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Galerkin approximation for H^∞-control of the stable parabolic system under Dirichlet boundary control[☆]

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ABSTRACT

In this paper, we explore state feedback control for the H^∞ disturbance-attenuation problem in stable parabolic systems with in-domain distributed disturbances under Dirichlet boundary control. Calculating the state feedback control involves solving an operator algebraic Riccati equation, which poses challenges in finding an analytic solution. A practical approach is to seek an approximate solution via finite-dimensional approximation. Specifically, we employ the Galerkin approximation, which generates a sequence of finite-dimensional systems that approximate the original infinite-dimensional system. All corresponding finite-dimensional disturbance-attenuation problems are solvable, and it is demonstrated that the sequence of solutions to the associated finite-dimensional algebraic Riccati equations converges in norm to the solution of the infinite-dimensional operator algebraic Riccati equation. Furthermore, the state feedback controls derived from the finite-dimensional algebraic Riccati equations are proven to be γ-admissible controls for the original system.

1. Introduction

H^∞ control addresses the challenge of managing extensive disturbances in control systems. Since the 1980s, there has been a vast amount of literature on H^∞ control problems for finite-dimensional systems, as seen in works [1–4]. Additionally, infinite-dimensional systems with bounded control operators have been extensively studied in [5,6], and others. When dealing with bounded control operators, the transition from lumped to distributed parameter systems does not introduce significant complexities. However, the scenario becomes considerably more intricate when the control operator is unbounded. The unboundedness of the control operator typically reduces the regularity of the solution to the dynamic equation, adding another layer of complexity to the problem. The monograph [7] specifically addressed the H^∞-control problem for a class of systems with unbounded control operators, commonly referred to as Pritchard-Salamon systems. This class encompasses delay equations as well as the one-dimensional heat equation with Neumann boundary control, yet it excludes heat equations subject to Dirichlet boundary control. On the other hand, [8] focused on H^∞-boundary control under state feedback for hyperbolic-like systems. In the context of finite-dimensional systems [1–4], state feedback for H^∞-control can be derived by solving a matrix algebraic Riccati equation, which is a relatively straightforward task with the aid of modern mathematical tools. For infinite-dimensional systems [5–8], finding state feedback requires solving an infinite-dimensional operator algebraic Riccati equation. This task has rarely been undertaken in the literature. [9] found an analytic optimal H^∞-control directly for a very specific heat equation without solving the algebraic Riccati equation. However, this approach is not readily applicable to general infinite-dimensional systems. Previous works [10–12] explored the approximation method for operator Riccati equations where the control operators were bounded. However, the literature on addressing unbounded control operators in this context is limited. The approach introduced in [10] cannot be directly applied to boundary control problems due to its heavy reliance on the boundedness and compactness of the control operator. This limitation provided the impetus for our current research, as the control operator in this paper lacks both boundedness and compactness, posing a significant challenge.

In infinite-dimensional systems governed by partial differential equations (PDEs), unbounded control operators primarily originate from boundary control, which stands out as one of the most practical control strategies in PDEs. This paper delves into the H^∞ disturbance-attenuation

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problem for stable parabolic systems under Dirichlet boundary control and distributed disturbance. First, we revisit the min–max game theory for parabolic-like systems, as a standard approach to tackling the H^∞ disturbance-attenuation problem often involves associating the system with a min–max game problem. Numerous studies have examined min–max game problems for PDEs with unbounded control operators, such as parabolic-like PDEs discussed in [13, Chapter 6] and [14], as well as hyperbolic-like PDEs in [15]. We will incorporate some findings from min–max game problems for parabolic-like systems. A crucial aspect is the explicit representation of solutions to the associated game problem and the corresponding algebraic Riccati equation, as outlined in [13, part II of Chapter 6]. Next, we employ the Galerkin approximation method, which generates a sequence of finite-dimensional systems that approximate the original parabolic system, along with a sequence of finite-dimensional algebraic Riccati equations that approximate the operator algebraic Riccati equation with an unbounded control operator. Our approach differs significantly from that presented in [10]. We leverage the uniform analyticity of approximation operators discussed in [16] for the original free dynamics generator, as well as results on Galerkin approximation for open-loop problems with nonhomogeneous Dirichlet boundary conditions. It is demonstrated that solutions to algebraic Riccati equations linked with finite-dimensional systems converge, in terms of norm, to the solution of the operator algebraic Riccati equation associated with the original infinite-dimensional system. Simultaneously, state feedback controls for the finite-dimensional systems converge in norm to the state feedback control of the original infinite-dimensional system. Furthermore, finite-dimensional feedback controls, constructed using solutions of finite-dimensional algebraic Riccati equations, are proven to be γ-admissible for the original infinite-dimensional system.

We proceed as follows. In the next section, Section 2, the problem is formulated. In Section 3, we sum up some results about the min–max game theory. The Galerkin approximation scheme of the H^∞ control problem is introduced in Section 4.1 where the convergence analysis is carried out in Section 4.2. A numerical simulation is presented in Section 5, followed up by conclusions in Section 6.

2. Problem statement

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Let $A(\xi,\partial)$ be a uniformly strongly elliptic operator of order two in Ω that satisfies

$$-A(\xi,\partial) = \sum_{i,j=1}^{n} \frac{\partial}{\partial \xi_i}\left(a_{ij}(\xi)\frac{\partial}{\partial \xi_j}\right) + \sum_{j=1}^{n} b_j(\xi)\frac{\partial}{\partial \xi_j} + c,$$

where the coefficients $a_{ij}(\xi)$ and $b_j(\xi)$ are sufficiently smooth functions, c is a constant and the matrix $[a_{ij}]$ is symmetric. Furthermore, there is a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(\xi)\xi_i\xi_j \ge \alpha|\xi|^2, \ \forall \xi = (\xi_1,\xi_2,\dots,\xi_n) \in \mathbb{R}^n.$$

We consider a parabolic control system of the following:

$$\begin{cases} \dfrac{\partial z(\xi,t)}{\partial t} = -A(\xi,\partial)z(\xi,t) + d(\xi,t) & \text{in } \Omega\times(0,\infty], \\ z(\sigma,t) = u(\sigma,t) & \text{in } \Gamma\times(0,\infty], \\ z(\xi,0) = z_0(\xi), & \xi \in \Omega, \end{cases} \tag{2.1}$$

where $u \in L^2(0,\infty;L^2(\Gamma))$ is the boundary control, $d \in L^2(0,\infty;L^2(\Omega))$ is an in-domain disturbance and the initial value $z_0 \in L^2(\Omega)$. Suppose that $d = Gw$ where $G \in \mathcal{L}(V, L^2(\Omega))$, V is a Hilbert space and $w \in L^2(0,\infty;V)$. Let

$$\mathcal{X} = L^2(0,\infty;L^2(\Omega)),\ \mathcal{U} = L^2(0,\infty;L^2(\Gamma)),\ \mathcal{W} = L^2(0,\infty;V).$$

Let $A : \mathcal{D}(A)(\subset L^2(\Omega)) \to L^2(\Omega)$ be the operator $-A(\xi,\partial)$ with the homogeneous Dirichlet boundary condition. From [17, Theorem 2.7, p.211], A generates a strongly continuous semigroup e^{At} on $L^2(\Omega)$ and e^{At} is analytic in a triangular sector containing the positive real line. Throughout the paper, we assume that A is exponentially stable, i.e., there are constants $M_0, \omega_0 > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(L^2(\Omega))} \le M_0 e^{-\omega_0 t},\ \forall t \ge 0. \tag{2.2}$$

By [17, Theorem 6.13, p.74], a more general form than (2.2) holds:

$$\|(-A)^\rho e^{At}\|_{\mathcal{L}(L^2(\Omega))} \le \frac{C_\rho e^{-\omega_0 t}}{t^\rho},\ \forall t > 0,\ 0 \le \rho \le 1, \tag{2.3}$$

where the fractional power of $-A$ is defined in [17, p. 69] and C_ρ is a constant depending on ρ only. In addition, it follows from [18, p. 187] that

$$\mathcal{D}(A) = \mathcal{D}(A^*),\ \text{hence}\ \mathcal{D}((-A)^\theta) = \mathcal{D}((-A)^{*\theta}),\ 0 < \theta < 1.$$

The operator A can be extended isomorphically from $L^2(\Omega)$ to $[\mathcal{D}(A)]'$, which is still denoted by A. Let a Dirichlet map D be defined by

$$h = Dg \Leftrightarrow Ah = 0 \text{ in } \Omega,\ h|_\Gamma = g.$$

It is well known from the elliptic theory that [18, Theorem 7.3, p.187]

$$D \in \mathcal{L}(L^2(\Gamma), H^{1/2}(\Omega)).$$

Referring to [13, p. 181], we can reformulate (2.1) into the following abstract form:

$$\dot z(t) = Az(t) + Bu(t) + Gw(t) \text{ in } [\mathcal{D}(A)]',\ z(0) = z_0 \in L^2(\Omega), \tag{2.4}$$

where $B = -AD \in \mathcal{L}(L^2(\Gamma), [\mathcal{D}(A)]')$. By [19], $H^{1/2}(\Omega) \subset \mathcal{D}((-A)^{1/4-\beta}) = H^{1/2-2\beta}(\Omega)$ for any $0 < \beta < \frac{1}{4}$. Hence

$$(-A)^{-(3/4+\beta)}B = (-A)^{1/4-\beta}D \in \mathcal{L}(L^2(\Gamma), L^2(\Omega)). \tag{2.5}$$

Moreover, from [13, Remark 3.1.2, p.183], the following properties hold

$$B^* f = -\frac{\partial}{\partial \nu_{A^*}} f,\ \forall f \in \mathcal{D}(A) \text{ and } B^* f = -\frac{\partial}{\partial \nu} f \text{ whenever } -A(\xi, \partial) = \Delta, \tag{2.6}$$

$$\|B^* f\|_{L^2(\Gamma)} \le C \left\| \frac{\partial}{\partial \nu} f \right\|_{L^2(\Gamma)},\ \forall f \in \mathcal{D}(A), \tag{2.7}$$

where $C > 0$ is a constant independent of f and $\frac{\partial}{\partial \nu_{A^*}}$ is the co-normal derivative with respect to A^*.

Now, we introduce the class $\mathcal{F}$ of admissible feedback operators $F \in \mathcal{L}(\mathcal{D}(A), L^2(\Gamma))$ satisfying the following conditions: $F \in \mathcal{F}$ if and only if $A_F = A + BF$ generates an exponentially stable C_0-semigroup $e^{A_F t}$ on $L^2(\Omega)$; $\mathcal{D}(A_F) \subset \mathcal{D}(F)$; and F is an infinite-admissible observation operator for $e^{A_F t}$, meaning that $F e^{A_F \cdot} \in \mathcal{L}(L^2(\Omega), \mathcal{U})$. Let

$$R \in \mathcal{L}(L^2(\Omega), Y)$$

where Y is a Hilbert space. Define

$$(S_F w)(t) = \left(R \int_0^t e^{A_F (t-\tau)} G w(\tau) \mathrm{d}\tau,\ F \int_0^t e^{A_F (t-\tau)} G w(\tau) \mathrm{d}\tau \right).$$

By [20],

$$S_F \in \mathcal{L}(\mathcal{W}, L^2(0, \infty; Y \times L^2(\Gamma))),\ \forall F \in \mathcal{F}.$$

Definition 2.1. A γ-admissible state feedback for (2.4) is an operator $F \in \mathcal{F}$ such that $\|S_F\| < \gamma$. Here $\|S_F\|$ is the norm of the operator $S_F \in \mathcal{L}(\mathcal{W}, L^2(0, \infty; Y \times L^2(\Gamma)))$.

The present paper considers the problem of constructing a γ-admissible state feedback for (2.4). This is equivalent to construct a stabilizing feedback control $u(t) = F z(t), F \in \mathcal{F}$, with which there is a $\delta > 0$ independent of w such that the closed-loop solution $z(\cdot)$ of (2.4) with $z_0 = 0$ satisfies

$$\int_0^\infty (\|R z(t)\|_Y^2 + \|F z(t)\|_{L^2(\Gamma)}^2) \mathrm{d}t \le (\gamma^2 - \delta) \int_0^\infty \|w(t)\|_V^2 \mathrm{d}t,\ \forall w \in \mathcal{W}. \tag{2.8}$$

This is a standard H^∞ disturbance-attenuation problem with state feedback for the system (2.4). Moreover, we shall estimate the optimal disturbance attenuation $\hat{\gamma}$ defined by

$$\hat{\gamma} = \inf \gamma \tag{2.9}$$

over all $\gamma \ge 0$ such that there is a γ-admissible state feedback for (2.4).

Remark 2.1. Eq. (2.1) bears similarities to the one examined in [21,22], where linear quadratic regulator problems were discussed. We point out that the linear quadratic regulator problem may be regarded as a limiting case of H^∞ disturbance-attenuation problem when $\gamma \to \infty$. The paper [22] introduced a Galerkin approximation technique to tackle the regulator problem for the system (2.4) under the condition $w \equiv 0$. In contrast, the current paper delves into the Galerkin approximation of the H^∞ disturbance-attenuation problem for the same system (2.4). It will be observed that the findings of these two distinct problems exhibit numerous parallels.

3. Min–max game theory over an infinite time interval

The result presented in this section originates from [13, part II of Chapter 6], which holds significance for our discussions in the upcoming sections. Given a fixed value of $\gamma \ge 0$, define a cost functional:

$$J(u, w; z_0) = \int_0^\infty (\|R z(t)\|_Y^2 + \|u(t)\|_{L_2(\Gamma)}^2 - \gamma^2 \|w(t)\|_V^2) \mathrm{d}t \tag{3.1}$$

where $z(t) = z(t; z_0)$ is the mild solution of (2.4). A standard approach to handle the H^∞ disturbance-attenuation problem is to associate the system (2.4) with a game problem:

$$\sup_{w \in \mathcal{W}} \inf_{u \in \mathcal{U}} J(u, w; z_0) \tag{3.2}$$

where the infimum is taken over all $u \in \mathcal{U}$ for fixed w, and the supremum is taken over all $w \in \mathcal{W}$. Before giving related results, we need some preparations. Define

$$\begin{cases} (Lu)(t) = \displaystyle\int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau,\ \forall u \in \mathcal{U}, \\ (W w)(t) = \displaystyle\int_0^t e^{A(t-\tau)} G w(\tau) \mathrm{d}\tau,\ \forall w \in \mathcal{W}. \end{cases} \tag{3.3}$$

Then the mild solution of (2.4) can be given explicitly by

$$z(t; z_0) = e^{At} z_0 + (Lu)(t) + (W w)(t).$$

We have the following regularity properties of L and W.

Lemma 3.1. *Let the operators L and W be defined by (3.3). Then,*

$$L \in \mathcal{L}(\mathcal{U}, \mathcal{X}), \quad W \in \mathcal{L}(\mathcal{W}, \mathcal{X}). \tag{3.4}$$

Proof. The second assertion is a consequence of (2.2) and $G \in \mathcal{L}(V, L^2(\Omega))$ from Young's inequality. Here we only need to prove the first assertion. Notice that

$$\begin{aligned}
\|Lu\|_{\mathcal{X}} &\le \left(\int_0^\infty \left\| \int_0^t e^{A(t-\tau)}(-A)^{3/4+\beta}(-A)^{-(3/4+\beta)}Bu(\tau)\mathrm{d}\tau \right\|^2_{L^2(\Omega)} \mathrm{d}t \right)^{1/2} \\
&\le \int_0^\infty \|e^{At}(-A)^{3/4+\beta}\|_{\mathcal{L}(L^2(\Omega))}\mathrm{d}t \left(\int_0^\infty \|(-A)^{-(3/4+\beta)}Bu(t)\|^2_{L^2(\Omega)}\mathrm{d}t \right)^{1/2} \\
&\le C \int_0^\infty \frac{e^{-\omega_0 t}}{|t|^{3/4+\beta}}\mathrm{d}t \left(\int_0^\infty \|u(t)\|^2_{L^2(\Gamma)}\mathrm{d}t \right)^{1/2} \\
&\le C\|u\|_{\mathcal{U}},
\end{aligned}$$

where C is a constant independent of $u(\cdot)$, the second inequality was from Young's inequality and the third inequality was induced from (2.3) and (2.5). ■

We can therefore define L^* and W^* as the $L^2(0,\infty;\cdot)$-adjoint of L and W respectively. A straightforward computation shows that

$$(L^* f)(t) = B^* \int_t^\infty e^{A^*(\tau-t)} f(\tau)\mathrm{d}\tau, \ \forall f \in \mathcal{X}$$
$$(W^* f)(t) = G^* \int_t^\infty e^{A^*(\tau-t)} f(\tau)\mathrm{d}\tau, \ \forall f \in \mathcal{X}.$$

Now we briefly review some results on the game problem (3.2). We only give a brief proof for later usage. The details can be found in [13, part II of Chapter 6].

Theorem 3.1. *For the system (2.4) with assumptions (2.3) and (2.5), there is a critical value γ_c (determined in (3.13) later) such that:*

(a) If $\gamma_c > 0$ and $0 < \gamma < \gamma_c$, then

$$\sup_{w\in\mathcal{W}} \inf_{u\in\mathcal{U}} J(u, w; z_0) = +\infty, \ \forall z_0 \in L^2(\Omega).$$

(b) If $\gamma > \gamma_c$, then there exists a unique solution $\{w^*, u^*, z^*\}$ of the game problem (3.2) with explicit expressions:

$$\begin{cases} w^*(\cdot; z_0) = E_\gamma^{-1} W^* R^* [I + RLL^*R^*]^{-1} R e^{A\cdot} z_0, \\ u^*(\cdot; z_0) = -[I + L^*R^*RL]^{-1} L^*R^*R[e^{A\cdot} z_0 + Ww^*(\cdot; z_0)], \\ z^*(\cdot; z_0) = e^{A\cdot} z_0 + Lu^*(\cdot; z_0) + Ww^*(\cdot; z_0), \end{cases} \tag{3.5}$$

where $E_\gamma \in \mathcal{L}(\mathcal{W})$ is defined in (3.9) later and there is an operator $P = P^* \ge 0 \in \mathcal{L}(L^2(\Omega))$ satisfying the following algebraic Riccati equation, that is, for any $f, g \in \mathcal{D}((-A)^\varepsilon)$ with $\varepsilon > 0$,

$$(Pf, Ag)_{L^2(\Omega)} + (Af, Pg)_{L^2(\Omega)} + (Rf, Rg)_Y = (B^*Pf, B^*Pg)_{L^2(\Gamma)} - \gamma^2 (G^*Pf, G^*Pg)_V; \tag{3.6}$$

$A_{P,\gamma} = A - BB^*P + \gamma^{-2}GG^*P$ with $\mathcal{D}(A_{P,\gamma}) \subset \mathcal{D}((-A)^{1/4-\beta})$ generates an analytic exponentially stable C_0-semigroup $\Phi(t)$ on $L^2(\Omega)$:

$$\|\Phi(t)\|_{\mathcal{L}(L^2(\Omega))} \le M_1 e^{-\omega_1 t}, \tag{3.7}$$

for some constants $M_1, \omega_1 > 0$; moreover,

$$u^*(t; z_0) = -B^*Pz^*(t; z_0), \ B^*P \in \mathcal{L}(L^2(\Omega), L^2(\Gamma)), \ z^*(t, z_0) = \Phi(t)z_0$$

and $A_P = A - BB^*P$ generates an analytic exponentially stable C_0-semigroup $e^{A_P t}$ on $L^2(\Omega)$:

$$\|e^{A_P t}\|_{\mathcal{L}(L^2(\Omega))} \le M_2 e^{-\omega_2 t},$$

for some constants $M_2, \omega_2 > 0$.

Proof. We split the proof into three steps.

Step 1. First, for a given $w \in \mathcal{W}$, consider the following minimization problem:

$$\inf_{u\in\mathcal{U}} J(u, w; z_0).$$

It is obvious that the optimal problem is a standard quadratic problem in u for fixed $w \in \mathcal{W}$. Applying the Lagrange multiplier as in the proof of [13, Theorem 6.20.1.1, p.613], we can get

$$\inf_{u\in\mathcal{U}} J(u, w; z_0) = J(-L^*R^*Rz, w; z_0). \tag{3.8}$$

Denote $J(-L^*R^*Rz, w; z_0)$ by $J^0_w(z_0)$ and

$$E_\gamma = \gamma^2 I - S \in \mathcal{L}(\mathcal{W}), \tag{3.9}$$

where S is defined by

$$S = W^*R^*[I + RLL^*R^*]^{-1} RW \in \mathcal{L}(\mathcal{W}). \tag{3.10}$$

A simple calculation shows that

$$J_w^0(z_0) = J_{w=0}^0(z_0) + J_w^0(z_0 = 0) + \chi_{z_0,w}, \tag{3.11}$$

where

$$J_{w=0}^0(z_0) = (Re^{A\cdot}z_0, [I + RLL^*R^*]^{-1}Re^{A\cdot}z_0)_{L^2(0,\infty;Y)},$$
$$J_w^0(z_0 = 0) = -(w, E_\gamma w)_{\mathcal{W}},$$
$$\chi_{z_0,w} = 2(Re^{A\cdot}z_0, [I + RLL^*R^*]^{-1}RWw)_{L^2(0,\infty;Y)}.$$

Step 2. Next, we consider the following maximization problem:

$$\sup_{w\in\mathcal{W}} J_w^0(z_0).$$

which is equivalent to the minimization problem

$$\inf_{w\in\mathcal{W}} -J_w^0(z_0). \tag{3.12}$$

We know from (3.10) that S is self-adjoint and $S \geq 0$. Define the critical value γ_c of γ by:

$$\gamma_c^2 = \|S\|_{\mathcal{L}(\mathcal{W})} = \sup_{\|w\|_{\mathcal{W}}=1} (W^*R^*[I + RLL^*R^*]^{-1}RWw, w)_{\mathcal{W}}. \tag{3.13}$$

(3.11) tells us that $-J_w^0(z_0)$ consists of three terms: a quadratic term in w, given by $-J_w^0(z_0 = 0) = (w, E_\gamma w)$ which, by (3.13), satisfies

$$(E_\gamma w, w)_{\mathcal{W}} \geq (\gamma^2 - \gamma_c^2)\|w\|_{\mathcal{W}}^2; \tag{3.14}$$

a linear term in w, given by $-\chi_{z_0,w}$ satisfying

$$\begin{aligned}|\chi_{z_0,w}| &\leq 2\|w\|_{\mathcal{W}}\|Qz_0\|_{\mathcal{W}}\\ &\leq \varepsilon\|w\|_{\mathcal{W}}^2 + \frac{1}{\varepsilon}\|Q\|_{\mathcal{L}(L^2(\Omega),\mathcal{W})}^2\|z_0\|_{L^2(\Omega)}^2, \qquad \forall\varepsilon > 0,\end{aligned}$$

with $Q = W^*R^*[I + RLL^*R^*]^{-1}Re^{A\cdot} \in \mathcal{L}(L^2(\Omega), \mathcal{W})$; and finally a constant term in w, given by $-J_{w=0}^0(z_0)$. Thus,

$$-J_w^0(z_0) \geq [\gamma^2 - (\gamma_c^2 + \varepsilon)]\|w\|_{\mathcal{W}}^2 - J_{w=0}^0(z_0) - \frac{1}{\varepsilon}\|Q\|_{\mathcal{L}(L^2(\Omega),\mathcal{W})}^2\|z_0\|_{L^2(\Omega)}^2.$$

Therefore for $\gamma > \gamma_c$, $-J_w^0(z_0)$ is a quadratic functional with respect to w and is bounded from below. As a result, for the minimization problem (3.12), there exists a unique solution denoted by $w^*(\cdot; z_0) \in \mathcal{W}$, i.e.,

$$\sup_{w\in\mathcal{W}} J_w^0(z_0) = -\inf_{w\in\mathcal{W}} -J_w^0(z_0) = J_{w^*}^0(z_0).$$

If $\gamma_c > 0$ and $\gamma < \gamma_c$, then it follows from

$$\inf_{\|w\|_{\mathcal{W}}=1}(w, E_\gamma w) \leq \gamma^2 - \gamma_c^2 < 0,$$

that

$$\sup_{w\in\mathcal{W}} J_w^0(z_0) = +\infty,\ \forall z_0 \in L^2(\Omega).$$

Step 3. We omit the remaining proof but it needs to be pointed out that after getting the explicit of $z^*(t; z_0)$, there holds

$$z^*(t; z_0) = \Phi(t)z_0,\ \forall z_0 \in L^2(\Omega),$$

where $\Phi(t)$ is an analytic and exponentially stable C_0-semigroup. Define $P \in \mathcal{L}(L^2(\Omega))$ by:

$$Pf = \int_0^\infty e^{A^*t}R^*Rz^*(t; f)\mathrm{d}t = \int_0^\infty e^{A^*t}R^*R\Phi(t)\mathrm{d}t f. \tag{3.15}$$

Then $A_{P,\gamma} = A - BB^*P + \gamma^{-2}GG^*P$ generates the same semigroup $\Phi(t)$ on $L^2(\Omega)$ and the operator $P \geq 0$ satisfies the algebraic Riccati Eq. (3.6). ■

While the solution to Eq. (3.6) is typically not unique, the following Theorem 3.2 establishes that, under certain conditions, a unique solution to (3.6) does exist.

Theorem 3.2. *If one operator $0 \leq P \in \mathcal{L}(L^2(\Omega))$ satisfies (3.6) such that $B^*P \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$ and $A_{P,\gamma} = A - BB^*P + \gamma^{-2}GG^*P$ is exponentially stable, then P is unique in $\mathcal{L}(L^2(\Omega))$ and is given by (3.15).*

Proof. Suppose that P_1 and P_2 are two solutions satisfying the conditions in the theorem. Then, for any $f, g \in \mathcal{D}((-A)^\varepsilon)$ with $\varepsilon > 0$, one has

$$(A_{P_1,\gamma}f, (P_1 - P_2)g)_{L^2(\Omega)} + ((P_1 - P_2)f, A_{P_2,\gamma}g)_{L^2(\Omega)} = 0,$$

which leads to

$$\frac{d}{dt}(e^{A_{P_1,\gamma}t}f, (P_1 - P_2)e^{A_{P_2,\gamma}t}g)_{L^2(\Omega)} = 0,\ \forall t \geq 0.$$

Since $e^{A_{P_1,\gamma}t}$ and $e^{A_{P_2,\gamma}t}$ are exponentially stable, it has

$$(f, (P_1 - P_2)g)_{L^2(\Omega)} = 0,\ \forall f, g \in \mathcal{D}((-A)^\varepsilon), \forall\varepsilon > 0.$$

By density argument, $P_1 = P_2$. Moreover, from Theorem 3.1, we know that the $P \geq 0$ given by (3.15) satisfies (3.6), $B^*P \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$ and $A_{P,\gamma} = A - BB^*P + \gamma^{-2}GG^*P$ is exponentially stable. ■

Pursuant to [13, Theorem 6.19.2, p. 612], the parameter γ_c also represents a pivotal threshold for the algebraic Riccati Eq. (3.6). Precisely, if $\gamma < \gamma_c$, there is no solution to (3.6) in the sense outlined in Theorem 3.2. Conversely, when $\gamma > \gamma_c$, a unique solution emerges, adhering to the conditions stipulated in Theorem 3.2. To conclude this section, we extend a result from the realm of finite-dimensional systems, as demonstrated in [2, p. 150], indicating that the critical value γ_c precisely corresponds to the optimal disturbance attenuation $\hat{\gamma}$.

Theorem 3.3. *The γ_c defined by (3.13) is equal to $\hat{\gamma}$ defined by (2.9):*

$$\hat{\gamma} = \gamma_c.$$

Proof. The proof consists of two steps. First, we show that $\hat{\gamma} \le \gamma_c$. From Theorems 3.2 and 3.3, for any $\gamma > \gamma_c$, there exists a unique operator $P \in \mathcal{L}(L^2(\Omega))$ satisfying (3.6) for all $f, g \in \mathcal{D}((-A)^\varepsilon)$ with $\varepsilon > 0$ such that $B^*P \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$ and $e^{A_P t}$ is analytic and exponentially stable. Since $A_P = A - BB^*P = A_{P,\gamma} - \gamma^{-2}GG^*P$, we have $\mathcal{D}(A_P) = \mathcal{D}(A_{P,\gamma}) \subset \mathcal{D}((-A)^{1/4-\beta})$. As a result, $-B^*P \in \mathcal{F}$. Let $w \in C_0^1(0,\infty;V) \subset \mathcal{W}$, $u(t) = -B^*Pz(t)$ and $z_0 \in \mathcal{D}(A_P)$ in (2.4). Then,

$$\dot{z}(t) = (A - BB^*P)z(t) + Gw(t), \;\; z(0) = z_0 \in \mathcal{D}(A_P). \tag{3.16}$$

By [23, Theorem 4.1.6], $z \in C([0,\infty); \mathcal{D}(A_P)) \cap C^1([0,\infty); L^2(\Omega))$ and hence

$$\begin{aligned} \frac{d}{dt}(Pz(t), z(t))_{L^2(\Omega)} &= -\|Rz(t)\|_Y^2 - \|B^*Pz(t)\|_{L^2(\Gamma)}^2 - \gamma^{-2}\|G^*Pz(t)\|_V^2 + 2(G^*Pz(t), w(t))_V \\ &= -\|Rz(t)\|_Y^2 - \|B^*Pz(t)\|_{L^2(\Gamma)}^2 - \|\gamma w(t) - \gamma^{-1}G^*Pz(t)\|_V^2 + \gamma^2\|w(t)\|_V^2. \end{aligned} \tag{3.17}$$

Since $e^{A_P t}$ is exponentially stable, integrating the last equality of (3.17) from 0 to ∞ gives

$$\int_0^\infty (\|Rz(t)\|_Y^2 + \|B^*Pz(t)\|_{L^2(\Gamma)}^2 - \gamma^2\|w(t)\|_V^2)\mathrm{d}t = -\gamma^2\int_0^\infty \|\bar{w}(t)\|_V^2\mathrm{d}t + (Pz_0, z_0)_{L^2(\Omega)}, \tag{3.18}$$

where $\bar{w} = w - \gamma^{-2}G^*Pz$. On the other hand, if $z_0 = 0$, (3.16) is equivalent to

$$\dot{z}(t) = A_{P,\gamma}z(t) + G\bar{w}(t), \;\; z(0) = 0$$

which has solution

$$z(t) = \int_0^t \Phi(t-\tau)G\bar{w}(\tau)\mathrm{d}\tau.$$

By Young's inequality,

$$\int_0^\infty \|z\|_{L^2(\Omega)}^2\mathrm{d}t \le \frac{M_1^2\|G\|_{\mathcal{L}(V,L^2(\Omega))}^2}{\omega_1^2}\int_0^\infty \|\bar{w}(t)\|_V^2\mathrm{d}t.$$

Since $w = \bar{w} + \gamma^{-2}G^*Pz$, we obtain

$$\int_0^\infty \|w(t)\|_V^2\mathrm{d}t \le 2\left(1 + \gamma^{-4}\|G^*P\|_{\mathcal{L}(L^2(\Omega),V)}^2\frac{M_1^2\|G\|_{\mathcal{L}(V,L^2(\Omega))}^2}{\omega_1^2}\right)\int_0^\infty \|\bar{w}(t)\|_V^2\mathrm{d}t. \tag{3.19}$$

Hence for $z_0 = 0$ and any $w \in C_0^1(0,\infty)$, combining (3.18) and (3.19) gives

$$\int_0^\infty (\|Rz(t)\|_Y^2 + \|B^*Pz(t)\|_{L^2(\Gamma)}^2 - \gamma^2\|w(t)\|_V^2)\mathrm{d}t \le -\delta\int_0^\infty \|w(t)\|_V^2\mathrm{d}t, \tag{3.20}$$

where $\delta = \frac{1}{2}\gamma^2\left(1 + \gamma^{-4}\|G^*P\|_{\mathcal{L}(L^2(\Omega),V)}^2\frac{M_1^2\|G\|_{\mathcal{L}(V,L^2(\Omega))}^2}{\omega_1^2}\right)^{-1}$ is independent of w. Since $C_0^1(0,\infty;V)$ is dense in $\mathcal{W}$, (3.20) holds for all $w \in \mathcal{W}$. This means that $\hat{\gamma} < \gamma$ for any $\gamma > \gamma_c$. Therefore, $\hat{\gamma} \le \gamma_c$.

Next, we show $\hat{\gamma} \ge \gamma_c$. If $\gamma_c = 0$, then $\hat{\gamma} = \gamma_c = 0$. For $\gamma_c > 0$, we assume $\hat{\gamma} < \gamma_c$. Choose $\gamma > 0$ so that $\hat{\gamma} < \gamma < \gamma_c$. From the definition of $\hat{\gamma}$, there exist an operator $F \in \mathcal{F}$ and a constant $\delta > 0$ such that

$$\int_0^\infty (\|Rz(t)\|_Y^2 + \|Fz(t)\|_{L^2(\Gamma)}^2)\mathrm{d}t \le (\gamma^2 - \delta)\int_0^\infty \|w(t)\|_V^2\mathrm{d}t,$$

where $z(\cdot)$ is the solution of (2.4) with $u(t) = Fz(t)$ and $z_0 = 0$, i.e.,

$$J(Fz, w; 0) \le -\delta\int_0^\infty \|w(t)\|_V^2\mathrm{d}t.$$

From (3.8), it follows that

$$J_w^0(z_0 = 0) = J(-L^*R^*Rz, w; 0) \le J(Fz, w; 0), \forall w \in \mathcal{W}.$$

Hence,

$$\sup_{\|w\|_{\mathcal{W}}=1} J_w^0(z_0 = 0) = \sup_{\|w\|_{\mathcal{W}}=1} -(w, (\gamma^2 I - S)w)_{\mathcal{W}} = \gamma_c^2 - \gamma^2 \le -\delta.$$

Therefore $\gamma \ge \gamma_c$ which contradicts $\gamma < \gamma_c$. This proves $\hat{\gamma} \ge \gamma_c$. ■

4. Approximation and convergence analysis

In this section, we delve into the Galerkin approximation of the H^∞ disturbance-attenuation problem for the system described by Eq. (2.4). Initially, in Section 4.1, we introduce approximating subspaces and operators. These elements form the foundation for constructing the approximating dynamics and game problems. We rigorously confirm that these approximating equations adhere to the prerequisites of Theorem 3.1. As a result, the pertinent conclusions from the game theory are applicable.

Proceeding to Section 4.2, we undertake a comprehensive convergence analysis. This analysis encompasses the computation of the optimal disturbance attenuation $\hat{\gamma}$. Furthermore, we demonstrate that the solutions to the corresponding finite-dimensional algebraic Riccati equations converge, in terms of norm, towards the solution of the operator algebraic Riccati equation. Importantly, we establish that the finite-dimensional feedback controls, derived from the solutions of the finite-dimensional algebraic Riccati equations, exhibit γ-admissibility for the original infinite-dimensional system. This finding provides a feasible and computable approach to seeking the H^∞ state feedback control.

4.1. Approximating subspaces and operators

Firstly, we introduce approximating subspaces. Let $h \to 0$ be monotonically decreasing and $0 < h \le h_0$ for some constant $h_0 > 0$. For every h, let $X_h \subset H^2(\Omega) \cap H_0^1(\Omega) = \mathcal{D}(A)$ be a finite-dimensional subspace which is equipped with the induced norm of $L^2(\Omega)$ and let Π_h be the orthogonal projection of $L^2(\Omega)$ onto X_h. We assume that X_h possesses the following approximation properties:

$$\begin{aligned}&(i)\ \|x-\Pi_h x\|_{H^\alpha(\Omega)} \le Ch^{s-\alpha}\|x\|_{H^s(\Omega)},\ 0\le\alpha\le 1,\ 0\le s-\alpha,\ s\le 2,\\ &\ \text{where } x\in H_0^s(\Omega),\ 0\le s\le 1, s\ne \tfrac{1}{2};\ x\in H^s(\Omega)\cap H_0^1(\Omega),\ 1\le s\le 2;\end{aligned} \tag{4.1}$$

$$\begin{aligned}&(ii)\ \text{(inverse approximation property)}\\ &\left\|\tfrac{\partial}{\partial\nu}x_h\right\|_{L^2(\Gamma)} \le Ch^{-3/2}\|x_h\|_{L^2(\Omega)},\ x_h\in X_h;\end{aligned} \tag{4.2}$$

$$(iii)\ \left\|\frac{\partial}{\partial\nu}(x-\Pi_h x)\right\|_{L^2(\Gamma)} \le Ch^{s-3/2}\|x\|_{H^s(\Omega)},\ x\in H^s(\Omega)\cap H_0^1(\Omega),\ \frac{3}{2}<s\le 2; \tag{4.3}$$

$$(iv)\ \|x_h\|_{H^\alpha(\Omega)} \le Ch^{-\alpha}\|x_h\|_{L^2(\Omega)},\ 0\le\alpha\le 1,\ x_h\in X_h, \tag{4.4}$$

where and hereafter C denotes constant independent of h but may possibly depend on s, α, etc, although they may have different values in different contexts.

Remark 4.1. Here we choose the same class of approximating subspaces as that in [22]. The approximation properties (4.1)–(4.4) are satisfied by linear splines defined over a quasi-uniform mesh and vanishing on the boundary Γ.

Next, we introduce approximating operators. Let $A_h : X_h \to X_h$ be the approximation of A, which is given by

$$(A_h f_h, g_h)_{L^2(\Omega)} = (A f_h, g_h)_{L^2(\Omega)},\ f_h, g_h \in X_h.$$

By [16], A_h generates a uniformly exponentially stable analytic semigroup $e^{A_h t}$ on X_h, i.e.,

$$\|A_h^\rho e^{A_h t}\|_{\mathcal{L}(L^2(\Omega))} \le \frac{Ce^{-\omega_0 t}}{t^\rho},\ \forall t>0,\ 0\le\rho\le 1; \tag{4.5}$$

and the following estimate holds

$$\|e^{A_h t}\Pi_h - e^{At}\|_{\mathcal{L}(L^2(\Omega))} = \|e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t}\|_{\mathcal{L}(L^2(\Omega))} \le \frac{Ch^\alpha e^{-\omega_0 t}}{t^{\alpha/2}},\ \forall t>0,\ 0\le\alpha<2. \tag{4.6}$$

Let $B_h = \Pi_h B : L^2(\Gamma) \to X_h$ be the approximation of B. In fact, from $X_h \subset \mathcal{D}(A) \subset \mathcal{D}(A^{3/4+\beta}) \subset \mathcal{D}(B^*)$, we know that $B^*\Pi_h \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$ is well defined. Hence $B_h = \Pi_h B \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$ can be defined to be the adjoint of $B^*\Pi_h$, i.e.,

$$(B_h u, f)_{L^2(\Omega)} = (u, B^*\Pi_h f)_{L^2(\Gamma)} = -\left(u, \frac{\partial}{\partial\nu_{A^*}}\Pi_h f\right)_{L^2(\Gamma)},\ \forall u\in L^2(\Gamma), f\in L^2(\Omega). \tag{4.7}$$

It is readily seen that $B_h u \in X_h$ for any $u \in L^2(\Gamma)$. Hence $B_h = \Pi_h B \in \mathcal{L}(L^2(\Gamma), X_h)$ is well defined.

Furthermore, let $G_h = \Pi_h G \in \mathcal{L}(V, X_h)$ be the approximation of G and let $R_h = R\big|_{X_h} \in \mathcal{L}(X_h, Y)$ be the approximation of R. Then, $G_h^* = G^*\big|_{X_h} \in \mathcal{L}(X_h, V)$ and $R_h^* = \Pi_h R^* \in \mathcal{L}(Y, X_h)$.

From above preliminaries, an approximation of the system (2.4) can be defined by:

$$\begin{cases}\dot{z}_h(t) = A_h z_h(t) + B_h u(t) + G_h w(t) \text{ in } X_h,\\ z_h(0) = \Pi_h z_0,\end{cases} \tag{4.8}$$

i.e.,

$$\begin{cases}\dfrac{d}{dt}(z_h(t), f_h)_{L^2(\Omega)} = (A z_h(t), f_h)_{L^2(\Omega)} - \left(u(t), \dfrac{\partial}{\partial\nu_{A^*}} f_h\right)_{L^2(\Gamma)} + (w(t), G_h^* f_h)_V,\ \forall f_h\in X_h,\\ z_h(0) = \Pi_h z_0,\end{cases}$$

where $z_h(t) \in X_h$ is the approximation of $z(t)$. Finally, the approximating game problem on X_h corresponding to (3.2) now reads

$$\sup_{w\in\mathcal{W}}\inf_{u\in\mathcal{U}} J_h(u, w; \Pi_h z_0) = \sup_{w\in\mathcal{W}}\inf_{u\in\mathcal{U}} \int_0^\infty (\|R_h z_h(t)\|_Y^2 + \|u(t)\|_{L^2(\Gamma)}^2 - \gamma^2\|w(t)\|_V^2)\mathrm{d}t. \tag{4.9}$$

Likewise, the solution of (4.8) has the following explicit representation:

$$z_h(t) = e^{A_h t} z_h(0) + (L_h u)(t) + (W_h w)(t),$$

where L_h and W_h are defined by

$$(L_h u)(t) = \int_0^t e^{A_h(t-\tau)} B_h u(\tau)\mathrm{d}\tau, \ \forall u \in \mathcal{U},$$
$$(W_h w)(t) = \int_0^t e^{A_h(t-\tau)} G_h w(\tau)\mathrm{d}\tau, \ \forall w \in \mathcal{W}.$$

Since A_h is stable and B_h, G_h are bounded, it has $L_h \in \mathcal{L}(\mathcal{U}, L^2(0,\infty; X_h))$ and $W_h \in \mathcal{L}(\mathcal{W}, L^2(0,\infty; X_h))$. Let $L_h^* \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ be the adjoint of L_h and $W_h^* \in \mathcal{L}(\mathcal{X}, \mathcal{W})$ be the adjoint of W_h satisfying

$$(L_h u, f)_{\mathcal{X}} = (u, L_h^* f)_{\mathcal{U}}, \ (W_h w, f)_{\mathcal{X}} = (w, W_h^* f)_{\mathcal{W}}, \forall u \in \mathcal{U}, w \in \mathcal{W}, f \in \mathcal{X}.$$

A straightforward calculation shows that for $\forall f \in \mathcal{X}$,

$$(L_h^* f)(t) = \int_t^\infty B_h^* e^{A_h^*(\tau-t)} \Pi_h f(\tau)\mathrm{d}\tau, \ (W_h^* f)(t) = \int_t^\infty G_h^* e^{A_h^*(\tau-t)} \Pi_h f(\tau)\mathrm{d}\tau.$$

Define

$$S_h = W_h^* R_h^* [I + R_h L_h L_h^* R_h^*]^{-1} R_h W_h,$$
$$\gamma_{h,c}^2 = \|S_h\|_{\mathcal{L}(\mathcal{W})},$$

and let $E_{h,\gamma} = \gamma^2 - S_h$. By Theorem 3.1, for $\gamma > \gamma_{h,c}$, there exists a unique solution $\{w_h^*, u_h^*, z_h^*\}$ to the game problem (4.9), which can be represented as

$$\begin{cases} w_h^*(\cdot; \Pi_h z_0) = E_{h,\gamma}^{-1} W_h^* R_h^* [I + R_h L_h L_h^* R_h^*]^{-1} R_h e^{A_h \cdot} \Pi_h z_0, \\ u_h^*(\cdot; \Pi_h z_0) = -[I + L_h^* R_h^* R_h L_h]^{-1} L_h^* R_h^* R_h [e^{A_h \cdot} \Pi_h z_0 + W_h w_h^*(\cdot; \Pi_h z_0)], \\ z_h^*(\cdot; \Pi_h z_0) = e^{A_h \cdot} \Pi_h z_0 + L_h u_h^*(\cdot; \Pi_h z_0) + W_h w_h^*(\cdot; \Pi_h z_0). \end{cases} \tag{4.10}$$

In addition, there exists an operator $0 \le P_h = P_h^* \in \mathcal{L}(X_h)$ satisfying the following algebraic Riccati equation:

$$\begin{aligned} &(P_h f_h, A_h g_h)_{L^2(\Omega)} + (A_h f_h, P_h g_h)_{L^2(\Omega)} + (R_h f_h, R_h g_h)_Y \\ &= (B_h^* P_h f_h, B_h^* P_h g_h)_{L^2(\Gamma)} - \gamma^{-2} (G_h^* P_h f_h, G_h^* P_h g_h)_V, \ \forall f_h, g_h \in X_h, \end{aligned} \tag{4.11}$$

which is given by

$$P_h f_h = \int_0^\infty e^{A_h^* t} R_h^* R_h z_h^*(t; f_h)\mathrm{d}t = \int_0^\infty e^{A_h^* t} R_h^* R_h \Phi_h(t)\mathrm{d}t f_h, \ \forall f_h \in X_h, \tag{4.12}$$

where $A_{h,P_h,\gamma} t = A_h - B_h B_h^* P_h + \gamma^{-2} G_h G_h^* P_h$ generates the analytic exponentially stable C_0-semigroup $\Phi_h(t)$.

4.2. Convergence analysis

Theorem 4.1 following is a special case of [24, theorem 3.3]. Because the proof of Theorem 4.5 later needs a conclusion in the proof of Theorem 4.1, we give a detailed proof.

Theorem 4.1. *The following estimates hold true:*

$$(i) \ \|L_h - L\|_{\mathcal{L}(\mathcal{U},\mathcal{X})} = \|L_h^* - L^*\|_{\mathcal{L}(\mathcal{X},\mathcal{U})} \le C h^\theta, \ \forall \theta < \frac{1}{2}; \tag{4.13}$$

$$(ii) \ \|W_h - W\|_{\mathcal{L}(\mathcal{W},\mathcal{X})} = \|W_h^* - W^*\|_{\mathcal{L}(\mathcal{X},\mathcal{W})} \le C h^\alpha, \ \forall \alpha < 2. \tag{4.14}$$

Proof. For $f \in L^2(\Omega)$, one has

$$B^*(e^{A_h^* t} \Pi_h - e^{A^* t}) f = B^*(e^{A_h^* t} \Pi_h - \Pi_h e^{A^* t}) f + B^*(\Pi_h - I) e^{A^* t} f.$$

Since $e^{A^* t} f \in D(A^*) = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ for all $t > 0$, it follows from (2.7), (4.2),(4.3), (4.5) and (4.6) that for any $\frac{3}{2} < s \le 2, 0 \le \alpha < 2$,

$$\begin{aligned} \|B^*(e^{A_h^* t} \Pi_h - e^{A^* t}) f\|_{L^2(\Gamma)} &\le C \left(\left\| \frac{\partial}{\partial \nu} [e^{A_h^* t} \Pi_h - \Pi_h e^{A^* t}] f \right\|_{L^2(\Gamma)} + \left\| \frac{\partial}{\partial \nu} (\Pi_h - I) e^{A^* t} f \right\|_{L^2(\Gamma)} \right) \\ &\le C (h^{-3/2} \|[e^{A_h^* t} \Pi_h - \Pi_h e^{A^* t}] f\|_{L^2(\Omega)} + h^{s-3/2} \|e^{A^* t} f\|_{H^s(\Omega)}) \\ &\le C \left(h^{-3/2} \frac{h^\alpha e^{-\omega_0 t}}{t^{\alpha/2}} \|f\|_{L^2(\Omega)} + h^{s-3/2} \|(-A^*)^{s/2} e^{A^* t} f\|_{L^2(\Omega)} \right) \\ &\le C \left(\frac{h^{\alpha-3/2} e^{-\omega_0 t}}{t^{\alpha/2}} + \frac{h^{s-3/2} e^{-\omega_0 t}}{t^{s/2}} \right) \|f\|_{L^2(\Omega)}. \end{aligned}$$

Choosing $\frac{3}{2} < \alpha = s < 2$ gives

$$\|B^*(e^{A_h^* t} \Pi_h - e^{A^* t})\|_{\mathcal{L}(L^2(\Omega), L^2(\Gamma))} \le C h^{\alpha-3/2} \frac{e^{-\omega_0 t}}{t^{\alpha/2}}. \tag{4.15}$$

Thus, for any $f \in \mathcal{X}$ and $\frac{3}{2} < \alpha < 2$, we have

$$
\begin{aligned}
\|(L_h^* - L^*)f\|_{\mathcal{U}}^2 &= \int_0^\infty \left\| \int_t^\infty (B_h^* e^{A_h^*(\tau-t)}\Pi_h - B^* e^{A^*(\tau-t)}) f(\tau)\mathrm{d}\tau \right\|_{L^2(\Gamma)}^2 \mathrm{d}t \\
&\le Ch^{2\alpha-3} \int_0^\infty \left(\int_t^\infty \frac{e^{-\omega_0(\tau-t)}}{(\tau-t)^{\alpha/2}} \|f(\tau)\|_{L^2(\Omega)} \mathrm{d}\tau \right)^2 \mathrm{d}t \\
&\le Ch^{2\alpha-3} \int_0^\infty \left(\int_t^\infty \frac{e^{-\omega_0(\tau-t)}}{(\tau-t)^{\alpha/2}} \mathrm{d}\tau \right) \left(\int_t^\infty \frac{e^{-\omega_0(\tau-t)}}{(\tau-t)^{\alpha/2}} \|f(\tau)\|_{L^2(\Omega)}^2 \mathrm{d}\tau \right) \mathrm{d}t \\
&\le Ch^{2\alpha-3} \int_0^\infty \int_0^\tau \frac{e^{-\omega_0(\tau-t)}}{(\tau-t)^{\alpha/2}} \|f(\tau)\|_{L^2(\Omega)}^2 \mathrm{d}t\mathrm{d}\tau \\
&\le Ch^{2\alpha-3} \|f\|_{\mathcal{X}}^2.
\end{aligned}
$$

This gives (4.13) with $\theta = \alpha - \frac{3}{2} < \frac{1}{2}$. Next, since $G \in \mathcal{L}(V, L^2(\Omega))$, it follows from (4.6) that

$$
\|G^*(e^{A_h^* t}\Pi_h - e^{A^* t})\|_{\mathcal{L}(L^2(\Omega),V)} \le Ch^\alpha \frac{e^{-\omega_0 t}}{t^{\alpha/2}}, \ \forall \alpha < 2,
$$

which gives (4.14):

$$
\|W_h^* - W^*\|_{\mathcal{L}(\mathcal{X},\mathcal{W})} \le Ch^\alpha, \ \forall \alpha < 2. \quad \blacksquare
$$

Theorem 4.2. *For any $\theta < \frac{1}{2}$,*

$$
\|S_h - S\|_{\mathcal{L}(\mathcal{W})} \le Ch^\theta, \tag{4.16}
$$

which implies that

$$
|\gamma_{h,c} - \gamma_c| \le Ch^\theta \to 0. \tag{4.17}
$$

Proof. Since $R_h = R\big|_{X_h}$ and $R_h^* = \Pi_h R^*$, it follows that

$$
\begin{aligned}
S_h - S &= W_h^* R^*[I + RL_h L_h^* R^*]^{-1} RW_h - W^* R^*[I + RLL^* R^*]^{-1} RW \\
&= W_h^* R^*[I + RL_h L_h^* R^*]^{-1}(RW_h - RW) + W_h^* R^*([I + RL_h L_h^* R^*]^{-1} - [I + RLL^* R^*]^{-1})RW \\
&\quad + (W_h^* R^* - W^* R^*)[I + RLL^* R^*]^{-1} RW.
\end{aligned} \tag{4.18}
$$

By (4.13),

$$
\|RLL^*R^* - RL_h L_h^* R^*\|_{\mathcal{L}(L^2(0,\infty;Y))} \le Ch^\theta, \ \forall \theta < \frac{1}{2}.
$$

On the other hand,

$$
[I + RL_h L_h^* R^*]^{-1} - [I + RLL^* R^*]^{-1} = [I + RL_h L_h^* R^*]^{-1}(RLL^* R^* - RL_h L_h^* R^*)[I + RLL^* R^*]^{-1}.
$$

Thus

$$
\|[I + RL_h L_h^* R^*]^{-1} - [I + RLL^* R^*]^{-1}\|_{\mathcal{L}(L^2(0,\infty;Y))} \le Ch^\theta, \ \forall \theta < \frac{1}{2}. \tag{4.19}
$$

By (4.14),

$$
\|RW_h - RW\|_{\mathcal{L}(\mathcal{W},L^2(0,\infty;Y))} = \|W^* R^* - W_h^* R^*\|_{\mathcal{L}(L^2(0,\infty;Y),\mathcal{W})} \le Ch^\alpha, \ \forall \alpha < 2. \tag{4.20}
$$

Then (4.16) is obtained from (4.18)–(4.20).

Remark 4.2. From Theorem 3.3 and (4.17), we obtain $\hat{\gamma}_h \to \hat{\gamma}$, indicating that to compute the optimal disturbance attenuation $\hat{\gamma}$ of the system (2.4), it suffices to calculate the optimal disturbance attenuation $\hat{\gamma}_h$ of the approximating equations for sufficiently small h. This is significant because the direct computation of $\hat{\gamma}$ is nearly impossible, whereas the computation of $\hat{\gamma}_h$ is performed in a finite-dimensional space using matrix operations and can be solved effectively with modern mathematical tools.

Theorem 4.3. *For fixed $\gamma > \gamma_c$ and any $z_0 \in L^2(\Omega)$, the following properties hold true for any $\theta < \frac{1}{2}$:*

$$
(i)\ \|w_h^*(\cdot;\Pi_h z_0) - w^*(\cdot;z_0)\|_{\mathcal{W}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}; \tag{4.21}
$$

$$
(ii)\ \|u_h^*(\cdot;\Pi_h z_0) - u^*(\cdot;z_0)\|_{\mathcal{U}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}; \tag{4.22}
$$

$$
(iii)\ \|z_h^*(\cdot;\Pi_h z_0) - z^*(\cdot;z_0)\|_{\mathcal{X}} = \|\Phi_h(\cdot)\Pi_h z_0 - \Phi(\cdot)z_0\|_{\mathcal{X}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}. \tag{4.23}
$$

Proof. We first prove *(i)*. From the representations of w^* give by (3.5) and w_h^* given by (4.10), we obtain

$$
\begin{aligned}
w_h^*(\cdot;\Pi_h z_0) - w^*(\cdot;z_0) &= (E_{h,\gamma}^{-1} - E_\gamma^{-1}) W_h^* R^*[I + RL_h L_h^* R^*]^{-1} Re^{A_h \cdot}\Pi_h z_0 \\
&\quad + E_\gamma^{-1}(W_h^* R^*[I + RL_h L_h^* R^*]^{-1} - W^* R^*[I + RLL^* R^*]^{-1}) Re^{A_h \cdot}\Pi_h z_0 \\
&\quad + E_\gamma^{-1} W^* R^*[I + RLL^* R^*]^{-1} R(e^{A_h \cdot}\Pi_h z_0 - e^{A \cdot} z_0) \\
&= I_{1,h} + I_{2,h} + I_{3,h}.
\end{aligned} \tag{4.24}
$$

Since $\gamma_{h,c} \to \gamma_c$, there is a constant $0 < h_1 \le h_0$ such that when $h < h_1$, it has $|\gamma_{h,c} - \gamma_c| < \frac{\gamma-\gamma_c}{2}$ which implies that $\gamma - \gamma_{h,c} > \frac{\gamma-\gamma_c}{2}$. Then, for $h < h_1$,

$$E_{h,\gamma} \ge (\gamma^2 - \gamma_{h,c}^2)I \ge \gamma \frac{\gamma - \gamma_c}{2} I,$$

which means that for sufficiently small h, $E_{h,\gamma}^{-1}$ is well-defined. Moreover,

$$\|E_{h,\gamma}^{-1} - E_\gamma^{-1}\|_{\mathcal{L}(\mathcal{W})} = \|[\gamma^2 I - S_h]^{-1}(S_h - S)[\gamma^2 I - S]^{-1}\|_{\mathcal{L}(\mathcal{W})} \le Ch^\theta, \ \forall \theta < \frac{1}{2}. \tag{4.25}$$

On the other hand, choosing $\rho = 0$ in (4.5) gives

$$\int_0^\infty \|e^{A_h t}\Pi_h z_0\|_{L^2(\Omega)}^2 \mathrm{d}t \le C\|z_0\|_{L^2(\Omega)}^2. \tag{4.26}$$

From (4.19), (4.20), (4.25) and (4.26), we arrive at

$$\|I_{1,h}\|_{\mathcal{W}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}, \ \forall \theta < \frac{1}{2}, \tag{4.27}$$

and

$$\|I_{2,h}\|_{\mathcal{W}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}, \ \forall \theta < \frac{1}{2}. \tag{4.28}$$

Next, choosing $0 < \alpha < 1$ in (4.6) gives

$$\left(\int_0^\infty \|e^{A_h t}\Pi_h z_0 - e^{At}z_0\|_{L^2(\Omega)}^2 \mathrm{d}t\right)^{1/2} \le Ch^\alpha \left(\int_0^\infty \frac{e^{-2\omega_0 t}}{t^\alpha}\mathrm{d}t\right)^{1/2} \|z_0\|_{L^2(\Omega)} \le Ch^\alpha \|z_0\|_{L^2(\Omega)} \tag{4.29}$$

which, together with (4.19), (4.20) and (4.25), leads to

$$\|I_{3,h}\|_{\mathcal{W}} \le Ch^\alpha \|z_0\|_{L^2(\Omega)}, \ \forall \alpha < 1. \tag{4.30}$$

(4.21) is then concluded from (4.27), (4.28) and (4.30). We next show *(ii)*. Firstly,

$$\begin{aligned} u_h^*(\cdot;\Pi_h z_0) - u^*(\cdot; z_0) = {} & ([I + L^*R^*RL]^{-1}L^*R^*R - [I + L_h^*R^*RL_h]^{-1}L_h^*R^*R)[e^{A_h\cdot}\Pi_h z_0 + W_h w_h^*(\cdot;\Pi_h z_0)] \\ & + [I + L^*R^*RL]^{-1}L^*R^*R[(e^{A\cdot}z_0 - e^{A_h\cdot}\Pi_h z_0) + (Ww^*(\cdot;z_0) - W_h w_h^*(\cdot;\Pi_h z_0))]. \end{aligned}$$

Following the argument of (4.19), we have

$$\|[I + L^*R^*RL]^{-1} - [I + L_h^*R^*RL_h]^{-1}\|_{\mathcal{L}(\mathcal{U})} \le Ch^\theta, \ \forall \theta < \frac{1}{2}$$

which, together with (4.13), (4.14), (4.21) and (4.29), gives

$$\|u_h^*(\cdot;\Pi_h z_0) - u^*(\cdot;z_0)\|_{\mathcal{U}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}, \ \forall \theta < \frac{1}{2}.$$

Finally, we consider *(iii)*. Since

$$\begin{aligned} z_h^*(\cdot;\Pi_h z_0) - z^*(\cdot;z_0) = {} & (e^{A_h\cdot}\Pi_h z_0 - e^{A\cdot}z_0) + (L_h u_h^*(\cdot;\Pi_h z_0) - Lu^*(\cdot;z_0)) \\ & + (W_h w_h^*(\cdot;\Pi_h z_0) - Ww^*(\cdot;z_0)), \end{aligned}$$

it follows from (4.13), (4.14), (4.21), (4.22) and (4.29) that

$$\|z_h^*(\cdot;\Pi_h z_0) - z^*(\cdot;z_0)\|_{\mathcal{X}} \le Ch^\theta \|z_0\|_{L^2(\Omega)}, \ \forall \theta < \frac{1}{2}. \quad \blacksquare$$

Theorem 4.4. *If $\gamma > \gamma_c$, then for sufficiently small h, the algebraic Riccati Eq. (4.11) admits a nonnegative, self-adjoint solution P_h given by (4.12) and*

$$\|P_h\Pi_h - P\|_{\mathcal{L}(L^2(\Omega))} \le Ch^\theta, \ \forall \theta < \frac{1}{2}, \tag{4.31}$$

where P given by (3.15) is the unique solution of the algebraic Riccati Eq. (3.6) in the same sense as stated in Theorem 3.2.

Proof. Since $\gamma_{h,c} \to \gamma_c$ as $h \downarrow$ and $\gamma > \gamma_c$, it has $\gamma > \gamma_{h,c}$ for sufficiently small h. By Theorem 3.1, this implies that the algebraic Riccati Eq. (4.11) has a nonnegative, self-adjoint solution P_h given by (4.12). As for (4.31), by (2.2), (4.23) and (4.29), it follows that for $\forall f \in L^2(\Omega)$,

$$\begin{aligned} \|P_h\Pi_h f - Pf\|_{L^2(\Omega)} &= \left\| \int_0^\infty (e^{A_h^* t}\Pi_h - e^{A^*t})R^*Rz_h^*(t;\Pi_h f) + e^{A^*t}R^*R[z_h^*(t;\Pi_h f) - z^*(t;f)]\mathrm{d}t \right\|_{L^2(\Omega)} \\ &\le C\left(\int_0^\infty \|(e^{A_h^* t}\Pi_h - e^{A^*t})\|_{\mathcal{L}(L^2(\Omega))}\|z_h^*(t;\Pi_h f)\|_{L^2(\Omega)}\mathrm{d}t \right. \\ &\quad \left. + \int_0^\infty e^{-\omega_0 t}\|z_h^*(t;\Pi_h f) - z^*(t;f)\|_{L^2(\Omega)}\mathrm{d}t \right) \\ &\le C(h^\alpha\|f\|_{L^2(\Omega)} + h^\theta\|f\|_{L^2(\Omega)}) \le Ch^\theta\|f\|_{L^2(\Omega)}, \ \forall \theta < \frac{1}{2}. \quad \blacksquare \end{aligned}$$

Theorem 4.5.

$$\lim_{h\downarrow 0} \|B_h^*P_h\Pi_h - B^*P\|_{\mathcal{L}(L^2(\Omega),L^2(\Gamma))} = 0. \tag{4.32}$$

Proof. From (3.15) and (4.12), we have

$$\begin{aligned} B_h^* P_h \Pi_h - B^* P &= \int_0^\infty B^* e^{A_h^* t} \Pi_h R^* R \Phi_h(t) \Pi_h \mathrm{d}t - \int_0^\infty B^* e^{A^* t} R^* R \Phi(t) \mathrm{d}t \\ &= \int_0^\infty B^* (e^{A_h^* t} \Pi_h - e^{A^* t}) R^* R \Phi_h(t) \Pi_h \mathrm{d}t + \int_0^\infty B^* e^{A^* t} R^* R (\Phi_h(t) \Pi_h - \Phi(t)) \mathrm{d}t \\ &= I_{4,h} + I_{5,h}. \end{aligned} \tag{4.33}$$

By (4.15),

$$\int_0^\infty \| B^* (e^{A_h^* t} \Pi_h - e^{A^* t}) \|_{\mathcal{L}(L^2(\Omega), L^2(\Gamma))} \mathrm{d}t \le C h^{\alpha - 3/2} \int_0^\infty \frac{e^{-\omega_0 t}}{t^{\alpha/2}} \mathrm{d}t, \ \forall \frac{3}{2} < \alpha < 2.$$

Note that (4.23) implies

$$\int_0^\infty \| \Phi_h(t) z_0^h \|^2_{L^2(\Omega)} \le C \| z_0^h \|^2_{L^2(\Omega)}, \ \forall z_0^h \in X_h. \tag{4.34}$$

If there exist constants $\mu, \nu \ge 0$ independent of h such that

$$\| \Phi_h(t) \|_{\mathcal{L}(X_h)} \le \mu e^{\nu t}, \tag{4.35}$$

then by [13, Theorem 4 A.2, p.489] and (4.34), we can obtain

$$\| \Phi_h(t) \|_{\mathcal{L}(X_h)} \le M_3 e^{-\omega_3 t},$$

for some $M_3, \omega_3 > 0$ independent of h. Thus

$$\| I_{4,h} \|_{\mathcal{L}(L^2(\Omega), L^2(\Gamma))} \le C h^{\alpha - 3/2} \int_0^\infty \frac{e^{-(\omega_0 + \omega_3) t}}{t^{\alpha/2}} \mathrm{d}t \le C h^\theta, \ \forall \theta = \alpha - \frac{3}{2} < \frac{1}{2}. \tag{4.36}$$

Now we prove (4.35). By [13, Theorem 6.21.2], we have

$$u_h^*(\cdot; z_0^h) = -L_h^* R^* R \Phi_h(\cdot) z_0^h, \ w_h^*(\cdot; z_0^h) = \gamma^{-2} W_h^* R^* R \Phi_h(\cdot) z_0^h.$$

It then follows that

$$\Phi_h(t) z_0^h = e^{A_h t} z_0^h - L_h [L_h^* R^* R \Phi_h(\cdot) z_0^h](t) + \gamma^{-2} W_h [W_h^* R^* R \Phi_h(\cdot) z_0^h](t). \tag{4.37}$$

From [22, Lemma 4.8], we get, for any $p > 4$, that

$$\begin{aligned} &\sup_{h>0} \| L_h^* \|_{\mathcal{L}(L^2(0,\infty; L^2(\Omega)), L^p(0,\infty; L^2(\Gamma)))} < \infty, \\ &\sup_{h>0} \| L_h \|_{\mathcal{L}(L^p(0,\infty; L^2(\Gamma)), C([0,\infty]; L^2(\Omega)))} < \infty, \end{aligned}$$

which are also valid for the operators W_h and W_h^*. With (4.5) and (4.34) at hand, taking L^2-norm in both sides of (4.37) gives

$$\| \Phi_h(t) z_0^h \|_{L^2(\Omega)} \le C \| z_0^h \|_{L^2(\Omega)}, \ \forall z_0^h \in X_h,$$

which leads to (4.35).

To handle $I_{5,h}$, we first consider $\| \Phi_h(t) \Pi_h - \Phi(t) \|_{\mathcal{L}(L^2(\Omega))}$. From [23, Proposition 2.3.1] and (4.23),

$$\begin{aligned} \| [R(\lambda, A_{h,P_h,\gamma}) \Pi_h - R(\lambda, A_{P,\gamma})] f \|_{L^2(\Omega)} &= \left\| \int_0^\infty e^{-\lambda t} (\Phi_h(t) \Pi_h - \Phi(t)) f \, \mathrm{d}t \right\|_{L^2(\Omega)} \\ &\le C h^\theta \| f \|_{L^2(\Omega)}, \ \forall \theta < \frac{1}{2}, \end{aligned}$$

for all $\mathrm{Re}(\lambda) > \max\{0, -\omega_1, -\omega_3\}$ and $\forall f \in L^2(\Omega)$. Hence, for any $\mathrm{Re}(\lambda) > 0$,

$$\| R(\lambda, A_{h,P_h,\gamma}) \Pi_h - R(\lambda, A_{P,\gamma}) \|_{\mathcal{L}(L^2(\Omega))} \to 0 \ \text{ as } h \downarrow 0.$$

By the Trotter–Kato Theorem, it follows that

$$\| \Phi_h(t) \Pi_h - \Phi(t) \|_{\mathcal{L}(L^2(\Omega))} \to 0 \text{ as } h \downarrow 0$$

uniformly in $t \in [0, T]$ for any $T > 0$. Recalling $B^*(-A^*)^{-(3/4+\beta)} \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$, we get from (4.6) that

$$\begin{aligned} \| I_{5,h} \|_{\mathcal{L}(L^2(\Omega), L^2(\Gamma))} &= \left\| \int_0^\infty B^*(-A^*)^{-(3/4+\beta)} (-A^*)^{3/4+\beta} e^{A^* t} R^* R (\Phi_h(t) \Pi_h - \Phi(t)) \mathrm{d}t \right\|_{\mathcal{L}(L^2(\Omega). L^2(\Gamma))} \\ &\le C \int_0^\infty \frac{e^{-\omega_0 t}}{t^{3/4+\beta}} \| \Phi_h(t) \Pi_h - \Phi(t) \|_{\mathcal{L}(L^2(\Omega))} \mathrm{d}t \to 0, \end{aligned} \tag{4.38}$$

where the convergence to zero was guaranteed by the Lebesgue dominated convergence theorem. The desired conclusion then follows from (4.33), (4.36) and (4.38). ■

Theorem 4.6. *For fixed $\gamma > \gamma_c$, if we choose $u(t) = -B_h^* P_h \Pi_h z(t)$ for the original system (2.4) where P_h given by (4.12) is the solution of the algebraic Riccati Eq. (4.11) in the same sense as stated in Theorem 3.2, then for sufficiently small h, the closed-loop solution $z(\cdot)$ with $z(0) = 0$ satisfies*

$$\int_0^\infty (\| R z(t) \|_Y^2 + \| B_h^* P_h \Pi_h z(t) \|^2_{L^2(\Gamma)}) \mathrm{d}t \le (\gamma^2 - \delta) \int_0^\infty \| w(t) \|_V^2 \mathrm{d}t, \ \forall w \in \mathcal{W},$$

for some $\delta > 0$ independent of w, i.e., $u(t) = -B_h^* P_h \Pi_h z(t)$ is a γ-admissible state feedback control for the system (2.4).

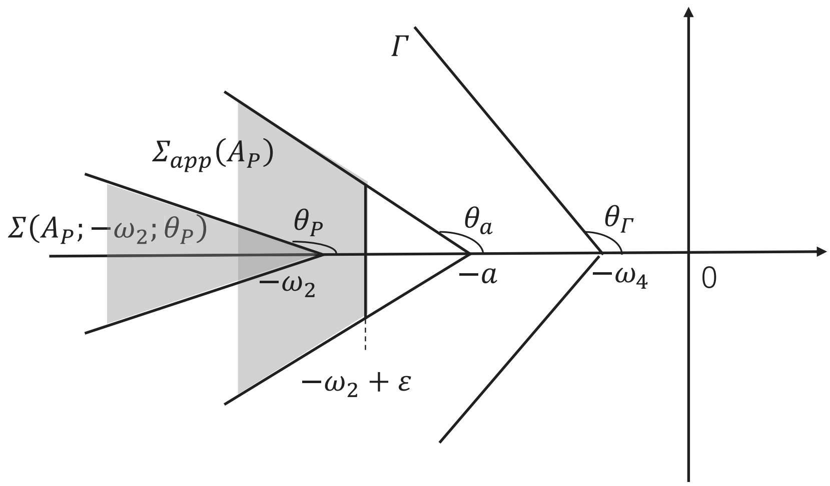

Fig. 1. The positional relationship between $\sum_{app}(A_P)$, $\sum(A_P)$ and Γ.

Proof. Firstly, the closed-loop system can be written as

$$\dot{z}(t) = (A - BB_h^* P_h \Pi_h)z(t) + Gw(t),\ z(0) = 0.$$

Let $A_{P_h} = A - BB_h^* P_h \Pi_h$. Then A_{P_h} generates a C_0-semigroup $e^{A_{P_h}t}$ due to the fact that $B_h^* P_h \Pi_h \in \mathcal{L}(L^2(\Omega))$. Additionally, according to [22, Theorem 4.1] and (4.32), for any given $\varepsilon > 0$, there are constants $M_\varepsilon > 0$ and $h_\varepsilon > 0$ such that, for any sufficiently small $h < h_\varepsilon$, there holds

$$\|e^{A_{P_h}t}\|_{\mathcal{L}(L^2(\Omega))} \le M_\varepsilon e^{(-\omega_2+\varepsilon)t},\ \forall t \ge 0,$$

and

$$\|R(\lambda, A_{P_h})\|_{L^2(\Omega)} \le C\frac{1}{|\lambda + \omega_2 - \varepsilon|},\ \forall \lambda \in \Sigma_{app}^c(A_P) = \text{complement of } \Sigma_{app}(A_P), \tag{4.39}$$

where $\Sigma_{app}(A_P) = \Sigma_{app}(A) \cap \{\lambda | Re(\lambda) \le -\omega_2 + \varepsilon\}$, and $\Sigma_{\text{app}}(A) \equiv \Sigma_{\text{app}}(A; -a; \theta_a)$ is defined as the closed triangular sector containing the axis $[-\infty, -a]$ for some $0 < a < \omega_0$ and being delimited by two rays $-a + \rho e^{\pm i\theta_a}$ for some θ_a with $\pi/2 < \theta_a < \pi$ (see Fig. 1). Let $\bar{z}(\cdot)$ satisfy the following equation

$$\dot{\bar{z}}(t) = (A - BB^* P)\bar{z}(t) + Gw(t),\ \bar{z}(0) = 0,$$

where P, given by (3.15), is the solution of algebraic Riccati Eq. (3.6) in the same sense as in Theorem 3.2. It then follows from the proof of Theorem 3.3 that there exists a constant $\delta > 0$ independent of w such that

$$\int_0^\infty (\|R\bar{z}(t)\|_Y^2 + \|B^* P\bar{z}(t)\|_{L^2(\Gamma)}^2)\mathrm{d}t \le (\gamma^2 - 2\delta)\int_0^\infty \|w(t)\|_V^2 \mathrm{d}t,\ \forall w \in \mathcal{W}. \tag{4.40}$$

By Young's inequality,

$$\begin{aligned}\int_0^\infty \|z(t) - \bar{z}(t)\|_{L^2(\Omega)}^2 \mathrm{d}t &= \int_0^\infty \left\|\left(\int_0^t \left(e^{A_{P_h}(t-\tau)}Gw(\tau) - e^{A_P(t-\tau)}Gw(\tau)\right)\mathrm{d}\tau\right)\right\|_{L^2(\Omega)}^2 \mathrm{d}t \\ &\le C\left(\int_0^\infty \|e^{A_{P_h}t} - e^{A_P t}\|_{\mathcal{L}(L^2(\Omega))}\mathrm{d}t\right)^2 \|w\|_{\mathcal{W}}^2.\end{aligned} \tag{4.41}$$

Notice that

$$R(\lambda, A_P) - R(\lambda, A_{P_h}) = R(\lambda, A_P)B(B_h^* P_h \Pi_h - B^* P)R(\lambda, A_{P_h}), \tag{4.42}$$

Since

$$R(\lambda, A_P)B = [I + R(\lambda, A)BB^* P]^{-1} R(\lambda, A)A^{1/4-\beta}A^{3/4+\beta}B,$$

it follows [16, Lemma A.1.] that for sufficiently large $|\lambda|$,

$$\|R(\lambda, A_P)B\|_{\mathcal{L}(L^2(\Gamma), L^2(\Omega))} \le \frac{C}{|\lambda|^{1/4-\beta}}. \tag{4.43}$$

Since $e^{A_P t}$ is analytic and $\|e^{A_P t}\|_{\mathcal{L}(Y)} \le M_2 e^{-\omega_2 t}$, it has $\sigma(A_P) \subset \Sigma(A_P)$ where $\Sigma(A_P) \equiv \Sigma(A_P; -\omega_2; \theta_P)$ is defined as the closed triangular sector containing the axis $[-\infty, -\omega_2]$ and being delimited by the two rays $-\omega_2 + \rho e^{\pm i\theta_P}$ for some θ_P, with $\pi/2 < \theta_P < \pi$. Define Γ_P be the path $-\omega_4 + \rho e^{\pm i\theta_\Gamma}, 0 \le \rho < \infty$ where $0 < \omega_4 < \min\{\omega_2, a\}$ and $\pi/2 < \theta_\Gamma < \min\{\theta_a, \theta_P\}$. It is easy to see that $\Gamma_P \subset \Sigma^c(A_P) \cap \Sigma_{app}^c(A_P)$. The first resolvent equation on $R(\lambda, A_P)$ leads to (4.43) for all $\lambda \in \Gamma_P$. Combining (4.39), (4.42) and (4.43) gives

$$\|R(\lambda, A_P) - R(A_{P_h})\|_{\mathcal{L}(L^2(\Omega))} \le \frac{C}{|\lambda|^{5/4-\beta}}\|B_h^* P_h \Pi_h - B^* P\|_{\mathcal{L}(L^2(\Omega), L^2(\Gamma))},\ \forall \lambda \in \Gamma_P.$$

For any $t > 0$, the Dunford integral representation [25, p. 487] gives

$$\begin{aligned}\|e^{A_P t} - e^{A_{P_h} t}\|_{\mathcal{L}(L^2(\Omega))} &= \left\| \int_{\Gamma_P} e^{\lambda t}[R(\lambda, A_P) - R(\lambda, A_{P_h})]\mathrm{d}\lambda \right\|_{\mathcal{L}(L^2(\Omega))} \\ &\le C \int_{\Gamma_P} |e^{\lambda t}| \frac{1}{|\lambda|^{5/4-\beta}} \|B_h^* P_h \Pi_h - B^* P\|_{\mathcal{L}(L^2(\Omega),L^2(\Gamma))} |\mathrm{d}\lambda| \\ &\le C e^{-\omega_4 t} \|B_h^* P_h \Pi_h - B^* P\|_{\mathcal{L}(L^2(\Omega),L^2(\Gamma))},\end{aligned}$$

which, together with (4.41), leads to

$$\int_0^\infty \|z(t) - \bar{z}(t)\|^2_{L^2(\Omega)} \mathrm{d}t \le C \|B_h^* P_h \Pi_h - B^* P\|^2_{\mathcal{L}(L^2(\Omega),L^2(\Gamma))} \|w\|^2_{\mathcal{W}}. \tag{4.44}$$

By (4.32), (4.40) and (4.44), we have

$$\begin{aligned}&\int_0^\infty \|B_h^* P_h \Pi_h z(t) - B^* P \bar{z}(t)\|^2_{L^2(\Gamma)} \mathrm{d}t \\ &\le 2 \int_0^\infty \|B_h^* P_h \Pi_h (z(t) - \bar{z}(t))\|^2_{L^2(\Gamma)} \mathrm{d}t + 2 \int_0^\infty \|(B_h^* P_h \Pi - B^* P)\bar{z}(t)\|^2_{L^2(\Gamma)} \mathrm{d}t \\ &\le C \|B_h^* P_h \Pi_h - B^* P\|^2_{\mathcal{L}(L^2(\Omega),L^2(\Gamma))} \|w\|^2_{\mathcal{W}}.\end{aligned} \tag{4.45}$$

Finally, by (4.32), (4.40),(4.44) and (4.45), we have, for sufficiently small h, that

$$\int_0^\infty (\|Rz(t)\|^2_Y + \|B_h^* P_h \Pi_h z(t)\|^2_{L^2(\Gamma)}) \mathrm{d}t \le (\gamma^2 - \delta) \int_0^\infty \|w(t)\|^2_V \mathrm{d}t, \ \forall w \in \mathcal{W}. \quad \blacksquare$$

5. Numerical simulation

In this section, we conduct numerical simulations to validate the obtained results. The system under investigation is a one-dimensional heat equation describing a uniform thin rod with a unit length.:

$$\begin{cases} z_t(x,t) = z_{xx}(x,t) + g(x)w(t), \ 0 < x < 1, t > 0, \\ z(0,t) = u_1(t), \ t \ge 0, \\ z(1,t) = u_2(t), \ t \ge 0, \\ z(x,0) = z_0(x), \end{cases} \tag{5.1}$$

where $z(x,t)$ represents the temperature at a given time t and position x. The initial state of the temperature is denoted by $z_0 \in L^2(0,1)$. The temperature profile is perturbed by the term $g(x)w(t)$, where $w \in L^2(0,\infty;\mathbb{R})$ represents an external disturbance. This disturbance can be regulated through the boundary control input $u(t) = [u_1(t), u_2(t)]^\top$. The cost functional is given by

$$J(u,w) = \int_0^\infty \left[\int_0^1 |z(x,t)|^2 \mathrm{d}x + \|u(t)\|^2_{\mathbb{R}^2} - \gamma^2 |w(t)|^2 \right] \mathrm{d}t, \tag{5.2}$$

i.e., $R = I$ in (3.1). The approximating spaces $X_h \in H_0^1(0,1)$ consist of linear splines: $X_h = \{\phi_i^N\}_{i=1}^N$ where $h = \frac{1}{N+1}$, $N = 1, 2, \ldots$, and

$$\phi_i^N = \begin{cases} 1 - (N+1)\left| x - \dfrac{i}{N+1} \right|, & x \in \left(\dfrac{i-1}{N+1}, \dfrac{i+1}{N+1} \right], \\ 0, & \text{elsewhere over } [0,1]. \end{cases}$$

Let $z_h(x,t) = \sum_{i=1}^N z_{h,i}(t)\phi_i^N(x) = \Phi_N^\top(x) Z_N(t)$ where

$$\Phi_N(x) = [\phi_1^N(x), \phi_2^N(x), \ldots, \phi_N^N(x)]^\top, \ Z_N(t) = [z_{h,1}(t), z_{h,2}(t), \ldots, z_{h,N}(t)]^\top.$$

The approximating dynamic of (5.1) is equivalent to

$$M_N \dot{Z}_N(t) = A_N Z_N(t) + B_N u(t) + G_N w(t), \ Z_N(0) = \left[\int_0^1 z_0(x)\phi_i^N(x) \mathrm{d}x \right], \tag{5.3}$$

and the approximating cost functional is

$$J_N(u,w) = \int_0^\infty [Z^\top(t) M_N Z(t) + \|u(t)\|^2_{\mathbb{R}^2} - \gamma^2 |w(t)|^2] \mathrm{d}t, \tag{5.4}$$

where

$$M_N = \left[\int_0^1 \phi_i^N(x)\phi_j^N(x) \mathrm{d}x \right], \quad A_N = \left[\int_0^1 \phi_i^N(x)(\phi_j^N(x))'' \mathrm{d}x \right],$$

$$B_N = [(\phi_i^N(x))'\big|_{x=0}, -(\phi_i^N(x))'\big|_{x=1}], \ G_N = \left[\int_0^1 g(x)\phi_i^N(x) \mathrm{d}x \right],$$

i.e.,

$$M_N = \frac{1}{6(N+1)} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{bmatrix}_{N\times N}, A_N = (N+1) \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix}_{N\times N}, B_N = (N+1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \\ 0 & 1 \end{bmatrix}_{N\times 2}.$$

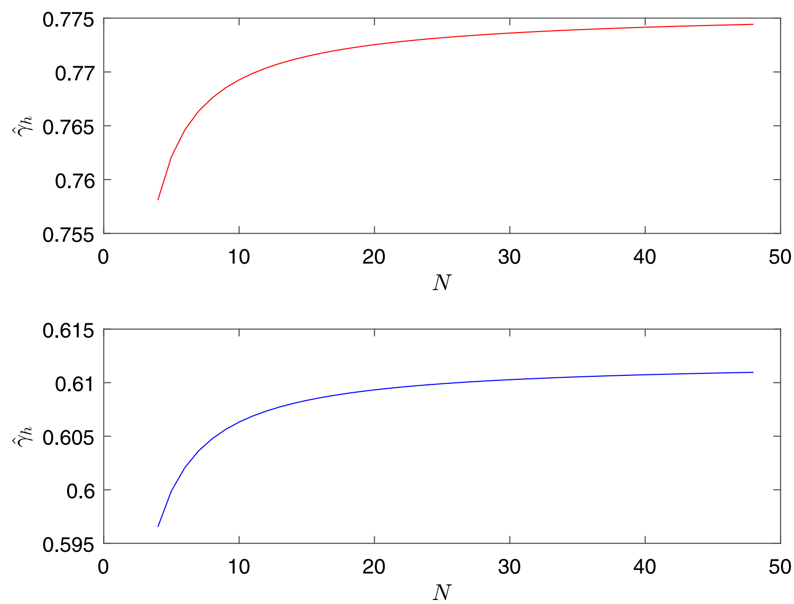

Fig. 2. Convergence of optimal disturbance $\hat{\gamma}_h$ for $g(x) = 10$ (above) and $g(x) = 10\sin\pi x$ (below).

Firstly, we compute the optimal disturbance attenuation $\hat{\gamma}_h$ of (5.3), (5.4) for $g(x) = 10$ and $g(x) = 10\sin\pi x$. In other words, compute the minimum value of γ such that there exists an nonnegative self-adjoint solution to the following algebraic Riccati equation:

$$A_N^\top M_N^{-1} P_N + P_N M_N^{-1} A_N + M_N - P_N M_N^{-1} B_N B_N^\top M_N^{-1} P_N + \gamma^{-2} P_N M_N^{-1} G_N G_N^\top M_N^{-1} P_N = 0. \tag{5.5}$$

Fig. 2 illustrates the evolution of the optimal disturbance attenuation $\hat{\gamma}_h$ as $N \uparrow \infty$. The figure clearly demonstrates that $\hat{\gamma}_h$ converges in both scenarios, validating the conclusion of Theorem 4.2. Furthermore, by correlating Fig. 2 with Theorem 4.2, we can infer that the optimal disturbance attenuation $\hat{\gamma}$ of (5.1) and (5.2) satisfies

$$0.77 < \hat{\gamma} < 0.775 \text{ (for } g(x) = 10),\ 0.61 < \hat{\gamma} < 0.615 \text{ (for } g(x) = 10\sin\pi x).$$

Next, for some fixed $\gamma > \hat{\gamma}$, choose sufficiently large N and set P_N to be the solution of the algebraic Riccati Eq. (5.5). Define $\pi_N : L^2(\Omega) \to \mathbb{R}^N$ as

$$\pi_N f = [(\Pi_h f, \phi_1), (\Pi_h f, \phi_2), \dots, (\Pi_h f, \phi_N)]^\top,\ \forall f \in L^2(\Omega),$$

where Π_h is the orthogonal projection of $L^2(\Omega)$ onto X_h. Also set B_h and P_h be defined as in Section 4.1. Then,

$$B_h^* P_h \Pi_h = B_N^\top M_N^{-1} P_N \pi_N.$$

Let

$$u(t) = -B_N^\top M_N^{-1} P_N \pi_N z(\cdot, t) \tag{5.6}$$

in Eq. (5.1) and define

$$\eta = \left(\frac{\int_0^\infty [\int_0^1 |z(x,t)|^2 \mathrm{d}x + \|u(t)\|^2_{\mathbb{R}^2}] \mathrm{d}t}{\int_0^\infty |w(t)|^2 \mathrm{d}t} \right)^{\frac{1}{2}},$$

where $z(\cdot, t)$ is the closed-loop solution with $z_0 = 0$. Table 1 provides a comparison of the open-loop and closed-loop values of η for the case where $g(x) = 10$, $N = 15$, and $\gamma = 0.78$. Similarly, Table 2 compares the open-loop and closed-loop values of η for the case where $g(x) = 10\sin\pi x$, $N = 15$, and $\gamma = 0.62$. Consider different disturbance scenarios:

$$w(t) \in \{\sin(0.5t)\chi_{\{t\le 50\}}, 1\chi_{\{t\le 50\}}, t^2\chi_{\{t\le 50\}}\},$$

where

$$\chi_{\{t\le 50\}} = \begin{cases} 1, & 0 \le t \le 50, \\ 0, & t > 50. \end{cases}$$

From Tables 1 and 2, it is evident that under the feedback control (5.6), the value of $\eta < \gamma$ for all three disturbances considered. This observation validates Theorem 4.6, indicating that the feedback control (5.6) with $N = 15$ is γ-admissible for the system defined by (5.1) and (5.2), specifically, for the first case, $\gamma = 0.78$, and for the second case, $\gamma = 0.62$. Furthermore, Fig. 3 illustrates the open-loop and closed-loop time evolutions of $z(0.5, t)$ in both scenarios described in Tables 1 and 2, respectively. Here, the disturbance is given as $w(t) = \sin 0.5t\chi_{\{t\le 50\}}$. It is clearly seen that the absolute value of the closed-loop response $|z(0.5, t)|$ is smaller than that of the open-loop response. This observation intuitively demonstrates the significant attenuation effect of the feedback control (5.6) on the sine function form of disturbance.

Table 1
The values of η for $g(x) = 10$, $N = 15$, $\gamma = 0.78$.

$u(t)$	$w(t)$		
	$\sin 0.5t\chi_{\{t\le 50\}}$	$1\chi_{\{t\le 50\}}$	$t^2\chi_{\{t\le 50\}}$
0	0.9119	0.9117	0.9106
$-B_N^\top M_N^{-1} P_N \pi_N z(\cdot,t)$	0.7789	0.7787	0.7779

Table 2
The values of η for $g(x) = 10\sin\pi x$, $N = 15$, $\gamma = 0.62$.

$u(t)$	$w(t)$		
	$\sin 0.5t\chi_{\{t\le 50\}}$	$1\chi_{\{t\le 50\}}$	$t^2\chi_{\{t\le 50\}}$
0	0.7166	0.7167	0.7157
$-B_N^\top M_N^{-1} P_N \pi_N z(\cdot,t)$	0.6157	0.6157	0.6150

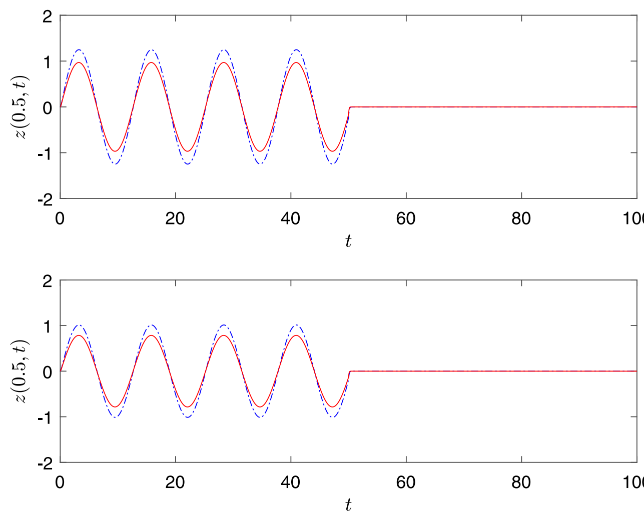

Fig. 3. Response $z(0.5,t)$ of open (-.) and closed loop (-) to $w(t) = \sin 0.5t\chi_{\{t\le 50\}}$ for $g(x) = 10$, $N = 15$, $\gamma = 0.78$ (above) and for $g(x) = 10\sin\pi x$, $N = 15$, $\gamma = 0.62$ (below).

6. Conclusions

This paper investigates the state feedback control for the H^∞ disturbance-attenuation problem of stable parabolic systems with in-domain distributed disturbances under Dirichlet boundary control. We employ a Galerkin approximation approach to approximate the original problem. It is demonstrated that, under four key assumptions, there exists a sequence of matrices P_h that converges in norm to the solution of the algebraic Riccati equation. These assumptions include (a) $A \in \mathcal{L}(D(A), L^2(\Omega))$ generates an analytic and exponentially stable C_0-semigroup e^{At}; (b) $B \in \mathcal{L}(L^2(\Gamma), [\mathcal{D}(A)]')$ satisfies $A^{-\eta_1}B \in \mathcal{L}(L^2(\Gamma).L^2(\Omega))$ with $0 < \eta_1 < 1$; (c) $G \in \mathcal{L}(V, L^2(\Omega))$; (d) $R \in \mathcal{L}(L^2(\Omega), Y)$.

While our discussion has focused on the case where the admissibility exponent η_1 of B satisfies $\frac{3}{4} < \eta_1 < 1$, it is worth noting that similar arguments can be applied to establish the same conclusion for the case of $0 < \eta_1 < 1$. This ensures the generality of our results across different admissibility ranges for the control operator. Utilizing the sequence P_h, we construct a state feedback control law $u(t) = -B_h^* P_h \Pi_h z(t)$ and demonstrate that it serves as a γ-admissible state feedback for the original PDE system. To the best of our knowledge, this represents the first result pertaining to the approximation theory of the algebraic Riccati equation with an unbounded control operator in the context of H^∞ control. Future work will explore extensions to cases where the semigroup e^{At} is not necessarily exponentially stable and where the disturbance operator G may be unbounded. Such extensions could potentially encompass boundary disturbances, further broadening the applicability of our approach. An intriguing issue worthy of further exploration in the future pertains to the output feedback control built upon the state feedback established in this paper.

CRediT authorship contribution statement

Bao-Zhu Guo: Writing – review & editing, Supervision, Methodology, Conceptualization. **Zheng-Qiang Tan:** Writing – original draft, Investigation, Formal analysis.

Declaration of competing interest

No interests conflicts

Data availability

No data was used for the research described in the article.

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