Dynamic and Static Feedback Control for Second Order Infinite-dimensional Systems *

Jing Wei\textsuperscript{a,†}  

\textit{School of Mathematical Sciences}  
\textit{Shanxi University, Taiyuan, Shanxi, 030006, P.R. China}  

Bao-Zhu Guo\textsuperscript{b}  

\textit{Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing, China}  

Abstract  

This paper considers collocated dynamic feedback stabilization for abstract second-order systems, where the dynamic feedback controller is designed as another abstract second order infinite/finite-dimensional system. This makes the closed-loop system PDE-PDE or PDE-ODE coupled. The stability of the closed-loop system is found to have three different cases. We first consider the dynamic feedback control in a general Hilbert space which is usually different from the control space. It is shown that the stability of the closed-loop systems under the dynamic and static feedbacks are usually not equivalent. However, if the dynamic control law is a copy of the original system, we deduce, under some conditions, that the coupled system is exponentially stable if and only if the static feedback closed-loop system is exponentially stable. When the dynamic feedback is designed in the control space, the closed-loop system is asymptotically stable if and only if static feedback closed-loop system is asymptotically stable.  

\textbf{Keywords:} stability; dynamic feedback; static feedback.  

1 Introduction  

The notion of dynamic feedback controller or dynamic feedback compensator is widely used in modern control theory (see e.g. [8, 13]). It is well known that the robust controllers for output regulation exist only by dynamics feedback controls, which can be obtained by the internal model principle ([6, Lemma 1.21, p.19]). From the control point of view, an observer based feedback control is a special dynamic feedback control, which has been used in active disturbance rejection control theory ([5]), and adaptive control theory ([14]). Moreover, an observer based control leads

\textsuperscript{*}This work is supported by the Project of Department of Education of Guangdong Province (No. 2017KZDXM087), the National Natural Science Foundation of China (Nos. 61873153).  
\textsuperscript{†}Corresponding author: Jing Wei. Email: wuhenshuijing@163.com
naturally to a coupled closed-system. On the other hand, many papers were devoted to the study of
the coupled second order infinite-dimensional systems in both control field and material sciences.
Some examples can be found in [15] for coupled beam-wave system, [1] for coupled hyperbolic

It has been known that the asymptotic stability of a finite-dimensional system under the static
feedback controller and the dynamic feedback controller is equivalent ([3]). The same question could
be asked for linear infinite-dimensional systems and in paper [3], it was shown that the equivalence
of the asymptotic stability still holds true when the dynamic feedback control is chosen in the form
of the first-order system. It was proved in [3] that in all cases the closed-loop system under a first-
order dynamic output feedback is asymptotically stable if and only if the Hautus test is satisfied.
In this paper, we try to develop this equivalence for the dynamic feedback control which is chosen
in the form of the second-order system. We show that if the controlled system satisfies the Haustus
condition, the second order dynamic output feedback in the control space and the static output
feedback are equivalent to achieve the asymptotic stability, which is consistent with the conclusion
of the first order dynamic feedback developed in [3]. However, if we design a second-order feedback
compensator in a general Hilbert space, the results become more complicated because the Haustus
condition is no longer sufficient for the stability of the dynamic feedback closed-loop systems. And
in addition, we achieve exponential stability of the closed-loop system under the dynamic feedback,
which is stronger than asymptotic stability of [3].

To the best of our knowledge, there are few works focus on the study of the dynamic feedback
control for infinite-dimensional systems. The present paper is therefore significant to the design of
dynamic controller for infinite-dimensional systems.

The paper is organized as follows. In section 2, we design a second order dynamic feedback
control in a Hilbert space which is different from the control space. We establish the well-posedness,
asymptotic stability and the controllability of the closed-loop system. When the dynamic feedback
has a special form, we obtain, in section 3, the equivalence of the exponential stability under
dynamic feedback and static feedback. In section 4, by designing a dynamic feedback controller
in the control space, we obtain similar conclusions for asymptotic stability. Some simulations are
presented in Section 5 to illustrate the theoretical results, following up concluding remarks in section
6.

2 Dynamic output feedback in different spaces

Consider the following second order infinite-dimensional system in a complex Hilbert space $\mathcal{H}$
and control space $U$:

\[
\begin{aligned}
\dot{x}(t) + Ax(t) + Bu(t) &= 0, \\
y(t) &= B^*\dot{x}(t),
\end{aligned}
\]  

(2.1)
where \( u(t) \) is the control and \( y(t) \) is the measurement. Hereafter, the norm or the dual pairing on a Hilbert space are denoted respectively by the same symbols \( \| \cdot \| \) and \( \langle \cdot , \cdot \rangle \). For the operators \( A \) and \( B \), we make assumptions as follows:

\( \textbf{(H1).} \) \( A \) is a linear unbounded, densely defined, self-adjoint and strictly positive operator in \( \mathcal{H} \). Let \( D(A^{1/2}) \) be the domain of \( A^{1/2} \). \( A \) satisfies the following Gelfand’s triple compact inclusions:

\[
D(A^{1/2}) \hookrightarrow \mathcal{H} \hookrightarrow D(A^{1/2})'.
\]  

\( \textbf{(H2).} \) The control operator \( B \in \mathcal{L}(U, D(A^{1/2}')) \). The observation operator \( B^* \in \mathcal{L}(D(A^{1/2}), U) \) satisfies

\[
\langle B^* z , u \rangle_U = \langle z, Bu \rangle_{D(A^{1/2}), D(A^{1/2})'}, \quad \forall \ z \in D(A^{1/2}), \ u \in U.
\]  

\( \textbf{(H3).} \) An extension \( \check{A} \in \mathcal{L}(D(A^{1/2}), D(A^{1/2}')) \) of \( A \) is defined by

\[
\langle \check{A} x , z \rangle_{D(A^{1/2})', D(A^{1/2})} = \langle A^{1/2} x , A^{1/2} z \rangle_{\mathcal{H}}, \quad \forall \ x , z \in D(A^{1/2})
\]  

and hence \( \check{A}^{-1} B \in \mathcal{L}(U, D(A^{1/2})) \).

\( \textbf{(H4).} \) \( B \in \mathcal{L}(U, D(A^{1/2}')) \) or \( \check{A}^{-1} B \in \mathcal{L}(U, D(A^{1/2})) \) is compact.

We first propose the following static proportional feedback:

\[
u(t) = kB^* \dot{x}(t),
\]

where \( \text{Re}(k) > 0 \). Then, the closed-loop of system (2.1) under feedback (2.5) is

\[
\ddot{x} + \check{A} x + k B B^* \dot{x} = 0 \text{ in } D(A^{1/2})'.
\]  

We consider system (2.6) in the state space \( \mathcal{X}_s = D(A^{1/2}) \times \mathcal{H} \) equipped with the usual inner product

\[
\langle (f_1, g_1)^\top , (f_2, g_2)^\top \rangle_{\mathcal{X}_s} = \langle A^{1/2} f_1 , A^{1/2} f_2 \rangle_{\mathcal{H}} + \langle g_1 , g_2 \rangle_{\mathcal{H}}, \quad \forall \ (f_i, g_i)^\top \in \mathcal{X}_s, \ i = 1, 2.
\]  

Denote the operator \( A_s : D(A_s)(\subset \mathcal{X}_s) \rightarrow \mathcal{X}_s \) as

\[
\begin{cases}
A_s(f, g)^\top = (g, -\check{A} f - k B B^* g)^\top, \quad \forall \ (f, g)^\top \in D(A_s),
\end{cases}
\]

\[
D(A_s) = \{(f, g)^\top \in D(A^{1/2}) \times D(A^{1/2})\mid -\check{A} f - k B B^* g \in \mathcal{H}\}.
\]  

Then, system (2.6) can be written as an evolution equation in \( \mathcal{X}_s \):

\[
\dot{X}(t) = A_s X(t), \quad X(0) = X_0,
\]

where \( X(t) = (x(t), \dot{x}(t))^\top \).

\textbf{Lemma 2.1.} Suppose the assumptions (\textbf{H1})-(\textbf{H4}). Let the operator \( A_s \) be given by (2.8). Then, \( A_s \) generates an asymptotically stable \( C_0 \)-semigroup on \( \mathcal{X}_s \):

\[
\| e^{A_s t} x \|_{\mathcal{X}_s} \rightarrow 0 \text{ as } t \rightarrow \infty, \quad \forall \ x \in \mathcal{X}_s,
\]

if and only if the following Hautus test

\[
\text{Ker}(\lambda - A) \cap \text{Ker}(B^*) = \{0\}, \quad \forall \ \lambda > 0
\]  

holds.
Proof. See Theorem 2.2 of [3].

Now, we design a dynamic output feedback control for system (2.1) as

\[ u(t) = K^* \dot{z}(t), \quad (2.12) \]

where \( z(t) \) is governed by the following second order equation in a complex Hilbert space \( V \):

\[ \ddot{z}(t) + Gz(t) + KK^* \dot{z}(t) - KB \dot{z}(t) = 0. \quad (2.13) \]

Similarly as \( A \) and \( B \), the operators \( G \) and \( K \) are supposed to satisfy the following properties:

(H5). \( G \) is a linear densely defined, self-adjoint and strictly positive operator in \( V \) and \( G \) satisfies Gelfand's triple compact inclusions in \( V \): \( D(G^{1/2}) \hookrightarrow V \hookrightarrow D(G^{1/2})' \).

(H6). \( K \in \mathcal{L}(U, D(G^{1/2})') \) and \( K^* \in \mathcal{L}(D(G^{1/2}), U) \) satisfy

\[ \langle K^* z, u \rangle_U = \langle z, Ku \rangle_{D(G^{1/2}), D(G^{1/2})'}, \quad \forall \ z \in D(G^{1/2}), \ u \in U. \quad (2.14) \]

(H7). An extension \( \tilde{G} \in \mathcal{L}(D(G^{1/2}), D(G^{1/2})') \) of \( G \) is defined by

\[ \langle \tilde{G} x, z \rangle_{D(G^{1/2}), D(G^{1/2})'} = \langle G^{1/2} x, G^{1/2} z \rangle_V, \quad \forall \ x, z \in D(G^{1/2}). \quad (2.15) \]

(H8). \( K \in \mathcal{L}(U, D(G^{1/2})') \) or \( \tilde{G}^{-1} K \in \mathcal{L}(U, D(G^{1/2})) \) is compact.

The closed-loop of system (2.1) under feedback (2.12) is

\[
\begin{cases}
\ddot{x}(t) + \dot{A}x(t) + BK^* \dot{z}(t) = 0, & \text{in } D(A^{1/2})', \\
\ddot{z}(t) + \tilde{G}z(t) + KK^* \dot{z}(t) - KB^* \dot{z}(t) = 0, & \text{in } D(G^{1/2})'.
\end{cases}
\quad (2.16)
\]

We consider system (2.16) in the state space \( X_d = D(A^{1/2}) \times \mathcal{H} \times D(G^{1/2}) \times V \) equipped with the usual inner product

\[
\langle (f_1, g_1, \phi_1, \psi_1)^\top, (f_2, g_2, \phi_2, \psi_2)^\top \rangle_{X_d} = \langle A^{1/2} f_1, A^{1/2} f_2 \rangle_{\mathcal{H}} + \langle g_1, g_2 \rangle_{\mathcal{H}} + \langle G^{1/2} \phi_1, G^{1/2} \phi_2 \rangle_V + \langle \psi_1, \psi_2 \rangle_V,
\quad (2.17)
\]

for \( (f_i, g_i, \phi_i, \psi_i)^\top \in X_d, i = 1, 2 \). If we introduce the new variable \( X(t) = (x(t), \dot{x}(t), z(t), \dot{z}(t))^\top \), then, system (2.16) can be written as a first order system in \( X_d \)

\[
\begin{cases}
\dot{X}(t) = A_d X(t), \\
X(0) = X_0,
\end{cases}
\quad (2.18)
\]

where the operator \( A_d : D(A_d)(\subset X_d) \to X_d \) is defined by

\[
\begin{cases}
A_d(f, g, \phi, \psi)^\top = (g, -\dot{A}f - BK^* \psi, \psi, -\tilde{G} \phi - KK^* \psi + KB^* g)^\top, \\
D(A_d) = \{(f, g, \phi, \psi) \in [D(A^{1/2})]^2 \times [D(G^{1/2})]^2 | \\
\quad -\dot{A}f - BK^* \psi \in \mathcal{H}, -\tilde{G} \phi - KK^* \psi + KB^* g \in V \}, \forall (f, g, \phi, \psi)^\top \in D(A_d).
\end{cases}
\quad (2.19)
\]
Theorem 2.1. Suppose the assumptions (H1)-(H3),(H5)-(H7). Let $A_d$ be given by (2.19). Then, $A_d$ generates a $C_0$-semigroup of contractions $e^{A_d t}$ on $X_d$.

Proof. By a simple calculation, for any $(f,g,\phi,\psi)^\top \in D(A_d)$,

$$
\langle A_d(f,g,\phi,\psi)^\top, (f,g,\phi,\psi)^\top \rangle_{X_d}
\quad = \langle (g, -\tilde{A} f - BK^* \phi, \psi), (\hat{G} \phi - \mathcal{K} \mathcal{K}^* \psi + \mathcal{K} \mathcal{B}^* g) \rangle_{X_d}
\quad = \langle A^{1/2} g, A^{1/2} f \rangle_{\mathcal{H}} - \langle A^{1/2} f, A^{1/2} g \rangle_{\mathcal{H}} - \langle \mathcal{K}^* \psi, \mathcal{B}^* g \rangle_{\mathcal{U}}
\quad + \langle \mathcal{G}^{1/2} \psi, \mathcal{G}^{1/2} \phi \rangle_{\mathcal{V}} - \langle \mathcal{G}^{1/2} \phi, \mathcal{G}^{1/2} \psi \rangle_{\mathcal{V}} - \|\mathcal{K}^* \psi\|^2_{\mathcal{U}} + \|\mathcal{B}^* g, \mathcal{K}^* \psi\|_{\mathcal{U}},
$$

from which, it yields

$$
\text{Re}\langle A_d(f,g,\phi,\psi)^\top, (f,g,\phi,\psi)^\top \rangle_{X_d} = -\|\mathcal{K}^* \psi\|^2_{\mathcal{U}} \leq 0. \quad (2.21)
$$

Since $\tilde{A}$ is an isometric mapping from $D(A^{1/2})$ to $D(A^{1/2})'$ and $\tilde{G}$ is an isometric mapping from $D(G^{1/2})$ to $D(G^{1/2})'$, for any $(f_1,g_1,\phi_1,\psi_1)^\top \in X_d$, it has

$$
A^{-1}_d(f_1,g_1,\phi_1,\psi_1)^\top = (-\tilde{A}^{-1}(g_1 + B\mathcal{K}^* \phi_1), f_1, -\tilde{G}^{-1}(\psi_1 + \mathcal{K} \mathcal{K}^* \phi_1 - B \mathcal{B}^* f_1), \phi_1)^\top. \quad (2.22)
$$

Hence, $A^{-1}_d$ exists and is bounded on $X_d$. Since $D(A_d)$ is densely defined in $X_d$ by [12, Proposition 3.1.6,p. 71], we complete the proof by the Lumer-Phillips Theorem [10, Theorem 4.3,p. 14].

Theorem 2.2. Suppose the assumptions (H1)-(H8). Then, $A_d$ given by (2.19) generates an asymptotically stable $C_0$-semigroup on $X_d$:

$$
\|e^{A_d t} x\|_{X_d} \to 0 \text{ as } t \to \infty, \forall x \in X_d \quad (2.23)
$$

if and only if for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the following equations

$$
\begin{align*}
(\lambda^2 - A) f &= 0, \\
\mathcal{K}^* \phi &= 0, \\
(\lambda^2 - \mathcal{G}) \phi + i\lambda \mathcal{B}^* f &= 0,
\end{align*} \quad (2.24)
$$

has null solution only.

Proof. “Sufficiency”: Recall the assumptions (H4) and (H8), $A_d$ is compact on $X_d$. The proof will be accomplished if we can show that there is no eigenvalue of $A_d$ located on the imaginary axis ([9, Theorem 3.26, p. 130]). Suppose that

$$
A_d(f,g,\phi,\psi)^\top = i\omega (f,g,\phi,\psi)^\top, \forall (f,g,\phi,\psi)^\top \in D(A_d), \quad (2.25)
$$

where $\omega \in \mathbb{R}, \omega \neq 0$. It follows that

$$
\begin{align*}
g &= i\omega f, & \text{in } D(A^{1/2}), \\
\psi &= i\omega \phi, & \text{in } D(G^{1/2}), \\
-\tilde{A} f - i\omega B \mathcal{K}^* \phi &= -\omega^2 f, & \text{in } \mathcal{H}, \\
-\tilde{G} \phi - i\omega \mathcal{K} \mathcal{K}^* \phi + i\omega \mathcal{B}^* f &= -\omega^2 \phi, & \text{in } \mathcal{V}.
\end{align*} \quad (2.26)
$$
Since $\mathcal{A}_d$ is dissipative in (2.21), we obtain

$$K^*\phi = 0 \text{ in } U$$  \hspace{1cm} (2.27)

from

$$0 = \Re\langle i\omega(f, g, \phi, \psi)^\top, (f, g, \phi, \psi)^\top \rangle x_d$$

$$= \Re\langle \mathcal{A}_d(f, g, \phi, \psi)^\top, (f, g, \phi, \psi)^\top \rangle x_d$$

$$= -\|K^*\psi\|_U^2.$$  \hspace{1cm} (2.28)

Plug (2.27) back into equation (2.26) to conclude that $\tilde{\mathcal{A}} f = \omega^2 f$ and $(\omega^2 - \tilde{\mathcal{G}})\phi + i\omega KB^* f = 0$. Thus, $(f, \phi) \in \text{Ker}(\omega^2 - \mathcal{A}) \times \text{Ker}(K^*)$ is a solution of (2.24). This fact implies immediately that $f = 0$ and $\phi = 0$. Therefore, there is no eigenvalue of $\mathcal{A}_d$ on the imaginary.

“Necessity”: Suppose that $(f, \phi)$ satisfies (2.24) for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since

$$\begin{bmatrix} f \\ i\lambda f \\ \phi \\ i\lambda \phi \end{bmatrix} = \begin{bmatrix} i\lambda f \\ -\tilde{\mathcal{A}} f - i\lambda KB^* \phi \\ i\lambda \phi \\ -\tilde{\mathcal{G}} \phi - i\lambda KK^* \phi + i\lambda KB^* f \end{bmatrix} = i\lambda \begin{bmatrix} f \\ i\lambda f \\ \phi \\ i\lambda \phi \end{bmatrix},$$  \hspace{1cm} (2.29)

$i\lambda$ is an eigenvalue of $\mathcal{A}_d$. Since $e^{\mathcal{A}_d t}$ is asymptotically stable, it must have $f = \phi = 0$. This ends the proof.

**Remark 2.1.** It is worth mentioning that the condition in Theorem 2.2 is stronger than Hautus test (2.11). This means that there is no equivalence between asymptotic stability under static feedback (2.5) and second-order dynamic feedback (2.12) in general situation. However, by choosing a special dynamic feedback in next section, we can establish not only the equivalence of the asymptotic stability, but also the equivalence of the exponential stability. This is the key feature of our second-order dynamic feedback controller.

**Theorem 2.3.** Suppose the assumptions (H1)-(H8). Then, the following assertions are equivalent:

(i). System (2.1) is approximately observable;

(ii). $\text{Ker}(\lambda - \mathcal{A}) \cap \text{Ker}(B^*) = \{0\}, \forall \lambda > 0$;  \hspace{1cm} (2.30)

(iii). System (2.16) is approximately observable with the output $y_z(t) = \mathcal{G}^{1/2} y(t)$.

**Proof.** $(i) \Leftrightarrow (ii)$ is the consequence of [3, Theorem 2.3].

$(ii) \Leftrightarrow (iii)$. If the output $y_z(t) = \mathcal{G}^{1/2} z(t) \equiv 0$ for all $t \geq 0$, then, $z(t) \equiv 0$ for all $t \geq 0$. System (2.16) with the output $y_z(t) = \mathcal{G}^{1/2} z(t)$ is approximately observable if and only if the following system

$$\begin{aligned}
\dot{X}(t) &= AX(t), \\
y(t) &= BX(t),
\end{aligned}$$  \hspace{1cm} (2.31)
is approximately observable, where
\[ X(t) = (x(t), \dot{x}(t))^\top, A = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, B^* = (0, B^*)^\top. \]

According to [3, Theorem 2.3], \( \Sigma_o(B^*, A) \) is approximately observable if and only if (2.30) holds. This proves the theorem.

**Example 2.1.** Consider the following wave equation with boundary control
\[
\begin{cases}
w_{tt}(x, t) = w_{xx}(x, t), & x \in (0, 1), \ t \in (0, \infty), \\
w(0, t) = 0, & t \in [0, \infty), \\
w_x(1, t) = u(t), & t \in [0, \infty), \\
y(t) = -w_t(1, t), & t \in [0, \infty),
\end{cases}
\]
where \( u(t) \) is the control and \( y(t) \) is the measured output. Let \( \mathcal{H} = L^2(0, 1) \) and denote \( A : D(A) \to \mathcal{H} \) as
\[
\begin{cases}
Af = -f'', & \forall f \in D(A), \\
D(A) = \{ f \in H^2(0, 1) | f'(1) = f(0) = 0 \}.
\end{cases}
\]
The control space is \( U = \mathbb{C} \) and the control operator is \( B = -\delta(x - 1) \), where \( \delta(\cdot) \) is the Dirac distribution. Clearly, \( B^*f = -f(1), \forall f \in H^1(0, 1) \). We shall present two different stabilizing controllers to illustrate our results. First, we design the following direct proportional control law:
\[
u(t) = -kw_t(1, t), \ \forall \ t \geq 0,
\]
where \( k > 0 \). Then, the closed-loop of system (2.32) under the feedback (2.34) is
\[
\begin{cases}
w_{tt}(x, t) = w_{xx}(x, t), & x \in (0, 1), \ t \in (0, \infty), \\
w(0, t) = 0, & t \in [0, \infty), \\
w_x(1, t) = -kw_t(1, t), & t \in [0, \infty),
\end{cases}
\]
which is well known to be exponentially stable for all \( k > 0 \).

Next, we design a dynamic feedback controller satisfying Theorem 2.2. Define the operator \( \mathcal{G} : D(\mathcal{G}) \to \mathcal{V} = L^2(0, 1) \) by
\[
\begin{cases}
\mathcal{G}f = -f'', & \forall f \in D(\mathcal{G}), \\
D(\mathcal{G}) = \{ f \in H^2(0, 1) | f'(0) = qf(0), \ f'(1) = 0 \},
\end{cases}
\]
and \( \mathcal{K} = \mathcal{B} = -\delta(x - 1), \mathcal{K}^*f = -f(1), \forall f \in H^1(0, 1) \). It is seen that such defined \( \mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{K} \) satisfy the assumptions (H1)-(H8) and \( \mathcal{A} \neq \mathcal{G} \). The corresponding stabilizing controller is then designed as
\[
u(t) = -z_t(1, t), \ \forall \ t \geq 0,
\]
where \( z(x,t) \) is governed by

\[
\begin{align*}
  z_{tt}(x,t) &= z_{xx}(x,t), & x \in (0,1), \ t \in (0,\infty), \\
  z_x(0,t) &= qz(0,t), & t \in [0,\infty), \\
  z_x(1,t) &= -z_t(1,t) + w_t(1,t), & t \in [0,\infty).
\end{align*}
\]

Thus, the closed-loop system under feedback (2.37)-(2.38) reads:

\[
\begin{align*}
  w_{tt}(x,t) &= w_{xx}(x,t), & x \in (0,1), \ t \in (0,\infty), \\
  w(0,t) &= 0, & t \in [0,\infty), \\
  w_x(1,t) &= -z_t(1,t), & t \in [0,\infty), \\
  z_{tt}(x,t) &= z_{xx}(x,t), & x \in (0,1), \ t \in (0,\infty), \\
  z_x(0,t) &= qz(0,t), & t \in [0,\infty), \\
  z_x(1,t) &= -z_t(1,t) + w_t(1,t). & t \in [0,\infty).
\end{align*}
\]

We consider system (2.39) in the state space \( \mathcal{X}_d = H^1(0,1) \times L^2(0,1) \times H^1(0,1) \times L^2(0,1) \). Define the system operator \( \mathcal{A}_d : D(\mathcal{A}_d) \to \mathcal{X}_d \) for (2.39) as follows

\[
\begin{align*}
  \mathcal{A}_d(f,g,\phi,\psi)^\top &= (f''(x), \phi''(x))^\top, \forall (f,g,\phi,\psi)^\top \in D(\mathcal{A}_d), \\
  D(\mathcal{A}_d) &= \{(f,g,\phi,\psi)^\top \in [H^2(0,1) \times H^1(0,1)]^2 | f(0) = 0, \ f'(1) = -\psi(1), \ \phi'(0) = q\phi(0), \ \phi'(1) = -\psi(1) + g(1)\}.
\end{align*}
\]

We now consider the eigenvalue problem of \( \mathcal{A}_d \), i.e.

\[
\mathcal{A}_d(f,g,\phi,\psi)^\top = \lambda(f,g,\phi,\psi)^\top, \forall (f,g,\phi,\psi)^\top \in D(\mathcal{A}_d),
\]

which is equivalent to

\[
\begin{align*}
  f''(x) &= \lambda^2 f(x), \quad \phi''(x) = \lambda^2 \phi(x), & x \in [0,1], \\
  f(0) &= 0, \quad f'(1) = -\lambda \phi(1), \\
  \phi'(0) &= q\phi(0), \quad \phi'(1) = -\lambda \phi(1) + \lambda f(1).
\end{align*}
\]

Let

\[
f(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \quad \text{and} \quad \phi(x) = c_3 e^{\lambda x} + c_4 e^{-\lambda x},
\]

where \( c_1, c_2, c_3, c_4 \) are constants. Substituting (2.43) into the boundary conditions of (2.42) yields

\[
\begin{align*}
  c_1 + c_2 &= 0, \\
  (\lambda - q)c_3 - (\lambda + q)c_4 &= 0, \\
  (e^\lambda + e^{-\lambda})c_1 + e^\lambda c_3 + e^{-\lambda}c_4 &= 0, \\
  (e^{-\lambda} - e^\lambda)c_1 + 2e^\lambda c_3 &= 0.
\end{align*}
\]
Hence, (2.44) has nonzero solution if and only if the characteristic determinant \( \text{det}(\Delta(\lambda)) = 0 \), where

\[
\Delta(\lambda) = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & (\lambda - q) & -(\lambda + q) \\
(e^\lambda + e^{-\lambda}) & 0 & e^\lambda & e^{-\lambda} \\
(e^{-\lambda} - e^\lambda) & 0 & 2e^\lambda & 0
\end{bmatrix},
\tag{2.45}
\]

from which we find the characteristic equation as

\[
(q - \lambda)e^{-2\lambda} + 3(q + \lambda)e^{2\lambda} + 2\lambda = 0, \quad \forall \lambda \in \sigma(A_d).
\tag{2.46}
\]

**Proposition 2.1.** Let \( A_d \) be given by (2.40). Then, the \( C_0 \)-semigroup \( e^{A_dt} \) generated by \( A_d \) is asymptotically stable but not exponentially stable on \( X_d \).

**Proof.** If \((f, \phi)\) satisfies (2.24), then,

\[
\begin{cases}
f''(x) = -\lambda^2 f(x), & \phi''(x) = -\lambda^2 \phi(x), \quad x \in [0, 1], \\
f(0) = 0, & f'(1) = 0, \\
\phi'(0) = q\phi(0), & \phi(1) = 0, \\
\phi'(1) = i\lambda f(1).
\end{cases}
\tag{2.47}
\]

The general solution to system (2.47) is

\[
\begin{cases}
f(x) = c_1 \sin \lambda x + c_2 \cos \lambda x, & x \in [0, 1], \\
\phi(x) = c_3 \sin \lambda x + c_4 \cos \lambda x, & x \in [0, 1],
\end{cases}
\tag{2.48}
\]

where the coefficients \( c_j (j = 1, 2, 3, 4) \) satisfy

\[
\begin{cases}
c_2 = 0, & \lambda c_3 - qc_4 = 0, \\
c_3 \sin \lambda + c_4 \cos \lambda = 0, \\
c_3 \cos \lambda - c_4 \sin \lambda - ic_1 \sin \lambda = 0, \\
c_1 \cos \lambda = 0.
\end{cases}
\tag{2.49}
\]

If \( \cos \lambda \neq 0 \), one has \( c_1 = 0 \) and \( c_3 \sin \lambda + c_4 \cos \lambda = 0, c_3 \cos \lambda - c_4 \sin \lambda = 0. \) The last two equations yield \( c_3 = c_4 = 0. \) If \( \cos \lambda = 0 \) and hence \( \sin \lambda \neq 0, \) one has \( c_3 \sin \lambda = 0, \lambda c_3 - qc_4 = 0 \) and \( c_4 \sin \lambda + ic_1 \sin \lambda = 0. \) Thus, \( c_3 = c_4 = c_1 = 0. \) Therefore, \( c_1 = c_2 = c_3 = c_4 = 0 \) and hence \( e^{A_dt} \) is asymptotically stable.

On the other hand, by a simple calculation for (2.46), we obtain

\[
e^{2\lambda} = -\lambda \pm \sqrt{-3q^2 + 4\lambda^2} \over 3(q + \lambda) = -1 \pm 2 \over 3 + q \lambda + \mathcal{O} \left( \lambda^2 \right), \quad |\lambda| \to \infty.
\tag{2.50}
\]
Since the roots of $e^{2\lambda_1} = -1$ are $\lambda_1 = (n - \frac{1}{2})\pi i$. By Rouché’s theorem, the roots of $(2.50)$ have the following asymptotic expansion:

$$
\lambda_{1n} = \left(n - \frac{1}{2}\right) \pi i + O(n^{-1}), \quad n = 1, 2, \ldots.
$$

(2.51)

Thus, Re($\lambda_{1n}$) → 0 as $n \to \infty$. Therefore, $e^{A_d t}$ is not exponentially stable. □

It follows from Theorem 2.3 that system $(2.39)$ is approximately observable, that is,

$$
\int_0^t \left(\int_0^1 z^2(x, \tau) dx + qz^2(0, t)\right) d\tau = 0, \forall \; t \geq 0 \Rightarrow (w(\cdot, 0), w_i(\cdot, 0), z(\cdot, 0), z_1(\cdot, 0)) = 0. \quad (2.52)
$$

Remark 2.2. As indicated in Example 2.1, the exponential stability between the static feedback and the dynamic feedback is no equivalence in general case $A \neq G$.

### 3 Equivalence of exponential stability: $G = A$ and $K = B$

In this section, we discuss the relationship between the static feedback closed-loop system and the dynamic feedback closed-loop system under the condition $G = A$ and $K = B$. In this way, system $(2.16)$ is reduced to

$$
\begin{align*}
\begin{cases}
\ddot{x}(t) + \dot{A}x(t) + BB^* \dot{z}(t) = 0, & \text{in } D(A^{1/2})^\prime, \\
\ddot{z}(t) + \dot{A}z(t) + BB^* \dot{z}(t) - BB^* \dot{x}(t) = 0, & \text{in } D(A^{1/2})^\prime.
\end{cases}
\end{align*}
$$

(3.1)

We consider system $(3.1)$ in the state space $X_d = D(A^{1/2}) \times H \times D(A^{1/2}) \times H$. Define the operator $A_d : D(A_d) \subset X_d \rightarrow X_d$ by

$$
\begin{align*}
\begin{cases}
A_d(f, g, \phi, \psi)^\top = (g, -\dot{A}f - BB^* \psi, \psi, -\dot{\phi} - BB^* \psi + BB^* g)^\top, & \forall \; (f, g, \phi, \psi)^\top \in X_d, \\
D(A_d) = \{(f, g, \phi, \psi)^\top \in [D(A^{1/2})]^4] - \dot{A}f - BB^* \psi \in H, -\dot{\phi} - BB^* \psi + BB^* g \in H\}.
\end{cases}
\end{align*}
$$

(3.2)

Theorem 3.1. Suppose the assumptions (H1)-(H4). If $B$ is admissible, and the transfer function is bounded, that is, sup$_{Res=\alpha} \|sB^*(s^2 + \dot{A})^{-1}B\|_U < \infty$, for some $\alpha > 0$, then, the following assertions are equivalent:

(i). The $A_s$ defined by $(2.9)$ generates an exponentially stable $C_0$-semigroup $e^{A_st}$ on $X_s$;

(ii). The $A_d$ defined by $(3.2)$ generates an exponentially stable $C_0$-semigroup $e^{A_dt}$ on $X_d$;

(iii). $\Sigma_s(B^*, A)$ is exactly observable on $[0, \tau]$ for some $\tau > 0$.

Proof. $(i) \Leftrightarrow (iii)$. The case of $k = 1$ has been proved in [4, Theorems 2,3]. It is not difficult to prove that such a conclusion can be extended to Re($k$) > 0 analogously.

$(i) \Rightarrow (ii)$. Define the following new coordinates

$$
\begin{pmatrix}
\varepsilon \\
\delta
\end{pmatrix} =
\begin{pmatrix}
1 & -1 + \sqrt{3}i \\
1 & -1 - \sqrt{3}i
\end{pmatrix}
\begin{pmatrix}
x \\
z
\end{pmatrix} = P
\begin{pmatrix}
x \\
z
\end{pmatrix},
$$

(3.3)
where \( P = \begin{pmatrix} 1 & -\frac{1+\sqrt{3}i}{2} \\ 1 & -\frac{1-\sqrt{3}i}{2} \end{pmatrix} \) is an invertible operator. Then, \((\varepsilon, \delta)\) satisfies

\[
\begin{align*}
\ddot{\varepsilon} + \tilde{A}\varepsilon + \frac{1+\sqrt{3}i}{2}BB^*\dot{\varepsilon} &= 0, & \text{in } D(A^{1/2})', \\
\ddot{\delta} + \tilde{A}\delta + \frac{1-\sqrt{3}i}{2}BB^*\dot{\delta} &= 0, & \text{in } D(A^{1/2}).
\end{align*}
\] (3.4)

Note that \((\varepsilon, \delta)\) is decoupled and hence \((\varepsilon, \delta)\) is exponentially stable due to (i). Since \((x, z)^\top = P^{-1}(\varepsilon, \delta)^\top\), the \((x, z)\)-subsystem is exponentially stable as well.

(ii) \(\Rightarrow\) (iii). By (3.3), the \(\varepsilon\)-subsystem is exponentially stable. By “(i) \(\Rightarrow\) (iii)” \(\Sigma_o(B^* , A)\) is exactly observable on \([0, \tau]\). This ends the proof. \(\square\)

**Example 3.1.** We consider Example 2.1 in a multi-dimensional case. Let \(\Omega\) be an open bounded domain of \(\mathbb{R}^n (n \geq 1)\) with a smooth \(C^2\)-boundary \(\partial\Omega = \Gamma_0 \cup \Gamma_1\), where \(\Gamma_0\) and \(\Gamma_1\) are open and disjoint, with \(\text{meas}(\Gamma_0) > 0\) and \(\nu\) being the unit outer normal to \(\partial\Omega\). We now study the following multi-dimensional wave system

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} (x,t) - \Delta w(x,t) &= 0, & \text{in } \Omega \times (0, \infty), \\
w(x,t) &= 0, & \text{on } \Gamma_0 \times [0, \infty), \\
\frac{\partial w}{\partial \nu}(x,t) &= u(x,t), & \text{on } \Gamma_1 \times [0, \infty), \\
y(x,t) &= w_t(x,t), & \text{on } \Gamma_1 \times [0, \infty),
\end{align*}
\] (3.5)

where \(u(x,t)\) is the control, \(y(x,t)\) is the measurement (output). Let us first select the general proportional output control as

\[u(x,t) = -kw_t(x,t) \text{ on } \Gamma_1 \times [0, \infty),\] (3.6)

where \(k > 0\) is a constant. Then, the closed-loop takes the form:

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} (x,t) - \Delta w(x,t) &= 0, & \text{in } \Omega \times (0, \infty), \\
w(x,t) &= 0, & \text{on } \Gamma_0 \times [0, \infty), \\
\frac{\partial w}{\partial \nu}(x,t) &= -kw_t(x,t), & \text{on } \Gamma_1 \times [0, \infty).
\end{align*}
\] (3.7)

By applying the multiplier method ([7, Theorem 8.6]), we know that system (3.7) is exponentially stable in the state space \(X_s = H^1(\Omega) \times L^2(\Omega)\) if \(\Omega\) satisfies the geometric condition

\[ (x - x_0) \cdot \nu \leq 0 \text{ on } \Gamma_0, \quad (x - x_0) \cdot \nu \geq 0 \text{ on } \Gamma_1, \] (3.8)

where \(x_0\) is a fixed point in \(\mathbb{R}^n\).

Now, choose \(\mathcal{H} = L^2(\Omega)\) and \(U = L^2(\Gamma_1)\). The corresponding operator \(A\) is given by

\[
\begin{align*}
Af &= -\Delta f, & \forall f \in D(A), \\
D(A) &= \left\{ f \in H^2(\Omega) \times H^1(\Omega) \left| f|_{\Gamma_0} = 0, \frac{\partial f}{\partial \nu}|_{\Gamma_1} = 0 \right. \right\}.
\end{align*}
\] (3.9)
The control operator is defined by
\[ Bu = -\mathbf{A} \Upsilon u, \forall u \in U, \] (3.10)
where \( \Upsilon \in \mathcal{L}(L^2(\Gamma), H^{3/2}(\Omega)) \) is the Neumann map ([3, p.1942]). Then, \( \mathbf{B}^* \in \mathcal{L}(D(A^{1/2}), L^2(\Gamma)) \) satisfies \( \mathbf{B}^* f = -f|_{\Gamma_1} \), and \( \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{K} \) with \( \mathbf{G} = \mathbf{A}, \mathbf{K} = \mathbf{B} \) satisfy the assumptions of Theorem 3.1.

Under the feedback
\[ u(x,t) = -z_t(x,t) \text{ on } \Gamma_1 \times [0, \infty), \] (3.11)
where
\[ \begin{cases} 
  z_{tt}(x,t) - \Delta z(x,t) = 0, & \text{in } \Omega \times (0, \infty), \\
  z(x,t) = 0, & \text{on } \Gamma_0 \times [0, \infty), \\
  \frac{\partial z}{\partial \nu}(x,t) = -z_t(x,t) + w_t(x,t), & \text{on } \Gamma_1 \times [0, \infty),
\end{cases} \] (3.12)
the corresponding closed-loop system becomes
\[ \begin{cases} 
  w_{tt}(x,t) - \Delta w(x,t) = 0, & \text{in } \Omega \times (0, \infty), \\
  z_{tt}(x,t) - \Delta z(x,t) = 0, & \text{in } \Omega \times (0, \infty), \\
  w(x,t) = z(x,t) = 0, & \text{on } \Gamma_0 \times [0, \infty), \\
  \frac{\partial w}{\partial \nu}(x,t) = -z_t(x,t), & \text{on } \Gamma_1 \times [0, \infty), \\
  \frac{\partial z}{\partial \nu}(x,t) = -z_t(x,t) + w_t(x,t), & \text{on } \Gamma_1 \times [0, \infty),
\end{cases} \] (3.13)
with the state space \( \mathcal{X}_d = [H^1(\Omega) \times L^2(\Omega)]^2 \). By the equivalence between the exponential stability of the closed-loop system under the dynamic feedback (3.11) and the exponential stability of the closed-loop system under the static feedback (3.6), we have succeeding Proposition 3.1.

**Proposition 3.1.** For any initial value \((w(\cdot,0), w_t(\cdot,0), z(\cdot,0), z_t(\cdot,0)) \in \mathcal{X}_d\), if \( \Omega \) satisfies the geometric condition (3.8), then, the solution of system (3.13) is exponentially stable:
\[ \left\| \nabla w(\cdot,t) \right\|_{L^2(\Omega)}^2 + \left\| w_t(\cdot,t) \right\|_{L^2(\Omega)}^2 + \left\| \nabla z(\cdot,t) \right\|_{L^2(\Omega)}^2 + \left\| z_t(\cdot,t) \right\|_{L^2(\Omega)}^2 \leq C e^{-\omega t} \left[ \left\| \nabla w(\cdot,0) \right\|_{L^2(\Omega)}^2 + \left\| w_t(\cdot,0) \right\|_{L^2(\Omega)}^2 + \left\| \nabla z(\cdot,0) \right\|_{L^2(\Omega)}^2 + \left\| z_t(\cdot,0) \right\|_{L^2(\Omega)}^2 \right], \] (3.14)
where \( C \) and \( \omega \) are positive constants.

### 4 Dynamic output feedback in control space

In this section, we discuss dynamic output feedback control in the control space \( U \). Inspired by the controller design (2.12)-(2.13), a dynamic output feedback control is proposed for system (2.1) as
\[ u(t) = \dot{z}(t), \] (4.1)
where \( z(t) \) is given by the following second order equation in the control space \( U \):

\[
\ddot{z}(t) + z(t) + \mathcal{F}\dot{z}(t) - \mathcal{B}^*\dot{x}(t) = 0 \quad \text{in} \quad U,
\]  

(4.2)

where \( \mathcal{F} \) satisfies assumption \((\text{H9})\). \((\text{H9})\). The operator \( \mathcal{F} \in \mathcal{C}(U) \) is compact, strictly positive on \( U \).

Under the feedback control (4.1), closed-loop of system (2.1) is

\[
\begin{aligned}
\mathcal{A} x(t) + \mathcal{B} \dot{z}(t) &= 0, \\
\dot{z}(t) + \mathcal{F} \dot{z}(t) - \mathcal{B}^* \dot{x}(t) &= 0, \quad \text{in} \quad U.
\end{aligned}
\]

(4.3)

We consider system (4.3) in the state space \( \mathcal{X}_d = D(A^{1/2}) \times \mathcal{H} \times U^2 \) equipped with the usual inner product

\[
\langle (f_1, g_1, \phi_1, \psi_1)^T, (f_2, g_2, \phi_2, \psi_2)^T \rangle_{\mathcal{X}_d}
\]

\[
= \langle A^{1/2} f_1, A^{1/2} f_2 \rangle_{\mathcal{H}} + \langle g_1, g_2 \rangle_{\mathcal{H}} + \langle \phi_1, \phi_2 \rangle_U + \langle \psi_1, \psi_2 \rangle_U, \quad \forall \ (f_1, g_1, \phi_1, \psi_1)^T \in \mathcal{X}_d, i = 1, 2.
\]

(4.4)

Introduce the new variable \( X(t) = (x(t), \dot{x}(t), z(t), \dot{z}(t))^T \). Then, \( X(t) \) satisfies

\[
\begin{aligned}
\dot{X}(t) &= \mathcal{A}_d X(t), \\
X(0) &= X_0,
\end{aligned}
\]

(4.5)

where the operator \( \mathcal{A}_d : D(\mathcal{A}_d)(\subset \mathcal{X}_d) \to \mathcal{X}_d \) is defined by

\[
\begin{aligned}
\mathcal{A}_d(f, g, \phi, \psi)^T &= (g, -\dot{\mathcal{A}}f - B\psi, \psi, -\phi - \mathcal{F}\psi + \mathcal{B}^*g)^T, \\
D(\mathcal{A}_d) &= \{(f, g, \phi, \psi)^T \in [D(A^{1/2})]^2 \times U^2 | -\dot{\mathcal{A}}f - B\psi \in \mathcal{H}, -\phi - \mathcal{F}\psi + \mathcal{B}^*g \in U\}.
\end{aligned}
\]

(4.6)

**Theorem 4.1.** Suppose the assumptions \((\text{H1})\)-(\(\text{H4}\)) and \((\text{H9})\). Then, the following assertions are equivalent:

(i). The \( \mathcal{A}_d \) given by (4.6) generates an asymptotically stable \( C_0 \)-semigroup \( e^{\mathcal{A}_d t} \) on \( \mathcal{X}_d \);

(ii). The \( \mathcal{A}_s \) given by (2.8) generates an asymptotically stable \( C_0 \)-semigroup \( e^{\mathcal{A}_s t} \) on \( \mathcal{X}_s \);

(iii). \( \text{Ker}(\mathcal{A} - \lambda) \cap \text{Ker}(\mathcal{B}^*) = \{0\}, \quad \forall \lambda > 0 \);

(4.7)

(iv). System (4.3) is approximately observable with the output \( y_z(t) = z(t) \).

**Proof.** (ii) \( \iff \) (iii) is a consequence of Lemma 2.1.

(iii) \( \Rightarrow \) (i). Let \( Y = (f, g, \phi, \psi)^T \in D(\mathcal{A}_d) \). Then,

\[
\text{Re}\langle \mathcal{A}_d Y, Y \rangle_{\mathcal{X}_d} = \text{Re}\langle (g, -\dot{\mathcal{A}}f - B\psi, \psi, -\phi - \mathcal{F}\psi + \mathcal{B}^*g)^T, (f, g, \phi, \psi)^T \rangle_{\mathcal{X}_d}
\]

\[
= \text{Re}\langle (A^{1/2} f, A^{1/2} f)_{\mathcal{H}} + (-\dot{\mathcal{A}}f - B\psi, g)_{\mathcal{H}} + (\psi, \phi)_{U} + (-\phi - \mathcal{F}\psi + \mathcal{B}^*g, \psi)_{U} \rangle
\]

\[
= -\langle \mathcal{F}\psi, \psi \rangle_{U} \leq 0.
\]

(4.8)
For any \((f_1, g_1, \phi_1, \psi_1)^T \in \mathcal{X}_d\), solve

\[
\mathcal{A}_d(f, g, \phi, \psi)^T = (f_1, g_1, \phi_1, \psi_1)^T,
\]

to obtain

\[
(f, g, \phi, \psi)^T = \mathcal{A}_d^{-1}(f_1, g_1, \phi_1, \psi_1)^T = (-\tilde{A}^{-1}(g_1 + \mathcal{B}\phi_1), f_1, -\psi_1 - \mathcal{F}\phi_1 + \mathcal{B}^*f_1, \phi_1)^T.
\]

Hence, \(\mathcal{A}_d^{-1}\) exists and is bounded on \(\mathcal{X}_d\). Note that \(D(\mathcal{A}_d)\) is densely defined in \(\mathcal{X}_d\) by [12, Proposition 3.1.6, p.71]. Thus, \(\mathcal{A}_d\) given by (4.6) generates a \(C_0\)-semigroup of contractions on \(\mathcal{X}_d\) by a the Lumer-Phillips Theorem [10, Theorem 4.3, p.14].

If condition (4.7) holds, we show the strong stability of \(\mathcal{A}_d\) on \(\mathcal{X}_d\). Under the assumption, \(\mathcal{A}_d^{-1}\) is compact on \(\mathcal{X}\). By [9, Theorem 3.26, p. 130], we only need to prove that there is no eigenvalue of \(\mathcal{A}_d\) located on the imaginary axis.

Suppose that \((f, g, \phi, \psi) = (0, 0, 0, 0)^T \in D(\mathcal{A}_d)\),

\[
\mathcal{A}_d(f, g, \phi, \psi)^T = i\omega(f, g, \phi, \psi)^T, \quad \forall (f, g, \phi, \psi)^T \in D(\mathcal{A}_d),
\]

where \(\omega \in \mathbb{R}, \omega \neq 0\). Then,

\[
ge = \omega f, \quad \text{in } D(A^{1/2}),
\]

\[
\psi = \omega \phi, \quad \text{in } \mathcal{U},
\]

\[
-i\tilde{A}f - i\omega \mathcal{B}\phi = -\omega^2 f, \quad \text{in } \mathcal{H},
\]

\[
-\phi - i\omega \mathcal{F}\phi + i\omega \mathcal{B}^* f = -\omega^2 \phi, \quad \text{in } \mathcal{U}.
\]

Since \(\mathcal{A}_d\) is dissipative in (4.8), we perform the same as (2.28) to obtain

\[
(\mathcal{F}\phi, \psi)^T_U = 0.
\]

Hence, \(\phi = 0\). Then, (4.12) is reduced to \(\tilde{A}f = \omega^2 f\) and \(\mathcal{B}^* f = 0\). By assumption (4.7), \(f = 0\) and hence \((f, g, \phi, \psi) = 0\). Therefore, there is no eigenvalue of \(\mathcal{A}_d\) located on the imaginary axis.

\((i) \Rightarrow (iii)\). Suppose that \(f \in \text{Ker}(\lambda - A) \cap \text{Ker}(\mathcal{B}^*), \forall \lambda > 0\). It implies immediately from

\[
\mathcal{A}_d(f, i\sqrt{\lambda} f, 0, 0)^T = (i\sqrt{\lambda} f, -\tilde{A}f, 0, i\sqrt{\lambda} \mathcal{B}^* f)^T = i\sqrt{\lambda}(f, i\sqrt{\lambda} f, 0, 0)^T
\]

that \(i\sqrt{\lambda}\) is an eigenvalue of \(\mathcal{A}_d\). Since \(e^{A_d t}\) is asymptotically stable, there is no eigenvalue of \(\mathcal{A}_d\) located on the imaginary axis, we thus obtain \(f = 0\).

\((iv) \Leftrightarrow (iii)\). If the output \(y_z(t) = z(t) \equiv 0\) for all \(t \geq 0\), then, being approximately observable for system (4.3) with \(y_z(t) = z(t)\) is equivalent to the approximately observable of the system \(\Sigma_o(\mathcal{B}^*, \mathcal{K})\) defined by (2.31). Thus, system (2.16) is approximately observable with the output \(y_z(t) = z(t)\) from the same arguments presented in the proof of Theorem 2.3.

\[\square\]

**Example 4.1.** We still consider the controlled system (2.32) in Example 2.1. Let \(\mathcal{F} = I\). It is obvious that \(\mathcal{F} \in \mathcal{L}(\mathbb{C})\) is compact and system (2.32) satisfies the assumptions (H1)-(H4). Now, we design a dynamic feedback in control space \(U = \mathbb{C}\) as follows:

\[
u(t) = \dot{z}(t), \quad \forall t \geq 0,
\]

(4.15)
where $z(t)$ satisfies the second order ODE

$$
\ddot{z}(t) + z(t) + \dot{z}(t) + w_t(1, t) = 0
$$

(4.16)

in $\mathbb{C}$. Then, the closed-loop system reads:

$$
\begin{align*}
  w_t(x, t) &= w_xx(x, t), & x \in (0, 1), & t \in (0, \infty), \\
  w(0, t) &= 0, & t \in [0, \infty), \\
  w_x(1, t) &= \dot{z}(t), & t \in [0, \infty), \\
  \ddot{z}(t) + z(t) + \dot{z}(t) + w_t(1, t) &= 0, & t \in [0, \infty).
\end{align*}
$$

(4.17)

We consider the closed-loop system (4.17) in the state space $X_d = H^1(0, 1) \times L^2(0, 1) \times \mathbb{C}^2$. Define the system operator $A_d : D(A_d) \to X_d$ for (4.17) as follows

$$
\begin{align*}
  A_d(f, g, \phi, \psi)^\top &= (g, f'' \psi, -\phi - \psi - g(1))^\top, \quad \forall (f, g, \phi, \psi)^\top \in D(A_d), \\
  D(A_d) &= \{(f, g, \phi, \psi)^\top \in [H^2(0, 1) \times H^1(0, 1) \times \mathbb{C}^2] | f(0) = 0, \ f'(1) = \psi\}.
\end{align*}
$$

(4.18)

Then, the following proposition is a direct result from Theorem 4.1 and [2, Example 1].

**Proposition 4.1.** Let $A_d$ be given by (4.18). Then, the $C_0$-semigroup $e^{A_d t}$ generated by $A_d$ is asymptotically stable but not exponentially stable on $X_d$.

From the perspective of approximate observability, we obtain

$$
\int_0^t |z(\tau)|^2 d\tau = 0, \ \forall \ t \geq 0 \Rightarrow (w(\cdot, 0), w_t(\cdot, 0), z(0), \dot{z}(0)) = 0.
$$

(4.19)

**Remark 4.1.** As indicated in Example 4.1, Theorem 4.1 can not achieve the equivalence of the exponential stability of $e^{A_s t}$ and $e^{A_d t}$. Recall Remark 2.2, we thereby pay more attention to the equivalence of asymptotic stability in Sections 2 and 4.

5 Numerical simulation

In this section, we present some numerical simulations for Examples 2.1-4.1 with space dimension $n = 1$. The finite difference method is applied in simulation. The time and space steps are taken as 0.005 and 0.007, respectively. Here the initial value of $w$-system in three examples is chosen as

$$
  w(x, 0) = \cos 2\pi x - 1, \ w_t(x, 0) = 0.
$$

(5.1)

The initial value of auxiliary system “$z$” is chosen as

$$
\begin{align*}
  z(x, 0) &= \cos 2\pi x - 1, \ z_t(x, 0) = 0, \\
  z(0) &= 1, \ \dot{z}(0) = 0.
\end{align*}
$$

(5.2)
The solution of closed-loop system (2.35) with static feedback control is depicted in Figure 1. The solution of three closed-loop systems with dynamic feedback control are depicted in Figure 1-3. It can be seen that system (2.35) and (3.13) are indeed exponentially stable and (2.39) and (4.17) are asymptotically stable.

Figure 1: Simulations for system (2.35) and (3.13).

Figure 2: Simulations for system (2.44).

6 Concluding remarks

This paper addresses the difference between static output feedback control and dynamic feedback control for general linear collocated infinite-dimensional systems. The control \( u = B^* \dot{z} \) designed in Section 3 is applicable to establish the equivalence of exponential stability of \( e^{A_s t} \) and \( e^{A_d t} \), but it may require some strong geometric assumptions for instance the condition (3.8). In Section 2, we can design a controller \( u = K^* \dot{z} \) in a more general Hilbert space than in Section 3 without geometric assumptions and the stability result of closed-loop under this feedback works for various cases. It is seen that the control \( u = \dot{z} \) designed in Section 4 is simplest to use. Nevertheless, the equivalence of exponential stability can not be obtained in general by controller \( u = K^* \dot{z} \) or \( u = \dot{z} \).
Figure 3: Simulations for system (4.17).

References


