Asymptotic stabilization for a wave equation with periodic disturbance

JING WEI AND HONGYINPING FENG*
School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, P.R. China
*Corresponding author. Email: fhyp@sxu.edu.cn

AND

BAO-ZHU GUO
Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing, China
Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

[Received on 10 April 2019; revised on 27 July 2019; accepted on 24 September 2019]

In this paper, we consider boundary stabilization for a one-dimensional wave equation subject to periodic disturbance. By regarding the periodic signal as a boundary output of a free wave equation, we transform the controlled plant into a coupled wave system. We first design a state observer for the coupled system to estimate the disturbance and the system state simultaneously. An output feedback control is then designed to stabilize the original system. As an application, the result is applied to the stabilization of a wave equation with periodic disturbance suffering in output. Finally, some simulations are presented to validate the theoretical results.

Keywords: stability; periodic disturbance; coupled system.

1. Introduction

Periodic disturbance rejection occurs in many practical problems in industrial fields like magnetic spacecraft attitude control and picking operation of manipulator. There are several methods to eliminate/track periodic disturbances. In Byrnes et al. (2001), the periodic reference is represented as an output of an infinite-dimensional exosystem. The output regulation problem is then solved by solving a corresponding regulator equation.

A periodic output regulation problem for general linear heterodirectional hyperbolic systems was considered in Deutscher & Gabriel (2019) where the periodic disturbances represented by an infinite-dimensional Fourier series were taken into account by making use of finite-dimensional time-varying exosystems. However, the method used in Deutscher & Gabriel (2019) cannot be extended to the case of the measurement disturbances directly.

The internal-model-based repetitive control (RC) that was first presented in Inoue et al. (1981) for finite dimensional systems is another approach to cope with the periodic output regulation. This method has been applied to finite dimensional control systems in Weiss & Häfele (1999), Chung & Chen (2012), Wu et al. (2014) and Fedele (2018), to name just a few. For infinite dimensional systems, Chauvin (2011) designed an exponentially convergent observer for a wave equation driven by a finite sum of $T$-periodic harmonics. The novelty of Chauvin (2011) lies that it provides an invertible transmission function from the observer error system and the stable target system. Another effective method for
periodic signal tracking/rejection is the adaptive control approach. Unlike internal-model-based RC, the adaptive algorithm is used to estimate unknown frequencies or unknown parameters. Combining the internal-model-based RC and adaptive compensation, Marino & Tomei (2014) obtained a robust periodic disturbance compensator for a single-input single-output system. As for infinite dimensional systems, Guo & Guo (2013a) designed an infinite-dimensional adaptive observer for a wave equation subject to boundary harmonic disturbance of finite sum, by which an observer-based output feedback law has been designed to achieve asymptotic stability.

There are many other methods to be used for disturbance rejection of PDE systems. In Gugat & Schultz (2018), the Lyapunov method was used to stabilize a quasilinear wave equation with a non-periodic disturbance. Recently, a powerful method in dealing with general disturbances was presented in Feng & Guo (2017) where the output measurements are only three boundary signals. Subsequently, Zhou & Weiss (2018) improved the results of Feng & Guo (2017) by using only two boundary signals to achieve the exponential stability that is stronger than the asymptotic stability presented in Feng & Guo (2017). If we adopt the approaches of Feng & Guo (2017) and Zhou & Weiss (2018) to deal with the problem discussed in this paper, the measured three signals as in Feng & Guo (2017) or two signals as in Zhou & Weiss (2018) are also needed. What is more, these approaches are not easy to be extended to the case of the measurement disturbance. In this paper, however, by adopting a different approach, we need only one boundary measured signal, which improves, in this case, the result of Feng & Guo (2017) and Zhou & Weiss (2018).

In this paper, we consider the periodic disturbance $d(t)$ with period $T = 4\ell (\ell > 0)$, to be represented by the following Fourier series:

$$d(t) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{(2n-1)\pi}{2\ell} t \right),$$  \hspace{1cm} (1.1)

where $a_n (n = 1, 2, \ldots)$ are Fourier coefficients given by

$$a_n = \frac{2}{\ell} \int_{0}^{\ell} d(s) \cos \left( \frac{(2n-1)\pi}{2\ell} s \right) ds, \hspace{0.2cm} n = 1, 2, \ldots.$$ \hspace{1cm} (1.2)

This is an odd harmonic dynamic signal that is widely used in industry. For instance, the reference or disturbance signals of the power electronic systems are periodic signals with odd harmonics frequencies only (see, e.g., Costa-Castelló et al., 2004; Griñó & Costa-Castelló, 2005). However, the most of the traditional methods cannot be used to deal with infinite sum of harmonic signals. The present paper is therefore to devote to disturbance rejection and control design for a wave equation with odd-harmonic periodic disturbances.

The paper is organized as follows. In Section 2, we consider a one-dimensional wave equation subject to boundary input periodic disturbance. An observer is designed to estimate the uncertainty and the system state simultaneously. An observer-based output feedback controller is then designed in Section 3 and the stability of the closed-loop is concluded. In Section 4, we investigate asymptotic stability for a wave system with corrupted periodic disturbance output by the same method introduced in Section 2. Finally, some numerical experiments are carried out for illustration in Section 5.
2. Boundary input periodic disturbance

Consider the following one-dimensional wave equation with control matched disturbance:

\[
\begin{align*}
\begin{cases}
  w_{tt}(x,t) &= w_{xx}(x,t), & x \in (0,1), & t \in (0, \infty), \\
  w(0,t) &= 0, & t \in [0, \infty), \\
  w_x(1,t) &= U(t) + d(t), & t \in [0, \infty), \\
  y_{out}(t) &= w_t(1,t), & t \in [0, \infty),
\end{cases}
\end{align*}
\]

(2.1)

where \(U(t)\) is the input (control), \(y_{out}(t)\) is the output (measurement) and \(d(t)\) is the periodic disturbance with expression (1.1) that can be written as an output of the following exosystem:

\[
d(t) = z_t(\ell, t), \quad t \in [0, \infty),
\]

(2.2)

with

\[
\begin{align*}
\begin{cases}
  z_{tt}(x,t) &= z_{xx}(x,t), & x \in (0, \ell), & t \in (0, \infty), \\
  z(0,t) &= 0, & z_x(\ell, t) &= 0, & t \in [0, \infty), \\
  z(x,0) &= 0, & z_t(x,0) &= -d(x + \ell), & x \in [0, \ell].
\end{cases}
\end{align*}
\]

(2.3)

In terms of (2.2) and (2.3), the control plant with the input disturbance can be written as a coupled system:

\[
\begin{align*}
\begin{cases}
  w_{tt}(x,t) &= w_{xx}(x,t), & x \in (0,1), & t \in (0, \infty), \\
  w(0,t) &= 0, & t \in [0, \infty), \\
  w_x(1,t) &= U(t) + z_t(\ell, t), & t \in [0, \infty), \\
  z_{tt}(x,t) &= z_{xx}(x,t), & x \in (0, \ell), & t \in (0, \infty), \\
  z(0,t) &= z_x(\ell, t) = 0, & t \in [0, \infty), \\
  w(x,0) &= w_0(x), & w_t(x,0) &= w_1(x), & x \in [0, 1], \\
  z(x,0) &= 0, & z_t(x,0) &= -d(x + \ell), & x \in [0, \ell], \\
  y_{out}(t) &= w_t(1,t), & t \in [0, \infty).
\end{cases}
\end{align*}
\]

(2.4)
We aim at designing an output feedback law to stabilize system (2.4). An observer for system (2.4) can be designed as:

\[
\begin{align*}
\dot{w}_i(x,t) &= \dot{\hat{w}}_{xx}(x,t), & x \in (0,1), & t \in (0,\infty), \\
\dot{w}(0,t) &= 0, & t \in [0,\infty), \\
\dot{w}_i(1,t) &= U(t) + k_1[w_i(1,t) - \dot{\hat{w}}_i(1,t)] + \dot{\tilde{z}}_i(\ell,t), & t \in [0,\infty), \\
\dot{\hat{w}}_i(x,t) &= \dot{\hat{w}}_{xx}(x,t), & x \in (0,\ell), & t \in (0,\infty), \\
\dot{\hat{w}}(0,t) &= 0, & t \in [0,\infty), \\
\dot{\hat{w}}_i(\ell,t) &= w_i(1,t) - \dot{\hat{w}}_i(1,t), & t \in [0,\infty),
\end{align*}
\]  

(2.5)

where \(k_1\) is a positive tuning parameter. The motivation of this observer design comes from Feng & Guo (2015) where the coupled system is decoupled as the control plant and its dynamic feedback. If we let

\[
\begin{align*}
\ddot{w}(x,t) &= w(x,t) - \dot{\hat{w}}(x,t), & x \in [0,1], & t \in [0,\infty), \\
\ddot{z}(x,t) &= z(x,t) - \dot{\hat{z}}(x,t), & x \in [0,\ell], & t \in [0,\infty),
\end{align*}
\]

(2.6)

then, the observation error (\(\ddot{w}, \ddot{z}\)) is governed by

\[
\begin{align*}
\ddot{w}_i(x,t) &= \ddot{\hat{w}}_{xx}(x,t), & x \in (0,1), & t \in (0,\infty), \\
\ddot{w}(0,t) &= 0, & t \in [0,\infty), \\
\ddot{w}_i(1,t) &= -k_1\ddot{w}_i(1,t) + \ddot{\hat{z}}_i(\ell,t), & t \in [0,\infty), \\
\ddot{\hat{w}}_i(x,t) &= \ddot{\hat{w}}_{xx}(x,t), & x \in (0,\ell), & t \in (0,\infty), \\
\ddot{\hat{w}}(0,t) &= 0, & t \in [0,\infty), \\
\ddot{\hat{w}}_i(\ell,t) &= -\ddot{\hat{w}}_i(1,t), & t \in [0,\infty). 
\end{align*}
\]  

(2.7)

Let \(\mathcal{V} = \mathcal{H}_1 \times \mathcal{H}_{s'}\), where for \(s > 0\),

\[
\mathcal{H}_{s} = \{ (f,g) \in H^1(0,s) \times L^2(0,s) \mid f(0) = 0 \}. 
\]

(2.8)

We consider system (2.7) in the state space \(\mathcal{V}\) with the inner product

\[
\langle X_1, X_2 \rangle_{\mathcal{V}} = \int_0^1 [f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)}]dx + \int_0^\ell [\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}]dx
\]

(2.9)

for any \(X_i = (f_i,g_i,\phi_i,\psi_i) \in \mathcal{V}, \ i = 1,2\). Define the operator \(A_1 : D(A_1) \subset \mathcal{V} \rightarrow \mathcal{V}\) by

\[
\begin{align*}
A_1(f,g,\phi,\psi) &= (g,f'',\psi,\phi''), & (f,g,\phi,\psi) \in D(A_1), \\
D(A_1) &= \{ (f,g,\phi,\psi) \in [H^2(0,1) \times H^1(0,1) \times H^2(0,\ell) \times H^1(0,\ell)] \cap \mathcal{V} \mid f(0) = 0, \phi(0) = 0, f'(1) = -k_1g(1) + \psi(\ell), \phi'(\ell) = -g(1) \}.
\end{align*}
\]

(2.10)
Lemma 2.1 For any \( k_1 > 0, \ell \in \mathbb{R}_+ \setminus J_1 \) with \( \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x > 0 \} \) and

\[
J_1 := \left\{ \frac{2m + 1}{2n} \bigg| m, n \in \mathbb{Z} \right\},
\]

(2.11)

the operator \( \mathcal{A}_1 \) defined by (2.10) generates an asymptotically stable \( C_0 \)-semigroup on \( \mathcal{V} \). Therefore, for any initial value \((\tilde{\omega}(\cdot,0), \tilde{\omega}_t(\cdot,0), \tilde{z}(\cdot,0), \tilde{z}_t(\cdot,0)) \in \mathcal{V} \), there exists a unique solution \((\tilde{\omega}, \tilde{\omega}_t, \tilde{z}, \tilde{z}_t) \in C(0, \infty; \mathcal{V}) \) such that

\[
\lim_{t \to \infty} \|(\tilde{\omega}(\cdot, t), \tilde{\omega}_t(\cdot, t), \tilde{z}(\cdot, t), \tilde{z}_t(\cdot, t))\|_{\mathcal{V}} = 0.
\]

(2.12)

Proof. For any given \((f, g, \phi, \psi) \in D(\mathcal{A}_1) \), a simple computation shows that

\[
\text{Re}\langle \mathcal{A}_1(f, g, \phi, \psi), (f, g, \phi, \psi) \rangle_{\mathcal{V}} = \text{Re} \left[ \int_0^1 [g'(x)f''(x) + f''(x)g'(x)]dx + \int_0^\ell \left[ \phi'(x)g'(x) + \phi''(x)g(x) \right]dx \right]
\]

\[
= -k_1 |g(1)|^2 \leq 0, \quad \forall (f, g, \phi, \psi) \in D(\mathcal{A}_1),
\]

(2.13)

which implies that \( \mathcal{A}_1 \) is dissipative in \( \mathcal{V} \). Therefore, \( D(\mathcal{A}_1) \) is densely defined in \( \mathcal{V} \) by Tucsnak & Weiss (2009, p. 71, Proposition 3.1.6). For any \((f_1, g_1, \phi_1, \psi_1) \in \mathcal{V} \), we solve \( \mathcal{A}_1(f, g, \phi, \psi) = (f_1, g_1, \phi_1, \psi_1) \) to obtain

\[
\begin{align*}
g(x) &= f_1(x), \quad x \in [0, 1], \\
\psi(x) &= \phi_1(x), \quad x \in [0, \ell], \\
f(x) &= [\phi(\ell) - k_1f_1(1)]x - \int_0^x \int_s^1 g_1(\tau)d\tau ds, \quad x \in [0, 1], \\
\phi(x) &= -f_1(1)x - \int_0^x \int_s^\ell \psi_1(\tau)d\tau ds, \quad x \in [0, \ell].
\end{align*}
\]

(2.14)

By the Sobolev embedding theorem, \( \mathcal{A}_1^{-1} \) exists and is compact on \( \mathcal{V} \). Hence, \( \sigma(\mathcal{A}_1) \), the spectrum of \( \mathcal{A}_1 \), consists of the isolated eigenvalues of finite algebraic multiplicity only. By the Lumer–Phillips theorem (Pazy, 1983, p. 14, Theorem 4.3), \( \mathcal{A}_1 \) generates a \( C_0 \)-semigroup of contractions on \( \mathcal{V} \).

Now, we prove the asymptotic stability. Since the operator \( \mathcal{A}_1 \) is dissipative, we only need to prove that there is no eigenvalue on the imaginary axis (Luo & Guo, 1999, p. 130, Theorem 3.26). Suppose that \( \mathcal{A}_1 X = i\omega X \) with \( 0 \neq \omega \in \mathbb{R} \) and \( X = (f, g, \phi, \psi) \in D(\mathcal{A}_1) \). Then, \( g = i\omega f \). It follows from (2.13) that

\[
0 = \text{Re}\langle i\omega X, X \rangle_{\mathcal{V}} = \text{Re}\langle \mathcal{A}_1 X, X \rangle_{\mathcal{V}} = k_1 \omega^2 f^2(1).
\]

(2.15)
Thus, \( f(1) = 0 \) and

\[
\begin{align*}
    f''(x) &= -\omega^2 f(x), \quad x \in [0, 1], \\
    \phi''(x) &= -\omega^2 \phi(x), \quad x \in [0, \ell], \\
    f(0) &= f(1) = 0, \quad f'(1) = i\omega \phi(\ell), \\
    \phi(0) &= \phi'(\ell) = 0.
\end{align*}
\]

The general solutions to (2.16) are

\[
\begin{align*}
    f(x) &= c_1 \cos \omega x + c_2 \sin \omega x, \quad x \in [0, 1], \\
    \phi(x) &= c_3 \cos \omega x + c_4 \sin \omega x, \quad x \in [0, \ell],
\end{align*}
\]

where \( c_j \in \mathbb{C}, \ j = 1, 2, 3, 4 \). The boundary conditions of (2.16) imply that \( c_1 = c_3 = 0 \) and

\[
\begin{align*}
    c_2 \sin \omega &= 0, \quad c_4 \cos \omega \ell = 0, \\
    c_2 \cos \omega - ic_4 \sin \omega \ell &= 0.
\end{align*}
\]

Thanks to the assumption on \( \ell \), we can conclude that \( \sin^2 \omega + \cos^2 \omega \ell \neq 0 \). When \( \sin \omega \neq 0 \), it follows that \( c_2 = 0, \ c_4 \sin \omega \ell = 0 \) and \( c_4 \cos \omega \ell = 0 \). The last two equations yield \( c_4 = 0 \). Similarly, when \( \cos \omega \ell \neq 0 \), it follows that \( c_4 = 0 \) and hence \( c_2 = 0 \). As a result, system (2.16) admits a unique solution \((f, g, \phi, \psi) \equiv 0 \) for any \( \omega \neq 0 \). Therefore, there is no eigenvalue on the imaginary axis. This completes the proof. \( \square \)

**Lemma 2.2** If \( k_1 > 0, \ k_1 \neq 2, \ \ell \in Q_+ \setminus J_1 \), where \( Q_+ = \{x \in \mathbb{R}_+ | x \text{ is rational number} \} \), then, the operator \( A_1 \) defined by (2.10) generates an exponentially stable \( C_0 \)-semigroup \( e^{A_1 t} \). There are two positive constants \( L_{A_1} \) and \( \omega_{A_1} \) such that

\[
\|e^{A_1 t}\|_{\mathcal{L}} \leq L_{A_1} e^{-\omega_{A_1} t}, \quad \forall \ t \in [0, \infty).
\]

**Proof.** Since proof is independent of the controller design, we leave it to the Appendix. \( \square \)

### 3. Controller design and closed-loop system

We design the following feedback control law:

\[
U(t) = -k_1 w_\ell(1, t) - \hat{z}_\ell(\ell, t), \ \forall \ t \in [0, \infty),
\]

where \( k_1 \) and \( \hat{z}_\ell(\ell, t) \) come from the observer (2.5). The first term of (3.1) is the damper and the second term is used to compensate the disturbance. Since we only consider the stabilization problem, all the disturbances that affect the state must be compensated. This is different from the output regulation where we only need to compensate the disturbance that affects the regulated output.
Under the feedback (3.1), the closed-loop of system (2.4) reads

\[
\begin{align*}
  w_t(x,t) &= w_{xx}(x,t), & x \in (0,1), \ t \in (0, \infty), \\
  w(0,t) &= 0, & t \in [0, \infty), \\
  w_x(1,t) &= -k_1 w_t(1,t) - \hat{z}_x(\ell,t) + z_x(\ell,t), & t \in [0, \infty), \\
  z_{tt}(x,t) &= z_{xx}(x,t), & x \in (0, \ell), \ t \in (0, \infty), \\
  z(0,t) &= z_x(\ell,t) = 0, & t \in [0, \infty), \\
  \hat{w}_t(x,t) &= \hat{w}_{xx}(x,t), & x \in (0,1), \ t \in (0, \infty), \\
  \hat{w}(0,t) &= 0, \quad \hat{w}_x(1,t) = -k_1 \hat{w}_t(1,t), & t \in [0, \infty), \\
  \hat{z}_t(x,t) &= \hat{z}_{xx}(x,t), & x \in (0, \ell), \ t \in (0, \infty), \\
  \hat{z}(0,t) &= 0, \quad \hat{z}_x(\ell,t) = w_x(1,t) - \hat{w}_t(1,t), & t \in [0, \infty).
\end{align*}
\]

(3.2)

The state space of system (3.2) is chosen as \( \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_\ell \times \mathcal{H}_1 \times \mathcal{H}_\ell \). The stability of system (3.2) is stated in succeeding Theorem 3.1.

**Theorem 3.1** Suppose that \( k_1 > 0, \ \ell \in \mathbb{R}_+ \setminus J_1 \). For any initial value \((w(\cdot,0), w_t(\cdot,0), z(\cdot,0), z_x(\cdot,0), z_t(\cdot,0), \hat{w}(\cdot,0), \hat{w}_t(\cdot,0), \hat{z}(\cdot,0), \hat{z}_t(\cdot,0)) \) \( \in \mathcal{H} \), the closed-loop system (3.2) admits a unique solution \((w, w_t, z, z_t, \hat{w}, \hat{w}_t, \hat{z}, \hat{z}_t) \) \( \in C(0, \infty; \mathcal{H}) \) such that

\[
\lim_{t \to \infty} \| (w(\cdot,t), w_t(\cdot,t)) \|_{\mathcal{H}_1} = 0. \tag{3.3}
\]

If we suppose specially \( k_1 \neq 2 \) and \( \ell \in Q_+ \setminus J_1 \), then, there exist two positive constants \( \omega \) and \( L \) such that

\[
\| (w(\cdot,t), w_t(\cdot,t)) \|_{\mathcal{H}_1} \leq L e^{-\omega t} \| (w(\cdot,0), w_t(\cdot,0)) \|_{\mathcal{H}_1}, \quad \forall \ t \in [0, \infty). \tag{3.4}
\]

**Proof.** Since the \((z, \hat{w})\)-subsystem of (3.2) is independent of other subsystems, for any initial state \((\hat{w}(\cdot,0), \hat{w}_t(\cdot,0), z(\cdot,0), z_x(\cdot,0)) \) \( \in \mathcal{V} \), the solution of \((z, \hat{w})\)-subsystem \((\hat{w}, \hat{w}_t, z, z_t) \) \( \in C(0, \infty; \mathcal{V}) \) is well defined. Moreover, there exist two positive constants \( L_1 \) and \( \omega_1 \) such that

\[
\| (\hat{w}(\cdot,t), \hat{w}_t(\cdot,t)) \|_{\mathcal{H}_1} \leq L_1 e^{-\omega_1 t} \| (\hat{w}(\cdot,0), \hat{w}_t(\cdot,0)) \|_{\mathcal{H}_1}, \quad \forall \ t \geq 0. \tag{3.5}
\]

In terms of the initial state of system (3.2), define

\[
\begin{align*}
  \hat{w}(\cdot,0) &= w(\cdot,0) - \hat{w}(\cdot,0), & \hat{w}_t(\cdot,0) &= w_t(\cdot,0) - \hat{w}_t(\cdot,0), \\
  \hat{z}(\cdot,0) &= z(\cdot,0) - \hat{z}(\cdot,0), & \hat{z}_t(\cdot,0) &= z_t(\cdot,0) - \hat{z}_t(\cdot,0).
\end{align*}
\]

(3.6)
Then, \((\tilde{w}(\cdot, 0), \tilde{\dot{w}}(\cdot, 0), \tilde{z}(\cdot, 0), \tilde{\dot{z}}(\cdot, 0)) \in V\), and it follows from Lemma 2.1 that system (2.7) admits a unique solution \((\tilde{w}(\cdot, t), \tilde{\dot{w}}(\cdot, t), \tilde{z}(\cdot, t), \tilde{\dot{z}}(\cdot, t)) \in C(0, \infty; V)\). Define

\[
\begin{pmatrix}
    w \\
    \dot{w} \\
    \hat{w} \\
    \hat{\dot{w}} \\
    z \\
    \dot{z} \\
    \hat{z} \\
    \hat{\dot{z}}
\end{pmatrix} =
\begin{pmatrix}
    1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
    \tilde{w} \\
    \tilde{\dot{w}} \\
    \tilde{z} \\
    \tilde{\dot{z}} \\
    \hat{z} \\
    \hat{\dot{z}}
\end{pmatrix}.
\]

(3.7)

It is easy to check that such a defined \((w, w_t, z, z_t, \hat{w}, \hat{\dot{w}}, \hat{z}, \hat{\dot{z}}) \in C(0, \infty; H)\) solves system (3.2), and moreover, (3.3) and (3.4) hold due to Lemmas 2.1 and 2.2, respectively.

\[\square\]

4. Boundary output periodic disturbance

In this section, as an application, we consider stabilization for a one-dimensional wave equation suffering from output disturbance as follows:

\[
\begin{aligned}
    w_{tt}(x, t) &= w_{xx}(x, t), \quad x \in (0, 1), \ t \in (0, \infty), \\
    w(0, t) &= 0, \quad t \in [0, \infty), \\
    w_x(1, t) &= U(t), \quad t \in [0, \infty), \\
    y_{out}(t) &= w_t(1, t) + d(t), \quad t \in [0, \infty),
\end{aligned}
\]

(4.1)

where \(U(t)\) is the input (control), \(y_{out}(t)\) is the output (measurement) and \(d(t)\) is the periodic disturbance with the same expression (1.1). We aim at constructing an observer-based controller to achieve asymptotic stability under the corrupted boundary observation. Same as (2.4), the control plant becomes

\[
\begin{aligned}
    w_{tt}(x, t) &= w_{xx}(x, t), \quad x \in (0, 1), \ t \in (0, \infty), \\
    w(0, t) &= 0, \quad w_x(1, t) = U(t), \quad t \in [0, \infty), \\
    z_{tt}(x, t) &= z_{xx}(x, t), \quad x \in (0, \ell), \ t \in (0, \infty), \\
    z(0, t) &= 0, \quad z_x(\ell, t) = 0, \quad t \in [0, \infty), \\
    z(x, 0) &= 0, \quad z_t(x, 0) = -d(x + \ell), \quad x \in [0, \ell], \\
    y_{out}(t) &= w_t(1, t) + z_t(\ell, t), \quad t \in [0, \infty).
\end{aligned}
\]

(4.2)

It is worth to mention that our method is different from Guo & Guo (2013b) and Feng & Guo (2014). On the one hand, the periodic disturbance is generated by an infinite-dimensional system and hence the stability problem of a single system is transformed into the stability problem of a coupled wave system.
On the other hand, such coupled system with similar structure can simplify the difficulty of the control problem. We design the following observer for system (4.2)

\[
\begin{align*}
\dot{\hat{w}}_y(x, t) &= \hat{w}_x(x, t), & x \in (0, 1), & t \in (0, \infty), \\
\hat{w}(0, t) &= 0, & t \in [0, \infty), \\
\hat{w}_x(1, t) &= U(t) + k_2y_{out}(t) - k_2\hat{w}_y(1, t) - k_2\hat{z}_y(\ell, t), & t \in [0, \infty), \\
\hat{z}_y(x, t) &= \hat{z}_x(x, t), & x \in (0, \ell), & t \in (0, \infty), \\
\hat{z}(0, t) &= 0, & t \in [0, \infty), \\
\hat{z}_x(\ell, t) &= k_2y_{out}(t) - k_2\hat{w}_y(1, t) - k_2\hat{z}_y(\ell, t), & t \in [0, \infty),
\end{align*}
\]

where \(k_2 > 0\) is a design parameter. The motivation is obvious. Actually, let

\[
\begin{align*}
\tilde{w}(x, t) &= w(x, t) - \hat{w}(x, t), & x \in [0, 1], & t \in [0, \infty), \\
\tilde{z}(x, t) &= z(x, t) - \hat{z}(x, t), & x \in [0, \ell], & t \in [0, \infty),
\end{align*}
\]

be the errors. Then, it is easy to verify that \((\tilde{w}, \tilde{z})\) is governed by

\[
\begin{align*}
\dot{\tilde{w}}_y(x, t) &= \tilde{w}_x(x, t), & x \in (0, 1), & t \in (0, \infty), \\
\tilde{w}(0, t) &= 0, & t \in [0, \infty), \\
\dot{\tilde{w}}_x(1, t) &= -k_2\hat{w}_y(1, t) - k_2\hat{z}_y(\ell, t), & t \in [0, \infty), \\
\dot{\tilde{z}}_y(x, t) &= \tilde{z}_x(x, t), & x \in (0, \ell), & t \in (0, \infty), \\
\dot{\tilde{z}}(0, t) &= 0, & t \in [0, \infty), \\
\dot{\tilde{z}}_x(\ell, t) &= -k_2\hat{w}_y(1, t) - k_2\hat{z}_y(\ell, t), & t \in [0, \infty).
\end{align*}
\]

We consider system (4.5) in the state space \(\mathcal{V}\). Define the operator \(A_2 : D(A_2) (\subset \mathcal{V}) \to \mathcal{V}\) by

\[
A_2(f, g, \phi, \psi) = (g, f'', \phi'', \psi''), \quad (f, g, \phi, \psi) \in D(A_2), \\
D(A_2) = \{(f, g, \phi, \psi) \in [H^2(0, 1) \times H^1(0, 1) \times H^2(0, \ell) \times H^1(0, \ell)] \cap \mathcal{V}| f(0) = \phi(0) = 0, f'(1) = \phi'(\ell) = -k_2g(1) - k_2\psi(\ell)\}.
\]

**Lemma 4.1** Suppose \(k_2 > 0, \ell \in \mathbb{R}_+ \setminus J_2\) with

\[
J_2 := \left\{ \frac{2m+1}{2n+1} \middle| m, n \in \mathbb{Z} \right\}.
\]

For any initial value \((\hat{w}(\cdot, 0), \hat{w}_y(\cdot, 0), \hat{z}(\cdot, 0), \hat{z}_y(\cdot, 0)) \in \mathcal{V}\), there exists a unique solution \((\hat{w}, \hat{w}_y, \hat{z}, \hat{z}_y) \in C(0, \infty; \mathcal{V})\) such that

\[
\lim_{t \to \infty} \| (\hat{w}(\cdot, t), \hat{w}_y(\cdot, t), \hat{z}(\cdot, t), \hat{z}_y(\cdot, t)) \|_{\mathcal{V}} = 0.
\]
Proof. For any given \((f, g, \phi, \psi) \in D(A_2)\), a simple computation shows that

\[
\text{Re}(A_2(f, g, \phi, \psi), (f, g, \phi, \psi))_\mathcal{V} = -k_2 |g(1) + \psi(\ell)|^2 \leq 0, \quad \forall \ (f, g, \phi, \psi) \in D(A_2),
\]

which means that \(A_2\) is dissipative in \(\mathcal{V}\). Moreover, \(D(A_2)\) is densely defined in \(\mathcal{V}\) due to (Tucsnak & Weiss, 2009, p. 71, Proposition 3.1.6). For any \((f_1, g_1, \phi_1, \psi_1) \in \mathcal{V}\), we solve \(A_2(f, g, \phi, \psi) = (f_1, g_1, \phi_1, \psi_1)\) to obtain

\[
\begin{align*}
  g(x) &= f_1(x), & x \in [0, 1], \\
  \psi(x) &= \phi_1(x), & x \in [0, 1], \\
  f(x) &= -k_2[f_1(1) + \phi_1(\ell)]x - \int_0^x \int_s^1 g_1(\tau)d\tau ds, & x \in [0, 1], \\
  \phi(x) &= -k_2[f_1(1) + \phi_1(\ell)]x - \int_0^x \int_s^\ell \psi_1(\tau)d\tau ds, & x \in [0, 1].
\end{align*}
\]

(4.10)

By the Sobolev embedding theorem, \(A_2^{-1}\) exists and is compact on \(\mathcal{V}\). By the Lumer–Phillips theorem (Pazy, 1983, p. 14, Theorem 4.3), \(A_2\) generates a \(C_0\)-semigroup of contractions on \(\mathcal{V}\).

Now, we prove the asymptotic stability. Since the operator \(A_2\) is dissipative, we only need to prove that there is no eigenvalue on the imaginary axis (Luo & Guo, 1999, p. 130, Theorem 3.26). Suppose that \(A_2X = i\omega X\) with \(0 \neq \omega \in \mathbb{R}\) and \(X = (f, g, \phi, \psi) \in D(A_2)\). Then, \(g = i\omega f\) and \(\psi = i\omega \phi\). It follows from (4.9) that

\[
0 = \text{Re}(i\omega X, X)_{\mathcal{V}} = \text{Re}(A_2X, X)_{\mathcal{V}} = k_2 \omega^2 |f(1) + \phi(\ell)|^2.
\]

(4.11)

Thus, \(f(1) + \phi(\ell) = 0\) and

\[
\begin{align*}
  f''(x) &= -\omega^2 f(x), & x \in [0, 1], \\
  \phi''(x) &= -\omega^2 \phi(x), & x \in [0, 1], \\
  f(0) &= \phi(0) = 0, & f'(0) = \phi'(0) = 0, \\
  f(1) + \phi(\ell) &= 0.
\end{align*}
\]

(4.12)

Same as the proof of the Lemma 2.1, the general solution to system (4.12) is (2.17). Thus, the constants of (2.17) satisfy

\[
\begin{align*}
  c_1 &= c_3 = 0, \\
  c_2 \cos \omega &= 0, & c_4 \cos \omega \ell &= 0, \\
  c_2 \sin \omega + c_4 \sin \omega \ell &= 0.
\end{align*}
\]

(4.13)

Since \(\ell \in \mathbb{R}_+ \setminus J_2, \cos^2 \omega + \cos^2 \omega \ell \neq 0\). When \(\cos \omega \neq 0\), it follows that \(c_2 = 0, \ c_4 \sin \omega \ell = 0\) and \(c_4 \cos \omega \ell = 0\). Hence, \(c_4 = 0\) due to the last two equations. Similarly, when \(\cos \omega \ell \neq 0\), it follows that \(c_2 = 0\) and hence \(c_2 = 0\). Therefore, system (4.12) admits a unique solution \((f, g, \phi, \psi) \equiv 0\) for any \(0 \neq \omega \in \mathbb{R}\) and hence there is no eigenvalue on the imaginary axis. \(\square\)
We design the following feedback control law

\[ U(t) = -k_2 y_{out}(t) + k_2 \hat{z}_j(\ell, t), \quad \forall \ t \in [0, \infty), \]  

(4.14)

where \( k_2 \) and \( \hat{z}_j(\ell, t) \) come from observer (4.3). Then, system (4.2) under (4.14) becomes

\[
\begin{aligned}
&\frac{w_{tt}(x, t)}{w_{xx}(x, t)}, \quad x \in (0, 1), \ t \in (0, \infty), \\
&w(0, t) = 0, \quad t \in [0, \infty), \\
&\frac{w_t(1, t)}{k_2 [w_t(1, t) + z_t(\ell, t) + k_2 \hat{z}_j(\ell, t)]}, \quad t \in [0, \infty), \\
&z_{tt}(x, t) = z_{xx}(x, t), \quad x \in (0, \ell), \ t \in (0, \infty), \\
&z(0, t) = 0, \quad z_{\ell}(\ell, t) = 0, \\
&\hat{w}_{tt}(x, t) = \hat{w}_{xx}(x, t), \quad x \in (0, 1), \ t \in (0, \infty), \\
&\hat{w}(0, t) = 0, \quad \hat{w}_{\ell}(1, t) = -k_2 \hat{w}_t(1, t), \quad t \in [0, \infty), \\
&\hat{z}_{tt}(x, t) = \hat{z}_{xx}(x, t), \quad x \in (0, \ell), \ t \in (0, \infty), \\
&\hat{z}(0, t) = 0, \\
&\hat{z}_{\ell}(\ell, t) = k_2 w_t(1, t) + k_2 z_t(\ell, t) - k_2 \hat{w}_t(1, t) - k_2 \hat{z}_t(\ell, t), \quad t \in [0, \infty). \\
\end{aligned}
\]

(4.15)

The main result for plant system (4.15) is stated as following Theorem 4.1.

**Theorem 4.1** Suppose that \( k_2 > 0, \ell \in \mathbb{R}_+ \setminus J_2 \). For any initial value \((w(\cdot, 0), w_t(\cdot, 0), z(\cdot, 0), z_t(\cdot, 0), \hat{w}(\cdot, 0), \hat{w}_t(\cdot, 0), \hat{z}(\cdot, 0), \hat{z}_t(\cdot, 0)) \in \mathcal{H}\), the closed-loop system (4.15) admits a unique solution \((w, w_t, z, z_t, \hat{w}, \hat{w}_t, \hat{z}, \hat{z}_t) \in C(0, \infty; \mathcal{H})\) such that

\[
\lim_{t \to \infty} \| (w(\cdot, t), w_t(\cdot, t)) \|_{\mathcal{H}_1} = 0. 
\]

(4.16)

**Proof.** The proof is quite similar to the proof of Theorem 3.1. We omit the details here. \(\square\)

5. Numerical simulation

In this section, we present some numerical simulations to validate the theoretical results. The finite difference method is applied to compute the solution numerically by choosing the time and space steps as 0.005 and 0.007, respectively.
The periodic disturbance $d(t)$ with period $T = 4\ell$ is chosen as:

$$
d(t) = \begin{cases} 
-1, & -2\ell \leq t < -\ell, \\
1, & -\ell \leq t < 0, \\
1, & 0 \leq t < \ell, \\
-1, & \ell \leq t < 2\ell,
\end{cases} \quad (5.1)
$$

which has infinite Fourier series representation:

$$
d(t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{(2n-1)\pi} \cos \left( \frac{(2n-1)\pi}{2\ell} \right) t. \quad (5.2)
$$
According to (2.3), the initial values of the $z$-part are
\[ z(x,0) = 0, \quad z_t(x,0) = 1, \quad x \in [0, \ell]. \]  
(5.3)

In closed-loop system (3.2), we choose $\ell = 1, k_1 = 1$. Thus, the disturbance $d(t)$ generated by system (2.3) is
\[ d(t) = z_t(\ell, t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{(2n-1)\pi} \cos \frac{(2n-1)\pi}{2} t. \]  
(5.4)

The remaining initial values are taken as
\[ w(x,0) = \hat{w}(x,0) = \cos(2\pi x) - 1, \quad w_t(x,0) = \hat{w}_t(x,0) = 0, \quad x \in [0, \ell], \]  
(5.5)
ASYMPTOTIC STABILIZATION FOR A WAVE EQUATION WITH PERIODIC DISTURBANCE

Fig. 3. disturbance estimation.

\[ \hat{z}(x,0) = 0, \quad \hat{z}_t(x,0) = -\sin 5.5\pi x, \quad x \in [0, \ell]. \]  \hspace{1cm} (5.6)

We consider the case \( \ell = 2, k_2 = 1 \) in the closed-loop system (4.15) with the initial values (5.3), (5.5) and (5.6), respectively. The disturbance \( d(t) \) is then taken as

\[ d(t) = z_t(\ell,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{(2n-1)\pi} \cos \frac{(2n-1)\pi}{4} t. \]  \hspace{1cm} (5.7)

The solution of the closed-loop system (3.2) is depicted in Fig. 1 and the solution of the closed-loop system (4.15) is depicted in Fig. 2. The numerical results for the disturbance \( d(t) \) and its estimation \( \hat{z}_t(\ell,t) \) of two cases are plotted in Fig. 3. It is seen that \( (w, w_t) \) is stable and the state observer is effective in both cases. Since all parameters are taken the same for Figs 1 and 2, the convergence of Fig. 1 is faster than Fig. 2, which might suggest that the convergence of the former is exponentially and the latter is asymptotically, confirmed by the theoretical results.

6. Concluding remarks

In this paper, we consider boundary stabilization for a one-dimensional wave equation subject to periodic disturbance. The disturbance is regarded as the output of an infinite dimensional exosystem, and we transform the controlled plant into a coupled wave system. We design an state observer to estimate uncertainty by using of the dynamics of the coupled system. An output feedback controller is then designed. Two different cases are discussed. For the first case, the control and external disturbance are matched and it is shown that the designed output feedback control can stabilize exponentially the system under the condition \( \ell \in \mathbb{R}_+ \setminus J_1 \), where \( T = 4\ell \) is the period of the external disturbance. The second case considers the situation where the measured output is contaminated by the external disturbance. By the designed output feedback, we show that the closed-loop system is asymptotically stable under the condition \( \ell \in \mathbb{R}_+ \setminus J_2 \), which might be exponentially in some cases but not generally suggested by the numerical simulations.
Funding

National Natural Science Foundation of China (61873153, 61873260 and 11671240) and the Project of Department of Education of Guangdong Province (2017KZDXM087).

REFERENCES


A. Appendix

In this appendix, we give a proof of Lemma 2.2 by the spectral analysis. First, we consider the eigenvalue problem of \( A_1 \), i.e.

\[
A_1(f, g, \phi, \psi) = \lambda(f, g, \phi, \psi), \quad \forall (f, g, \phi, \psi) \in D(A_1). \tag{A.1}
\]

By the definition of (2.10),

\[
\begin{aligned}
  f''(x) &= \lambda^2 f(x), \quad x \in [0, 1], \\
  \phi''(x) &= \lambda^2 \phi(x), \quad x \in [0, \ell], \\
  f(0) &= 0, \quad f'(1) = -k_1 \lambda f(1) + \lambda \phi(\ell), \\
  \phi(0) &= 0, \quad \phi'(\ell) = -\lambda f(1).
\end{aligned} \tag{A.2}
\]

The general solution of (A.2) is

\[
\begin{aligned}
  f(x) &= c_1 e^{\lambda x} + c_2 e^{-\lambda x}, \quad x \in [0, 1], \\
  \phi(x) &= c_3 e^{\lambda x} + c_4 e^{-\lambda x}, \quad x \in [0, \ell],
\end{aligned} \tag{A.3}
\]

where \( c_j (j = 1, 2, 3, 4) \) are complex constants. Substituting (A.3) into the boundary condition of (A.2) gives

\[
\begin{aligned}
  c_1 + c_2 &= 0, \quad c_3 + c_4 = 0, \\
  (e^{\lambda \ell} - e^{-\lambda})c_1 + (e^{\lambda \ell} + e^{-\lambda \ell})c_3 &= 0, \\
  [(1 + k_1) e^{\lambda} + (1 - k_1) e^{-\lambda}]c_1 - (e^{\lambda \ell} - e^{-\lambda \ell})c_3 &= 0.
\end{aligned} \tag{A.4}
\]

Then, (A4) has non-zero solution if and only if the characteristic determinant \( \det(\Delta(\lambda)) = 0 \), where

\[
\Delta(\lambda) = \begin{bmatrix}
  1 & 1 & 0 & 0 \\
  0 & 0 & 1 & 1 \\
  e^{\lambda} - e^{-\lambda} & 0 & e^{\lambda \ell} + e^{-\lambda \ell} & 0 \\
  (1 + k_1) e^{\lambda} + (1 - k_1) e^{-\lambda} & 0 & -(e^{\lambda \ell} - e^{-\lambda \ell}) & 0
\end{bmatrix}. \tag{A.5}
\]
Thus, the characteristic equation reads
\[
\det(\Delta(\lambda)) = (2 + k_1)\ell^{(\ell + 1)} + k_1\ell^{-(\ell - 1)} - k_1\ell^{(\ell - 1)} + (2 - k_1)\ell^{-\ell(\ell + 1)} = 0, \quad \forall \lambda \in \sigma(A_1).
\] (A6)

**Proposition A.1** Let \( A_1 \) be given by (2.10) and \( \det(\Delta(\lambda)) \) be given by (A.6). Then, for any \( \lambda \in \rho(A_1) \) (resolvent set of \( A_1 \)), the resolvent operator \( R(\lambda, A_1) \) can be formulated as
\[
R(\lambda, A_1)(\xi, \zeta, \theta, \vartheta) = \frac{G(\lambda, (\xi, \zeta, \theta, \vartheta))}{\det(\Delta(\lambda))}, \quad \forall (\xi, \zeta, \theta, \vartheta) \in \mathcal{V},
\] (A.7)

where
\[
G(\lambda, (\xi, \zeta, \theta, \vartheta))^T = \begin{pmatrix}
\lambda^{-1} p_3(\lambda) \sinh \lambda x - \lambda^{-1} \det(\Delta(\lambda)) G_1(x) \\
p_3(\lambda) \sinh \lambda x - \det(\Delta(\lambda)) G_1(x) - \det(\Delta(\lambda)) \xi(x) \\
\lambda^{-1} p_4(\lambda) \sinh \lambda x - \lambda^{-1} \det(\Delta(\lambda)) G_3(x) \\
p_4(\lambda) \sinh \lambda x - \det(\Delta(\lambda)) G_4(x)
\end{pmatrix},
\] (A.8)

\[
\begin{align*}
G_1(x) &= \int_0^x \sinh \lambda(x - s)[\lambda \xi(s) + \zeta(s)] ds, \quad x \in [0, 1], \\
G_2(x) &= \int_0^x \cosh \lambda(x - s)[\lambda \xi(s) + \zeta(s)] ds, \quad x \in [0, 1], \\
G_3(x) &= \int_0^x \sinh \lambda(x - s)[\lambda \theta(s) + \vartheta(s)] ds, \quad x \in [0, \ell], \\
G_4(x) &= \int_0^x \cosh \lambda(x - s)[\lambda \theta(s) + \vartheta(s)] ds, \quad x \in [0, \ell],
\end{align*}
\] (A.9)

and
\[
\begin{align*}
p_1(\lambda) &= k_1 G_1(1) + G_2(1) - G_3(\ell) - \theta(\ell) + k_1 \xi(1), \\
p_2(\lambda) &= G_1(1) + G_4(\ell) + \xi(1), \\
p_3(\lambda) &= 4[p_1(\lambda) \cosh \lambda \ell + p_2(\lambda) \sinh \lambda \ell], \\
p_4(\lambda) &= 4[p_2(\lambda)(\cosh \lambda + k_1 \sinh \lambda) - p_1(\lambda) \sinh \lambda].
\end{align*}
\] (A.10)
**Proof.** For any \((\xi, \zeta, \theta, \vartheta) \in \mathcal{V}\), let

\[
(f, g, \phi, \psi) = R(\lambda, A_1) (\xi, \zeta, \theta, \vartheta). \tag{A.11}
\]

Then, \((f, g, \phi, \psi)\) satisfies

\[
\begin{aligned}
f''(x) - \lambda^2 f(x) &= -\lambda \xi(x) - \zeta(x), \quad x \in [0, 1], \\
g(x) &= \lambda f(x) - \xi(x), \quad x \in [0, 1], \\
\phi''(x) - \lambda^2 \phi(x) &= -\lambda \theta(x) - \vartheta(x), \quad x \in [0, \ell], \\
\psi(x) &= \lambda \phi(x) - \theta(x), \quad x \in [0, \ell], \\
f(0) &= \phi(0) = 0, \\
f'(1) &= \lambda \phi(\ell) - k_1 \lambda f(1) - \theta(\ell) + k_1 \xi(1), \\
\phi'(\ell) &= -\lambda f(1) + \xi(1). \\
\end{aligned} \tag{A.12}
\]

The general solution of (A.12) is given by

\[
\begin{aligned}
f(x) &= c_1 e^{\lambda x} + c_2 e^{-\lambda x} - \lambda^{-1} G_1(x), \quad x \in [0, 1], \\
\phi(x) &= c_3 e^{\lambda x} + c_4 e^{-\lambda x} - \lambda^{-1} G_3(x), \quad x \in [0, \ell], \\
\end{aligned} \tag{A.13}
\]

where \(c_1, c_2, c_3, c_4\) are constants to be determined. Substituting (A.13) into the boundary condition of system (A.12) yields

\[
\begin{aligned}
c_1 + c_2 &= 0, \quad c_3 + c_4 = 0, \\
2c_1 (\lambda \cosh \lambda + k_1 \lambda \sinh \lambda) - 2c_3 \lambda \sinh \lambda \ell &= p_1(\lambda), \\
2c_1 \lambda \sinh \lambda + 2c_3 \lambda \cosh \lambda \ell &= p_2(\lambda), \\
\end{aligned} \tag{A.14}
\]

where \(p_1(\lambda)\) and \(p_2(\lambda)\) are defined in (A.10). Hence,

\[
\begin{aligned}
c_1 &= \frac{2[p_1(\lambda) \cosh \lambda \ell + p_2(\lambda) \sinh \lambda \ell]}{\lambda \det(\Delta(\lambda))} = \frac{p_3(\lambda)}{2 \lambda \det(\Delta(\lambda))}, \\
c_3 &= \frac{2[p_2(\lambda)(\cosh \lambda + k_1 \sinh \lambda) - p_1(\lambda) \sinh \lambda]}{\lambda \det(\Delta(\lambda))} = \frac{p_4(\lambda)}{2 \lambda \det(\Delta(\lambda))}, \\
\end{aligned} \tag{A.15}
\]

where \(p_3(\lambda)\) and \(p_4(\lambda)\) are defined in (A.10). Therefore, the expression of solution \((f, g, \phi, \psi)\) of (A.12) is given by (A.7). \(\square\)

**Proposition A.2** Let \(A_1\) be given by (2.10) and \(\det(\Delta(\lambda))\) be given by (A.6). Then, the following assertions hold:

(i) If \(\ell > 0, k_1 > 0\), then, all eigenvalues of \(A_1\) are geometrically simple.

(ii) If \(\ell > 0, k_1 > 0\), then, the multiplicity of each root of \(\det(\Delta(\lambda)) = 0\) is at most two, and hence the algebraic multiplicity of each eigenvalue of \(A_1\) is at most two.
(iii) There is a constant $M > 0$ such that for any $\lambda \in \sigma(A_1)$, \(|\text{Re}\lambda| \leq M\).

(iv) If $\ell > 0$, $k_1 > 0$, then, the eigenvalues of $A_1$ are separated in the sense that
\[
\inf_{\lambda_m, \lambda_n \in \sigma(A_1), \lambda_m \neq \lambda_n} |\lambda_m - \lambda_n| > 0. \tag{A.16}
\]

**Proof.** (i). If system (A.2) has two linearly independent solutions $(f_1, \phi_1)$, $(f_2, \phi_2)$, then, there exist two constants $C_1$, $C_2$ with $|C_1| + |C_2| \neq 0$ such that $C_1f_1(0) + C_2f_2(0) = 0$. It is seen that $(f, \phi) = C_1(f_1, \phi_1) + C_2(f_2, \phi_2)$ is a solution of system (A.2) with $f'(0) = 0$. Following the steps in (A.2)–(A.4), we obtain
\[
\begin{cases}
    c_1 + c_2 = 0, & c_3 + c_4 = 0, & 2c_1 = 0, \\
    (e^{\lambda} - e^{-\lambda})c_1 + (e^{\lambda} + e^{-\lambda})c_3 = 0, \\
    [(1 + k_1)e^{\lambda} + (1 - k_1)e^{-\lambda}]c_1 - (e^{\lambda} - e^{-\lambda})c_3 = 0.
\end{cases} \tag{A.17}
\]
Thus, $(e^{\lambda} - e^{-\lambda})c_3 = 0$, $(e^{\lambda} + e^{-\lambda})c_3 = 0$ due to $c_1 = 0$ and hence $c_3 = 0$. Thus, $c_1 = c_2 = c_3 = c_4 = 0$ and $(f, \phi) = C_1(f_1, \phi_1) + C_2(f_2, \phi_2) = 0$. This contradicts with the assumption that $(f_1, \phi_1)$ and $(f_2, \phi_2)$ are linearly independent.

(ii). Recall the definition of $\det(\Delta(\lambda))$ in (A.6) and rewrite it as
\[
\frac{1}{4} \det(\Delta(\lambda)) = \cosh \lambda \ell (\cosh \lambda + k_1 \sinh \lambda) + \sinh \lambda \ell \sinh \lambda = 0. \tag{A.18}
\]
When $\cosh \lambda \ell \sinh \lambda \neq 0$, we divide (A.18) by $\cosh \lambda \ell \sinh \lambda$ to obtain
\[
\tanh \lambda \ell = -(\coth \lambda + k_1). \tag{A.19}
\]
If $\det(\Delta(\lambda)) = [\det(\Delta(\lambda))]' = 0$, we claim that $[\det(\Delta(\lambda))]' \neq 0$. Otherwise, taking the first and the second order derivatives of $\det(\Delta(\lambda))$ with respect to $\lambda$ respectively, we obtain
\[
0 = \frac{[\det(\Delta(\lambda))]'}{4}
\]
\[
= \ell \sinh \lambda \ell (\cosh \lambda + k_1 \sinh \lambda) + \cosh \lambda \ell (\sinh \lambda + k_1 \cosh \lambda)
\]
\[
+ \ell \cosh \lambda \ell \sinh \lambda + \sinh \lambda \ell \cosh \lambda
\]
\[
= \sinh \lambda \ell [k_1 \ell \sinh \lambda + (\ell + 1) \cosh \lambda] + \cosh \lambda \ell [(\ell + 1) \sinh \lambda + k_1 \cosh \lambda], \tag{A.20}
\]
and
\[
0 = \frac{[\det(\Delta(\lambda))]''}{4}
\]
\[
= \ell \cosh \lambda \ell [k_1 \ell \sinh \lambda + (\ell + 1) \cosh \lambda] + \sinh \lambda \ell [k_1 \ell \cosh \lambda + (\ell + 1) \sinh \lambda]
\]
\[
+ \ell \sinh \lambda \ell [(\ell + 1) \sinh \lambda + k_1 \cosh \lambda] + \cosh \lambda \ell [(\ell + 1) \cosh \lambda + k_1 \sinh \lambda]
\]
\[
= \cosh \lambda \ell [(\ell + 1)^2 \cosh \lambda + k_1 (\ell^2 + 1) \sinh \lambda] + \sinh \lambda \ell [(\ell + 1)^2 \sinh \lambda + 2k_1 \ell \cosh \lambda]. \tag{A.21}
\]
Dividing (A.20) and (A.21) by $\cosh \lambda \ell \sinh \lambda$ and using (A.19) produce

$$\begin{cases}
\frac{[\det(\Delta(\lambda))]'}{4 \cosh \lambda \ell \sinh \lambda} = -(\ell + 1)(\coth^2 \lambda + 2k_1 \hat{\ell} \coth \lambda + k_1^2 \hat{\ell} - 1) = 0, \\
\frac{[\det(\Delta(\lambda))]''}{4 \cosh \lambda \ell \sinh \lambda} = -2k_1 \ell (\coth^2 \lambda + k_1 \coth \lambda + 1) = 0,
\end{cases} \tag{A.22}$$

where $\hat{\ell} = \frac{\ell+1}{\ell} > 1$. Solve (A.22) to obtain

$$\coth \lambda = -k_1 \pm \sqrt{k_1^2 - 4}. \tag{A.23}$$

Substituting (A.23) into the first equation of (A.22) yields

$$\pm k_1 \sqrt{k_1^2 - 4} \mp 2k_1 \hat{\ell} \sqrt{k_1^2 - 4} = k_1^2 - 4. \tag{A.24}$$

The solution of (A.24) is found to be $\hat{\ell} = \frac{k_1 + \sqrt{k_1^2 - 4}}{2k_1}$, that is contradictory to the assumption $\hat{\ell} > 1$. Thus, $[\det(\Delta(\lambda))]'' \neq 0$. When $\cosh \lambda \ell = 0$, one has $\sinh \lambda \ell \neq 0$ and $\frac{1}{4} \det(\Delta(\lambda)) = \sinh \lambda \ell \sinh \lambda = 0$. Hence, $\sinh \lambda = 0$ and then $\frac{1}{4} \det(\Delta(\lambda))' = \sinh \lambda \ell \cosh \lambda \neq 0$. Similarly, when $\sinh \lambda = 0$, one has $\frac{1}{4} \det(\Delta(\lambda))' = \ell \sinh \lambda \ell \cosh \lambda \neq 0$. Therefore, the multiplicity of $\lambda$, as the root of $\det(\Delta(\lambda))$, is at most two.

Note that a general formula of the following (see, e.g., Luo & Guo, 1999, p. 148)

$$m_{(a)}(\lambda) \leq p_{\lambda} \cdot m_{(g)}(\lambda), \tag{A.25}$$

where $m_{(a)}(\lambda)$, $m_{(g)}(\lambda)$ represent the algebraic multiplicity and the geometric multiplicity respectively, and $p_{\lambda}$ denotes the order of the pole of $R(\lambda, A_1)$ at $\lambda$. The expression of $R(\lambda, A_1)$ in (A.7) shows that $p_{\lambda}$ is no more than the multiplicity of $\det(\Delta(\lambda))$ at $\lambda$. Since we have proved that the multiplicity of each root of $\det(\Delta(\lambda))$ is at most two and $A_1$ is geometrically simple, we can conclude that

$$\sup_{\lambda \in \sigma(A_1)} m_{(a)}(\lambda) \leq 2. \tag{A.26}$$

(iii). Suppose that $\text{Re} \lambda \to -\infty$. If $\ell \geq 1$ and $k_1 \neq 2$, then,

$$| \det(\Delta(\lambda)) | \geq | 2 - k_1 | e^{-(\ell+1)|\text{Re} \lambda|} - k_1 e^{-(\ell-1)|\text{Re} \lambda|} - (2 + k_1) e^{\ell+1|\text{Re} \lambda|} - k_1 e^{\ell-1|\text{Re} \lambda|} |
= e^{-(\ell-1)|\text{Re} \lambda|} (| 2 - k_1 | e^{-2|\text{Re} \lambda|} - k_1 ) + \mathcal{O}(e^{\text{Re} \lambda}) \to +\infty, \tag{A.27}$$

as $\text{Re} \lambda \to -\infty$. The same argument applies to other cases yields

$$| \det(\Delta(\lambda)) | \geq \begin{cases}
| 2 - k_1 | e^{-(\ell+1)|\text{Re} \lambda|} - k_1 e^{-(\ell-1)|\text{Re} \lambda|} + \mathcal{O}(e^{\text{Re} \lambda}), & \text{when } k_1 \neq 2, \\
2 e^{-|\ell-1|\text{Re} \lambda} + \mathcal{O}(e^{\text{Re} \lambda}), & \text{when } k_1 = 2,
\end{cases} \tag{A.28}$$

as $\text{Re} \lambda \to -\infty$. Thus, $| \det(\Delta(\lambda)) | \to +\infty$ as $\text{Re} \lambda \to -\infty$. This is a contradiction with $\det(\Delta(\lambda)) = 0$. Therefore, there is a constant $M > 0$ such that for any $\lambda \in \sigma(A_1)$, $| \text{Re} \lambda | \leq M$.

(iv). Suppose that $\{ \lambda_n \}_{n=1}^{\infty}$ is the solution of the characteristic equation $\det(\Delta(\lambda)) = 0$ in (A.6). If the multiplicity of each root of $\det(\Delta(\lambda)) = 0$ is two, which means that $\det(\Delta(\lambda_n)) = | \det(\Delta(\lambda_n)) |' = 0$.


0, \[\det(\Delta(\lambda_n))\] \(\neq 0\), by Theorem 3 of Xu & Guo (2003), it suffices to prove that \(\inf_n |\det(\Delta(\lambda_n))| > 0\). We now suppose that there exists a subsequence \(\{\lambda_{n_k}\}\) of \(\{\lambda_n\}\) satisfying

\[
\det \Delta(\lambda_{n_k}) = |\det \Delta(\lambda_{n_k})| > 0, \quad |\det(\Delta(\lambda_{n_k}))| \to 0 \text{ as } k \to \infty,
\]

then, \(\{\coth \lambda_{n_k}\}\) satisfies (A.19), the first equation of (A.22) and \(-2k_1 \ell (\coth^2 \lambda_{n_k} + k_1 \coth \lambda_{n_k} + 1) \to 0 (k \to \infty)\). Since all \(\{\lambda_{n_k}\}\) lie in vertical strip \(-M \leq \Re \lambda < 0\) for some \(M > 0\) by (iii), \(\coth \lambda_{n_k}\) is uniformly bounded. Thus, there exists an accumulation point \(\eta\) of \(\{\coth \lambda_{n_k}\}\) : \(\coth \lambda_{n_k} \to \eta\) as \(k \to \infty\), and \(\eta\) satisfies \(\eta^2 + 2k_1 \ell \eta + k_1^2 \ell - 1 = 0\) and \(\eta^2 + k_1 \eta + 1 = 0\). Based on the proof in (ii), such a solution \(\eta\) does not exist. Thus, the zeros of \(\det(\Delta(\lambda)) = 0\) are separated.

If the multiplicity of each root \(\det(\Delta(\lambda)) = 0\) is simple, which means \(\det(\Delta(\lambda)) = 0\), it is sufficient to prove that \(\inf_n |\det(\Delta(\lambda_{n_k}))| > 0\) by Theorem 3 of Xu & Guo (2003). When \(\cosh \lambda \ell \sinh \lambda \neq 0\), the same conclusion holds along the same line. When \(\cosh \lambda \ell = 0\), it is seen from the proof of (ii) that \(\frac{1}{2} |\det(\Delta(\lambda))| = \sinh \lambda \ell \sinh \lambda = 0\) and \(\frac{1}{2} |\det(\Delta(\lambda))| = \sinh \lambda \ell \cosh \lambda \neq 0\). Suppose that there exists a subsequence \(\{\lambda_n\}\) such that \(\det(\Delta(\lambda_n)) \to 0\) as \(n \to \infty\). Since \(\sinh \lambda\) is bounded uniformly, we denote \(\eta\) to be an accumulation point of \(\sinh \lambda_n : \sinh \lambda_n \to \eta\) as \(n \to \infty\). Then, \(\eta\) satisfies \(\eta = 0\) and \((1 + \eta^2) = 0\). This is a contradiction. When \(\sinh \lambda = 0\), same results holds also. Therefore, in all cases, the zeros of \(\det(\Delta(\lambda)) = 0\) are separated.

Before establishing the completeness of the root space \(\text{Sp}(A_1)\) that is the closed subspace spanned by all generalized eigenvectors of \(A_1\), we present some basic facts of entire function. A \(\mathcal{V}\)-value entire function \(f(z)\) is said to be of order \(\alpha\) if

\[
\|f(z)\|_{\mathcal{V}} = \mathcal{O}(e^{\alpha|z|^n}) \text{ as } |z| \to \infty
\]

(A.30)

holds for a real number \(\alpha\). An entire function \(f(z)\) is said to be of exponential type if

\[
|f(z)| \leq c_1 e^{c_2|z|}
\]

(A.31)

holds for some positive constants \(c_1, c_2\) and complex \(z\). An entire function \(f(z)\) of exponential type is said to be of sine type (Wang et al., 2011) if

(a) the zero of \(f(z)\) lie in a strip \(|z| \in \mathbb{C} |\Re z| \leq c\) for some \(c > 0\);

(b) there exist positive constants \(c_1, c_2\) and \(x_0 \in \mathbb{R}\) such that \(c_1 \leq |f(x_0 + iy)| \leq c_2\) for all \(y \in \mathbb{R}\).

It is obviously that \(\det(\Delta(\lambda))\), given by (A.6), is an entire function of exponential type of order 1. Furthermore, from Proposition A.2 (iii), \(\det(\Delta(\lambda))\) is a sine type function.

**Proposition A.3** If \(k_1, \ell > 0\) and \(k_1 \neq 2\), then, \(\text{Sp}(A_1) = \mathcal{V}\).

**Proof.** Define an operator \(A_0 : D(A_0) \to \mathcal{V}\) as

\[
\begin{cases}
A_0(f_1, g_1, \phi_1, \psi_1) = (g_1, f_1''(\lambda), \phi_1''(\lambda), \psi_1''), & \forall (f_1, g_1, \phi_1, \psi_1) \in D(A_0), \\
D(A_0) = \{ (f_1, g_1, \phi_1, \psi_1) \in [H^2(0, 1) \times H^1(0, 1) \times H^2(0, \ell) \times H^1(0, \ell)] \cap \mathcal{V} | f_1(0) = 0, \phi_1(0) = 0, f_1'(1) = \psi_1(\ell), \phi_1'(\ell) = -g_1(1) \}. 
\end{cases}
\]

(A.32)

Then, \(A_0\) is a skew-adjoint operator in \(\mathcal{V} : A_0^* = -A_0\) and \(A_0\) generates a unitary group. Therefore, the resolvent \(R(\lambda, A_0)\) satisfies

\[
\|R(\lambda, A_0)\|_{\mathcal{V}} \leq \frac{1}{|\lambda|}, \quad \forall \lambda \in \mathbb{R}.
\]

(A.33)
For any \((\xi, \zeta, \theta, \vartheta) \in \mathcal{V}, \lambda \in \rho(A_1) \cap \rho(A_0)\), let
\[
\begin{aligned}
(f_1, g_1, \phi_1, \psi_1) &= R(\lambda, A_0)(\xi, \zeta, \theta, \vartheta), \\
(f, g, \phi, \psi) &= R(\lambda, A_1)(\xi, \zeta, \theta, \vartheta).
\end{aligned}
\] (A.34)

Then,
\[
(\lambda - A_0)(f_1, g_1, \phi_1, \psi_1) = (\lambda - A_1)(f, g, \phi, \psi) = (\xi, \zeta, \theta, \vartheta),
\] (A.35)

and hence \((f_2, g_2, \phi_2, \psi_2) = (f, g, \phi, \psi) - (f_1, g_1, \phi_1, \psi_1)\) satisfies
\[
\begin{aligned}
f''_2(x) - \lambda^2 f_2(x) &= 0, \quad g_2(x) = \lambda f_2(x), \quad x \in [0, 1], \\
\phi''_2(x) - \lambda^2 \phi_2(x) &= 0, \quad \psi_2(x) = \lambda \phi_2(x), \quad x \in [0, \ell], \\
f_2(0) = \phi_2(0) = 0, \quad \phi'_2(\ell) = -\lambda \phi_2(1), \\
f'_2(1) = \lambda \phi_2(\ell) - k_1 \lambda f_2(1) - k_1 \lambda \phi_1(1) + k_1 \xi(1).
\end{aligned}
\] (A.36)

Solving (A.36) gives
\[
\begin{aligned}
f'_2(x) &= \frac{-k_1 \lambda f_1(1) + k_1 \xi(1)}{\det(\Delta(\lambda))} 4 \cosh \lambda \ell \cosh x, \quad x \in [0, 1], \\
g_2(x) &= \frac{-k_1 \lambda f_1(1) + k_1 \xi(1)}{\det(\Delta(\lambda))} 4 \cosh \lambda \ell \sinh x, \quad x \in [0, 1], \\
\phi'_2(x) &= \frac{-k_1 \lambda f_1(1) + k_1 \xi(1)}{\det(\Delta(\lambda))} 4 \sinh \lambda \cosh x, \quad x \in [0, \ell], \\
\psi_2(x) &= \frac{-k_1 \lambda f_1(1) + k_1 \xi(1)}{\det(\Delta(\lambda))} 4 \sinh \lambda \sinh x, \quad x \in [0, \ell].
\end{aligned}
\] (A.37)

By (A.33), we obtain
\[
| - k_1 \lambda f_1(1) | \leq k_1 | \lambda | \| f'_1 \|_{L^2(0,1)} \leq k_1 | \lambda | \| R(\lambda, A_0)(\xi, \zeta, \theta, \vartheta) \|_\mathcal{V} \leq k_1 \| (\xi, \zeta, \theta, \vartheta) \|_\mathcal{V},
\] (A.38)

\[
| k_1 \xi(1) | \leq k_1 \| (\xi, \zeta, \theta, \vartheta) \|_\mathcal{V}.
\] (A.39)

Notice that
\[
\| \cosh \lambda \ell \cosh \lambda x \|_{L^2(0,1)} + \| \cosh \lambda \ell \sinh \lambda x \|_{L^2(0,1)} \quad \text{and} \quad \frac{\| \sinh \lambda \cosh \lambda x \|_{L^2(0,\ell)} + \| \sinh \lambda \sinh \lambda x \|_{L^2(0,\ell)}}{\| \det(\Delta(\lambda)) \|}
\]

\[
\rightarrow \frac{c_1 e^{-\ell+1} + c_2}{|2 - k_1 e^{-\ell+1} - k_1 e^{-\ell-1}|} \leq c_3 \quad \text{as} \quad \lambda \rightarrow -\infty,
\] (A.40)

where \(c_1, c_2, c_3\) are positive constants. Combining (A.37)–(A.40) yields
\[
\| (f_2, g_2, \phi_2, \psi_2) \|_\mathcal{V} = \| (f'_2, g_2, \phi'_2, \psi_2) \|_{L^2(0,1)^2 \times [L^2(0,\ell)]^2} \leq M \quad \text{as} \quad \lambda \rightarrow -\infty,
\] (A.41)
where $M > 0$ is a constant. By (A.33) and (A.34), we conclude that
\[
\| (f_1, g_1, \phi_1, \psi_1) \|_V \leq \frac{1}{|\lambda|} \| (\xi, \zeta, \theta, \vartheta) \|_V \to 0 \text{ as } \lambda \to -\infty. \tag{A.42}
\]
Since $\| (f, g, \phi, \psi) \|_V \leq \| (f_1, g_1, \phi_1, \psi_1) \|_V + \| (f_2, g_2, \phi_2, \psi_2) \|_V$, we see that $\| (f, g, \phi, \psi) \|_V$ is uniformly bounded as $\lambda \to -\infty$. From proposition A.1, we have $(f, g, \phi, \psi) = \frac{G(\lambda, (\xi, \zeta, \theta, \vartheta))}{\det(\Delta(\lambda))}$, where $G(\lambda, (\xi, \zeta, \theta, \vartheta))$ is a $V$-value entire function with order less than or equal to 1, and $\det(\Delta(\lambda))$ is a scalar function with order equal to 1. Therefore, all conditions of Theorem 4 of Xu & Guo (2003) are satisfied with $\rho = 1, n = 2, \gamma_1 = \{\lambda| \arg \lambda = \pi\}$ and the result follows.

**Proposition A.4** Assume $k_1, \ell > 0$ and $k_1 \neq 2$. Let $A_1$ be given by (2.10) and let $\det(\Delta(\lambda))$ be given by (A6). Then, the following assertions hold:
(i) There is a set of generalized eigenfunctions of $A_1$, which forms a Riesz basis in $V$.
(ii) The spectrum-determined growth condition holds, namely,
\[
S(A_1) = \omega(A_1), \tag{A.43}
\]
where $S(A_1) := \sup \{\Re \lambda | \lambda \in \sigma(A_1)\}$ is the spectral bound of $A_1$, and $\omega(A_1) := \inf \{\omega| \exists M > 0 \text{ such that } \|e^{A_1t}\| \leq Me^{\omega t}, \forall t \geq 0\}$ is the growth order of $A_1$.

**Proof**. Since $\det(\Delta(\lambda))$ is a sine-type function, by (Guo & Xu, 2006, Proposition 2.1) and Proposition A.2 (ii), the set of its zeros that consists of the eigenvalues of $A_1$ can be decomposed into two separable sets, (a multiple eigenvalue is repeated in a number of times of its algebraic multiplicity), i.e.
\[
\text{zeros of } \det(\Delta(\lambda)) = \text{eigenvalues of } A_1 = \{\lambda_i, i \in \mathbb{Z}\} = \Lambda = \bigcup_{n=1}^{2} \Lambda_n, \tag{A.44}
\]
where $\inf_{\lambda_m, \lambda_n \in \Lambda_n, m \neq n} |\lambda_m - \lambda_n| > 0$ for all $1 \leq n \leq 2$. Note that $\Lambda = \{\lambda_i, i \in \mathbb{Z}\}$ is located in a strip parallel to the imaginary axis by Proposition A.2 (iii), and there is no finite accumulation point of $\Lambda$. Hence, $\Lambda$ can be assumed without loss of generality that $\{\Im \lambda_i\}$ forms a nondecreasing sequence. Define
\[
D^+(\Lambda) = \lim_{r \to \infty} \frac{n^+(r)}{r}, \tag{A.45}
\]
where
\[
n^+(r) = \sup_{x \in \mathbb{R}} \{\text{the number of } \Im(A) \cap [x, x + r]\}. \tag{A.46}
\]
It is evident (see from the proof in Proposition 1 of Guo & Xu, 2004) that $D^+(\Lambda) < \infty$. By Guo & Xu (2006), the generalized divided difference (GDD) produced by $\lambda_i$ is just $E(\lambda_i) = \{e^{\lambda_i t}\}$ or $E(\lambda_i) = \{e^{\lambda_i t}, te^{\lambda_i t}\}$. According to Proposition 3.2 of Guo & Xu (2006), for any $T > 2\pi D^+(\Lambda)$, the family of GDD $\{E(\lambda_i), i \in \mathbb{Z}\}$ constitutes a Riesz basis in the closed subspace of $L^2(0, T)$ spanned by itself. This, together with $\text{Sp}(A_1) = V$ by Proposition A.3 deduces (i). Then (ii) is a consequence of (i).

**Proof** of Lemma 2.2 If $\ell \in Q_+ \setminus J_1$, we show that exponential stability is true by proving that the imaginary axis is not an asymptote of $\sigma(A_1)$. Let $\ell = \frac{m}{n}$ for $m, n \in \mathbb{N}, m, n \neq 0$ and define
\[
F(s) = (2 + k_1)s^{n+m} + k_1 s^{(m-n)} - k_1 s^{m-n} + (2 - k_1)s^{-(m+n)} = 0, \quad \forall \ s \in \mathbb{C}, \tag{A.47}
\]
where $k_1 > 0$. Then, the characteristic equation (A.6) is
\[
F(e^{\lambda \mathbf{i}}) = 0, \quad \forall \ \lambda \in \sigma(A_1). \tag{A.48}
\]
Suppose by contradiction that there exists a sequence \( \{ \lambda_j \}_{j=1}^{\infty} \subset \sigma(A_1) \) such that \( \text{Re} \lambda_j \to 0 \) as \( j \to \infty \). Let \( y_j = e^{\lambda_j/n} \). Then,
\[
F(y_j) = 0 \quad \text{and} \quad |y_j| \to 1 \quad \text{as} \quad j \to \infty.
\]
Hence, there exists a positive constant \( M \) such that \( |y_j| \leq M \) for all \( j \in \mathbb{N} \). Let \( y \) be an accumulation point of sequence \( y_j \). We may assume without loss of generality that \( y_j \to y \) as \( j \to \infty \). Owing to the continuity of \( F \), we have
\[
0 = \lim_{j \to \infty} F(y_j) = F(y) \quad \text{and} \quad |y| = 1.
\]
Now we solve \( y \). Since \( |y| = 1 \), it has \( y = |y| (\cos \theta + i \sin \theta) = \cos \theta + i \sin \theta \), where \( \theta = \text{Arg} y \) and \( \text{Arg} \) denotes the argument of \( y \).

Take it back to \( F(y) = 0 \) to yield
\[
(2 + k_1)(\cos \theta + i \sin \theta)^{m+n} + k_1(\cos \theta + i \sin \theta)^{-(m-n)} - k_1(\cos \theta + i \sin \theta)^{m-n} + (2 - k_1)(\cos \theta + i \sin \theta)^{-(m+n)} = 0.
\]
We deduce from De Moivre's formula that
\[
2 \cos(m + n)\theta + ik_1[\sin(m + n)\theta - \sin(m - n)\theta] = 0.
\]
Compare the two sides of (A.52) to obtain
\[
\begin{align*}
\cos(m + n)\theta &= 0, \\
\sin(m + n)\theta - \sin(m - n)\theta &= 0,
\end{align*}
\]
and hence
\[
\cos 2n\theta = 1 \quad \text{and} \quad \cos(m + n)\theta = 0.
\]
Solve (A.54) to get \( \ell = \frac{2m+1}{2n} - 1 \in J_1 \), \( \forall m, n \in \mathbb{Z}, n \neq 0 \).

This is a contradiction with the assumption on \( \ell \). Thus, the imaginary axis is not an asymptote of \( \sigma(A_1) \). Combining with the spectrum-determined growth condition (A.43), we conclude that system (2.7) is exponentially stable. \( \square \)