \left[ \frac{Z(s)}{\eta(s)} \right]^{T} \left[ \begin{array}{l} 0 \\ 0 \\ \eta(s) \\ \eta(s) \end{array} \right] Q_{e} \left[ \begin{array}{l} 0 \\ 0 \\ \eta(s) \\ \eta(s) \end{array} \right] ds.
\end{align*}
$$

(30)

It is also easily obtained that

$$
\lambda_{\min}(Q_{e}) |Z|^2 + |\eta|^2 \leq V(Z, \eta) \leq \lambda_{\max}(Q_{e}) |Z|^2 + |\eta|^2.
$$

(31)

In this way,

$$
\frac{d}{dt} \mathbb{E} V(Z(t), \eta(t)) \leq \frac{1 - \mu_{c} \sigma^2}{\lambda_{\max}(Q_{e})} \mathbb{E} V(Z(t), \eta(t)),
$$

(32)

where $\mu_{c}$ is defined as that in (19). The (20) can follow from (31) and (32) that

$$
\begin{align*}
& \mathbb{E} |Z|^2 + |\eta|^2 \leq \mathbb{E} V(Z(0), \eta(0)) \\
& \leq \frac{1 - \mu_{c} \sigma^2}{\lambda_{\max}(Q_{e})} \mathbb{E} V(Z(t), \eta(t)) \\
& \leq \frac{1 - \mu_{c} \sigma^2}{\lambda_{\min}(Q_{e})} \mathbb{E} |Z|^2 + |\eta|^2.
\end{align*}
$$

(33)

Since both the drift term and diffusion term of system (17) satisfy the linear growth condition, by [22, Theorem 4.2, p. 128], (21) can be directly concluded from (20), where the relevant parameters are specified as $\lambda = \frac{1 - \mu_{c} \sigma^2}{\lambda_{\min}(Q_{e})}$, $p = 2$. This completes the proof of the lemma. □

APPENDIX B: Proof of Lemma 3.3. Since the admissibility theory for stochastic setting is not available for our case, the well-posedness cannot be simply proved. To derive the well-posedness of (23), we split the proof into three steps.

Step 1: Suppose that $\dot{w} \in C_{0}^{2}(0, T; L^{2}(\Omega; \mathbb{R}))$. Consider the boundary value problem of a second order PDE subject to random boundary value $\dot{w}(t)$ as follows:

$$
\begin{align*}
p_{xx}(x, t) - cp(x, t) = 0, & \quad x \in (0, 1), \\
p_{x}(0, t) = 0, & \quad \dot{w}(t).
\end{align*}
$$

(34)

We first claim that $p \in C_{0}^{2}(0, T; L^{2}(\Omega; H^{1}(0, 1)))$. Actually, by the classical theory of elliptic equations with Neumann boundary condition, for any $T > 0$, there exists a positive constant $C(T)$ depending on $T$ such that

$$
|p|_{H^{2}(0, 1)}^2 \leq C(T) |\dot{w}|^2, \quad \forall t \in [0, T] \quad \text{almost surely}.
$$

(35)

This, together with the fact $\dot{w}(t) \in L^{2}_{\mathcal{F}}(\Omega; \mathbb{R})$ for all $t \geq 0$ which is concluded from (20) and $\dot{w}(t) = -\mathcal{C}(t)$, implies that $p(\cdot, t) \in L^{2}_{\mathcal{F}}(\Omega; H^{1}(0, 1))$, and

$$
|p(\cdot, t) - p(\cdot, s)|_{H^{1}(0, 1)}^2 \leq C(T) |\dot{w}(t) - \dot{w}(s)|^2.
$$

(36)

Since $\dot{w} \in C_{0}^{2}(0, T; L^{2}(\Omega; \mathbb{R}))$, it follows that $p \in C_{0}^{2}(0, T; L^{2}(\Omega; H^{1}(0, 1)))$. Next, we consider the following stochastic heat equation:

$$
\begin{align*}
dv (x, t) & = \left[ \nu_{x}(x, t) - cp(x, t) - p(x, t) \right] dt \\
& \quad + \sigma \left[ \nu(x, t) + p(x, t) \right] dB(t), \\
\nu_{x}(0, t) & = 0, \quad \nu_{x}(1, t) = 0, \quad t \geq 0, \\
v(0, t) & = \gamma_{0}(t) - p(x, t), \quad 0 \leq x \leq 1,
\end{align*}
$$

(37)

which can be rewritten as

$$
\begin{align*}
dv (\cdot, t) & = \left[ \nu(\cdot, t) - p(\cdot, t) \right] dt + \sigma \left[ \nu(\cdot, t) + p(\cdot, t) \right] dB(t), \\
\text{where the linear operator } \nu & \text{ is defined by}
\end{align*}
$$

(38)

$$
\begin{align*}
\nu f(x) & = f'(x) - \sigma f(x), \\
D(\nu) & \left\{ \psi \in H^{2}(0, 1) \mid f'(0) = f'(1) = 0 \right\}.
\end{align*}
$$

(39)

It is well known that $\nu$ generates an exponentially stable $C_{0}$-semigroup on $L^{2}(0, 1)$ with the decay rate $-c$. This, together with [21, Theorem 3.5], shows that system (37) admits a unique weak solution $v \in C_{0}(0, +\infty; L^{2}(\Omega; L^{2}(0, 1)))$. Define $z(x, t) = v(x, t) + p(x, t)$. It is easy to see that $z(x, t)$ is a unique weak solution to (23).

Step 2: Suppose $\dot{w} \in C_{0}^{2}(0, T; L^{2}(\Omega; \mathbb{R}))$ and establish an energy estimate for the solution of (23). By virtue of Itô’s formula,

$$
\begin{align*}
d^2 \mathbb{E} |z|^2 & = (\sigma^2 - 2c)\mathbb{E} |z|^2 dt + 2z \mathbb{E} \nu(z) dt + 2c^2 \mathbb{E} |z|^2 dB(t),
\end{align*}
$$

(40)

By the Cauchy-Schwarz inequality, Young’s inequality and the classical embedding theorem in Sobolev spaces, we obtain

$$
\begin{align*}
\mathbb{E} \int_{0}^{T} \mathbb{E} |z(s)|^2 ds & \leq \mathbb{E} \left| \mathbb{E} \int_{0}^{\tau_{0}} \mathbb{E} \mathbb{E} |\dot{w}(s)|^2 ds \right| \\
& \quad + \frac{\varepsilon}{2} \mathbb{E} \int_{0}^{T} \mathbb{E} |\dot{w}(s)|^{2} ds,
\end{align*}
$$

(41)

where $0 < \varepsilon < 1$. The following proof is divided into two cases.

• Case 1: $\sigma^2 - 2c + 2\varepsilon \leq 0$. In this case, by (42) and (43), it follows that

$$
\begin{align*}
\mathbb{E} |z(t \wedge \tau_{0})|^2_{L^{2}(0, 1)} & \leq \mathbb{E} |z(0)|^2_{L^{2}(0, 1)} + \frac{1}{\varepsilon} \mathbb{E} \int_{0}^{\tau_{0}} \mathbb{E} |\dot{w}(s)|^{2} ds.
\end{align*}
$$

(44)

• Case 2: $\sigma^2 - 2c + 2\varepsilon > 0$. In this case, similarly to case 1, $\tau_{0} \uparrow T$ almost surely, and by Fatou’s Lemma and passing to the limit as $n \to \infty$ for (44), we have, for all $t \in [0, T]$, that

$$
\begin{align*}
\mathbb{E} |z(t)_{L^{2}(0, 1)}^2 & \leq \mathbb{E} |z(0)|^2_{L^{2}(0, 1)} + \frac{1}{\varepsilon} \int_{0}^{T} \mathbb{E} |\dot{w}(s)|^{2} ds,
\end{align*}
$$

(45)

which implies (42):
By Gronwall’s inequality, for all \( t \in [0, T] \), we have
\[
\mathbb{E}[\xi(t)]^2 \leq \mathbb{E}[\xi(0)]^2 e^{\sigma_2 t},
\]
which implies (24).

**Step 3:** It follows from (20) in Lemma 3.2 that \( \bar{w} \in L^2(0, T; L^2(\Omega, \mathbb{R}^n)) \). We can then find a sequence \( \{w_n\}_{n=1}^\infty \subseteq C([0, T]; L^2(\Omega, \mathbb{R}^n)) \) such that \( \lim_{n \to \infty} w_n = \bar{w} \) in \( L^2(0, T; L^2(\Omega, \mathbb{R}^n)) \). Denote by \( z_n(t) \) the weak solution to (23) with the initial value \( z_0(x) \) and random boundary value \( \bar{w}(t) \). Then, \( \{z_n(t)\}_{n=1}^\infty \) is a Cauchy sequence in \( C([0, T]; L^2(\Omega, \mathbb{R}^n)) \). Thus, there exists a unique \( z \in C([0, T]; L^2(\Omega, \mathbb{R}^n)) \) such that \( \lim_{n \to \infty} z_n = z \) in \( C([0, T]; L^2(\Omega, \mathbb{R}^n)) \). From the definition of \( z_n(t) \), we have, for all \( t \in [0, T] \) and \( \phi \in H^1(1, 1) \), that
\[
\int_0^1 z_n(t, x) \phi(x) \, dx - \int_0^1 z_n(0, x) \phi(x) \, dx = \int_0^t \int_0^1 \int_0^1 \phi(x, s, t) \, dx \, ds \, dt - \int_0^t \int_0^1 \phi(x, s, t) \, dx \, ds \, dt - c \int_0^t \int_0^1 \int_0^1 \phi(x, s, t) \, dx \, ds \, dt + \int_0^t \int_0^1 \sigma z_n(t, s) \phi(x) \, dx \, dB(s). \tag{48}
\]
This yields, for all \( t \in [0, T] \), that
\[
\int_0^1 z(t, x) \phi(x) \, dx = \int_0^1 z(0, x) \phi(x) \, dx = \int_0^1 \int_0^1 \phi(x, s, t) \, dx \, ds \, dt - \int_0^t \int_0^1 \phi(x, s, t) \, dx \, ds \, dt - c \int_0^t \int_0^1 \int_0^1 \phi(x, s, t) \, dx \, ds \, dt + \int_0^t \int_0^1 \sigma z(t, s) \phi(x) \, dx \, dB(s). \tag{49}
\]
Therefore, \( z(t) \) is a weak solution to (23) and satisfies (24). \( \square \)

**APPENDIX C: Proof of Lemma 3.4.** Let \( \beta(t) = z(t, x) + \frac{1}{2} C(t) \). Clearly, \( d\beta(t) = dz(t, x) + \frac{1}{2} C(t) \, dt \). By (13), a direct computation shows that satisfies the following stochastic PDE:
\[
\begin{cases}
\frac{d\beta(t)}{dt} = \beta_x(t) \phi(x) - \frac{1}{2} C(t) \phi(x) + \frac{1}{2} \nabla \phi(x) \cdot \nabla z(t, x) + \frac{1}{2} \nabla \cdot (A + L) \phi(x) \, dt \\
+ \sigma \sqrt{\frac{1}{2} C(t) \phi(x) + \frac{1}{2} \nabla \phi(x) \cdot \nabla z(t, x)} \, dB(t),
\end{cases}
\tag{50}
\]
By Itô’s formula, a direct computation shows that
\[
d\beta^2(t) = 2 \beta(t) \, d\beta(t) + \frac{1}{2} \sigma^2 \frac{x^2}{C(t)} \, dt + \beta(t) \, d\beta(t) + \frac{1}{2} \sigma^2 \frac{x^2}{C(t)} \, dt = 2 \beta(t) \, d\beta(t) + \frac{1}{2} \sigma^2 \frac{x^2}{C(t)} \, dt.
\tag{51}
\]
Since from Lemmas 3.2 and 3.3, \( \sigma \int_0^1 \beta(x, t) \, dx \) is a martingale for all \( t \geq 0 \). Integrating on both sides of (51) with respect to \( x \) and \( t \) and taking mathematical expectations, we obtain
\[
\mathbb{E}\int_0^1 \beta^2(t, x) \, dx = \mathbb{E}\int_0^1 \beta^2(0, x) \, dx - 2c \int_0^t \mathbb{E}\int_0^1 \beta^2(x, s) \, dx \, ds \\
- 2\int_0^t \mathbb{E}\int_0^1 \beta^2(x, s) \, dx \, ds + 2 \int_0^t \mathbb{E}\int_0^1 \beta(x, s) \, dx \, ds
\tag{52}
\]
\[
+ \int_0^t \mathbb{E}\int_0^1 \left( \frac{x^2}{2} C(t) + \beta(t, s) - \frac{1}{2} C(t) \right)^2 \, dx \, ds,
\]
which implies that
\[
\mathbb{E}\int_0^1 \beta^2(t, x) \, dx \leq -2c \mathbb{E}\int_0^1 \beta^2(0, x) \, dx + 2 \mathbb{E}\int_0^1 \beta^2(x, t) \, dx + 2 \mathbb{E}\int_0^1 \beta^2(x, t) \, dx + 2 \mathbb{E}\int_0^1 \beta^2(x, t) \, dx + 2 \mathbb{E}\int_0^1 \beta^2(x, t) \, dx.
\tag{53}
\]
By (20) in Lemma 3.2 and (53), we conclude that
\[
\mathbb{E}\int_0^1 \beta^2(t, x) \, dx \leq e^{-\left(2c-3\sigma^2\right)t} \mathbb{E}\int_0^1 \beta^2(0, x) \, dx + \int_0^t \mathbb{E}\int_0^1 \beta^2(x, s) \, dx \, ds + \int_0^t \mathbb{E}\int_0^1 \beta^2(x, s) \, dx \, ds + \int_0^t \mathbb{E}\int_0^1 \beta^2(x, s) \, dx \, ds.
\tag{54}
\]
where \( \theta^* \) is given in (25), \( \Gamma_1 = \max \{1 + \frac{1}{2} \sigma^2, \mathbb{E}[|Z(0)|^2] \}, \Gamma_2 = \mathbb{E}\int_0^1 \beta^2(x, t) \, dx + \frac{1}{2} \mathbb{E}[|Z(0)|^2] \mathbb{E}[|Z(0)|^2] \) if \( 2c-3\sigma^2 \neq 1-\mu^2 \) and \( \Gamma_2 = \mathbb{E}\int_0^1 \beta^2(x, t) \, dx + \sup_{0 \leq t \leq 1} e^{-\left(2c-3\sigma^2\right)t \theta^*} \mathbb{E}[|Z(0)|^2] \) with \( \theta \) given in (25). Furthermore, from (20) in Lemma 3.2 and (54), there holds
\[
\mathbb{E}\int_0^1 \beta^2(t, x) \, dx \leq -2c \mathbb{E}\int_0^1 \beta^2(0, x) \, dx + \frac{1}{2} \mathbb{E}[|Z(0)|^2] \mathbb{E}[|Z(0)|^2] \] otherwise with \( \theta \) given in (25).

**APPENDIX D: Proof of Theorem 3.1.** The existence of the solution \( y \in C^0([0, +\infty); L^2(\Omega, \mathbb{R}^n)) \) can be concluded directly from Lemma 3.3. In addition, it follows from (7), Lemma 3.4 and the Hölder inequality that
\[
\mathbb{E}\int_0^1 \gamma^2(t, x) \, dx \
\leq 2 \mathbb{E}\int_0^1 \beta^2(t, x) \, dx + 2 \mathbb{E}\int_0^1 \left( \int_0^t \left| \phi(x, t) \right|^2 \, dt \right)^2 \, dx \
\leq 2 \mathbb{E}\int_0^1 \beta^2(t, x) \, dx + 2 \max_{0 \leq x \leq 1} \mathbb{E}\int_0^1 \left| \phi(x, t) \right|^2 \, dx \\
\leq 2 \mathbb{E}\int_0^1 \beta^2(t, x) \, dx + 2 \max_{0 \leq x \leq 1} \mathbb{E}\int_0^1 \left| \phi(x, t) \right|^2 \, dx \\
= \Gamma e^{-\theta^*},
\tag{55}
\]
where \( \Gamma = 2 \Gamma_2 + \frac{1}{2} \mathbb{E}[|Z(0)|^2] \mathbb{E}[|Z(0)|^2] \). \( \square \)
APPENDIX E: Proof of Theorem 3.2. Let $n = 1, 2, \cdots$. Similarly to
the techniques in (40), (43), it follows from Itô’s formula that for
$n - 1 \leq t \leq n$,
\[
\begin{align*}
|\gamma(t)|_{L^2(0,1)} & = |\gamma(n - 1)|_{L^2(0,1)} + (\sigma^2 - 2c + 2\epsilon) \int_{n - 1}^{t} |\gamma(s)|_{L^2(0,1)} ds \\
& + \frac{1}{E} \int_{n - 1}^{t} \tilde{w}^2(s) ds + \frac{1}{E} \int_{n - 1}^{t} 2\sigma |\gamma(s)|_{L^2(0,1)} dB(s) \\
& \leq |\gamma(n - 1)|_{L^2(0,1)} + \frac{1}{E} \int_{n - 1}^{t} \tilde{w}^2(s) ds \\
& + \int_{n - 1}^{t} 2\sigma |\gamma(s)|_{L^2(0,1)} dB(s),
\end{align*}
\]
where $0 < \epsilon < 1$. Thus,
\[
\begin{align*}
E(\sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)}^2) & \leq E|\gamma(n - 1)|_{L^2(0,1)}^2 + \frac{1}{E} \int_{n - 1}^{n} \tilde{w}^2(s) ds \\
& + E(\sup_{n - 1 \leq t \leq n - 1} \int_{n - 1}^{t} 2\sigma |\gamma(s)|_{L^2(0,1)} dB(s)).
\end{align*}
\]
By the Burkholder-Davis-Gundy inequality (see, e.g., [22, Theorem
1.7.3, p. 40])
\[
\begin{align*}
E(\sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)}^2) & \leq 4\sqrt{2} E\left(\int_{n - 1}^{n} 4\sigma^2 |\gamma(s)|_{L^2(0,1)}^2 ds\right)^{\frac{1}{2}} \\
& \leq 4\sqrt{2} E\left(\sup_{n - 1 \leq t \leq n} |\gamma(s)|_{L^2(0,1)}^2 \int_{n - 1}^{n} 4\sigma^2 |\gamma(s)|_{L^2(0,1)}^2 ds\right)^{\frac{1}{2}} \\
& \leq \frac{1}{2} E\sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)}^2 + 64\sigma^2 \int_{n - 1}^{n} E|\gamma(s)|_{L^2(0,1)}^2 ds.
\end{align*}
\]
After substitution of (61) into (60), we obtain from Lemmas 3.2 and
3.4 that
\[
\begin{align*}
E(\sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)}^2) & \leq 2E|\gamma(n - 1)|_{L^2(0,1)}^2 \\
& + \frac{1}{E} \int_{n - 1}^{n} \tilde{w}^2(s) ds + 128\sigma^2 \int_{n - 1}^{n} E|\gamma(s)|_{L^2(0,1)}^2 ds \\
& \leq \Theta e^{-\theta(n - 1)},
\end{align*}
\]
where $\Theta = (2 + \frac{128\sigma^2}{\theta}) \Gamma + \frac{E}{\Gamma} \left(\int_{0}^{\infty} e^{-\phi(s)} ds\right)^{2} \int_{0}^{\infty} E|\gamma(s)|_{L^2(0,1)}^2 ds.$
Let $\epsilon \in (0, \theta^n)$, By (62) and Chebyshev’s inequality, it
follows that
\[
\begin{align*}
\sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)} & \leq e^{-\Theta(n - 1)} \\
\leq e^{-\epsilon(n - 1)} \sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)},
\end{align*}
\]
Applying the Borel-Cantelli lemma ([22, Lemma 2.4, p.7]), we
obtain for almost all $\omega \in \Omega$, that
\[
\sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)} \leq \Gamma \sup_{n - 1 \leq t \leq n} |\gamma(t)|_{L^2(0,1)},
\]
which holds for all but finitely many $n$ with $\Gamma^n$ given in (58). Hence,
there exists a random variable $n_0 = n_0(\omega)$, such that for almost all
$\omega \in \Omega$, (64) holds whenever $n \geq n_0$. Hence, for almost all $\omega \in \Omega$,
\[
\begin{align*}
\limsup_{t \to \infty} \frac{1}{t} \log |\gamma(t)|_{L^2(0,1)} & \leq \limsup_{t \to \infty} \frac{\log |\gamma(t)|_{L^2(0,1)}}{2n} \\
& \leq \limsup_{t \to \infty} \frac{\log \Gamma^n}{2n} = \frac{(\theta^n - \epsilon)}{2n} \\
& \leq \frac{(\theta^n - \epsilon)}{2n}
\end{align*}
\]
almost surely when $n - 1 \leq t \leq n$. Therefore,
\[
\limsup_{t \to \infty} \frac{1}{t} \log |\gamma(t)|_{L^2(0,1)} \leq - \frac{\epsilon}{2} \quad \text{almost surely.}
\]