Finite dimensional control of multichannel systems

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Abstract

We consider the control system $y_{tt} = Ay + Bu$ where $Ay = R(x)y'' + P(x)y' + Q(x)y$ is a differential operator, with $R(x)$, $P(x)$, and $Q(x)$ being $N \times N$ matrix valued functions, $R(x)$ diagonal. Assuming that the eigenfunctions of $A$ form a Riesz basis and using the moment approach and the theory of exponential families with vector coefficients, we obtain two controllability results. i) The system is exactly controllable by the boundary control if the control $u_j(t)$ in the $j$th channel of the control $u(t) = (u_1(t), \ldots, u_N(t))$ lasts longer than the double travel time $T_j$ of a wave in this channel; ii) for any state space $\text{Dom}(A^{r}) \times \text{Dom}(A^{r-1/2})$, it is possible to find the vector valued profile functions $b_j(x)$ such that the system is exactly controllable with the distributed control $Bu = \sum_{1}^{N} u_j(t) b_j(x)$. We apply this approach to the damped Timoshenko beam and prove that the generalized eigenfunctions of the operator of the closed-loop system forms a Riesz basis. New controllability results are developed on the distributed control for Timoshenko beam.

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1. Introduction and main results

In this paper, we consider the controllability of the following second order control system:

\[
\frac{d^2}{dt^2} y(t) = Ay(t) + Bu(t),
\]

where \( y(t) \) is a function from a Hilbert space \( H \), \( A \) is a differential operator acting in \( H \), and \( B : U \rightarrow H \) is a control operator. The control \( u(t) = (u_1(t), u_2(t), \ldots, u_N(t))^\top \) takes values in a finite-dimensional space \( U = \mathbb{C}^N \). The \( j \)-th channel control \( u_j(t) \) acts time \( \tau_j \) and belongs to \( L^2(0, \tau_j) \). We may set \( T = \max \tau_j \) and consider controls in a subspace \( \mathcal{U} \) of \( L^2(0, T; \mathbb{C}^N) \).

The “elliptical” part of the control system, namely, \( A \) is a differential operator:

\[
Ay = R(x)y'' + P(x)y' + Q(x)y, \quad x \in (0, 1),
\]

where \( R(x), P(x), \) and \( Q(x) \) are sufficiently smooth \( N \times N \) matrix valued functions, and \( R(x) = \text{diag}(r_j(x)) \) is a diagonal matrix with positive entries.

We can consider this system as \( N \) coupled channel wave systems with different velocities \( c_j(x), c_j(x) = \sqrt{r_j(x)} \) the propagation of singularities (modes). Channels are interacting in domain \( x \in (0, 1) \) via the \( P \)- and \( Q \)- terms.

We consider two types of controls: i) Boundary control, that is, the controls act in the channels at the boundary entering the Dirichlet boundary condition (BC) at one end or at both ends; ii) Distributed control, \( Bu = \sum_1^H u_j(t)b_j(x) \), where \( b_j(x) \) are the fixed profiles taking values in \( \mathbb{C}^N \).

The unperturbed system with \( P = Q = 0 \) represents \( N \) uncoupled channels governed by the string equations, and the travel time (the optical length) of a wave from the start to the end of the \( j \)-th channel is

\[
T_j = \int_0^1 \frac{dx}{\sqrt{r_j(x)}}.
\]

Such systems are boundary controllable because each channel is exactly controllable in time \( 2T_j \) (or \( T_j \) for controls at both ends).

We show that the system is exactly controllable in the state space \( L^2(0, 1; \mathbb{C}^N) \oplus H^{-1}(0, 1; \mathbb{C}^N) \) if for all \( j \), the control in the \( j \)-th channel lasts longer than \( 2T_j \) (or longer than \( T_j \) for controls on both ends). In other words, the control space \( \mathcal{U} \) is

\[
\mathcal{L}_\varepsilon = L^2(0, 2T_1 + \varepsilon) \oplus L^2(0, 2T_2 + \varepsilon) \oplus \cdots \oplus L^2(0, 2T_N + \varepsilon)
\]

for any \( \varepsilon > 0 \) or

\[
L^2(0, T_1 + \varepsilon) \oplus L^2(0, T_2 + \varepsilon) \oplus \cdots \oplus L^2(0, T_N + \varepsilon).
\]

Actually, we show more: The minimal norm of controls transferring the system from the zero state to \((y(\cdot, T), y_1(\cdot, T))^\top\) is equivalent to the norm of the final state. We call this type of controllability as basis-controllability because it takes place when the exponential family arising in
the moment method forms a Riesz basis for its span, [2, Ch III.3]. In our approach we reduce the controllability problem to a moment one with respect to vector exponentials (a general theory can be found in [2]). To obtain a Riesz basis, we find asymptotic expansion of eigenfrequencies and eigenfunctions, which corresponds to an uncoupled system. In contrast to the paper [10], we do not construct the so called generating function. The application of the moment approach and the nonharmonic Fourier series to control problem can be found also in [8]. For the nonharmonic Fourier series we refer to [23].

In general we proceed as follows. First, we suppose that the set of eigenfunctions of the operator \( A \) forms a Riesz basis for \( L^2(0, 1; \mathbb{C}^N) \). Note that the Riesz generation property is a separate problem that is less relevant to our major concern in this paper.

The main assumptions allow us to use the moment approach. This approach reduces the controllability problem to the moment one with respect to the vector exponential family \( \mathcal{E} = \{ \eta_\lambda e^{-i\lambda t} \}_{\lambda \in \sigma} \), where the vector coefficients \( \eta_\lambda \) are expressed in terms of the eigenfunctions of \( A^* \) corresponding to the eigenvalues \( \tilde{\lambda} \). The moment problem is always solvable in \( L^2 \) if \( \mathcal{E} \) forms a Riesz basis for its span, referring to as an \( \mathcal{L} \)-basis thereafter. A general theory of vector exponentials can be found in [2].

The unperturbed exponential family forms an \( \mathcal{L} \)-basis for

\[
\mathcal{L}_0 = L^2(0, 2T_1) \oplus L^2(0, 2T_2) \oplus \cdots \oplus L^2(0, 2T_N)
\]  

(with \( N \)-dim defect). Because the perturbed family asymptotically close to the unperturbed exponential family, it forms an \( \mathcal{L} \)-basis for \( \mathcal{L}_\varepsilon \) for any positive \( \varepsilon \). With the theory, this leads to the controllability of the system when controls are from space \( \mathcal{L}_0 \).

Also we consider a distributed control of the form \( Bu = \sum_{j=1}^{N} u_j(t)b_j(x) \) with fixed (vector valued) profiles \( b_j(x) \), and the controls \( u_1(t), u_2(t), \ldots, u_N(t) \). In the scalar case, the general results for a selfadjoint operator \( A \) can be found in [2, Th. V.1.3]. In present paper, we show that it is possible to choose the profiles in such a way that the system is controllable for the control in \( \mathcal{L}_0 \), with the state space \( \text{Dom}(A^{r+1}) \oplus \text{Dom}(A^r) \), for any positive \( r \).

The main feature of the system (1) is the presence of different wave modes. For the case of one mode (which means that \( R(x) \) in (2) is an identity matrix multiplied by a scalar function) the controllability results are similar to ones for the scalar system. In [1], the control system governed by the equation with potential:

\[
\begin{cases}
y_{tt}(x, t) = y_{xx}(x, t) + V(x)y(x, t), x, t > 0, & y(x, t) \in \mathbb{C}^N, \\
y|_{t=0} = 0, & y(0, t) = u(t),
\end{cases}
\]

was studied. The shape–controllability in the occupied domain was proved: for \( u \) runs \( L^2(0, T; \mathbb{C}^N) \), the first component \( y(., T) \) of the state runs all over \( L^2(0, T; \mathbb{C}^N) \). If the system is considered on an spatial interval \((0, 1)\) with the Dirichlet BC at \( x = 1 \), we have basis-controllability in time \( T = 2 \) as in the case of uncoupled channels.

In [16], an interesting case of the system of parallel coupled damped identical strings was considered. It was shown that the system is exactly controllable and there exist sustained oscillations despite the damping.

More interesting dynamics and more difficult control problems arise in the system possessing two-mode oscillations. In a series of papers, see, for instance [3,4], the dynamics and the shape–controllability in occupied domain have been studied.
A multichannel model arises in several fields. A two channel model can be used to describe the small vibrations of the system, composed by a steel beam and the reinforced concrete slab connected by some connectors, see [13]. The inverse problem corresponding to this model was studied in [14,5].

In [2, Ch VII.3], the controllability of a multichannel system governed by

\[
\begin{align*}
U(x) \text{diag}[\lambda_j(x)] U^*(x) y_{tt}(x,t) &= y_{xx}(x,t), \quad x \in (0,1), \\
y(0,t) &= u(t), \quad y(1,t) = 0,
\end{align*}
\]

was studied, where \( U(x) \) is a unitary matrix and \( \lambda_j(x) \) are positive on the segment. The basis-controllability was proved at time

\[
T = 2 \int_0^1 \max\{\lambda_j(x)\} \, dx,
\]

which is the double travel time of the ‘lowest’ wave. In the proof, a factorization of the generating function has been used. In [10] the controllability of a multichannel system governed by

\[
y_{tt}(x,t) = Dy_{xx}(x,t) + Q(x)y(x,t) + P(x)y_t(x,t), \quad x \in (0,1),
\]

was investigated, where \( D \) is a constant diagonal \( N \times N \) matrix, \( Q(x) \) and \( P(x) \) are smooth matrix valued functions. The controls act in the channels at the boundary entering the Dirichlet boundary condition at one end. At the other end, the homogeneous Dirichlet boundary condition was imposed:

\[
y(0,t) = u(t), \quad y(1,t) = 0.
\]

All channel controls \( u_j(t) \) are square integrable at any interval. It was shown that the system is exactly controllable in the state space in two cases: a) For all \( j \), the control in the \( j \)th channel lasts longer than \( T_j \); b) the whole control \( u(t) \) lasts longer than or equal to \( T_{\text{max}} = \max T_j \) which is the double “maximal” travel time for singularities. In the proof, the moment approach and factorization of the generating function were applied.

A classical example of a two-channel system with interacting modes is a Timoshenko beam model in elasticity theory. The free motion of a Timoshenko beam is governed by the equation:

\[
\begin{align*}
(\rho(x)w_{tt}(x,t) - (K(w_x(x,t) - \psi(x,t)))_x)_x &= 0, \\
(I\rho(x)\psi_{tt}(x,t) - (EI(x)\psi_x(x,t))_x - K(w_x(x,t) - \psi(x,t)))_x &= 0,
\end{align*}
\]

for which the velocities are

\[
v_1(x) = \sqrt{\frac{K(x)}{\rho(x)}}, \quad v_2(x) = \sqrt{\frac{EI(x)}{I\rho(x)}}.
\]

In [7,17], the system (4) with a distributed (interior) ‘1-d’ control (meaning that control is a scalar) of the form \( u(t) b(x) \), \( b(x) = (b_1(x), b_2(x)) \top \) was studied, where \( u(\cdot) \) is a scalar function and \( g_1(\cdot) \) and \( g_2(\cdot) \) are fixed functions (profiles). At any time the control acts in \( \mathbb{C} \). To obtain the
controllability, the authors used a non-physical condition that the constant velocities \( v_1 \) and \( v_2 \) are commensurable in the sense of \( v_1 / v_2 \) being a rational number. The reason is that the authors applied the moment approach which leads to a moment problem with respect to the exponential family \( \exp(i\omega_n t) \) where the real set of frequencies \( \{\omega_n\} \) is not separable. When we apply to a two-dimensional control

\[
 u_1(t) b_1(x) + u_2(t) b_2(x)
\]  

(at any time the control acts in \( C^2 \)), it gives a moment problem with respect to a ‘vector’ exponentials \( \eta_n \exp(i\omega_n t) \) and the vector coefficients \( \eta_n \) ‘separate’ the frequencies.

In [18], the controllability under control of the form (5) was studied for a damped Timoshenko beam (see below (18)). For the case when one controllers is zero and under natural assumption on the profiles, the controllability at \( T = 2T_1 + 2T_2 \) has been established there, in which

\[
 T_1 = \int_0^1 \frac{1}{v_1(x)} \, dx, \quad T_2 = \int_0^1 \frac{1}{v_2(x)} \, dx,
\]

are the optical lengths of \((0, 1)\) for two types of waves.

There are papers where the reduction of the control time for several controllers has been proved without using the moment approach with respect to vector potentials. We can mention the paper [12], where a control system similar to the Timoshenko model was studied on an interval of length \( L \). Using the multiplier method, the exact controllability at time

\[
 T > 2L \max \left\{ \frac{1}{\min_x(v_1(x))}, \frac{1}{\min_x(v_2(x))} \right\}
\]

has been proved.

In [21], a sharp result for the controllability of the Timoshenko model has been proved in time

\[
 T > 2 \max \{T_1, T_2\},
\]

in which the author used a so-called smoothing property of auxiliary problems consisting of a semi-infinite beam.

Because the damping in Timoshenko model involves equation of the first order in time, to apply the moment approach, we also need the assumption that the generalized eigenfunctions of the system operator form a Riesz basis for the state space, which is always true for the system without damping. Since there is a gap in the proof of the Riesz basis property for Timoshenko beam with variable coefficients in [19], which has been indicated in [9] (the same author claimed in [20] that she re-proved it by a different approach than [19]), we give a new and much simpler proof of this fact. Note that the Riesz basis property for the system with constant coefficients has been proved in [22].

Using the theory of exponential families with vector coefficients, we obtain, in this paper, almost sharp results about exact controllability of the Timoshenko beam. Precisely, under some conditions for distributed control, we obtain controllability for the control space \( L^2(0, 2T_1 + \varepsilon) \oplus L^2(0, 2T_2 + \varepsilon) \). In addition, we do not need some restrictions on damping to separate the eigenfrequencies.
2. Asymptotics of fundamental solutions and of eigenfunctions and eigenvalues

Let $A$ be the operator in (2) acting in the space

$$H = L^2(0, 1; \mathbb{C}^N)$$

and its domain is $H_0^1(0, 1; \mathbb{C}^N)$. Let us consider the root subspace $\mathcal{R}_\lambda$ of $A$ corresponding to an eigenvalue $\lambda$:

$$\mathcal{R}_\lambda = \bigcup_{n=1}^{\infty} \text{Ker}(A - \lambda)^n.$$ 

For the ordinary differential operators, the dimension of the root subspace is finite. The functions in the root subspace are usually referred to as generalized eigenfunctions.

**Main assumption.** There is a family of generalized eigenfunctions of $A$, which forms a Riesz basis for $L^2(0, 1; \mathbb{C}^N)$.

For simplicity we assume further that the eigenvalues of $A$ are semisimple. This means that any eigensubspace $\text{Ker}(A - \lambda)$ coincides with the corresponding root subspace $\mathcal{R}_\lambda$.

Now, we find the eigenmodes and eigenfrequencies of (1) by letting

$$y(x, t) = e^{i\omega t} v(x).$$

Then,

$$-\omega^2 v = Av. \quad (6)$$

Setting

$$z = \begin{pmatrix} v \\ \omega^{-1} v' \end{pmatrix}, \quad (7)$$

we obtain equation of the first order in $x$:

$$z' = \left\{ \omega \begin{pmatrix} 0 & 1 \\ -R^{-1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\omega^{-1} R^{-1} Q & R^{-1} P \end{pmatrix} \right\} z. \quad (8)$$

Consider the similar equation for the matrix $Z$. Set

$$L^2 = 1/R.$$ 

Take the transformation matrix

$$V = \begin{pmatrix} I & -I \\ iL & iL \end{pmatrix},$$

and change the dependent variable to obtain the matrix system which has the diagonal main part in $\omega$: 

6
Also consider the matrix equation for $W$ with the Cauchy data at $x = 0$. After simple calculations (see, e.g., [11], [2, Ch III.3]), we obtain the system:

$$W' = \left\{ \begin{pmatrix} i\omega L & 0 \\ 0 & -i\omega L \end{pmatrix} + Q_1 \right\} W, \ W|_{t=0} = I_{2N},$$

(9)

where

$$Q_1 = V^{-1} \begin{pmatrix} 0 & 0 \\ 0 & R^{-1}P \end{pmatrix} V - V^{-1}V' + O(1/\omega)$$

$$= \left( \frac{1}{2}L^{-1}R^{-1}PL - \frac{1}{2}L^{-1}L' \right) \begin{pmatrix} I_N & I_N \\ I_N & I_N \end{pmatrix} + O(1/\omega).$$

In what follows, we need only diagonal terms of $Q_1$, because we will see that the non-diagonal entries of $Q_1$ do not contribute the main term of the asymptotic expression:

$$\text{diag } Q_1 = \text{diag}[\tilde{q}_j], \ \tilde{q}_j = q_j + O(1/\omega), \ q_j = P_{jj}/r_j - \frac{1}{2}L^{-1}_j L'_j.$$

As an ‘etalon’ equation, we take the system with the diagonal system matrix equal to the diagonal part of the system matrix in (9) up to $O(1/\omega)$:

$$W'_0 = \begin{pmatrix} \text{diag}[i\omega L_j + q_j] & 0 \\ 0 & \text{diag}[-i\omega L_j + q_j] \end{pmatrix} W_0, \ W_0|_{t=0} = I_{2N}.$$

(10)

If we introduce the functions

$$t_j(x) = \int_0^x L_j(s) \, ds, \ \gamma_j(x) = \exp \left( \int_0^x q_j(s) \, ds \right),$$

and the diagonal $N \times N$ matrices

$$W_{\pm}(x, \omega) = \text{diag} [\gamma_j(x)e^{\pm i\omega t_j(x)}],$$

then, the solution can be written as

$$W_0(x, \omega) = [W_+(x, \omega), W_-(x, \omega)].$$

By $A_{nd}$, we denote the non-diagonal part of the matrix of system (9) and the terms $O(1/\omega)$ in the diagonal part. We can write

$$W' = \begin{pmatrix} \text{diag}[i\omega L_j + q_j] & 0 \\ 0 & \text{diag}[-i\omega L_j + q_j] \end{pmatrix} W + A_{nd} W, \ W|_{t=0} = I_{2N}.$$

(10)
Lemma 1. (i) All functions $L_i(x) - L_j(x) = 1/\sqrt{r_i(x)} - 1/\sqrt{r_j(x)}$, $i \neq j$, have finite number of zeros and the maximal order of zeros be $m - 1$.

(ii) In a neighborhood of any zero $x_0$ of $L_i(x) - L_j(x)$, $i \neq j$, the order of smoothness of $L_i(x) - L_j(x)$ is larger than the order $n(x_0)$ of $x_0$ as a zero of $L_i(x) - L_j(x)$. In other words, $L_i(x) - L_j(x)$ can be written as

$$L_i(x) - L_j(x) = p_{ij}(x, x_0)(x - x_0)^{n(x_0)},$$

(11)

where $p_{ij}(x_0, x_0) \neq 0$, $p_{ij}(x, x_0)$ is continuous at $x_0$. As a result, in the strip $|\Im \omega| \leq \text{Const}$,

$$W(x, \omega) = W_0(x, \omega) + O(\omega^{-q}), \ q = 1/m.$$

Proof. Equation (10) can be written as an integral equation of the Volterra type

$$W = W_0 + KW,$$

where

$$(KW)(x, \omega) = \int_0^x W_0(x, \omega) W_0^{-1}(s, \omega) A_{nd}(s, \omega) W(s, \omega) ds.$$

Evidently, $K$ is a bounded operator in the space of continuous functions. If $\Im \omega$ is bounded, the operator is uniformly bounded with respect to $\omega$.

Consider the first iteration $KW_0$ of the Neumann series. The diagonal elements of this iteration are $O(1/\omega)$ and for non-diagonal entries with $i \neq j$ and $i, j \leq N$,

$$(KW_0)_{ij} = \int_0^x e^{i\omega t_i(x) + \gamma_i(x) - i\omega t_i(s) - \gamma_i(s)} (A_{nd})_{ij}(s) e^{i\omega t_j(s) + \gamma_j(s)} ds$$

$$= \int_0^x e^{i\omega \varphi_{ij}(x, s)} C_{ij}(x, s) ds,$$

where

$$\varphi_{ij}(x, s) = \int_s^x L_i(\xi) d\xi + \int_0^s L_j(\xi) d\xi,$$

$$C_{ij}(x, s) = e^{\gamma_i(x) + q_i(x) + \gamma_j(s) + q_j(s)} (A_{nd})_{ij}(s).$$

We have an oscillatory integral and the critical points of $\varphi_{ij}(x, s)$ are the zeros of $L_j(x) - L_i(x)$, or the turning points of the system (9). The method of stationary phase concludes Lemma 1 (see, e.g., [15]).

For the case of $i, j > N$, we have the integral of the same type with the phase function of the opposite sign:
\[(K\mathbf{W}_0)_{ij} = \int_0^x e^{-i\omega_\tilde{\varphi}_{j-N,N} - N(x,s)} C_{ij}(x,s) \, ds.\]

These entries are also \(O(\omega^{-q})\). For the cases of \(i \leq N, j > N\) or \(i > N, j \leq N\), the phase functions are

\[\int_s^x L_i(\xi) \, d\xi - \int_0^s L_{j-N}(\xi) \, d\xi,\]

and

\[\int_s^x L_{i-N}(\xi) \, d\xi + \int_0^s L_j(\xi) \, d\xi,\]

respectively. These functions have no turning point and give the contribution \(O(1/\omega)\).

Now we come to the asymptotic expansion of (8).

**Theorem 1.** Under condition (11), the matrix solution \(Z(x,\omega)\) of (8) with the condition \(Z(0,\omega) = I_{2N}\) has the asymptotic expansion:

\[Z(x,\omega) = \frac{1}{2} \begin{pmatrix} \text{diag}[\gamma_j(x) \cos(\omega t_j(x))] & \text{diag}[\gamma_j(x) \sin(\omega t_j(x))/L_j(0)] \\ -\text{diag}[L_j(x) \sin(\omega t_j(x))] & \text{diag}[L_j(x) \gamma_j(x) \cos(\omega t_j(x))/L_j(0)] \end{pmatrix} + O(\omega^{-q}),\]

where \(q\) is defined in Lemma 1.

**Proof.** Notice that

\[Z(x,\omega) = V(x)W(x,\omega)V^{-1}(0),\]

for

\[V^{-1}(0) = \frac{1}{2} \begin{pmatrix} I_N & iL^{-1}(0) \\ -iL^{-1}(0) & -iL^{-1}(0) \end{pmatrix}.\]

Hence,

\[Z(x,\omega) = \frac{1}{2} \begin{pmatrix} I & -iL \\ iL & iL \end{pmatrix} \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix} \begin{pmatrix} I_N & -iL^{-1}(0) \\ -iL^{-1}(0) & -iL^{-1}(0) \end{pmatrix} + O(\omega^{-q}),\]

and

\[Z = \frac{1}{2} \begin{pmatrix} W_+ + W_- & -i(W_+ - W_-)L^{-1}(0) \\ -iL(W_+ - W_-) & L(W_+ + W_-)L^{-1}(0) \end{pmatrix} + O(\omega^{-q}).\]

This completes the proof of the theorem. \(\square\)
We need this result to find the asymptotic expansions of the eigenvalues and eigenfunctions. For the Dirichlet BC, the following Theorem 2 gives the asymptotic expansion. Let us recall that \( q = 1/m \) where \( m - 1 \) is the maximal order of zeros of functions in Lemma 1.

**Theorem 2.** Under condition (11), the eigenfrequencies of \( A \) have \( N \) subsequences:

\[
\omega_{n,j} = \frac{n\pi}{T_j} + O(n^{-q}), \quad j = 1, 2, ..., N, \quad n \in -\mathbb{N} \cup \mathbb{N},
\]

(12)

where \( q \) is defined in Lemma 1. The corresponding eigenfunctions are

\[
\varphi_{n,j} = \gamma_j(x) \sin \left( \omega_{n,j} \int_0^x \frac{ds}{\sqrt{T_j(s)}} \right) \zeta_j + O(n^{-q}), \quad j = 1, 2, ..., N, \quad n \in -\mathbb{N} \cup \mathbb{N},
\]

(13)

where \( \{\zeta_j\} \) is the standard basis of \( \mathbb{C}^N \).

**Proof.** We find a \( 2N \times N \) matrix solution \( \tilde{Z}(x, \omega) \) of (8) with \( \tilde{Z}|_{t=0} = \left( \begin{array}{c} 0 \\ I \end{array} \right) \). The first \( N \times N \) block \( \tilde{Z}_1 \) is the matrix solution to (8). Let

\[
\det \tilde{Z}_1(1, \omega) = 0,
\]

and for nonzero \( a \in \mathbb{C}^N \)

\[
\tilde{Z}_1(1, \omega)a = 0.
\]

Then, \( \omega \) is an eigenmode and \( \tilde{Z}_1(x, \omega)a \) is a corresponding eigenfunction. The initial data for the solution to (9) is

\[
\tilde{Z}|_0 = \left( \begin{array}{cc} I & -I \\ iL & iL \end{array} \right)^{-1} \left( \begin{array}{c} 0 \\ I \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} I & -iL^{-1} \\ -I & iL^{-1} \end{array} \right) \left( \begin{array}{c} 0 \\ I \end{array} \right) = -\frac{iL^{-1}}{2} \left( \begin{array}{c} I \\ I \end{array} \right),
\]

which gives

\[
\tilde{Z}(x, \omega) = \left( \begin{array}{cc} I & -I \\ iL & iL \end{array} \right) W \tilde{Z}|_0 = \left( \begin{array}{c} \text{diag}[\gamma_j e^{i\omega t_j(x)}] \\ \text{diag}[\gamma_j e^{-i\omega t_j(x)}] \end{array} \right) \left[ \begin{array}{c} 0 \\ \frac{-iL^{-1}}{2} \left( \begin{array}{c} I \\ I \end{array} \right) \end{array} \right] + O(\omega^{-q})
\]

\[
= \left( \begin{array}{cc} I & -I \\ iL & iL \end{array} \right) \left[ \begin{array}{c} \text{diag}[L_j \gamma_j \sin(\omega t_j(x))] \\ \text{diag}[L_j^2 \gamma_j \cos(\omega t_j(x))] \end{array} \right] + O(\omega^{-q}).
\]

From here we obtain the asymptotic expansions: For large \( \omega \), the eigenfunctions and eigenfrequencies are described by (12) and (13). By the assumption, we can claim that all eigenfunctions and eigenfrequencies are enumerated in this way. \( \Box \)
The adjoint operator $A^*$ has the same BC and the differential expression:

$$A^*z = Rz'' - P^*z' + \tilde{Q}z$$

with

$$\tilde{Q} = -R(P^*/R)' + Q^*.$$  

**Corollary 1.** The eigenfrequencies and eigenfunctions of $A^*$ have $N$ subsequences with asymptotic expansions:

$$\tilde{\omega}_{n,j} = \frac{n\pi}{T_j} + O(\omega^{-q}), \ j = 1, 2, ..., N, \ n \in -\mathbb{N} \cup \mathbb{N},$$

and

$$\psi_{n,j}(x) = \tilde{\gamma}_j(x) \sin \left( \omega_{n,j} \int_0^x \frac{ds}{\sqrt{r_j(s)}} \right) \xi_j + O(\omega^{-q}), \ j = 1, 2, ..., N, \ n \in -\mathbb{N} \cup \mathbb{N}$$

with

$$\tilde{\gamma}_j(x) = \int_0^x (\tilde{P}_{jj}/r_j + L_j^1)dx.$$  

**Remark 1.** If $P(x)$ is a real matrix, the adjoint operator has the same asymptotic expansions of the eigenfrequencies and eigenfunctions.

**Remark 2.** The main assumption can be fulfilled in one of the following cases:

(i). The differential operator $A$ is formally selfadjoint, which takes place for $L^2(0,1)$ when $P^* = -P$ and $Q^* = Q$, or the operator may be selfadjoint in a weighted $L^2(0,1)$ space.

(ii). The term $P(x)$ by $d/dx$ is absent. In this case, the operator $A$ differs from the selfadjoint operator by a bounded operator and can apply Keldysh’ result (see, [6, Theorem 4.1, pp.170]).

The same approach can be used for the equation in the divergent form, see the Timoshenko beam model later.

(iii). There are no turning points ($R(x)$ has no equal entries everywhere on the interval). In this case, the perturbed family is quadratically close to a Riesz basis and is $\omega$-linearly independent because the family consists of eigenfunctions.

3. Moment problem

We can apply the moment approach to the controllability problem of the system (1), (2). The details are well known and can be found in [2], in particular in a scalar case, it is available in [8]. Let the solution to the equation under zero initial conditions be written as the series in terms of the family of eigenfunctions of $A$:
\[
\begin{aligned}
   y(x, t) &= \sum_{n=1}^{\infty} \sum_{j=1}^{N} y_{n,j}(t) \varphi_{n,j}(x), \\
   y(x, 0) &= y_t(x, 0) = 0.
\end{aligned}
\]

In what follows, we omit the subscript \( j \) when there is no confusion. Using the Fourier representation of the control term:

\[
Bv = \sum (Bv)_n(t) \varphi_n(x), \quad (Bv)_n = B_n = (Bv, \varphi_n)_H.
\]

Write (1) as the set of ordinary differential equations for the coefficient \( y_n(t) \):

\[
y_n(t) = -\omega_n^2 y_n(t) + B_n, \quad y_n(0) = 0, \quad n \in \mathbb{N}, \quad t \in [0, T].
\]

Let us introduce the sine-cosine family:

\[
\zeta_n^{(1)}(t) = \frac{1}{\omega_n} B^* \psi_n \sin \alpha_n t, \quad \zeta_n^{(2)} = B^* \psi_n \cos \alpha_n t.
\]

Setting \( \tilde{u}(t) = u(T - t) \) gives

\[
y_n(T) = (\tilde{u}, \zeta_n^{(1)})_{L^2(0,T;\mathbb{C}^N)}, \quad \dot{y}_n(T) = (\tilde{u}, \zeta_n^{(2)})_{L^2(0,T;\mathbb{C}^N)}.
\]

Now we introduce the powers of \( A \):

\[
A^r \sum a_n \varphi_n = \sum \lambda_n^r a_n \varphi_n.
\]

The domain \( \mathcal{H}^r \) of \( A^r \) coincides with the domain of \( A_0^r \) where \( A_0 = -d^2/dx^2 \) with the Dirichlet boundary conditions. Evidently,

\[
H_0^r(0,1) \subset \mathcal{H}^r(0,1) \subset H^r(0,1).
\]

Here we take the main branch of a power

\[
z^r = |z|^r e^{i \arg z}, \quad -\pi \leq \arg(z) \leq \pi.
\]

Let us introduce the spaces \( \ell_2^r \) and \( \ell_2'^1 \) of one-side and two-side sequences in \( \ell^2 \) with weights:

\[
a = \{a_n\}_1^\infty, \quad c = \{c_k\}_{k \in \mathbb{N} \cup \mathbb{N}},
\]

with the norms

\[
\|a\|_{\ell_2^r}^2 = \sum_{n=1}^{\infty} |a_n|^2 |\omega_n|^{2r}, \quad \|c\|_{\ell_2'^1}^2 = \sum_{k=\pm 1, \pm 2, \ldots} |c_k|^2 |\omega_k|^{2r}.
\]

Set \( \omega_n = -\omega_n \) and define the map \( \gamma_r : \)
\( \ell_r^2 \oplus \ell_{r-1}^2 \ni (a, b) \rightarrow \gamma_r(a, b) = c \in \widetilde{\ell}_{r-1}^2, \ c_k = -i\omega_k a[k] + b[k]. \)

Evidently, it is an isomorphism of \( \ell_r^2 \oplus \ell_{r-1}^2 \) and \( \widetilde{\ell}_{r-1}^2 \). For the spaces \( \mathcal{H}' \), this gives the isomorphism:

\[
\mathcal{H}' \oplus \mathcal{H}'^{-1} \ni \left( \sum a_n \varphi_n, \sum b_n \varphi_n \right) \gamma_r \rightarrow \sum_{k=1}^{\infty} \left( c_k \right) \varphi_k \in \widetilde{\mathcal{H}}^{-1},
\]

where

\[
\left\| \sum c_k \varphi[k] \right\|^2_{\mathcal{H}'^{-1}} = \sum_{k=\pm 1, \pm 2, \ldots} |c_k|^2 |\omega_k|^{2r-2}.
\]

Now, we go back to the vector exponential family. Take

\[
c = \gamma_r \left( \{y_n(T)\}, \{\dot{y}_n(T)\} \right).
\]

If we set

\[
\mathcal{E}^r = \{e_k^{(r)}\}_{k \in -\mathbb{N} \cup \mathbb{N}}, \ e_k^{(r)} = \tilde{\omega}_k e^{-i\tilde{\omega}_k t} B^* \psi[k],
\]

then, using the Euler formulas, we obtain the moment equalities

\[
d_k = \omega_k c_k = \omega_k' \left[ -i\omega_k y_n + \dot{y}_n \right] = \omega_k' \left[ -i\omega_k (\bar{u}, \xi_n^{(1)})_{L^2(0, T; \mathbb{C}^N)} + (\bar{u}, \xi_n^{(2)})_{L^2(0, T; \mathbb{C}^N)} \right]
\]

\[
= \omega_k' (\bar{u}, -i\omega_k \xi_n^{(1)} + \xi_n^{(2)})_{L^2(0, T; \mathbb{C}^N)} = (\bar{u}, e_k^{(r)})_{L^2(0, T; U)}, \ k \in -\mathbb{N} \cup \mathbb{N}.
\]  

The following Proposition 1 comes from [2, Th III.3.3].

**Proposition 1.** The following assertions hold true:

(i) The reachability set \( \mathcal{R}(T) \) of the system (1) considered in the space \( \mathcal{H}^{r+1} \oplus \mathcal{H}' \) is isometric to the set of sequences \( \{d_k\} \) in \( \ell_r^2 \) for which the moment problem (14) has a solution in \( L^2(0, T; \mathbb{C}^N) \).

(ii) The system is basis-controllable in time \( T \) in \( \mathcal{H}^{r+1} \oplus \mathcal{H}' \) if and only if \( \mathcal{E}^{(r)} \) forms a Riesz basis for its span.

We now clarify the relations with \( r \). From (14), we see that \( c \in \ell_r^2 \) if \( d \in \ell^2 \). As a result, the corresponding coefficients \( \{y_n(T)\} \) and \( \{\dot{y}_n(T)\} \) are in \( \ell_r^{2+1} \) and \( \ell_r^2 \) correspondingly. Or the state of the system at \( t = T \) is in \( \mathcal{H}^{r+1} \oplus \mathcal{H}' \).

Now we take \( r = -1 \) and obtain a moment problem with respect to

\[
\mathcal{E}^{-1} = \{e_k^{-1}\}_{k \in -\mathbb{N} \cup \mathbb{N}}, \ e_k^{-1} = \frac{1}{\omega_k} e^{-i\tilde{\omega}_k t} B^* \psi[k],
\]

and omit the superscript ‘-1’.

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4. Controllability

4.1. Boundary control

Firstly, we consider the control at the left end,

\[ y(0, t) = u, \quad y(1, t) = 0. \]

(15)

In this case, as well known,

\[ B^* f = f'(0) \]

and the exponential family is

\[ E = \left\{ \frac{1}{\bar{\omega}_{n,j}} \psi'_{n,j}(0) e^{\mp i\bar{\omega}_{n,j}t} \right\}_{n \in \mathbb{N}, j = 1, 2, \ldots, N}. \]

By Theorem 2, and Corollary 1, we conclude that this approximately normalized family is asymptotically close to the family

\[ E_0 = \left\{ \frac{1}{\sqrt{r_j(0)}} e^{\mp i\pi n/T_j \xi_j} \right\}_{n \in \mathbb{N}, j = 1, 2, \ldots, N}. \]

Add to this family \( N \) elements, precisely, the constant \( \mathbb{C}^N \)-basis elements \( \zeta_1, \zeta_2, \ldots, \zeta_N \).

The extended family forms an orthogonal basis in

\[ \mathcal{L}_0 = L^2(0, 2T_1) \oplus L^2(0, 2T_2) \oplus \cdots \oplus L^2(0, 2T_N). \]

These two facts give the Riesz basis property of \( \mathcal{E} \).

The following Proposition 2 is available in [10, Theorem 2] and [2, Theorem II.5.9].

**Proposition 2.** Suppose that the exponential family

\[ \mathcal{E}^0 = \left\{ e^{i\lambda_n^0 t} \eta_n^0 \right\} \]

forms a Riesz basis for \( \mathcal{L}_0 \), and the exponential family

\[ \mathcal{E} = \left\{ e^{i\lambda_n t} \eta_n \right\} \]

is asymptotically close to \( \mathcal{E}^0 \):

\[ |\lambda_n - \lambda_n^0| \rightarrow 0, \quad ||\eta_n - \eta_n^0||_{\mathbb{C}^N} \rightarrow 0. \]

Then, for any \( \varepsilon > 0 \), the family \( \mathcal{E} \) forms an \( \mathcal{L} \)-basis for the space

\[ \mathcal{L}_\varepsilon = L^2(0, 2T_1 + \varepsilon) \oplus L^2(0, 2T_2 + \varepsilon) \oplus \cdots \oplus L^2(0, 2T_N + \varepsilon) \].
By Proposition 1, we have the following Theorem 3, a first main result of the paper.

**Theorem 3.** For any $\varepsilon > 0$, the system (1), (2), (15) is controllable in the state space $L^2(0, 1; \mathbb{C}^N) \oplus H^{-1}(0, 1; \mathbb{C}^N)$ for $u \in \mathcal{L}_\varepsilon$ (the $j$-th channel control $u_j(t)$ supported on $[0, 2T_j + \varepsilon]$).

Next, we consider the control at both ends,

$$y(0, t) = u(t), \quad y(1, t) = v(t). \quad (16)$$

In this case, the controls take values in $U = \mathbb{C}^{2N}$ and the control operator $B^* : L^2(0, 1; \mathbb{C}^N) \rightarrow U$ is

$$B^* f = \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix}.$$

The exponential family arising in the moment approach is

$$\mathcal{E} = \{ \mathcal{E}_j \}_{j=1,2,\ldots,N}, \quad \mathcal{E}_j = \left\{ \frac{1}{\partial n,j} \left( \psi_{n,j}^{(0)}(0) - \psi_{n,j}^{(1)}(1) \right) e^{\mp i\bar{\omega}_{n,j} t} \right\}_{n \in \mathbb{N}}.$$

By Theorem 2, and Corollary 1, we conclude that this family is asymptotically close to the family

$$\mathcal{E}_0^{(\text{both})} = \{ \mathcal{E}_j^0 \}_{j=1,2,\ldots,N}, \quad \mathcal{E}_j^0 = \left\{ \frac{1}{\sqrt{r_j(0)}} e^{\pm \pi n t / T_j} \left( \begin{array}{c} \xi_j \\ \bar{\xi}_j \end{array} \right) \right\}_{n \in \mathbb{N}}.$$

The latter family forms an orthogonal $\mathcal{L}$-basis for

$$\mathcal{L}_0^{(\text{both})} = L^2(0, T_1) \oplus L^2(0, T_2) \oplus \cdots \oplus L^2(0, T_N)$$

with codimension $2N$. If we add $2N$ basis elements

$$\left( \begin{array}{c} \xi_j \\ \bar{\xi}_j \end{array} \right), \quad j = 1, 2, \ldots, N,$$

we will have an orthogonal $\mathcal{L}$-basis for $\mathcal{L}_0^{(\text{both})}$. Similarly to Theorem 3, we obtain succeeding Theorem 4.

**Theorem 4.** For any $\varepsilon > 0$, the system (1), (2), (16) is controllable in the state space $L^2(0, 1; \mathbb{C}^N) \oplus H^{-1}(0, 1; \mathbb{C}^N)$ for

$$u \in L^2(0, T_1 + \varepsilon) \oplus L^2(0, T_2 + \varepsilon) \oplus \cdots \oplus L^2(0, T_N + \varepsilon)$$

(the $j$-th channel controls $u_j$ and $v_j$ supported on $[0, T_j + \varepsilon]$).
4.2. Distributed control

Consider a control of the form

\[ Bu = \sum_{j=1}^{N} u_j(t) b_j(x) \]  

(17)

with fixed profiles \( b_j(x) \in \mathbb{C}^N \) and controls \( u_1(t), u_2(t), \ldots, u_N(t) \). In the scalar case, the general results for a selfadjoint operator \( A \) can be found in [2, Th. V.1.3]. Here we choose the profiles in order to obtain sharp result for the multichannel system.

**Theorem 5.** For any \( r \geq 0 \), there are profiles \( b_1(x), b_2(x), \ldots, b_N(x) \) such that the system (1), (2), (15), (17) is basis-controllable in the state space \( \mathcal{H}^{r+1} \oplus \mathcal{H}^r \) for the control in \( L_0^0 \) given by (3).

**Proof.** Take

\[ b_j(x) = \sum_{n=1}^{\infty} \frac{1}{n^{r}} f_{n,j}(x). \]

and find the operator \( B^* : H \rightarrow \mathbb{C}^N \). For \( a = (a_1, a_2, \ldots, a_N)^\top \in \mathbb{C}^N \) and \( f \in H \),

\[ \langle Ba, f \rangle_H = \int_0^1 \left\langle \sum b_j(x) a_j, f(x) \right\rangle_{\mathbb{C}^N} dx = \sum a_j \int_0^1 \left\langle b_j(x), f(x) \right\rangle_{\mathbb{C}^N} dx = \langle a, B^* f \rangle_{\mathbb{C}^N} \]

with

\[ B^* f = \begin{pmatrix} \int_0^1 \langle b_1(x), f(x) \rangle_{\mathbb{C}^N} dx \\ \int_0^1 \langle b_2(x), f(x) \rangle_{\mathbb{C}^N} dx \\ \vdots \\ \int_0^1 \langle b_N(x), f(x) \rangle_{\mathbb{C}^N} dx \end{pmatrix}^\top. \]

This gives

\[ B^* \psi_{n,j} = \frac{1}{n^{r}} \zeta_j. \]

The exponential family \( \mathcal{E}^r \) is an union of \( N \) orthogonal families:

\[ \mathcal{E}^r = \bigcup_{j=1}^{N} \mathcal{E}_j, \quad \mathcal{E}_j = \left\{ \frac{\tilde{\omega}_k,j}{|k|^{r}} e^{-i\tilde{\omega}_k,j t} \zeta_j : k \in \mathbb{N} \cup \mathbb{N} \right\}. \]

The factors \( \omega_k,j/|k|^r \) are bounded and separated from zero, and the scalar exponential family \( \{ e^{i\omega_k,j t} \} \) forms a Riesz basis for its span in \( L^2(0, 2T_j) \) (with unit codimension) by (12). Therefore, the family \( \mathcal{E}^r \) with \( N \mathbb{C}^N \)-basis elements forms a Riesz basis for \( \mathcal{L}_0 \). The result then follows from Proposition 1 by the fact that the exponentials are close to harmonics. \( \Box \)
5. Controllability of Timoshenko beam

In previous sections, we have discussed a class of second order systems. Similar approach can be used to study Timoshenko beam which is not obviously transformable to be the form of (1) yet can be dealt with by the similar approach after transforming into a first order system.

5.1. Problem statement and spectral analysis

On the span \((0, 1)\) of the spatial variable, we consider a Timoshenko beam equation with two-dimensional distributed control and loading:

\[
\begin{align*}
\rho(x)w_{tt}(x,t) - (K((w_x(x,t) - \psi(x,t)))_x &= g_1(x)u_1(t), \\
I_\rho(x)\psi_{tt}(x,t) - (EI\psi_x(x,t))_x - K(w_x(x,t) - \psi(x,t)) &= g_2(x)u_2(t),
\end{align*}
\]

where the function \(w(x,t)\) is the transverse displacement of the beam and \(\psi(x,t)\) is the rotation angle of a filament of the beam. The physical parameters are \(\rho(x)\), the mass density per unit length, \(E\), Young’s modulus of elasticity, \(I(x)\), the moment of inertia of a cross section of the beam, \(I_\rho(x)\), the polar moment of inertia of a cross section, and \(K(x)\), the shear modulus. As above we impose the Dirichlet boundary condition at the left end:

\[w(0, t) = \psi(0, t) = 0.\]

At \(x = 1\), we take the following boundary conditions:

\[K(\psi - w_x)|_{x=1} = \alpha w_t|_{x=1}, \quad EI\psi_x|_{x=1} = -\beta \psi_t|_{x=1},\]

(18)

where \(\alpha\) and \(\beta\) are complex parameters, and when \(\alpha = \beta = 0\), it is the free end. In particular, if \(\Re \alpha \geq 0\), \(\Re \beta \geq 0\) and \(\Re \alpha + \Re \beta > 0\), then, the homogeneous system is dissipative with respect to the following energy

\[
E(t) = \frac{1}{2} \int_0^1 \left[ \rho|w_t|^2 + I_\rho|\psi_t|^2 + K|\psi - w_x|^2 + EI|\psi_x|^2 \right] dx.
\]

(19)

In the previous section, we show how to choose the profiles to obtain a basis controllability. Now we will study the control time \(T\) and condition on the initial values

\[
w|_{t=0} = w_0, \quad w_t|_{t=0} = w_1, \quad \psi|_{t=0} = \psi_0, \quad \psi_t|_{t=0} = \psi_1,
\]

such that the system can be steered to rest in time \(T\) (the null controllability). We assume that the functions \(I_\rho(x)\), \(K(x)\), \(I(x)\) are positive, bounded from zero:

\[I_\rho(x), K(x), I(x) \geq C > 0\]

and are smooth enough.
In contrast to the control system considered in previous sections, the time derivatives here enter the boundary condition (18), and it is therefore natural to consider the system of the first order in time by introducing

\[
y(x, t) = \begin{pmatrix} w(x, t) \\ \psi(x, t) \end{pmatrix}, \quad v(x, t) = \begin{pmatrix} y(x, t) \\ y_\psi(x, t) \end{pmatrix} = \begin{pmatrix} v_1(x, t) \\ v_2(x, t) \\ v_3(x, t) \\ v_4(x, t) \end{pmatrix} = \begin{pmatrix} w(x, t) \\ \psi(x, t) \\ \psi_t(x, t) \end{pmatrix}.
\]

We have an evolution system of the first order in time:

\[
v_t = Av = \begin{pmatrix} y_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ A & 0 \end{pmatrix} \begin{pmatrix} y \\ \psi \end{pmatrix},
\]

where \( A \) is the differential operator with respect to \( x \):

\[
Ay = \begin{pmatrix} \frac{[K(w_x - \psi)_x/\rho}{[(EI\psi_x)_x + K(w_x - \psi)]/\rho} \\ [(EI\psi)_x + K((v_1)_x - v_2)]/\rho \end{pmatrix} = \begin{pmatrix} \frac{[K((v_1)_x - v_2)_x/\rho}{[(EI\psi)_x + K((v_1)_x - v_2)]/\rho} \end{pmatrix}.
\]

The operator \( A \) acts in the state space \( H \):

\[
H = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \right\} \subset H^1(0, 1; \mathbb{C}^2), \quad \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = 0, \quad \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} \in L^2(0, 1; \mathbb{C}^2),
\]

with the norm given in (19). The domain \( D(A) \) is

\[
D(A) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H^2(0, 1; \mathbb{C}^2), \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(0) = 0, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(1) = 0 \right\} \subset H^1(0, 1; \mathbb{C}^2),
\]

at \( x = 1 \) \( K(v_2 - (v_1)_x) = \alpha v_3, \quad EI(v_2)_x = \beta v_4 \} \}

It is more convenient to consider the operator

\[
\mathcal{A} = -i A,
\]

with which, the evolution system now becomes

\[
v_t = i\mathcal{A}v, \quad v(0) = v_0.
\]

Recall that the velocity of the system are

\[
\sqrt{\frac{K(x)}{\rho(x)}}, \quad \sqrt{\frac{EI(x)}{I_\rho(x)}}.
\]

In the previous sections, we allow the turning points, but here we assume that the velocities are different.
\[ K(x)/\rho(x) \neq EI(x)/I_{\rho}(x), \quad x \in [0, 1]. \quad (22) \]

This assumption is natural from the elasticity theory. Evidently, this assumption does not change the main terms in the asymptotic expansion, but the infinitesimal error terms – \( O(1/n) \) can be necessary for the main assumption about the Riesz generation.

The following Proposition 3 is straightforward because its proof consists of direct calculations of the adjoint operator.

**Proposition 3.** The adjoint operator \( \mathcal{A}^* \) is a differential one with the same differential expression as \( \mathcal{A} \); its domain has the form (20) if we replace \( \alpha \) by \( -\bar{\alpha} \) and \( \beta \) by \( -\bar{\beta} \).

Now, we find the asymptotic expansions of the eigenvalues and the eigenfunctions of \( \mathcal{A} \) along the line of Theorem 1 because we have the another different boundary conditions (18). A different method used in [19] seems too complicated for our problem.

Comparing the following 2 \( \times \) 2 matrix equation

\[ Af = -\omega^2 f, \quad f|_0 = 0, \quad f_x|_0 = \omega I_2, \quad (23) \]

with (6) and (7), we find from Theorem 1 that

\[ f = \begin{pmatrix} L_1^{-1}(0)\gamma_1(x) \sin \omega t_1(x) + O(\omega^{-1}) \\ O(\omega^{-1}) \end{pmatrix}, \quad L_2^{-1}(0)\gamma_1(x) \sin \omega t_1(x) + O(\omega^{-1})] \quad (24) \]

Recall the notation

\[ L_1(x) = \sqrt{\frac{\rho(x)}{K(x)}}, \quad L_2 = \sqrt{\frac{I_{\rho}(x)}{EI(x)}}, \]
\[ \gamma_1(x) = \frac{K(x)}{K(0)} e^{L_1(x) - L_1(0)}, \quad \gamma_2(x) = \frac{I(x)}{I(0)} e^{L_2(x) - L_2(0)}, \]
\[ t_j(x) = \int_0^x L_j(s) \, ds. \]

It is more convenient to consider 6 \( \times \) 2 matrix

\[ (h^{(1)} \quad h^{(2)}) = \begin{pmatrix} f \\ f_x \\ i\omega f \end{pmatrix}, \]

with \( f \) being given from (23).
Then,

\[
\begin{pmatrix}
    h^{(1)} & h^{(2)} \\
    L_1^{-1}(0)\gamma_1(x)\sin \omega t_1(x) + O(\omega^{-1}) & O(\omega^{-1}) \\
    O(\omega^{-1}) & L_2^{-1}(0)\gamma_2(x)\sin \omega t_2(x) + O(\omega^{-1}) \\
    \omega L_1(x)L_1^{-1}(0)\gamma_1(x)\cos \omega t_1(x) + O(1) & \omega L_2(x)L_2^{-1}(0)\gamma_2(x)\cos \omega t_2(x) + O(\omega^{-1}) \\
    O(1) & O(1) \\
    i\omega L_1^{-1}(0)\gamma_1(x)\sin \omega t_1(x) + O(1) & i\omega L_2^{-1}(0)\gamma_2(x)\sin \omega t_2(x) + O(1)
\end{pmatrix}
\]

(25)

The functionals \(l_1(\cdot)\) and \(l_2(\cdot)\) corresponding the boundary condition (18) are acting on the 6-tuples \(f^{(1)}, f^{(2)}\) as follows:

\[
\begin{align*}
    l_1(h^{(j)}) &= K(1)(h_2^{(j)}(1) - h_3^{(j)}(1)) - \alpha h_5^{(j)(1)} \\
    l_2(h^{(j)}) &= EI(1)h_4^{(j)(1)} + \beta h_6^{(j)(1)}. 
\end{align*}
\]

The eigenfunction \(\varphi_\omega\) of the operator \(\mathcal{A}\) corresponding to the eigenvalue \(\varphi_\omega\) is a linear combination of \(h^{(1)}\) and \(h^{(2)}\), satisfying

\[
l_1(\varphi_\omega) = l_2(\varphi_\omega) = 0.
\]

This is possible if and only if the determinant of the matrix

\[
\mathfrak{E} = \begin{pmatrix}
    l_1(h^{(1)}) & l_1(h^{(2)}) \\
    l_2(h^{(1)}) & l_2(h^{(2)})
\end{pmatrix}
\]

is zero. Introduce the optical length of the channels with respect to velocities.

\[
T_\alpha = t_1(1), \quad T_\beta = t_2(1).
\]

From (25), we obtain

\[
\begin{align*}
    \mathfrak{E}_{11} &= -\omega K(1)L(1)L_1^{-1}(0)\gamma_1(1)\cos \omega T_\alpha - i\omega \alpha L_1^{-1}(0)\gamma_1(1)\sin \omega T_\alpha + O(1), \\
    \mathfrak{E}_{12} &= O(1), \\
    \mathfrak{E}_{21} &= O(1), \\
    \mathfrak{E}_{22} &= \omega EI(1)L_2(1)L_2^{-1}(0)\gamma_2(1)\cos \omega T_\beta + i\alpha L_1^{-1}(0)\gamma_2(1)\sin \omega T_\beta + O(1),
\end{align*}
\]

from which, we find

\[
\det \mathfrak{E}(\omega) = \mathfrak{E}_{11}\mathfrak{E}_{22} + O(1) = \omega^2 c X_1(\omega)X_2(\omega) + O(1),
\]

where \(c = -\gamma_1(1)\gamma_2(1)L_1^{-1}(0)L_2^{-1}(0)\) and

\[
\begin{align*}
    X_1(\omega) &= K(1)L(1)\cos \omega T_\alpha + i\alpha \sin \omega T_\alpha \\
    &= \sqrt{K(1)\rho(1)}\cos \omega T_\alpha + i\alpha \sin \omega T_\alpha, \\
    X_2(\omega) &= EI(1)L_2(1)\cos \omega T_\beta + i\beta \sin \omega T_\beta \\
    &= \sqrt{T_\beta(1)EI(1)}\cos \omega T_\beta + i\beta \sin \omega T_\beta.
\end{align*}
\]
Let $X_1(\omega) = 0$. If

$$\sqrt{K(1)\rho(1)} \neq \pm \alpha, \quad (26)$$

then its roots satisfy

$$e^{2i\omega T_\alpha} = \frac{\alpha - \sqrt{K(1)\rho(1)}}{\alpha + \sqrt{K(1)\rho(1)}}. \quad (27)$$

For the second factor for

$$\sqrt{I_\rho(1)EI(1)} \neq \pm \beta \quad (28)$$

we obtain the relation

$$e^{2i\omega T_\beta} = \frac{\beta - \sqrt{I_\rho(1)EI(1)}}{\beta + \sqrt{I_\rho(1)EI(1)}}. \quad (29)$$

**Theorem 6.** Let the parameters satisfy (22), (26), (28). The set $\Omega$ of the eigenvalues of the operator $Q$ can be written as two sequences $\Omega = \Omega_\alpha \cup \Omega_\beta$, $\Omega_\alpha = \left\{ \frac{\pi}{T_\alpha} n + \frac{i}{2T_\alpha} \ln \frac{\alpha + \sqrt{K(1)\rho(1)}}{\alpha - \sqrt{K(1)\rho(1)}} + O(1/n), \ n \in \mathbb{Z} \right\}$, $\Omega_\beta = \left\{ \frac{\pi}{T_\beta} n + \frac{i}{2T_\beta} \ln \frac{\beta + \sqrt{I_\rho(1)EI(1)}}{\beta - \sqrt{I_\rho(1)EI(1)}} + O(1/n), \ n \in \mathbb{Z} \right\} \quad (30)$

The corresponding set $\Phi = \{\varphi_\omega\}$ of approximately normalized eigenfunctions is

$$\Phi = \Phi_\alpha \cup \Phi_\beta,$$

$$\Phi_\alpha = \left\{ \begin{pmatrix} \omega^{-1} \gamma_1(x) \sin \omega t_1(x) + O(\omega^{-2}) \\ O(\omega^{-2}) \\ i \gamma_1(x) \sin \omega t_1(x) + O(\omega^{-1}) \\ O(\omega^{-1}) \end{pmatrix} \right\}, \ \omega \in \Omega_\alpha \quad (31)$$

$$\Phi_\beta = \left\{ \begin{pmatrix} O(\omega^{-2}) \\ \omega^{-1} \gamma_2(x) \sin \omega t_2(x) + O(\omega^{-2}) \\ O(\omega^{-1}) \\ i \gamma_2(x) \sin \omega t_2(x) + O(\omega^{-1}) \end{pmatrix} \right\}, \ \omega \in \Omega_\beta \quad (32)$$

**Proof.** The main term in the asymptotic expansion of the eigenvalues coincides with the zeros of functions (27) and (29). Similarly, from the asymptotic expansions of $h(1)$ and $h^{(2)}$, we may conclude that the linear combination $\varphi_\omega$ satisfying $l_1(\varphi_\omega) = l_2(\varphi_\omega) = 0$ is the function in (31) up to a normalization. □

From Proposition 3 we have subsequent Corollary 2.
Corollary 2. (i) The set $\Omega^*$ of the eigenvalues of the operator $\mathcal{A}^*$ is $-\overline{\Omega}$. 
(ii) The asymptotic expansion of eigenfunctions $\Psi = \{\psi_\omega\} = \{\Phi_{\alpha}^*, \Phi_{\beta}^*\}$ of $\mathcal{L}^*$ is given by (31) if we replace $\alpha$ by $-\bar{\alpha}$ and $\beta$ by $-\bar{\beta}$.

Lemma 2. Let the family $\mathcal{F}$ be defined by

$$\mathcal{F} = \left\{ \begin{pmatrix} e_n \\ e_{-n} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ i \sin \frac{\pi}{T} n \xi \end{pmatrix} : n \in \mathbb{Z} \right\}. \quad (32)$$

Then, $\mathcal{F}$ forms a Riesz basis for $L^2(0, T; \mathbb{C}^2)$.

Proof. Under the isomorphism

$$O \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u + v \\ u - v \end{pmatrix},$$

the family $\mathcal{F}$ becomes

$$\tilde{\mathcal{F}} = \left\{ \begin{pmatrix} e_n \\ e_{-n} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ i \sin \frac{\pi}{T} n \xi \end{pmatrix} : n \in \mathbb{Z} \right\}. \quad (33)$$

Since the sine- and cosine-families

$$\left\{ \cos \frac{\pi}{T} n \xi, n = 0, 1, 2, \ldots \right\}, \left\{ \sin \frac{\pi}{T} n \xi, n = 1, 2, \ldots \right\}, \quad (33)$$

are basis of $L^2(0, T)$, we can write any $(f, g) \in L^2(0, T; \mathbb{C}^2)$ as

$$f(\xi) = \sum_{n=0}^{\infty} f_n \cos \frac{\pi}{T} n \xi, \quad g(\xi) = \sum_{n=1}^{\infty} g_n \sin \frac{\pi}{T} n \xi, \quad \{a_n\}, \{b_n\} \in \ell^2. \quad (34)$$

Suppose

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{n=-\infty}^{\infty} a_n \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ i \sin \frac{\pi}{T} n \xi \end{pmatrix}, \quad (35)$$

which is amount to

$$\sum_{n=-\infty}^{-1} a_n \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ i \sin \frac{\pi}{T} n \xi \end{pmatrix} + \sum_{n=0}^{\infty} a_n \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ i \sin \frac{\pi}{T} n \xi \end{pmatrix} = \sum_{n=1}^{\infty} a_{-n} \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ -i \sin \frac{\pi}{T} n \xi \end{pmatrix} + \sum_{n=0}^{\infty} a_n \begin{pmatrix} \cos \frac{\pi}{T} n \xi \\ i \sin \frac{\pi}{T} n \xi \end{pmatrix}. \quad (36)$$

Or

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} a_0 + \sum_{n=1}^{\infty} (a_n + a_{-n}) \cos \frac{\pi}{T} n \xi \\ \sum_{n=1}^{\infty} i(a_n - a_{-n}) \sin \frac{\pi}{T} n \xi \end{pmatrix}. \quad (37)$$
Comparing (34) and (35), we find that \( a_0 = f_0 \) and for \( n \neq 0 \),

\[
\begin{align*}
a_{-n} + a_n &= f_n, \\
i(-a_{-n} + a_n) &= g_n,
\end{align*}
\]

which have solutions

\[
\begin{pmatrix} a_{-n} \\ a_n \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix}
\]

and hence \( \{a_{-n}\}, \{a_n\} \in \ell^2 \).

This completes the proof of the lemma. \( \Box \)

**Theorem 7.** All eigenvalues of \( A \) with large modulus are semisimple with multiplicity not exceeding two, and there is a family of generalized eigenfunctions of \( A \), which forms a Riesz basis for \( H \).

**Proof.** Let us introduce the family

\[
\Phi^0 = \Phi^0_\alpha \cup \Phi^0_\beta,
\]

\[
\Phi^0_\alpha = \left\{ \begin{pmatrix} \omega^{-1} \gamma_1(x) \sin \omega t_1(x) \\ 0 \\ i \gamma_1(x) \sin \omega t_1(x) \\ 0 \end{pmatrix}, \omega \in \Omega^0_\alpha \right\},
\]

\[
\Phi^0_\beta = \left\{ \begin{pmatrix} 0 \\ \omega^{-1} \gamma_2(x) \sin \omega t_2(x) \\ 0 \\ i \gamma_2(x) \sin \omega t_2(x) \end{pmatrix}, \omega \in \Omega^0_\beta \right\}.
\]

We will show that the family \( \Phi^0 \) forms a Riesz basis for \( H \). For simplicity, set

\[
\mathcal{F}_0 = \left\{ \begin{pmatrix} \omega^{-1} \gamma(x) \sin \omega t(x) \\ i \gamma(x) \sin \omega t(x) \end{pmatrix}, \omega \in \Omega^0 \right\},
\]

\[
\Omega^0 = \frac{\pi}{T} n + i \tau, \: n \in \mathbb{Z}
\]

Here \( \gamma(x) \) is a smooth function, bounded from zero and \( t(x) \) is a smooth function such that \( t(0) = 0, t'(x) > 0 \) for \( x \in (0, 1) \) and \( T = t(1) \). In other words, we omit the subscripts \( j \) in \( \gamma_j(x) \) and \( t_j(x) \) in the family \( \Phi^0 \). Introduce also

\[
H_0 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix}, \: u \in H^1(0, 1), \: u(0) = 0; \: v \in L^2(0, 1) \right\}.
\]

We claim that

- The family \( \mathcal{F}_0 \) forms a Riesz basis for \( H_0 \)


which leads obviously to that $\Phi_0^0$ forms a Riesz basis for $H$.

The proof of the claim consists of several isomorphic transformations in Hilbert spaces, which give finally a family of harmonics of the exponentials of $i\pi n/T$. First, the map

$$\mathcal{O}_1 \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \frac{u}{\gamma} \right) \left( \begin{array}{c} \frac{u}{\gamma} \\ \frac{v}{\gamma} \end{array} \right),$$

is an isomorphism between the space $H_0$. Thus, $\mathcal{F}_0$ is a Riesz basis for $H_0$ if and only if the family

$$\mathcal{F}_1 = \mathcal{O}_1 \mathcal{F}_0 = \left\{ \left( \begin{array}{c} \omega^{-1} \sin \omega t(x) \\ i \sin \omega t(x) \end{array} \right), \ \omega \in \Omega^0 \right\},$$

is a Riesz basis for $H_0$.

The second map is the change of variable $\xi = \int_0^x t(y)dy$ (as in the Liouville transformation of a second order differential equation):

$$\left[ \mathcal{O}_2 \left( \begin{array}{c} u \\ v \end{array} \right) \right](\xi) = \left( \begin{array}{c} u \\ v \end{array} \right)(t(-1)(\xi)).$$

This map is an isomorphism of the spaces $H_0$ and $H_1$ where

$$H_1 = \left( \begin{array}{c} u \\ v \end{array} \right), \ u \in H^1(0, T), \ u(0) = 0; \ v \in L^2(0, T).$$

In this step, we have the family

$$\mathcal{F}_2 = \mathcal{O}_2 \mathcal{F}_1 = \left\{ \left( \begin{array}{c} \omega^{-1} \sin \omega \xi \\ i \sin \omega \xi \end{array} \right), \ \omega \in \Omega^0 \right\}.$$

The third map is

$$\mathcal{O}_3 \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} u' \\ v \end{array} \right),$$

which is an isomorphism between spaces $H_1$ and $L^2(0, T, \mathbb{C}^2)$. The obtained family after third map is

$$\mathcal{F}_3 = \mathcal{O}_3 \mathcal{F}_2 = \left\{ \left( \begin{array}{c} \cos \omega \xi \\ i \sin \omega \xi \end{array} \right), \ \omega \in \Omega^0 \right\}.$$

Let us go further from sines and cosines family to the family of exponentials. The fourth map

$$\mathcal{O}_4 \left( \begin{array}{c} u \\ v \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right)$$

is an isometry of the space $H_3$. Evidently
\[ F_4 = \mathcal{O}_4 F_3 = \frac{1}{2} \left\{ \left( e^{i\omega \xi}, \omega \in \Omega \right) \right\}. \]

The fifth map gives exponentials of imaginary values:
\[ \mathcal{O}_5 \left( \begin{array}{c} u \\ v \end{array} \right)(\xi) = 2 \left( \begin{array}{c} u(\xi) e^{i\xi} \\ v(\xi) e^{-i\xi} \end{array} \right), \]
which is an isomorphism in the space \( L^2(0, T; \mathbb{C}^2) \). Evidently
\[ \tilde{F} = \mathcal{O}_4 F_4 = \left\{ \left( e_n, e_{-n} \right) \right\} = \left\{ \left( e^{i\pi n \xi}, e^{-i\pi n \xi} \right), n \in \mathbb{Z} \right\}. \]

All the maps are isomorphic and then \( F_0 \) forms a Riesz basis for \( H_0 \) if and only if \( \tilde{F} \) forms a Riesz basis for \( L^2(0, T; \mathbb{C}^2) \) which has been confirmed by Lemma 2.

Therefore, under (22), the set of eigenfunctions \( \{ \Phi \} \) is quadratically close to the Riesz basis \( \{ \Phi^0 \} \). By [8, Theorem 2.38], there is a family of generalized eigenfunctions of \( A \), which forms a Riesz basis for \( H \). Since \( \{ \Phi \} \) consists of eigenfunctions only for large modulus, the eigenvalues with large modulus must be semisimple. From (30), the multiplicity of all eigenvalues with large modulus is less than or equal to two. This completes the proof of the theorem. \( \Box \)

### 5.2. Moment problem, exponentials, and controllability

Consider the following control problem:
\[ v_t = iA v + B u, \quad v|_{t=0} = v_0, \]
where the control operator \( B \) is defined as
\[ \mathbb{C}^2 \ni \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} 0 \\ 0 \\ c_1 g_1(x) \\ c_2 g_2(x) \end{pmatrix} \in H. \]

From the relation
\[ \langle B c, h \rangle_H = \begin{pmatrix} 0 \\ 0 \\ c_1 g_1(x) \\ c_2 g_2(x) \end{pmatrix} \begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \\ h_4(x) \end{pmatrix} = c_1 \langle g_1, h_3 \rangle_{L^2(0,1)} + c_2 \langle g_2, h_4 \rangle_{L^2(0,1)} \]
\[ = \left\langle c, \begin{pmatrix} (h_3, g_1)_{L^2(0,1)} \\ (h_4, g_2)_{L^2(0,1)} \end{pmatrix} \right\rangle_{\mathbb{C}^2}, \]
we see that
\[
\begin{pmatrix}
  h_1(x) \\
  h_2(x) \\
  h_3(x) \\
  h_4(x)
\end{pmatrix} \in H \rightarrow \begin{pmatrix}
  (h_3, g_1)_{L^2(0,1)} \\
  (h_4, g_2)_{L^2(0,1)}
\end{pmatrix} \in \mathbb{C}^2.
\]

Introduce the vector exponential family

\[
\mathcal{E} = \{e_{\omega}(t)\} = \left\{ \eta_\omega e^{-i\omega t} \right\} |_{\omega \in \Omega}, \quad \eta_\omega = B^* \varphi_\omega = \left( (\varphi_\omega)_3, g_1 \right)_{L^2(0,1)}, \varphi_\omega \in \Phi,
\]

where \( \Phi \) is the family of approximately normalized eigenfunctions with the asymptotic behavior given in Theorem 6. By Theorem 7, this family forms a Riesz basis for \( H \) and we can write the solution of (21) as

\[
v(x, t) = \sum_{\omega \in \Omega} c_{\omega}(t) \varphi_\omega(x), \quad v_0(x) = \sum_{\omega \in \Omega} c^0_{\omega} \varphi_\omega(x).
\]

Also write the control term as

\[
Bu(x, t) = \sum B_\omega(t) \varphi_\omega(x), \quad B_\omega(t) = \langle Bu, \psi_\omega \rangle_H = \langle u(t), B^* \psi_\omega \rangle_{L^2}, \psi_\omega \in \Psi,
\]

where \( \Psi \) was defined in Corollary 2. The coefficients \( c_{\omega}(t) \) satisfy ordinary differential equations:

\[
\dot{c}_{\omega}(t) = i \omega c_{\omega}(t) + B_\omega(t), \quad c_{\omega}(0) = c^0_{\omega}, \quad \omega \in \Omega.
\]

The solution meeting the condition \( c_{\omega}(T) = 0 \) satisfies the moment equalities

\[
\langle u, e_{\omega} \rangle_{L^2(0,T;C^2)} = c^0_{\omega}, \quad e_{\omega}(t) = e^{-i\omega t} \eta_\omega,
\]

with respect to the exponential family \( \mathcal{E} \).

In [18] the conditions on the profile function and the initial data were considered, for which the system can be steered to rest using only one control. It was found the sharp control time \( T = 2(T_\alpha + T_\beta) \). For the vector control, we can find the condition for controllability with the control space

\[
\mathcal{U}(\varepsilon) = \begin{pmatrix}
  L^2(0, 2T_\alpha + \varepsilon) \\
  L^2(0, 2T_\beta + \varepsilon)
\end{pmatrix}
\]

for any positive \( \varepsilon \).

**Theorem 8.** Let the initial data satisfy

(i) for \( \omega \in \Omega_\alpha \), the first component \( (\eta_\omega)_1 \) is not zero, and the sequence

\[
\{c^0_{\omega}/(\eta_\omega)_1\},
\]

is quadratically summarable, and
\[(\eta_\omega)_2 = o(|(\eta_\omega)_1|), \quad (38)\]

ii) for \( \omega \in \Omega_\beta \) the second component \((\eta_\omega)_2\) is not zero, and the sequence

\[\{e_\omega^0/(\eta_\omega)_2\},\]

is quadratically summable, and

\[(\eta_\omega)_1 = o(|(\eta_\omega)_2|). \quad (39)\]

Then, the system can be steered to rest with control from (37)

**Proof.** By Propositions 1, 2, and the proof is reduced to the study of the exponentials families. Let us introduce the exponential family

\[E_0 = E_0^\alpha \cup E_0^\beta, \quad E_{0\alpha} = \{e_\omega/(\eta_\omega)_1\}_{\omega \in \Omega_\alpha}, \quad E_{0\beta} = \{e_\omega/(\eta_\omega)_2\}_{\omega \in \Omega_\beta}.\]

The moment equalities (36) can be rewritten as a moment problem with respect to \(E_0\):

\[
\langle u, e_\omega/(\eta_\omega)_1 \rangle_{L^2(0,T;C^2)} = c_\omega^0/(\eta_\omega)_1, \quad \omega \in \Omega_\alpha
\]
\[
\langle u, e_\omega/(\eta_\omega)_2 \rangle_{L^2(0,T;C^2)} = c_\omega^0/(\eta_\omega)_2, \quad \omega \in \Omega_\beta. \quad (40)
\]

By the assumption, this family is approximately normalized because the imaginary parts of points from \(\Omega_\alpha\) and \(\Omega_\beta\) are bounded. Moreover, the family is asymptotically closed to the family

\[E_{0\alpha} = \left\{e^{-i\omega t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}_{\omega \in \Omega_{\alpha}^0} \cup \left\{e^{-i\omega t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}_{\omega \in \Omega_{\beta}^0},\]

where \(\Omega_{\alpha}^0\) and \(\Omega_{\beta}^0\) are the main parts of the asymptotics in Theorem 6:

\[\Omega_{\alpha}^0 = \left\{\frac{2\pi}{T_\alpha} n + c_\alpha, \ n \in \mathbb{Z}\right\},\]
\[\Omega_{\beta}^0 = \left\{\frac{2\pi}{T_\beta} n + c_\beta, \ n \in \mathbb{Z}\right\}.

Obviously, the latter family is a Riesz basis for the control space

\[\mathcal{U}(0) = \left(L^2(0, 2T_\alpha), L^2(0, 2T_\beta)\right)\]

By Proposition 2, we obtain a Riesz basis for its span in \(\mathcal{U}(\varepsilon)\). Then moment problem (40) has a solution in \(\mathcal{U}(\varepsilon)\), if and only if the RHS is in \(\ell^2\). \(\square\)

**Remark 3.** a) The condition (22) is used to have the Riesz basis property of the operator. If we know that the operator has a Riesz basis of eigenspaces, we can allow the turning points of the system as it was done in Lemma 1.
b) The conditions (38) and (39) are not too restrictive because, say, for $\omega \in \Omega_\alpha$, we have

$$(\eta_\omega)_1 = (g_1, (\varphi_\omega)_3)_{L^2}, \quad (\eta_\omega)_2 = (g_1, (\varphi_\omega)_4)_{L^2},$$

and $(\varphi_\omega)_4$ are asymptotically as small as $O(1/\omega)$ with respect to $(\varphi_\omega)_3$.

c) The eigenvalues lie near two lines parallel to the real axis. If

$$\Re \left[ \frac{1}{T_\beta} \log \frac{\beta + \sqrt{I_\rho(1)EI(1)}}{\beta - \sqrt{I_\rho(1)EI(1)}} \right] = \Re \left[ \frac{1}{T_\alpha} \log \frac{\alpha + \sqrt{K(1)\rho(1)}}{\alpha - \sqrt{K(1)\rho(1)}} \right],$$

then these lines coincide and the spectrum is not separated:

$$\inf_{\omega, \omega' \in \Omega, \omega \neq \omega'} |\omega - \omega'| = 0.$$

For the system with one control (one of $u_1(t)$ and $u_2(t)$ is zero), this leads to lack of the exact controllability because the scalar exponential family is not an $L$-basis. Nevertheless, in general case, the system is spectral controllable: It is possible to transfer any eigenmode $(\varphi_\omega)_i$ to rest. The case of ratio $T_\alpha/T_\beta$ being rational has been studied in [17] for a rotating Timoshenko beam with one control.

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References


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