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Brief paper

Output regulation for 1-D reaction-diffusion equation with a class of time-varying disturbances from exosystem ☆

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ABSTRACT

In this paper, we consider boundary output regulation for one-dimensional reaction–diffusion equation that has disturbances entering the system from in-domain and both boundaries. The reference signal and disturbances are generated from an exosystem with time varying coefficients. First, a feedforward control is designed on the basis of an infinite-dimensional regulator equation and a backstepping transformation. Second, with the measurement output, we design an observer to estimate both states of the plant and the external system. The output feedback boundary control is then designed by replacing the states with their estimates. As a result, we show that the output converges to the reference signal exponentially as time goes on.

1. Introduction

Output regulation is one of the central problems in control theory, which aims to design a feedback controller so that the output of control system tracks asymptotically reference signal in the presence of disturbances. However, most results on output regulation are conducted for linear time-invariant (LTI) exosystems, see, e.g. Deutscher (2015), Feng, Guo, and Wu (2020), Francis and Wonham (1976), Guo and Meng (2020), Isidori and Byrnes (1990) and Paunonen (2016). The solvability of such regulation problem is associated with the solvability of related regulator equations which are independent of time. The disturbance signals generated by a LTI exosystem are always smooth functions. However, in industry applications, it is desirable to take more general classes of exogenous disturbance signals into account, such as periodic signals (Anderson, 1977) and harmonic signals with time-varying frequencies (Hou, Liu, & Wang, 2018; Shim, Lee, Kim, & Back, 2007). All these types of disturbances can be generated by a linear time-varying (LTV) exosystem in a very natural way.

There are few results available for output regulation of LTV exosystems. The paper (Zhang & Serrani, 2006) considered output regulation for a finite-dimensional minimum-phase system driven by the periodic time-varying exosystem. It was shown that the problem can be solved if and only if a Sylvester differential equation has solution. In Shim, Kim, Kim, and Back (2010), the result of Zhang and Serrani (2006) was extended to the case of non-minimum phase systems. On the other hand, progress on LTV exosystem for distributed parameter systems has also been made in recent years particularly for periodic exogenous signals like the works in Deutscher and Gabriel (2019), Paunonen and Pohjolainen (2012). The periodic output regulation of an abstract infinite-dimensional system with bounded input and output operator was considered in Paunonen and Pohjolainen (2012). It was proved that the solvability of the periodic regulation problem can be characterized by the solvability of an infinite-dimensional Sylvester differential equation. A periodic output regulation problem for general linear heterodirectional hyperbolic systems was investigated in Deutscher and Gabriel (2019) by means of an observer-based output feedback regulator. The well-known Floquet representation theory implies that the periodic systems are the simplest form of LTV systems because they exhibit some similar properties of LTI systems (Zhou, 2016). However, it is not an easy job to extend output regulation problem with periodic exosystems to a class of LTV exosystems considered in our paper. This is because, for example, the state transition matrix of this kind of LTV system is difficult to be calculated analytically, except for some very special cases like periodic Sylvester matrix ODE needed in the observer design of Deutscher and Gabriel (2019).

In a recent paper (Deutscher, 2021), a cooperative output regulation problem for a network of parabolic systems with spatial

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and time-varying parameters has been investigated. Both the reference signal and disturbances are generated by LTI exosystems. Due to the time-varying parameters, the disturbance model of Deutscher (2021) can be regarded also as a class of LTV exosystems. However, a big difference between the paper (Deutscher, 2021) and our paper lies in the size of the class of the disturbances. In order to guarantee the observability of disturbance model, the time-varying parameters of Deutscher (2021) have to be in the Gevrey class which is much smaller than our class. Actually, our class of disturbances are assumed to be continuous or continuously differentiable only. Even for disturbances in Gevrey class, our paper still has the merits: a) our disturbance model does not need to be observable; b) our disturbance observer gain is independent of time, which is easily to achieve exponential stability for observer error system by the pole assignment argument whereas the disturbance observer design in Deutscher (2021) is based on a time-varying transformation, which is difficult to find a proper time-varying observer gain so that the observer error system is exponentially stable. On the other hand, the problem of Deutscher (2021) is a non-collocated output tracking control problem which is difficult even for solving the disturbance regulator equation. This constitutes a further investigation in the future to relax the assumptions by our proposed approach.

Consider the following reaction–diffusion equation:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + \lambda(x)w(x,t), \\ \qquad\qquad +f(x)d_0(t), \quad x \in (0,1),\ t \in (0,\infty), \\ w_x(0,t) = d_1(t), \quad t \in [0,\infty), \\ w_x(1,t) = u(t) + d_2(t), \quad t \in [0,\infty), \\ y_m(t) = w(1,t), \quad t \in [0,\infty), \\ w(x,0) = w_0(x), \quad x \in [0,1], \end{cases} \tag{1.1}$$

where $\lambda \in C^1[0,1], f \in C[0,1]$ are spatially varying coefficients, $u(t)$ is the control, $y_m(t)$ is the measurement, $w_0(x)$ is the initial state, $d_i(t)(i = 0, 1, 2)$ represent the external disturbances.

The disturbances and the reference signals are produced from a class of time-dependent exosystem of the form:

$$\begin{cases} \dot{v}(t) = Sv(t),\ v(0) = v_0 \in \mathbb{C}^{n\times 1}, & t \in (0,\infty), \\ d_i(t) = P_{d_i}(t)v(t),\ i = 0,1,2, & t \in [0,\infty), \\ y_{ref}(t) = P_r(t)v(t), & t \in [0,\infty), \end{cases} \tag{1.2}$$

where $y_{ref}(t)$ represents a given reference signal, S is a block diagonalizable matrix with all eigenvalues located on the imaginary axis, $P_{d_i}(t)(i = 0, 1, 2), P_r(t)$ are time-varying row vectors, v_0 is an unknown initial value which makes in turn the $d_i(t)(i = 0, 1, 2)$ be unknown. These types of exosystems can generate a variety of disturbance signals, such as periodic signals and harmonic signals with time-varying amplitudes. For $v = (v_r, v_d)^\top$, system (1.2) can be divided into two parts of the reference model:

$$\begin{cases} \dot{v}_r(t) = S_r v_r(t),\ \ v_r(0) = v_{r,0} \in \mathbb{C}^{n_r}, \\ y_{ref}(t) = E(t)v_r(t), \end{cases} \tag{1.3}$$

and the disturbance model:

$$\begin{cases} \dot{v}_d(t) = S_d v_d(t),\ \ v_d(0) = v_{d,0} \in \mathbb{C}^{n_d}, \\ d_i(t) = F_i(t)v_d(t),\ \ i = 0,1,2, \end{cases} \tag{1.4}$$

with $S = \mathrm{diag}(S_r, S_d)$ and $n = n_r + n_d$. In addition, the reference signal $y_{ref}(t)$ is assumed to be measurable.

Our aim is to design a tracking error feedback controller so that the tracking error $e(t) = w(1,t) - y_{ref}(t)$ converges exponentially to zero, precisely, there exists a $t_0 > 0$ such that

$$|e(t)| \le L_* e^{-\omega_* t}, \quad \forall\, t \ge t_0, \tag{1.5}$$

where $L_*, \omega_* > 0$ are constants. To achieve the control objective (1.5), we need the following Assumption **(A1)**.

Assumption (A1). There is a constant $C > 0$ such that

$$\begin{aligned} &\sup_{t\ge 0} \|P_{d_i}^\top(t)\|_{\mathbb{C}^{n_d}} + \sup_{t\ge 0}\|P_r^\top(t)\|_{\mathbb{C}^{n_r}} \le C, \\ &\sup_{t\ge 0} \|\dot{P}_{d_1}^\top(t)\|_{\mathbb{C}^{n_d}} + \sup_{t\ge 0}\|\dot{P}_r^\top(t)\|_{\mathbb{C}^{n_r}} \le C, \end{aligned} \tag{1.6}$$

where $i = 0, 1, 2$.

The paper is organized as follows. In Section 2, by proposing an infinite-dimensional regulator equation, a backstepping-based feedforward control is designed. A state observer for cascaded system (1.1)–(1.2) is designed in Section 3 by the reference signal and output. As a consequence, an output feedback control is developed in Section 4. Some simulations are presented in Section 5 to validate the theoretical results, followed up by concluding remarks in Section 6.

2. Feedforward control design

In this section, we design a feedforward control by the states of system (1.1) and exosystem (1.2). Likewise the periodic output regulation (Deutscher & Gabriel, 2019), we first introduce a transformation as follows:

$$\varepsilon(x,t) = w(x,t) - \alpha(x,t)v(t), \tag{2.1}$$

where $\alpha(x,t)$ satisfies

$$\begin{cases} \alpha_t(x,t) + \alpha(x,t)S = \alpha_{xx}(x,t) + \lambda(x)\alpha(x,t) \\ \qquad\qquad\qquad +f(x)P_{d_0}(t), \\ \alpha_x(0,t) = P_{d_1}(t), \quad \alpha(1,t) = P_r(t). \end{cases} \tag{2.2}$$

If system (2.2) has a solution, the transformation (2.1) brings system (1.1) into the following system:

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t) + \lambda(x)\varepsilon(x,t), \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = u(t) + [P_{d_2}(t) - \alpha_x(1,t)]v(t), \\ \varepsilon(1,t) = e(t). \end{cases} \tag{2.3}$$

Now, we design a feedforward control as

$$u(t) = \beta_0(t)v(t) + \int_0^1 \beta_1(y)\varepsilon(y,t)dy + \beta_2 e(t), \tag{2.4}$$

where $\beta_0(t) = -[P_{d_2}(t) - \alpha_x(1,t)]$, and the function β_1, the constant β_2 are the feedback gains to be determined. Introducing a backstepping transformation (Smyshlyaev & Krstic, 2010):

$$\bar{\varepsilon}(x,t) = \mathbb{P}\varepsilon(x,t) = \varepsilon(x,t) - \int_0^x p_1(x,y)\varepsilon(y,t)dy, \tag{2.5}$$

which is invertible and its inverse is given by

$$\varepsilon(x,t) = \bar{\varepsilon}(x,t) + \int_0^x q_1(x,y)\bar{\varepsilon}(y,t)dy, \tag{2.6}$$

where the gain kernels p_1 and q_1 satisfy:

$$\begin{cases} p_{1,xx}(x,y) - p_{1,yy}(x,y) = \lambda(y)p_1(x,y), \\ p_{1,y}(x,0) = 0, \quad p_1(x,x) = -\frac{1}{2}\int_0^x \lambda(y)dy, \\ q_{1,xx}(x,y) - q_{1,yy}(x,y) = -\lambda(x)q_1(x,y), \\ q_{1,y}(x,0) = 0, \quad q_1(x,x) = -\frac{1}{2}\int_0^x \lambda(y)dy. \end{cases} \tag{2.7}$$

System (2.7) has a unique bounded solution $(p_1, q_1) \in C^2(\overline{\mathcal{T}}) \times C^2(\overline{\mathcal{T}})$ (Smyshlyaev & Krstic, 2010), where $\mathcal{T} = \{x, y : 0 < y < x < 1\}$. Choose

$$\beta_1(y) = p_{1,x}(1,y) + cp_1(1,y),\ \beta_2 = p_1(1,1) - c, \tag{2.8}$$

where $c > 0$ is a tuning constant. The transformation (2.5) brings system (2.3) into the following system:

$$\begin{cases} \bar{\varepsilon}_t(x,t) = \bar{\varepsilon}_{xx}(x,t), \\ \bar{\varepsilon}_x(0,t) = 0, \quad \bar{\varepsilon}_x(1,t) = -c\bar{\varepsilon}(1,t), \\ e(t) = \bar{\varepsilon}(1,t) + \displaystyle\int_0^1 q_1(1,y)\bar{\varepsilon}(y,t)dy, \end{cases} \tag{2.9}$$

which is well known exponentially stable in the state space $\mathcal{H} = L^2(0,1)$ (Guo & Meng, 2020) and hence

$$|e(t)| \le L_* e^{-\omega_* t}, \quad \forall\, t \ge t_0, \tag{2.10}$$

for any given $t_0 > 0$ with some constants $L_*, \omega_* > 0$.

We turn to consider the well-posedness of the regulator Eq. (2.2). Let $\mathbb{H} = \{G = (g_1, g_2, \ldots, g_n)^\top | g_i \in L^2(0,1)(i = 1,2,\ldots,n)\}$ be the state space. Define the operator $\mathbb{A}_0 : D(\mathbb{A}_0)(\subset \mathbb{H}) \to \mathbb{H}$ by

$$\begin{cases} \mathbb{A}_0 G = G'', \ \forall\, G \in D(\mathbb{A}_0), \\ D(\mathbb{A}_0) = \{G \in \mathbb{H} | \mathbb{A}_0 G \in \mathbb{H}, \nabla G(0) = G(1) = 0\}. \end{cases}$$

It is well known that the operator $\mathbb{A}_0$ generates an analytic exponentially stable C_0-semigroup $e^{\mathbb{A}_0 t}$ on $\mathbb{H}$:

$$\|e^{\mathbb{A}_0 t}\|_{\mathbb{H}} \le L_0 e^{-\omega_0 t}, \ \forall\, t \ge 0, \tag{2.11}$$

where $L_0, \omega_0 > 0$ are two constants.

Lemma 2.1. *Suppose that Assumption* **(A1)** *holds. For any initial value $\alpha^\top(\cdot,0) \in \mathbb{H}$, system (2.2) admits a unique solution $\alpha^\top \in C(0,\infty;\mathbb{H})$ such that*

$$\|\alpha^\top(\cdot,t)\|_{\mathbb{H}} \le l_1 e^{\varpi_0 t} + l_2, \quad \forall\, t \ge 0, \tag{2.12}$$

where $l_1, l_2 > 0$ are constants and $\varpi_0 = -\omega_0 + L_0\|\lambda(\cdot)I - S^\top\|_{C([0,1];\mathbb{C}^n)}$. Moreover, $\alpha_x^\top(1,t)$ is exponentially bounded at most in the sense that

$$\|\alpha_x^\top(1,t)\|_{\mathbb{C}^n} \le l_3 e^{\varpi_0 t} + l_4, \quad \forall\, t \ge t_0, \tag{2.13}$$

where $l_3, l_4 > 0$ are constants and $t_0 > 0$ is any fixed constant.

Proof. It follows from the perturbation theory of operator semigroup ((Pazy, 1983, Theorem 1.1, p.76), Pazy (1983, Corollary 2.2, p.81) and Pazy (1983, Theorem 5.2, p.61)) that $\mathbb{A} \triangleq \mathbb{A}_0 + (\lambda(x)I - S^\top)$ generates an analytic C_0-semigroup $e^{\mathbb{A}t}$ on $\mathbb{H}$, which satisfies

$$\|e^{\mathbb{A}t}\|_{\mathbb{H}} \le L_0 e^{\varpi_0 t}, \quad \forall\, t \ge 0, \tag{2.14}$$

$$\|\mathbb{A}e^{\mathbb{A}t}\|_{\mathcal{L}(\mathbb{H})} \le C/t, \quad \forall\, t > 0, \tag{2.15}$$

where $\varpi_0 = -\omega_0 + L_0\|\lambda(\cdot)I - S^\top\|_{C([0,1];\mathbb{C}^n)}$, $C > 0$ is a constant and $\mathcal{L}(\mathbb{H})$ denotes the space of bounded linear operators from $\mathbb{H}$ to itself. Let

$$V(x,t) = \alpha^\top(x,t) - (x-1)P_{d_1}^\top(t) - x^2 P_r^\top(t), \tag{2.16}$$

which is governed by

$$\begin{cases} V_t(x,t) = V_{xx}(x,t) + (\lambda(x)I - S^\top)V(x,t) + F(x,t), \\ V_x(0,t) = V(1,t) = 0, \end{cases}$$

where $F(x,t) = f(x)P_{d_0}^\top(t) + (x-1)[-\dot{P}_{d_1}^\top(t) + (\lambda(x)I - S^\top)P_{d_1}^\top(t)] + x^2[-\dot{P}_r^\top(t) + (\lambda(x)I - S^\top)P_r^\top(t)] + 2P_r^\top(t)$. We rewrite V-system as an evolution equation in $\mathbb{H}$:

$$\frac{d}{dt}V(\cdot,t) = \mathbb{A}V(\cdot,t) + F(\cdot,t). \tag{2.17}$$

It is therefore for any initial value $V_0(x) = \alpha^\top(x,0) - (x-1)P_{d_1}^\top(0) - x^2 P_r^\top(0) \in \mathbb{H}$, system (2.17) admits a unique solution $V \in C(0,\infty;\mathbb{H})$, which takes the form

$$V(\cdot,t) = e^{\mathbb{A}t}V_0 + \int_0^t e^{\mathbb{A}(t-s)}F(\cdot,s)ds. \tag{2.18}$$

Since $\|F(\cdot,t)\|_{\mathbb{H}} \le M$ for all $t \ge 0$, we have

$$\|V(\cdot,t)\|_{\mathbb{H}} \le L_0 e^{\varpi_0 t}\|V_0\|_{\mathbb{H}} + \frac{ML_0}{\varpi_0}(e^{\varpi_0 t} - 1), \tag{2.19}$$

which implies (2.12).

By the Sobolev trace theorem and (2.17)–(2.18),

$$\begin{aligned} &\|V_x(1,t)\|_{\mathbb{C}^n} \le C_1(\|V_{xx}(\cdot,t)\|_{\mathbb{H}} + \|V(\cdot,t)\|_{\mathbb{H}}) \\ &\le C_2(\|\mathbb{A}V(\cdot,t)\|_{\mathbb{H}} + \|V(\cdot,t)\|_{\mathbb{H}}) \\ &\le C_2\left(\|\mathbb{A}e^{\mathbb{A}t}V_0\|_{\mathbb{H}} + \left\|\int_0^t \mathbb{A}e^{\mathbb{A}(t-s)}F(\cdot,s)ds\right\|_{\mathbb{H}}\right) \\ &\quad + C_2\|V(\cdot,t)\|_{\mathbb{H}}, \end{aligned} \tag{2.20}$$

where $C_i (i = 1,2)$ are positive constants. By Pazy (1983, Theorem 2.4, p. 4), (2.14) and (2.15),

$$\begin{aligned} \|\mathbb{A}e^{\mathbb{A}t}V_0\|_{\mathbb{H}} &= \|e^{\mathbb{A}(t-t_0)}(\mathbb{A}e^{\mathbb{A}t_0}V_0)\|_{\mathbb{H}} \\ &\le \frac{CL_0}{t_0}e^{\varpi_0(t-t_0)}\|V_0\|_{\mathbb{H}}, \quad \forall\, t \ge t_0, \end{aligned} \tag{2.21}$$

where $t_0 > 0$ is any fixed constant, and

$$\begin{aligned} &\left\|\int_0^t \mathbb{A}e^{\mathbb{A}(t-s)}F(\cdot,s)ds\right\|_{\mathbb{H}} \\ &\le \int_0^t \|e^{\mathbb{A}(t-t_0-s)}\|_{\mathbb{H}}\|\mathbb{A}e^{\mathbb{A}t_0}\|_{\mathcal{L}(\mathbb{H})}\|F(\cdot,s)\|_{\mathbb{H}}ds \\ &\le \frac{CML_0}{\varpi_0 t_0}[e^{\varpi_0(t-t_0)} - e^{-\varpi_0 t_0}], \quad \forall\, t \ge t_0. \end{aligned} \tag{2.22}$$

From (2.19)–(2.22), one obtains

$$\|V_x(1,t)\|_{\mathbb{C}^n} \le l_5 e^{\varpi_0 t} + l_6, \quad \forall\, t \ge t_0, \tag{2.23}$$

where $l_5, l_6 > 0$ are constants and $t_0 > 0$ is any fixed constant. Thus, (2.13) follows from (2.16) and (2.23). ■

Remark 2.1. If

$$\sup_{t\ge 0}\|\dot{P}_{d_0}^\top(t)\|_{\mathbb{C}^{n_d}} + \sup_{t\ge 0}\|\ddot{P}_r^\top(t)\|_{\mathbb{C}^{n_r}} + \sup_{t\ge 0}\|\ddot{P}_{d_1}^\top(t)\|_{\mathbb{C}^{n_d}} \le C,$$

then for any initial value $\alpha^\top(\cdot,0) \in \mathbb{H}$ satisfying $V_0(x) = \alpha^\top(x,0) - (x-1)P_{d_1}^\top(0) - x^2 P_r^\top(0) \in D(\mathbb{A})$, $\alpha_x^\top(1,t)$ grows exponentially at most in the sense that

$$\|\alpha_x^\top(1,t)\|_{\mathbb{C}^n} \le l_3 e^{\varpi_0 t} + l_4, \quad \forall\, t \ge 0, \tag{2.24}$$

where $l_3, l_4 > 0$ are constants. In fact, it follows from $V_0 \in D(\mathbb{A})$ that

$$\|\mathbb{A}e^{\mathbb{A}t}V_0\|_{\mathbb{H}} = \|e^{\mathbb{A}t}\mathbb{A}V_0\|_{\mathbb{H}} \le L_0 e^{\varpi_0 t}\|\mathbb{A}V_0\|_{\mathbb{H}}, \forall\, t \ge 0.$$

Since $\|F_t(\cdot,t)\|_{\mathbb{H}} \le M_1$ for all $t \ge 0$, we have

$$\begin{aligned} &\left\|\int_0^t \mathbb{A}e^{\mathbb{A}(t-s)}F(\cdot,s)ds\right\|_{\mathbb{H}} \\ \le\ & \|F(\cdot,t)\|_{\mathbb{H}} + \|e^{\mathbb{A}t}F(\cdot,0)\|_{\mathbb{H}} + \left\|\int_0^t e^{\mathbb{A}(t-s)}F_s(\cdot,s)ds\right\|_{\mathbb{H}} \\ \le\ & M(1 + L_0 e^{\varpi_0 t}) + M_1\frac{L_0}{\varpi_0}(e^{\varpi_0 t} - 1), \ \ \forall\, t \ge 0. \end{aligned}$$

A similarly estimation for $\alpha_x^\top(1,t)$ leads to (2.24).

3. Observer design

3.1. Observer for system (1.3)

In this subsection, we design a state observer for system (1.3) by introducing the coordinate transformation (the following (3.5)) in Shieh, Ganesan, and Navarro (1987). To begin with, we need some auxiliary definitions.

Definition 1 (*Silverman & Meadows, 1967, p. 66, D'Angelo, 1970, p. 117*)**.** Assume that $E(t)$ is an $(n_r - 1)$-order continuously differentiable matrix. The observability matrix of $\Sigma_o(E(t), S_r)$ is

$$N(t) = [N_1(t), N_2(t), \ldots, N_{n_r}(t)], \tag{3.1}$$

where

$$N_1(t) = E^\top(t),\ N_{k+1}(t) = S_r^\top N_k(t) + \dot{N}_k(t), \tag{3.2}$$

with $k = 1, 2, \dots, n_r - 1$.

Definition 2 (*Silverman & Meadows, 1967, Definition 4, D'Angelo, 1970, Definition 4.12, p. 118*). $\Sigma_o(E(t), S_r)$ is said to be uniformly observable on $[t_1, t_2]$, if

$$\text{rank } N(t) = n_r, \quad \forall\, t \in [t_1, t_2]. \tag{3.3}$$

Definition 3 (*Yuksel & Bongiorno, 1971, Definition 2*). The observability index $q (q \le n_r)$ of $\Sigma_o(E(t), S_r)$ is defined to be the smallest positive integer such that

$$\text{rank } \bar{N}(t) = n_r, \quad \forall\, t \in [t_1, t_2], \tag{3.4}$$

where $\bar{N}(t) = [N_1(t), N_2(t), \dots, N_q(t)]$.

Proposition 3.1 (*Shieh et al., 1987, Theorem 2*). *If $\Sigma_o(E(t), S_r)$ is uniformly observable on $[0, \infty)$ with the observability index $q = n_r$, there is a non-singular coordinate transformation*

$$v_r(t) = T_0(t) v_0(t), \tag{3.5}$$

such that $\Sigma_o(E(t), S_r)$ can be transformed into:

$$\dot{v}_0(t) = S_0(t) v_0(t),\ \ y_{ref}(t) = E_0 v_0(t), \tag{3.6}$$

where

$$\begin{cases} T_0(t) = (T_{0_1}(t), T_{0_2}(t), \dots, T_{0_{n_r}}(t)), \\ T_{0_1}(t) = (E_0 N^{-1}(t))^\top, \\ T_{0_{k+1}}(t) = S_r T_{0_k}(t) - \dot{T}_{0_k}(t), \end{cases} \tag{3.7}$$

with $k = 1, 2, \dots, n_r - 1$ and

$$\begin{cases} S_0(t) &= T_0^{-1}(t) S_r T_0(t) - T_0^{-1}(t)\dot{T}_0(t) \\ &= \begin{pmatrix} 0 & \cdots & 0 & -a_{n_r}(t) \\ 1 & \cdots & 0 & -a_{n_r-1}(t) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -a_1(t) \end{pmatrix}, \\ E_0 &= E(t) T_0(t) = (0, 0, \dots, 0, 1). \end{cases} \tag{3.8}$$

Remark 3.1. It is noted that the observability index for single-output LTV system is always having $q = n_r$. Thus, the assumption "$q = n_r$" can be removed from Proposition 3.1 for single-output system. As for multi-output LTV systems, Shieh et al. (1987, Theorem 2) remains valid if $q = \frac{n_r}{m}$ is an integer, where m is the dimension of the output.

Now we list the assumptions on $\Sigma_o(E(t), S_r)$.

Assumption (A2). $\Sigma_o(E(t), S_r)$ is uniformly observable on $[0, \infty)$.

Assumption (A3). $T_0(t)$ is at most exponentially bounded, i.e.,

$$\|T_0(t)\|_2 \le L e^{\omega t}, \quad \forall\, t \ge 0, \tag{3.9}$$

where $\|\cdot\|_2$ is defined by $\|A\|_2 = (\sum_{i,j} |a_{ij}|^2)^{\frac{1}{2}}, \forall\, A = (a_{ij}) \in M_{n_r,n_r}(\mathbb{C})$ and L, ω are some positive constants.

Remark 3.2. It is worth noting that the boundedness of $E(t)$ in Assumption (A1) does not provide the boundedness of $T_0(t)$. For example, let $S = \text{diag}\{i, -i\}$, $E(t) = (e^{-t}, 1)$, we have $T_0(t) = \frac{1}{2i-1}\begin{pmatrix} e^t & (i-1)e^t \\ -1 & i \end{pmatrix}$ is exponentially growth in the sense that $\|T_0(t)\|_2 = \frac{\sqrt{5}}{5}\sqrt{3e^{2t}+2} = \mathcal{O}(e^t)$. In applications, Assumption (A3) is valid for most LTV systems such as harmonic signals and polynomial functions. However, there is indeed an example which grows larger than the exponential growth. For example, let $S = \text{diag}\{i, -i\}$, $E(t) = (t^{-t}, 1)$, we have $T_0(t) = \frac{1}{2i+1}\begin{pmatrix} t^t & (i-1)t^t \\ -1 & i \end{pmatrix}$, which does not meet (3.9).

Based on the non-singular transformation (3.5)–(3.8), we design a state observer for system $\Sigma_o(E_0, S_0(t))$ as

$$\dot{\hat{v}}_0(t) = S_0(t)\hat{v}_0(t) - K_r(t)[y_{ref}(t) - E_0 \hat{v}_0(t)], \tag{3.10}$$

where $K_r(t) \in \mathbb{C}^{n_r}$ is a tuning vector. Let

$$\begin{cases} \tilde{v}_0(t) = v_0(t) - \hat{v}_0(t), \\ \hat{v}_r(t) = T_0(t)\hat{v}_0(t),\ \ \tilde{v}_r(t) = v_r(t) - \hat{v}_r(t). \end{cases} \tag{3.11}$$

Then, $\tilde{v}_0(\cdot)$ is governed by

$$\dot{\tilde{v}}_0(t) = [S_0(t) + K_r(t)E_0]\tilde{v}_0(t). \tag{3.12}$$

Theorem 3.1. *Suppose Assumptions (A2)–(A3) hold. Then, there exists a unique bounded vector $K_r(t) \in \mathbb{C}^{n_r}$ such that $S_0(t) + K_r(t)E_0$ is Hurwitz. Thus,*

$$\|\tilde{v}_r(t)\|_{\mathbb{C}^{n_r}} \le L_1 e^{-\omega_1 t}\|\tilde{v}_r(0)\|_{\mathbb{C}^{n_r}}, \quad \forall\, t \ge 0, \tag{3.13}$$

where $L_1 > 0$ and $\omega_1 > \max\{0, \varpi_0\}$ are constants.

Proof. Define the matrix

$$S_e = \begin{pmatrix} 0 & \cdots & 0 & \mu_1 \\ 1 & \cdots & 0 & \mu_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \mu_{n_r} \end{pmatrix}, \tag{3.14}$$

which is supposed to be Hurwitz, that is, the eigenvalues of S_e satisfy

$$\text{Re } \lambda_i(S_e) < -\max\{\omega, \varpi_0, 0\} \quad i = 1, 2, \dots, n_r, \tag{3.15}$$

where ω, ϖ_0 are given by (2.12) and (3.9), respectively. Let

$$K_r(t) = (k_{r_1}(t), k_{r_2}(t), \dots, k_{r_{n_r}}(t))^\top, \tag{3.16}$$

where $k_{r_i}(t) (i = 1, 2, \dots, n_r)$ are chosen as

$$k_{r_i}(t) = a_{n_r+1-i}(t) + \mu_i, \quad i = 1, 2, \dots, n_r, \tag{3.17}$$

with $a_i(t), \mu_i (i = 1, 2, \dots, n_r)$ being given by (3.8) and (3.14), respectively. A simple computation shows that $S_0(t) + K_r(t)E_0 = S_e$ is Hurwitz, which implies the convergence of $\tilde{v}_r(t)$ by Assumption (A3). ■

3.2. Observer for cascaded system (1.1) and (1.4)

In this subsection, we design the following observer for cascaded system (1.1) and (1.4):

$$\begin{cases} \dot{\hat{v}}_d(t) = S_d \hat{v}_d(t) - K_d[w(1, t) - \hat{w}(1, t)], \\ \hat{w}_t(x, t) = \hat{w}_{xx}(x, t) + \lambda(x)\hat{w}(x, t) \\ +f(x)F_0(t)\hat{v}_d(t) - \gamma(x)[w(1, t) - \hat{w}(1, t)], \\ \hat{w}_x(0, t) = F_1(t)\hat{v}_d(t), \\ \hat{w}_x(1, t) = u(t) + F_2(t)\hat{v}_d(t) \\ \qquad +k_0[w(1, t) - \hat{w}(1, t)], \end{cases} \tag{3.18}$$

where the observer gain $k_0 \in \mathbb{C}, K_d \in \mathbb{C}^{n_d}, \gamma(\cdot) \in C[0, 1]$ are chosen such that

- $k_0 = k + \frac{1}{2}\int_0^1 (\lambda(x) + \lambda_0)dx$, where $k > 0$, $\lambda_0 > \varpi_0$ are tuning constants;
- $\gamma(x) = b(x)K_d + \gamma_1(x)$, where the function γ_1 is to be determined and $b \in \mathbb{C}^{1\times n_d}$ satisfies:

$$\begin{cases} b''(x) = b(x)S_d - \lambda(x)b(x) - \gamma_1(x)Q, \\ b'(0) = 0, \quad b(1) = Q, \end{cases} \tag{3.19}$$

  where Q is chosen such that $\Sigma_o(b(1), S_d)$ is detectable;
- $S_d + K_d b(1)$ is Hurwitz.

Let

$$\begin{cases} \tilde{v}_d(t) = v_d(t) - \hat{v}_d(t), \\ \tilde{u}(x,t) = w(x,t) - \hat{w}(x,t) - b(x)\tilde{v}_d(t). \end{cases} \tag{3.20}$$

Then, $(\tilde{v}_d, \tilde{u})^\top$ is governed by

$$\begin{cases} \dot{\tilde{v}}_d(t) = [S_d + K_d b(1)]\tilde{v}_d(t) + K_d\tilde{u}(1,t), \\ \tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t) \\ \qquad + f(x)F_0(t)\tilde{v}_d(t) + \gamma_1(x)\tilde{u}(1,t), \\ \tilde{u}_x(0,t) = F_1(t)\tilde{v}_d(t), \\ \tilde{u}_x(1,t) = -k_0\tilde{u}(1,t) + \Delta(t)\tilde{v}_d(t), \end{cases} \tag{3.21}$$

with $\Delta(t) = F_2(t) - b'(1) - k_0 b(1)$. We introduce a backstepping transformation (Smyshlyaev & Krstic, 2010) as:

$$\tilde{u}(x,t) = \mathbb{L}z(x,t) = z(x,t) - \int_x^1 p_2(x,y)z(y,t)dy, \tag{3.22}$$

which is invertible and its inverse is given by

$$z(x,t) = \tilde{u}(x,t) + \int_x^1 q_2(x,y)\tilde{u}(y,t)dy, \tag{3.23}$$

where

$$\begin{cases} p_{2,xx}(x,y) - p_{2,yy}(x,y) = -(\lambda(x)+\lambda_0)p_2(x,y), \\ p_{2,x}(0,y) = 0, \quad p_2(x,x) = -\frac{1}{2}\int_0^x(\lambda(y)+\lambda_0)dy, \\ q_{2,xx}(x,y) - q_{2,yy}(x,y) = (\lambda(y)+\lambda_0)q_2(x,y), \\ q_{2,x}(0,y) = 0, \quad q_2(x,x) = -\frac{1}{2}\int_0^x(\lambda(y)+\lambda_0)dy. \end{cases} \tag{3.24}$$

System (3.24) has a unique bounded solution $(p_2, q_2) \in C^2(\overline{\mathcal{T}}) \times C^2(\overline{\mathcal{T}})$ (Smyshlyaev & Krstic, 2010) with $\mathcal{T} = \{x, y : 0 < y < x < 1\}$. Choose

$$\gamma_1(x) = p_{2,y}(x,1) + kp_2(x,1). \tag{3.25}$$

Under the transformation (3.22), we transform the system (3.21) into the following equivalent one:

$$\begin{cases} \dot{\tilde{v}}_d(t) = [S_d + K_d b(1)]\tilde{v}_d(t) + K_d z(1,t), \\ z_t(x,t) = z_{xx}(x,t) - \lambda_0 z(x,t) + g_0(x,t)\tilde{v}_d(t), \\ z_x(0,t) = F_1(t)\tilde{v}_d(t), \\ z_x(1,t) = -kz(1,t) + \Delta(t)\tilde{v}_d(t), \end{cases} \tag{3.26}$$

where

$$g_0(x,t) = \mathbb{L}^{-1}(f(x)F_0(t)) + q_2(x,1)\Delta(t). \tag{3.27}$$

Let $\Phi(x,s) = \begin{pmatrix} \phi_{11}(x,s) & \phi_{12}(x,s) \\ \phi_{21}(x,s) & \phi_{22}(x,s) \end{pmatrix}$, $x \in [s,1]$ be the transition matrix of the following equation:

$$\Phi'(x,s) = \begin{pmatrix} 0 & I_{n_d} \\ S_d^\top - \lambda(x)I_{n_d} & 0 \end{pmatrix}\Phi(x,s), \quad \Phi(s,s) = I_{2n_d}.$$

We now show the existence of system (3.19).

Lemma 3.1. *Assume that $\phi_{11}(1,0)$ is nonsingular. Then, system (3.19) admits a unique solution:*

$$\begin{aligned} (b(x), b'(x))^\top = \; & \Phi(x,0)(b(0),0)^\top \\ & + \int_0^x \Phi(x,s)(0, -\gamma_1(s)Q)^\top ds, \end{aligned}$$

where $b^\top(0) = \phi_{11}^{-1}(1,0)[Q^\top + \int_0^1 \phi_{12}(1,s)\gamma_1(s)Q^\top ds]$.

Since $S_d + K_d b(1)$ is Hurwitz, there exists a unique symmetric positive definite matrix $\hat{P}$ of the following equation:

$$[S_d + K_d b(1)]^\top \hat{P} + \hat{P}[S_d + K_d b(1)] = -mI, \tag{3.28}$$

where

$$\begin{cases} 0 < \delta_1 < k/2, 0 < \delta_2 < k/4, \\ 0 < \delta_3 < \min\{1, k/4\}, 0 < \delta_4 < 2(\lambda_0 - \varpi_0), \\ m > \dfrac{1}{2\delta_1}\sup_{t\ge 0}\|\Delta(t)\|_{\mathbb{C}^{n_d}}^2 + \dfrac{1}{\delta_2}\|K_d^\top \hat{P}\|_{\mathbb{C}^{n_d}}^2 \\ \quad + \dfrac{1}{2\delta_3}\sup_{t\ge 0}\|F_1(t)\|_{\mathbb{C}^{n_d}}^2 + 2\varpi_0\lambda_{\max}(\hat{P}) \\ \quad + \dfrac{1}{2\delta_4}\sup_{0\le x\le 1, t\ge 0}|g_0(x,t)|^2. \end{cases} \tag{3.29}$$

We now discuss system (3.26) in the state space $\mathcal{X}_1 = \mathbb{C}^{n_d} \times \mathcal{H}$ with the inner product

$$\begin{aligned} \langle (X_1,f_1)^\top, (X_2,f_2)^\top \rangle_{\mathcal{X}_1} = & \int_0^1 f_1(x)\overline{f_2(x)}dx \\ & + X_1^\top \hat{P}\overline{X_2}, \;\; \forall\, (X_i,f_i)^\top \in \mathcal{X}_1, \; i = 1,2. \end{aligned} \tag{3.30}$$

The next Lemma 3.2 gives the local existence result of system (3.26).

Lemma 3.2. *Suppose that Assumptions* **(A1)**–**(A3)** *hold. For any initial value $(\tilde{v}_d^0, z_0)^\top = (\tilde{v}_d(0), z(\cdot,0))^\top \in \mathcal{X}_1$, there exists $T > 0$ such that system (3.26) admits a unique solution $(\tilde{v}_d, z)^\top \in C(0,T;\mathcal{X}_1)$.*

Proof. Our method is based on the fixed point theorem. Define the operator $\mathcal{A}_1 : D(\mathcal{A}_1)(\subset \mathcal{H}) \to \mathcal{H}$ by

$$\begin{cases} \mathcal{A}_1 f = f'' - \lambda_0 f, \;\; \forall\, f \in D(\mathcal{A}_1), \\ D(\mathcal{A}_1) = \{f \in H^2(0,1) | f'(0) = 0, \\ \qquad f'(1) = -kf(1)\}. \end{cases} \tag{3.31}$$

It is seen that $\mathcal{A}_1$ generates an exponentially stable C_0-semigroup $e^{\mathcal{A}_1 t}$ on $\mathcal{H}$ and $-\delta(x), \delta(x-1)$ are admissible for $e^{\mathcal{A}_1 t}$. If "z-subsystem" of system (3.26) admits a unique solution at boundary $x = 1$, i.e., $z(1,\cdot) \in C[0,T]$ exists, it then follows from the admissibility of $-\delta(x)$ and $\delta(x-1)$ that system (3.26) admits a unique solution $(\tilde{v}_d(t), z(\cdot,t))^\top \in C(0,T;\mathcal{X}_1)$, which can be written as

$$\begin{aligned} & \tilde{v}_d(t) = e^{\hat{S}_d t}\tilde{v}_d^0 + \int_0^t e^{\hat{S}_d(t-s)}K_d z(1,s)ds, \\ & z(x,t) = e^{\mathcal{A}_1 t}z_0 + \int_0^t e^{\mathcal{A}_1(t-s)}g_1(x,s)e^{\hat{S}_d s}\tilde{v}_d^0 ds \\ & + \int_0^t e^{\mathcal{A}_1(t-s)}g_1(x,s)\int_0^s e^{\hat{S}_d(s-\tau)}K_d z(1,\tau)d\tau ds, \end{aligned} \tag{3.32}$$

where $\hat{S}_d = S_d + K_d b(1)$, $g_1(x,t) = \delta(x-1)\Delta(t) - \delta(x)F_1(t) + g_0(x,t)$.

Now, we show the existence of $z(1,t)$. Define the norm on $C[0,T]$ by $\|y\|_\infty = \sup_{t\in[0,T]}|y(t)|$ for any $y \in C[0,T]$. For a given initial value $(\tilde{v}_d^0, z_0)^\top \in \mathcal{X}_1$, we define a mapping

$$\mathbb{F} : C[0,T] \to C[0,T] \tag{3.33}$$

by

$$\begin{aligned} \mathbb{F}y(t) = & e^{\mathcal{A}_1 t}z_0(1) + \int_0^t e^{\mathcal{A}_1(t-s)}g_1(x,s)e^{\hat{S}_d s}\tilde{v}_d^0 ds\Big|_{x=1} \\ & + \int_0^t e^{\mathcal{A}_1(t-s)}g_1(x,s)\int_0^s e^{\hat{S}_d(s-\tau)}K_d y(\tau)d\tau ds\Big|_{x=1}. \end{aligned}$$

From Guo and Meng (2020), we obtain the following eigen-pairs of $\mathcal{A}_1$:

$$\begin{cases} \mu_n = -\lambda_0 - 4k - (n\pi)^2 + \mathcal{O}(n^{-1}), \\ q_n(x) = \sqrt{2}\cos(n\pi + \frac{2k}{n\pi})x + \mathcal{O}(n^{-2}). \end{cases} \tag{3.34}$$

It is easy to verify that $\mathcal{A}_1$ is self-adjoint with compact resolvent from which we consider that $\{q_n : n = 1,2,\ldots\}$ forms an orthonormal basis for $\mathcal{H} = L^2(0,1)$ by Tucsnak and Weiss (2009, Proposition 3.2.12). Therefore, $\mathbb{F}y(t)$ can be rewritten as

$$\begin{aligned}\mathbb{F}y(t) &= \sum_{n=1}^{\infty} a_n q_n(1) e^{\mu_n t} \\ &+ \sum_{n=1}^{\infty} q_n(1) \int_0^t b_n(s) e^{\mu_n(t-s)} e^{\hat{S}_d s} \tilde{v}_d^0 ds \\ &+ \sum_{n=1}^{\infty} q_n(1) \int_0^t b_n(s) e^{\mu_n(t-s)} \int_0^s e^{\hat{S}_d(s-\tau)} K_d y(\tau) d\tau ds,\end{aligned} \tag{3.35}$$

where

$$\begin{cases} a_n = \langle z_0, q_n \rangle_{\mathcal{H}}, \\ |b_n(t)| = |\langle g_1(\cdot,t), q_n\rangle_{\mathcal{H}}| \le \ell_1, \ \forall\, t \ge 0, \\ \ell_2 \|z_0\|_{\mathcal{H}}^2 \le \sum_{n=1}^{\infty} |a_n|^2 \le \ell_3 \|z_0\|_{\mathcal{H}}^2, \end{cases} \tag{3.36}$$

with constants $\ell_1, \ell_2, \ell_3 > 0$. According to the expression of $\mathbb{F}y(t)$, we obtain $\mathbb{F}y \in C[0,T]$, i.e., $\mathbb{F}(C[0,T]) \subset C[0,T]$. For any $y_1, y_2 \in C[0,T]$, a simple computation shows that

$$\begin{aligned}\mathbb{F}y_1(t) - \mathbb{F}y_2(t) &= \sum_{n=1}^{\infty} q_n(1) \int_0^t b_n(s) e^{\mu_n(t-s)} \\ &\cdot \int_0^s e^{\hat{S}_d(s-\tau)} K_d (y_1(\tau) - y_2(\tau)) d\tau ds.\end{aligned} \tag{3.37}$$

By Hölder's inequality,

$$\begin{aligned} &\left| \int_0^s e^{\hat{S}_d(s-\tau)} K_d (y_1(\tau) - y_2(\tau)) d\tau \right| \\ \le\ & \|e^{\hat{S}_d(s-\tau)}\|_{L^2(0,s)} \|K_d\|_{\mathbb{C}^{n_d}} \|y_1 - y_2\|_{L^2[0,T]} \\ \le\ & C_1 \|y_1 - y_2\|_{L^2[0,T]} \le C_1 T \|y_1 - y_2\|_{\infty}, \end{aligned} \tag{3.38}$$

where $C_1 > 0$ is a positive constant. By (3.36),

$$\left| \int_0^t b_n(s) e^{\mu_n(t-s)} ds \right| \le \frac{\ell_1}{|\mu_n|} = \mathcal{O}(n^{-2}), \tag{3.39}$$

which, together with (3.38), implies that

$$\|\mathbb{F}y_1 - \mathbb{F}y_2\|_{\infty} \le C_1 C_2 T \|y_1 - y_2\|_{\infty}, \tag{3.40}$$

where $C_2 = \sum_{n=1}^{\infty} \mathcal{O}(n^{-2})$. For any $T < \frac{1}{C_1 C_2}$, one can apply the contraction mapping theorem to obtain that (3.35) has a unique fixed point $z(1,\cdot) \in C[0,T]$. ■

The following Theorem 3.2 shows that the solution obtained in Lemma 3.2 is a global solution and is exponentially stable.

Theorem 3.2. *Suppose that Assumptions* **(A1)–(A3)** *hold. For any initial value $(\tilde{v}_d^0, z_0)^\top = (\tilde{v}_d(0), z(\cdot,0))^\top \in \mathcal{X}_1$, system (3.26) admits a unique solution $(\tilde{v}_d, z)^\top \in C(0,\infty; \mathcal{X}_1)$ such that*

$$\|(\tilde{v}_d, z)^\top\|_{\mathcal{X}_1} \le L_2 e^{-\omega_2 t} \|(\tilde{v}_d^0, z_0)^\top\|_{\mathcal{X}_1}, \forall\, t \ge 0, \tag{3.41}$$

where $L_2 > 0, \omega_2 > \max\{0, \varpi_0\}$ are constants.

Proof. First we note that the solution of system (3.26) on the interval $[0,T]$ can be extended to a larger interval $[T, T+\delta]$ with some $\delta > 0$. Indeed, by defining

$$\eta(t) = \tilde{v}_d(t+T), \quad u(x,t) = z(x, t+T), \tag{3.42}$$

then $(\eta(t), u(\cdot,t))$ is a solution of the following system:

$$\begin{cases} \dot{\eta}(t) = [S_d + K_d b(1)]\eta(t) + K_d u(1,t), \\ u_t(x,t) = u_{xx}(x,t) - \lambda_0 u(x,t) + g_0(x, t+T)\eta(t), \\ u_x(0,t) = F_1(t+T)\eta(t), \\ u_x(1,t) = -ku(1,t) + \Delta(t+T)\eta(t), \\ \eta(0) = \tilde{v}_d(T), \quad u(x,0) = z(x,T), \end{cases} \tag{3.43}$$

which has a local solution on interval $[T, T+\delta]$ ensured by Lemma 3.2.

Let $[0, T_{max})$ be the maximal interval to which the solution $(\tilde{v}_d(t), z(\cdot,t))^\top$ can be extended. We claim that if $T_{max} < \infty$, then

$$\lim_{t\uparrow T_{max}} \|(\tilde{v}_d(t), z(\cdot,t))^\top\|_{\mathcal{X}_1} = \infty.$$

If not, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \uparrow T_{max}$ such that $\|(\tilde{v}_d(t_n), z(\cdot,t_n))^\top\|_{\mathcal{X}_1} \le C$ for all n. Let $\{t_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{t_n\}_{n=1}^{\infty}$ such that t_{n_k} is close enough to T_{max}. It follows from Lemma 3.2 that $(\tilde{v}_d(t), z(\cdot,t))^\top$ defined on $[0, t_{n_k}]$ can be extended to $[0, t_{n_k} + \delta]$, where $\delta > 0$ is independent of t_{n_k}. In this way, $(\tilde{v}_d(t), z(\cdot,t))^\top$ can be extended beyond T_{max}, which contradicts with the definition of T_{max}. Next, we shall prove that $T_{max} = \infty$. For this purpose, it suffices to show

$$E(t) = \frac{1}{2}\int_0^1 z^2(x,t)dx + \tilde{v}_d^\top(t)\hat{P}\tilde{v}_d(t) < \infty. \tag{3.44}$$

Differentiating $E(t)$ along the solution of (3.26) yields

$$\begin{aligned}\dot{E}(t) &= -kz^2(1,t) - \int_0^1 z_x^2 dx - \lambda_0 \int_0^1 z^2 dx - m\tilde{v}_d^\top \tilde{v}_d \\ &+ \Delta(t)\tilde{v}_d z(1,t) + K_d^\top \hat{P}\tilde{v}_d z(1,t) + \tilde{v}_d^\top \hat{P} K_d z(1,t) \\ &- F_1(t)\tilde{v}_d z(0,t) + \int_0^1 z g_0(x,t) dx \tilde{v}_d.\end{aligned} \tag{3.45}$$

By Young's inequality and Poincaré's inequality, we have

$$|\Delta(t)\tilde{v}_d z(1,t)| \le \frac{m_1}{2\delta_1}\left(\tilde{v}_d^\top \tilde{v}_d\right) + \frac{\delta_1}{2}|z(1,t)|^2, \tag{3.46}$$

$$\left|K_d^\top \hat{P}\tilde{v}_d z(1,t)\right| \le \frac{m_2}{2\delta_2}\left(\tilde{v}_d^\top \tilde{v}_d\right) + \frac{\delta_2}{2}|z(1,t)|^2, \tag{3.47}$$

$$\begin{aligned}|F_1(t)\tilde{v}_d z(0,t)| &\le \frac{m_3}{2\delta_3}\left(\tilde{v}_d^\top \tilde{v}_d\right) + \frac{\delta_3}{2}|z(0,t)|^2 \\ &\le \frac{m_3}{2\delta_3}\left(\tilde{v}_d^\top \tilde{v}_d\right) + \delta_3|z(1,t)|^2 + \delta_3\int_0^1 |z_x|^2 dx,\end{aligned} \tag{3.48}$$

$$\left|\int_0^1 z g_0 dx \tilde{v}_d\right| \le \frac{m_4}{2\delta_4}\left(\tilde{v}_d^\top \tilde{v}_d\right) + \frac{\delta_4}{2}\int_0^1 |z|^2 dx, \tag{3.49}$$

where $m_1 = \sup_{t\ge 0} \|\Delta(t)\|_{\mathbb{C}^{n_d}}^2$, $m_2 = \|K_d^\top \hat{P}\|_{\mathbb{C}^{n_d}}^2$, $m_3 = \sup_{t\ge 0} \|F_1(t)\|_{\mathbb{C}^{n_d}}^2$, $m_4 = \sup_{0\le x\le 1, t\ge 0} |g_0(x,t)|^2$, $\delta_i (i = 1,2,3,4)$ are given by (3.29). It then follows from (3.45)–(3.49), (3.29) that

$$\begin{aligned}\dot{E}(t) &\le -\left(k - \delta_1/2 - \delta_2 - \delta_3\right)|z(1,t)|^2 \\ &- (1-\delta_3)\int_0^1 |z_x|^2 dx - \left(\lambda_0 - \frac{\delta_4}{2}\right)\int_0^1 |z|^2 dx \\ &- \left(m - \frac{m_1}{2\delta_1} - \frac{m_2}{\delta_2} - \frac{m_3}{2\delta_3} - \frac{m_4}{2\delta_4}\right)\tilde{v}_d^\top \tilde{v}_d \\ &\le -\mu E(t),\end{aligned} \tag{3.50}$$

where $\mu = \min\{2\lambda_0 - \delta_4, \frac{\mu_0}{\lambda_{\max}(\hat{P})}\} > 2\varpi_0$, with $\mu_0 = m - \frac{m_1}{2\delta_1} - \frac{m_2}{\delta_2} - \frac{m_3}{2\delta_3} - \frac{m_4}{2\delta_4}$ and ϖ_0 being given by (2.12). By using Gronwall's inequality, we get

$$E(t) \le E(0)e^{-\mu t}, \quad \forall\, t \ge 0, \tag{3.51}$$

which implies obviously (3.41) and (3.44). ■

4. Output regulation by output feedback

Based on the feedforward control (2.4), we design the following feedback control:

$$\begin{aligned}u(t) = &\ \beta_0(t)\hat{v}(t) + \beta_2 e(t) \\ &+ \int_0^1 \beta_1(y)[\hat{w}(y,t) - \alpha(y,t)\hat{v}(t)]dy,\end{aligned} \tag{4.1}$$

where $\hat{v}(t) = (T_0(t)\hat{v}_0(t), \hat{v}_d(t))^\top$ and $\hat{w}(x,t)$ are the state estimates of $v(t)$ and $w(x,t)$, respectively. Under the control (4.1), system (2.3) becomes:

$$\begin{cases} \bar{\varepsilon}_t(x,t) = \bar{\varepsilon}_{xx}(x,t), \\ \bar{\varepsilon}_x(0,t) = 0, \ \ \bar{\varepsilon}_x(1,t) = -c\bar{\varepsilon}(1,t) + g_2(t), \\ e(t) = \bar{\varepsilon}(1,t) + \int_0^1 q_1(1,y)\bar{\varepsilon}(y,t)dy, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} g_2(t) = & -\int_0^1 \beta_1(y)\mathbb{L}(z(y,t))dy - \beta_0(t)\tilde{v}(t) \\ & + \int_0^1 \beta_1(y)[(0,-b(y)) + \alpha(y,t)]dy\tilde{v}(t). \end{aligned}$$

We consider system (4.2) in the state space $\mathcal{H}$ as well.

Lemma 4.1. *Suppose that $\phi_{11}(1,0)$ is nonsingular and Assumptions **(A1)–(A3)** hold. For any initial value $(\bar{\varepsilon}(\cdot,0), \alpha(\cdot,0), \tilde{v}_r(0), \tilde{v}_d(0), z(\cdot,0))^\top \in \mathcal{H}\times\mathbb{H}\times\mathbb{C}^n\times\mathcal{H}$, system (4.2) admits a unique solution $\bar{\varepsilon}\in C(0,\infty;\mathcal{H})$ such that*

$$\|\bar{\varepsilon}(\cdot,t)\|_{\mathcal{H}} \le L_3 e^{-\omega_3 t}, \quad \forall\, t \ge t_0, \tag{4.3}$$

where $L_3, \omega_3 > 0$ are two constants and $t_0 > 0$ is any fixed constant. Moreover,

$$|\bar{\varepsilon}(1,t)| \le L_4 e^{-\omega_4 t}, \ \ \forall\, t > t_0, \tag{4.4}$$

where $L_4, \omega_4 > 0$ are two constants.

Proof. According to (2.13), (3.13) and (3.41), there exist two positive constants $C_1, \alpha_1 > 0$ such that

$$|g_2(t)| \le C_1 e^{-\alpha_1 t}, \qquad \forall\, t \ge t_0. \tag{4.5}$$

Define the operator $\mathcal{A}_2 : D(\mathcal{A}_2)(\subset\mathcal{H}) \to \mathcal{H}$ by

$$\begin{cases} \mathcal{A}_2 f = f'', \ \ \forall\, f\in D(\mathcal{A}_2), \\ D(\mathcal{A}_2) = \{f\in H^2(0,1) | f'(0)=0, f'(1) = -cf(1)\}. \end{cases}$$

System (4.2) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}\bar{\varepsilon}(\cdot,t) = \mathcal{A}_2\bar{\varepsilon}(\cdot,t) + \mathcal{B}_2 g_2(t), \tag{4.6}$$

with $\mathcal{B}_2 = \delta(x-1)$. It is seen that $\mathcal{A}_2$ generates an exponentially stable C_0-semigroup $e^{\mathcal{A}_2 t}$. Since $\mathcal{B}_2$ is admissible to $e^{\mathcal{A}_2 t}$, system (4.6) exists a unique solution $\bar{\varepsilon}\in C(0,\infty;\mathcal{H})$ of the form

$$\bar{\varepsilon}(\cdot,t) = \ e^{\mathcal{A}_2(t-t_0)}\bar{\varepsilon}(\cdot,t_0) + \int_{t_0}^t e^{\mathcal{A}_2(t-\tau)}\mathcal{B}_2 g_2(\tau)d\tau, \tag{4.7}$$

where $\forall\ t \ge t_0$. The admissibility of $\mathcal{B}_2$ and the convergence of g_2 implies that $\|\int_{t_0}^t e^{\mathcal{A}_2(t-\tau)}\mathcal{B}_2 g_2(\tau)d\tau\|_{\mathcal{H}} \to 0$, exponentially as $t_0 \le t\to\infty$. We have thus proved the conclusion (4.3).

We now show the exponentially convergence of $\bar{\varepsilon}(1,t)$ by the Riesz basis method. From Guo and Meng (2020), we obtain the following eigen-pairs of $\mathcal{A}_2$:

$$\begin{cases} \sigma_n = -4c - (n\pi)^2 + \mathcal{O}(n^{-1}), \\ \rho_n(x) = \sqrt{2}\cos(n\pi + \frac{2c}{n\pi})x + \mathcal{O}(n^{-2}). \end{cases} \tag{4.8}$$

Since $\mathcal{A}_2$ is self-adjoint, we may suppose that $\{\rho_n(x) : n = 1,2,\ldots\}$ forms an orthonormal basis for $\mathcal{H}$. As a consequence, (4.7) can be expressed as:

$$\begin{aligned} \bar{\varepsilon}(x,t) = & \sum_{n=1}^\infty \check{a}_n e^{\sigma_n(t-t_0)}\rho_n(x) \\ & + \sum_{n=1}^\infty \check{b}_n\rho_n(x)\int_{t_0}^t e^{\sigma_n(t-\tau)}g_2(\tau)d\tau, \end{aligned} \tag{4.9}$$

where

$$\begin{cases} \check{a}_n = \langle\bar{\varepsilon}(\cdot,t_0),\rho_n\rangle_{\mathcal{H}}, \check{b}_n = \rho_n(1) = \mathcal{O}(1), \\ l_1\|\bar{\varepsilon}(\cdot,t_0)\|^2_{\mathcal{H}} \le \sum_{n=1}^\infty|\check{a}_n|^2 \le l_2\|\bar{\varepsilon}(\cdot,t_0)\|^2_{\mathcal{H}}, \end{cases} \tag{4.10}$$

with constants $l_1, l_2 > 0$. Thus,

$$\begin{aligned} \bar{\varepsilon}(1,t) \ & = \sum_{n=1}^\infty \mathcal{O}(1)\check{a}_n e^{\sigma_n(t-t_0)} \\ & \quad + \sum_{n=1}^\infty \mathcal{O}(1)\int_{t_0}^t e^{\sigma_n(t-\tau)}g_2(\tau)d\tau \\ & := \hat{I}_1(t) + \hat{I}_2(t). \end{aligned} \tag{4.11}$$

First,

$$\begin{aligned} & |\hat{I}_1(t)|^2 \le c_1\sum_{n=1}^\infty|\check{a}_n|^2\sum_{n=1}^\infty e^{2\sigma_n(t-t_0)} \\ & \le c_1\sum_{n=1}^\infty|\check{a}_n|^2 e^{-8c(t-t_0)}\sum_{n=1}^\infty e^{(-2(n\pi)^2+\mathcal{O}(n^{-1}))(t-t_0)} \\ & \le c_2 e^{8ct_0}e^{-8ct}, \quad \forall\, t > t_0, \end{aligned} \tag{4.12}$$

where $c_1, c_2 > 0$ are constants. Next, from (4.5),

$$\begin{aligned} & \int_{t_0}^t e^{\sigma_n(t-\tau)}g_2(\tau)d\tau \\ & \le \frac{C_1}{-\sigma_n-\alpha_1}[e^{-\alpha_1 t} - e^{\sigma_n(t-t_0)-\alpha_1 t_0}], \ \forall\, t\ge t_0, \end{aligned} \tag{4.13}$$

which implies that

$$|\hat{I}_2(t)| \le C_2 e^{-\alpha_2 t}, \quad \forall\, t \ge t_0, \tag{4.14}$$

where C_2, α_2 are some positive constants. Combining (4.11)–(4.14), we arrive at the conclusion (4.4). ■

Combining (1.1)–(1.4), (2.2), (3.10), (3.18) and (4.1), we now turn to the closed-loop system:

$$\begin{cases} \dot{v}(t) = Sv(t), \\ w_t(x,t) = w_{xx}(x,t) + \lambda(x)w(x,t) + f(x)P_{d_0}(t)v(t), \\ w_x(0,t) = P_{d_1}(t)v(t), \\ w_x(1,t) = P_{d_2}(t)v(t) + \beta_0(t)\hat{v}(t) \\ \qquad + \int_0^1\beta_1(y)[\hat{w}(y,t) - \alpha(y,t)\hat{v}(t)]dy + \beta_2 e(t), \\ \dot{\hat{v}}_0(t) = S_0(t)\hat{v}_0(t) - K_r(t)[y_{ref}(t) - E_0\hat{v}_0(t)], \\ \dot{\hat{v}}_d(t) = S_d\hat{v}_d(t) - K_d[w(1,t) - \hat{w}(1,t)], \\ \hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \lambda(x)\hat{w}(x,t) + f(x)F_0(t)\hat{v}_d(t) \\ \qquad -\gamma(x)[w(1,t) - \hat{w}(1,t)], \\ \hat{w}_x(0,t) = F_1(t)\hat{v}_d(t), \\ \hat{w}_x(1,t) = F_2(t)\hat{v}_d(t) + k_0[w(1,t) - \hat{w}(1,t)] \\ \qquad +\beta_0(t)\hat{v}(t) + \beta_2 e(t) \\ \qquad + \int_0^1\beta_1(y)[\hat{w}(y,t) - \alpha(y,t)\hat{v}(t)]dy, \\ \alpha_t(x,t) + \alpha(x,t)S = \alpha_{xx}(x,t) + \lambda(x)\alpha(x,t) \\ \qquad + f(x)P_{d_0}(t), \\ \alpha_x(0,t) = P_{d_1}(t), \quad \alpha(1,t) = P_r(t). \end{cases} \tag{4.15}$$

We consider system (4.15) in the state space $\mathcal{X}_2 = (\mathbb{C}^n\times\mathcal{H})^2\times\mathbb{H}$. The main result of this paper is the following Theorem 4.1.

Theorem 4.1. *Suppose that $\phi_{11}(1,0)$ is nonsingular and Assumptions **(A1)–(A3)** hold. For any initial value $(v(0), w(\cdot,0), T_0(0)\hat{v}_0(0), \hat{v}_d(0), \hat{w}(\cdot,0), \alpha(\cdot,0))^\top \in \mathcal{X}_2$, system (4.15) admits a unique solution $(v, w, T_0\hat{v}_0, \hat{v}_d, \hat{w}, \alpha)^\top \in C(0,\infty;\mathcal{X}_2)$. Moreover, the tracking error $e(\cdot)$ satisfies*

$$|e(t)| \le L_5 e^{-\omega_5 t}, \quad \forall\, t > t_0, \tag{4.16}$$

where $L_5, \omega_5 > 0$ are constants, $t_0 > 0$ is any fixed constant.

Proof. Let

$$\begin{cases} \tilde{v}_r(0) = v_r(0) - T_0(0)\hat{v}_0(0), \\ \tilde{v}_d(0) = v_d(0) - \hat{v}_d(0), \\ z(\cdot,0) = \mathbb{L}^{-1}[w(\cdot,0) - \hat{w}(\cdot,0) - b(\cdot)\tilde{v}_d(0)], \\ \bar{\varepsilon}(\cdot,0) = \mathbb{P}[w(\cdot,0) - \alpha(\cdot,0)v(0)]. \end{cases} \tag{4.17}$$

By the assumption on initial value, we have $(v(0), \bar{\varepsilon}(\cdot,0), \tilde{v}_r(0), \tilde{v}_d(0), z(\cdot,0), \alpha(\cdot,0))^\top \in \mathcal{X}_2$. It follows from Theorems 3.1–3.2 and Lemmas 2.1, 4.1 that $(v, \bar{\varepsilon}, \tilde{v}_r, \tilde{v}_d, z, \alpha)^\top \in C(0,\infty;\mathcal{X}_2)$. Therefore, the existence of (4.15) is obtained by the following transformation

$$\begin{cases} (T_0(t)\hat{v}_0(t), \hat{v}_d(t))^\top = v(t) - (\tilde{v}_r(t), \tilde{v}_d(t))^\top, \\ w(\cdot,t) = \mathbb{P}^{-1}[\bar{\varepsilon}(\cdot,t)] + \alpha(\cdot,t)v(t), \\ \hat{w}(\cdot,t) = w(\cdot,t) - \mathbb{L}[z(\cdot,t)] - b(\cdot)\tilde{v}_d(t). \end{cases}$$

The (4.16) follows from (4.2)–(4.4) clearly. ■

Remark 4.1. When $\varpi_0 < 0$ in Lemma 2.1, the closed-loop system (4.15) has the following additional properties:

(i) $\sup_{t\geq t_0} \|(v, w, T_0\hat{v}_0, \hat{v}_d, \hat{w}, \alpha)^\top\|_{\mathcal{X}_2} < \infty$, where $t_0 > 0$ is any fixed constant;

(ii) The control $u(\cdot)$ is bounded in the sense that

$$\sup_{t>t_0} |u(t)| < \infty; \tag{4.18}$$

(iii) If $P_{d_i}(t)v(t)(i = 0,1,2) = P_r(t)v(t) = 0$, then

$$\|(w, \hat{w}, \alpha)^\top\|_{\mathcal{H}^2\times\mathbb{H}} \to 0, \tag{4.19}$$

exponentially as $t \to \infty$.

Remark 4.2. If $\alpha^\top(\cdot,0) - (x-1)P_{d_1}^\top(0) - x^2P_r^\top(0) \in D(\mathbb{A})$ and $\sup_{t\geq 0}\|\dot{P}_{d_0}^\top(t)\|_{\mathbb{C}^{n_d}} + \sup_{t\geq 0}\|\ddot{P}_r^\top(t)\|_{\mathbb{C}^{n_r}} + \sup_{t\geq 0}\|\ddot{P}_{d_1}^\top(t)\|_{\mathbb{C}^{n_d}} \leq C$, then the results of Lemma 4.1, Theorem 4.1 and Remark 4.1 can be concluded by replacing t_0 with zero.

5. Numerical simulation

In this section, we present some numerical simulations for closed-loop system (4.15) with $n_r = n_d = 2$. Choose $\lambda(x) = 3.2$, $f(x) = 0$, $S = \mathrm{diag}\{-i, i, 2i, -2i\}$, $E(t) = (e^{it}, 1)$, $F_1(t) = (\sin\frac{1}{t+5}, 0)$, $F_2(t) = (0, f(t))$, where $f(t)$ is a triangle signal with period $T = 2\pi$ and in the first period,

$$f(t) = \begin{cases} \frac{1}{\pi}t, & 0 \leq t < \pi, \\ -\frac{1}{\pi}t + 2, & \pi \leq t < 2\pi. \end{cases} \tag{5.1}$$

Let $S_e = \begin{pmatrix} 0 & -25 \\ 1 & -10 \end{pmatrix}$. By (3.1), (3.2), (3.7), (3.8), we have

$$N(t) = \begin{pmatrix} e^{it} & 0 \\ 1 & i \end{pmatrix}, T_0(t) = \begin{pmatrix} ie^{-it} & 0 \\ -i & 1 \end{pmatrix}, S_0(t) = \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix},$$

and hence $K_r(t) = (-25, -i-10)^\top$ by (3.17). The initial values and remaining coefficients in the closed-loop system (4.15) are taken as

$$\begin{cases} \lambda_0 = 0, k = c = 2, k_0 = k + \frac{1}{2}\int_0^1 \lambda(x)dx = 3.6, \\ K_d = (-5 - 11.5i, -5 + 11.5i)^\top, Q = (1,1), \\ w(\cdot,0) = 4\sin\pi x, \hat{w}(\cdot,0) = 6\cos x - 2 + 4i, \\ v_r(0) = v_d(0) = (1,1)^\top, \ \hat{v}_0(0) = \hat{v}_d(0) = (1,2)^\top, \\ \alpha(\cdot,0) = (0,0,0,2 + i\cos\pi x). \end{cases}$$

The finite difference method is applied in simulation. The time and space steps are taken as 4×10^{-5} and 1×10^{-1}, respectively. The boundary output $w(1,t)$ and the reference signal $y_{ref}(t)$ are presented in Fig. 1. It is seen that $w(1,t)$ tracks $y_{ref}(t)$ satisfactorily.

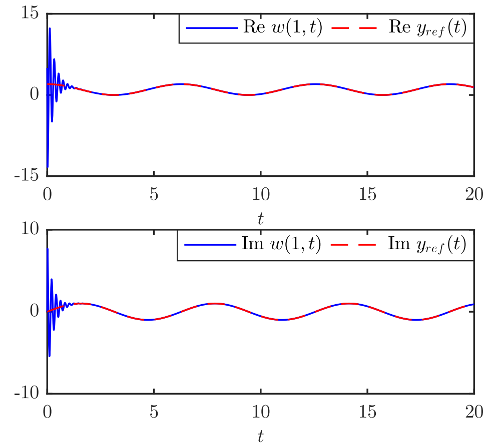

Fig. 1. $w(1,t)$ tracking $y_{ref}(t)$.

6. Concluding remarks

In this paper, a boundary output regulation for a one-dimensional reaction–diffusion equation is considered with disturbances and reference signals produced from a time-varying exosystem where the S is supposed to be constant matrix instead of $S(t)$. The main difficulty for time varying $S(t)$ lies in that the time-varying observer gain is hard to choose so that the error system between the disturbance and its observer is asymptotically stable. Another problem is the robustness. Although the problem can be solved perfectly by the internal model principle both for ODEs and PDEs with LTI exosystems, the relevant theory with LTV exosystems has unfortunately not been established yet. In this sense, the present paper is a first step toward the problem with a class of LTV exosystems. As pointed in Deutscher and Gabriel (2019), Paunonen and Pohjolainen (2012), the robust periodic regulator for infinite-dimensional systems remains open although the problem has been discussed for a finite-dimensional minimum-phase system in Zhang and Serrani (2006). Therefore, the robust output regulation for infinite-dimensional systems with time-varying exosystems is still a challenging topic in the future research.

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