Mathl. Comput. Modelling Vol. 18, No. 2, pp. 63–74, 1993
Printed in Great Britain.

0895-7177/93 $6.00 + 0.00

Rothe Approximation to an Ablation-Transpiration Cooling Control System

BAO ZHU GUO
Institute of Systems Science, Academia Sinica
Beijing 100080, P.R. China

(Received September 1992; accepted October 1992)

Abstract—A controlled ordinary differential equation model of an ablation-transpiration cooling system is considered, based on the Rothe approximation scheme for the corresponding partial differential equation model in which the thermal swelling is neglected. It is different from the by-lines approximation method proposed in [1]. The convergence is proved under the estimates of the approximation solutions, and the numerical results of an experimental example are presented.

Keywords—Ablation-transpiration cooling, Rothe approximation.

1. INTRODUCTION

When a space craft is flying very fast in the air, the higher temperature due to the friction of the front surface of the craft with the air may cause ablation of the material, which will cause damage to the structure of the craft. This problem is even more serious for side shooting of the electromagnetic gun which radiates massive amounts of electrical energy. In practice, a thermal shield must be designed to prevent this process [2–5]. In [2,3], this is implemented by coupling a transpiration cooling control design, which can be demonstrated by a one-dimensional version of a solid thermal shield, as shown in Figure 1.

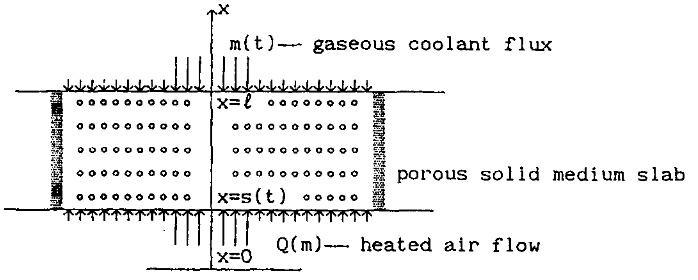

Figure 1. Schematic representation of the transpiration cooling control system of a porous medium slab.

The thermal shield consists of a porous solid structure of thickness ℓ. The gaseous coolant is input at $x = \ell$ and a heated air flow is input at $x = 0$. The coolant flows through the air hole of the slab and enters into the heated air flow $Q(m)$, a specified function of the coolant flux $m(t)$ (mass of coolant per unit time flowing through per unit area) on the outer layer of the

This work was carried out with the aid of a grant from the Postdoctoral Foundation and Science Foundation of China.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX structure [2,3]:

$$Q(m) = \bar{q}\Psi, \quad \bar{q} = q_0\left(1 - \frac{h_w}{h_r}\right), \tag{1}$$

where q_0 is the theoretical heat flow, h_r is the recovery enthalpy, and $h_w = c_\rho u_w$ is the enthalpy of the outer surface. The blocking coefficient Ψ is different for different coolants. For example, if helium is chosen to be the coolant and the condensation is not considered, then

$$\Psi = 1 - 0.724\,\frac{h_e}{q_0}\,m - 0.13\left(\frac{h_e}{q_0}m\right)^2. \tag{2}$$

When the temperature of the front face exceeds the melting temperature u_m of the material, the outer layer melts and recedes to the new position $x = s(t)$ after time t (Figure 1). Denote by $u(x,t)$ the temperature of the medium at the point x and at an instant of time t, and let u_c be the temperature of the coolant at the inside of tank. Then Yang [2] formulated this process by neglecting the thermal swelling as the following partial differential equation with moving boundary:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \alpha^2\dfrac{\partial u^2(x,t)}{\partial x^2} + \beta(t)\dfrac{\partial u(x,t)}{\partial x}, & s(t) < x < \ell, \\ -k\dfrac{\partial u(\ell,t)}{\partial x} = m(t)(\Delta h_N - \Delta h_C) = m(t)c_{pL}(u(\ell,t) - u_c), \\ \rho L s'(t) - k\dfrac{\partial u(s(t),t)}{\partial x} = Q(m(t)), \end{cases} \tag{3}$$

where L is the latent heat of melting. $\Delta h_N = c_{pL}u_N$, $\Delta h_C = c_{pL}u_c$ denote the enthalpy of the coolant at the inner face and at the tank storing coolant, respectively, and the thermal conductivity k, density ρ, and the specific heat c_ρ of the ablation material are all constants, c_{pL} is the specific heat of the coolant, $\alpha^2 = k/(\rho c_\rho)$, $\beta(t) = c_{pL}/(\rho c_\rho)m(t)$.

Since the melted part breaks away (by wind for example) from the material immediately after melting, the following Signorini boundary condition is necessary [6]:

$$u(s(t),t) \le u_m, \quad s'(t) \ge 0, \quad [u_m - u(s(t),t)]s'(t) = 0.$$

Noticing that $u_w = u(s(t),t)$, $u_N = u(\ell,t)$, we finally formulate the ablation transpiration cooling control process as the following Stefan-Signorini problem [7]:

$$\begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t) + \beta(t)u_x(x,t), & s(t) < x < \ell, \\ \alpha^2 u_x(\ell,t) + \beta(t)u(\ell,t) = \beta(t)u_c, & u_c > 0, \\ u(s(t),t) \le u_m, \quad s'(t) \ge 0, \quad [u_m - u(s(t),t)]s'(t) = 0, \\ s'(t) = \dfrac{k}{\rho L}[u_x(s(t),t) - a_1(\beta(t))u(s(t),t) + a_2(\beta(t))], & s(0) = 0, \end{cases} \tag{4}$$

where

$$a_1(\beta(t)) = \frac{q_0, c_\rho}{kh_r}\Psi, \quad a_2(\beta(t)) = \frac{q_0}{k}\Psi.$$

Since the following discussion also holds for the case of $\alpha^2 a_1(\beta) \neq 0$ and $k \neq \rho L$, for simplification, we set, without loss of generality, that $\alpha^2 = 1$, $k = \rho L$, and $a_1 = 0$. Thus, the simpler form of our model is:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \beta(t)u_x(x,t), & s(t) < x < \ell, \\ u_x(\ell,t) + \beta(t)u(\ell,t) = \beta(t)u_c, & u_c > 0, \\ u(s(t),t) \le u_m, \quad s'(t) \ge 0, \quad [u_m - u(s(t),t)]s'(t) = 0, \\ s'(t) = u_x(s(t),t) + Q(\beta(t)), \quad Q(\beta(t)) = \dfrac{q_0}{k}\Psi. \end{cases} \tag{5}$$

The initial condition $u(x,0) = u_0(x)$ satisfies the consistency condition:

$$\begin{cases} u_0 \in C^2[0,\ell], \quad u_c \le u_0 \le u_m, \quad \beta(t), Q(\beta(t)) \in C^1[0,\infty), \\ u_c \le u_0(x) \le u_m, \quad \beta(t) \ge 0, \quad \beta(t) \le \beta_M, \quad u_0'(\ell) + \beta(0)u_0(\ell) = u_c\beta(0), \\ u_0(0) \le u_m, \quad u_0'(0) + Q(\beta(0)) \ge 0, \\ [u_0(0) - u_m]\,[u_0'(0) + Q(\beta(0))] = 0. \end{cases} \tag{6}$$

We have proved in [7] the following theorem.

THEOREM 1. *There exists a unique classical solution $(u(x,t), s(t))$ to equation (5) under (6) for $(x,t) \in [s(t),\ell] \times [0,T]$ and either*

$$T = \infty \text{ and } s(t) < \ell \text{ for all } t > 0, \qquad \text{or}$$
$$T < \infty \text{ and } s(T) = \ell.$$

In Section 2, we will give a Rothe approximation scheme to equation (5) and establish the convergence result based on the estimates of the approximation solutions. A numerical result of an example is given in Section 3. Our results here provide a necessary preparation for further study of the control of this problem [8].

2. ROTHE APPROXIMATION SCHEME

Let

$$v(x,t) = u_m - u(x,t), \quad v_0(x) = u_m - u_0(x), \tag{7}$$

then equation (5) becomes

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + \beta(t)v_x(x,t) \\ v_x(\ell,t) + \beta(t)v(\ell,t) = q\beta(t), \quad q = u_m - u_c > 0, \\ v(s(t),t) \ge 0, \quad s'(t) \ge 0, \quad v(s(t),t)s'(t) = 0, \\ v(x,0) = v_0(x), \\ s'(t) = -v_x(s(t),t) + Q(\beta(t)). \end{cases} \tag{8}$$

In the following, we use $\|\cdot\|$ to denote the C^0 norm of a continuous function. The standard transformation

$$y = \frac{x - s(t)}{\ell - s(t)}\,\ell, \quad \vartheta(y,t) = v(x,t) \tag{9}$$

will map $[s(t),\ell]$ onto $[0,\ell]$, and equation (8) becomes a fixed domain problem:

$$\begin{cases} \vartheta_t(y,t) = \left(\dfrac{\ell}{\ell - s(t)}\right)^2 \vartheta_{yy}(y,t) + \left[\beta(t)\dfrac{\ell}{\ell - s(t)} + s'(t)\dfrac{\ell - y}{\ell - s(t)}\right]\vartheta_y(y,t), \\ \dfrac{\ell}{\ell - s(t)}\,\vartheta_y(\ell,t) + \beta(t)\vartheta(\ell,t) = q\beta(t), \\ \vartheta(0,t) \ge 0, \quad -\dfrac{\ell}{\ell - s(t)}\,\vartheta_y(0,t) + Q(\beta(t)) \ge 0, \\ \vartheta(0,t)\left[-\dfrac{\ell}{\ell - s(t)}\,\vartheta_y(0,t) + Q(\beta(t))\right] = 0, \\ s'(t) = -\dfrac{\ell}{\ell - s(t)}\,\vartheta_y(0,t) + Q(\beta(t)). \end{cases} \tag{10}$$

Let

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T, \quad t_n - t_{n-1} = \Delta t = \frac{T}{N}$$

be a partition of $[0,T]$, considering

$$\beta_n = \beta(t_n), \quad s_n = s(t_n),$$

and using the Rothe Approximation scheme to (10), we obtain a boundary value problem as follows:

$$\begin{cases} \dfrac{\vartheta_n(y) - \vartheta_{n-1}(y)}{\Delta t} = \left(\dfrac{\ell}{\ell - s_n}\right)^2 \vartheta_n''(y) + \left[\beta_n \dfrac{\ell}{\ell - s_n} + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - y}{\ell - s_n}\right] \vartheta_n'(y), \\ \dfrac{\ell}{\ell - s_n}\vartheta_n'(\ell) + \beta_n\vartheta_n(\ell) = q\beta_n, \quad \vartheta_0(y) = v_0(y), \\ \vartheta_n(0) \geq 0, \quad -\dfrac{\ell}{\ell - s_n}\vartheta_n'(0) + Q(\beta_n) \geq 0, \quad \vartheta_n(0)\left[-\dfrac{\ell}{\ell - s_n}\vartheta_n'(0)Q(\beta_n)\right] = 0, \end{cases} \tag{11}$$

Using the transformation

$$x = s_n + \frac{\ell - s_n}{\ell} y, \quad v_n(x) = \vartheta_n(y), \tag{12}$$

we finally obtain the Rothe approximation scheme of problem (8):

$$\begin{cases} \dfrac{v_n(x) - v_{n-1}(\vartheta(x))}{\Delta t} = v_n''(x) + \left[\beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}\right] v_n'(x), \quad s_n < x < \ell, \\ v_n'(\ell) + \beta_n v_n(\ell) = q\beta_n, \\ v_n(s_n) \geq 0, \quad -v_n'(s_n) + Q(\beta_n) \geq 0, \quad v_n(s_n)[-v_n'(s_n) + Q(\beta_n)] = 0, \\ v_0(x) = v_0(x), \quad \vartheta(x) = \dfrac{\ell - s_{n-1}}{\ell - s_n} x - \dfrac{s_n - s_{n-1}}{\ell - s_n}\ell, \end{cases} \tag{13}$$

where s_n is determined from

$$s_n = s_{n-1} + [-v_{n-1}'(s_{n-1}) + Q(\beta_{n-1})]\Delta t, \qquad s_0 = 0. \tag{14}$$

Note $\vartheta(s_n) = s_{n-1}$, $\vartheta(\ell) = \ell$.

LEMMA 1. *There exists a unique solution $\{v_n(x), 1 \leq n \leq N\}$ to (13) and*

$$0 \leq v_n(x) \leq q, \qquad c_1 \leq v_n'(x) \leq c_2, \tag{15}$$

where $c_1 = \min v_0'(x)$, $c_2 = \max\{\,\|v_0\|, q\beta_M, \|Q\|\,\}$, and hence,

$$0 \leq s_n - s_{n-1} \leq \|Q\|\,\Delta t. \tag{16}$$

PROOF. Equation (13) can be solved by the method of invariant imbedding (sweep method) [1]:

$$v_n'(x) = R_n(x)v_n(x) + z_n(x), \tag{17}$$

where

$$\begin{cases} R_n'(x) = -R_n^2(x) - \left[\beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}\right] R_n(x) + \dfrac{1}{\Delta t}, & R_n(\ell) = -\beta_n, \\ z_n'(x) = -\left[R_n(x) + \beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}\right] z_n(x) - \dfrac{v_{n-1}(\vartheta(x))}{\Delta t}, & z_n(\ell) = q\beta_n, \end{cases} \tag{18}$$

It should be noticed that by defining $p(x) = R_n + \beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}$,

$$R_n(x) = -\beta_n \exp\left(\int_x^\ell p(\tau)\,d\tau\right) - \frac{1}{\Delta t}\int_x^\ell \exp\left(\int_x^s p(\tau)\,d\tau\right) ds < 0.$$

Thus, $v_n(x)$ can be determined from (17) with initial condition

$$v_n(x)(s_n) = \max\left\{0, \frac{Q(\beta_n) - z_n(s_n)}{-R_n(s_n)}\right\}. \tag{19}$$

If $\pm v_n$ attains its maximum at $x_0 \in (s_n, \ell)$, then

$$\pm v_n(x_0) \le \pm v_{n-1}(\vartheta(x_0)).$$

If v_n attains negative minimum at $x = \ell$, by the maximum principle [9], $v_n'(\ell) < 0$, and hence, $v_n(\ell) > q$ from the boundary condition at $x = \ell$, a contradiction. If v_n attains positive maximum at $x = s_n$, then $v_n'(s_n) < 0$, and hence, $v_n(s_n) = 0$ by the boundary at $x = s_n$, a contradiction. If v_n attains positive maximum at $x = \ell$, by the maximum principle, $v_n'(\ell) > 0$, and hence, $v_n(\ell) < q$ from the boundary condition at $x = \ell$. Noticing that $v_n(s_n) \ge 0$ and $0 \le v_0(x) \le q$, one has $0 \le v_n(x) \le q$. Next, let $\phi_n(x) = v_n'(x)$, then

$$\begin{cases} \phi_n''(x) + \left[\beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}\right]\phi_n'(x) - \left[\dfrac{1}{\Delta t} + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{1}{\ell - s_n}\right]\phi_n(x) \\ \qquad = \dfrac{-\phi_{n-1}(\vartheta(x))}{\Delta t}\dfrac{\ell - s_{n-1}}{\ell - s_n}, \\ 0 \le \phi_n(\ell) = \beta_n(q - v_n(\ell)) \le \beta_M q, \\ 0 \le \phi_n(s_n) \le Q(\beta_n) \le \|Q\|. \end{cases}$$

The second inequality of (15) can be obtained by a direct application of the maximum principle for elliptic ordinary differential equations [9]. Equation (16) is a consequence of (15) using Definition (14). ∎

Next, take a family of nonpositive functions $\beta_\varepsilon \in C^2(-\infty, \infty)$ with $\beta_\varepsilon' \le 0$ and

$$\beta_\varepsilon(t) = \begin{cases} 0, & \text{if } t \ge 0, \\ (t + \varepsilon)/\varepsilon, & \text{if } t < -2\varepsilon, \end{cases} \tag{20}$$

and consider a penalized problem of (13):

$$\begin{cases} \dfrac{v_{\varepsilon n}(x) - v_{\varepsilon(n-1)}(\vartheta(x))}{\Delta t} = v_{\varepsilon n}''(x)\left[\beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}\right]v_{\varepsilon n}'(x), \qquad s_n < x < \ell, \\ v_{\varepsilon n}'(\ell) + \beta_n v_{\varepsilon n}(\ell) = q\beta_n, \\ -v_{\varepsilon n}'(s_n) + Q(\beta_n) + \beta_\varepsilon(v_{\varepsilon n}(s_n)) = 0, \\ v_{\varepsilon 0}(x) = v_0(x), \quad \vartheta(x) = \dfrac{\ell - s_{n-1}}{\ell - s_n}x - \dfrac{s_n - s_{n-1}}{\ell - s_n}. \end{cases} \tag{21}$$

LEMMA 2. *There exists a unique solution to (21) and the following results hold:*

(i) $-c_0\varepsilon \le v_{\varepsilon n}(x) \le q$, $c_0 = \max\{2, 1 + \|Q\|\}$;
(ii) $c_1 \le v_{\varepsilon n}'(x) \le c_2$, $c_1 = \max\{\min v_0'(x), (1 - c_0)\varepsilon\}$, $c_2 = \max\{\|Q\|, q\beta_M + c_0\varepsilon, \max v_0'(x)\}$;
(iii) $|v_{\varepsilon n}(x) - v_{\varepsilon(n-1)}(x)| \le C\Delta t$, $\|v_{\varepsilon n}''\| \le C$,

where C is a constant only depending on the C^1 norm of $\beta(t)$, $s(t)$, and $\ell - s(T)$, and the C_2 norm of v_0.

PROOF. Again we solve equation (21) by the method of invariant imbedding. The solution is related by the Riccati transformation:

$$v_{\varepsilon n}'(x) = R_n(x)v_{\varepsilon n}(x) + z_n(x). \tag{22}$$

Substituting $v'_{\varepsilon n}(x) = R_n(x)v_{\varepsilon n}(x) + z_n(x)$ into the boundary condition of (21) at s_n, we obtain

$$(-R_n(s_n) + a_1(\beta_n))v_{\varepsilon n}(s_n) - z_n(s_n) + Q(\beta_n) + \beta_\varepsilon(v_{\varepsilon n}(s_n)) = 0.$$

Define $f(t) = (-R_n(s_n) + a_1(\beta_n))t - z_n(s_n) + Q(\beta_n) + \beta_\varepsilon(t)$, then

$$f'(t) = (-R_n(s_n) + a_1(\beta_n)) + \beta'_\varepsilon(t) > 0, \qquad f(+\infty) = \infty, \quad f(-\infty) = -\infty,$$

and hence, there exists a unique t_0 such that $f(t_0) = 0$. Therefore, equation (21) has a unique solution with $v_{\varepsilon n}(s_n) = t_0$.

Now we use mathematical induction. For $n = 0$, the conclusions (i) and (ii) hold. Supposing that it is correct for $k < n$, we consider the case of $k = n$. By the maximum principle, if $v_{\varepsilon n}$ attains negative minimum at x_0, then either $v_{\varepsilon n}(x_0) \geq v_{\varepsilon(n-1)}(\vartheta(x_0))$ or $Q(\beta_n)+\beta_\varepsilon(v_{\varepsilon n}(x_0)) \geq 0$, depending on whether x_0 is located in the interior of (s_n, ℓ) or on the boundaries. So

$$\begin{aligned} &\text{either } v_{\varepsilon n}(x) \geq -2\varepsilon, \\ &\text{or } v_{\varepsilon n}(x) \geq -[1 + Q(\beta_n)]\varepsilon \geq -[1 + \|Q\|\,]\varepsilon, \end{aligned}$$

since $v_0(x) \geq 0$. This is the first part of (i). If $v_{\varepsilon n}$ attains positive maximum at x_0, then either $v_{\varepsilon n}(x_0) \leq v_{\varepsilon(n-1)}(\vartheta(x_0))$ or $v_{\varepsilon n}(x_0) \leq q$ or $v_{\varepsilon n}(x_0) \leq 0$. This is the second part of (i). For the proof of (ii), let $\psi_n(x) = v'_{\varepsilon n}(x)$, then

$$\begin{cases} \psi''_n(x) + \left[\beta_n + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{\ell - x}{\ell - s_n}\right]\psi'_n(x) - \left[\dfrac{1}{\Delta t} + \dfrac{s_n - s_{n-1}}{\Delta t}\dfrac{1}{\ell - s_n}\right]\psi_n(x) \\ \qquad = \dfrac{-\psi_{(n-1)}(\vartheta(x))}{\Delta t}\dfrac{\ell - s_{n-1}}{\ell - s_n}, \\ \psi_n(\ell) = \beta_n(q - v_{\varepsilon n}(\ell)), \\ \psi_n(s_n) = Q(\beta_n) + \beta_\varepsilon(v_{\varepsilon n}(s_n)). \end{cases} \tag{23}$$

Through the same discussion as above, we can get the second part of (ii). Now, we are in a position to prove (iii). Let

$$\eta_n(x) = \pm v''_{\varepsilon n}(x)e^{-\chi(x-\ell-\delta)^2}, \qquad \delta = \frac{\ell - s_N}{2}, \quad \chi > 0,$$

then $\eta_n(x)$ satisfies the following equation:

$$\begin{aligned} \eta''_n(x) &+ \left[4\chi(x - \ell + \delta) + \beta_n\frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}\right]\eta'_n(x) - \left[\frac{1}{\Delta t} - 2\chi - 4\chi^2(x - \ell + \delta)^2\right. \\ &- 2\,\chi(x - \ell + \delta)\left(\beta_n + \frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}\right) + 2\left.\frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n}\right]\eta_n(x) \\ &= \mp\frac{1}{\Delta t}\,\eta_{n-1}(\vartheta(x)) \pm \eta_{n-1}(\vartheta(x))K(x), \end{aligned} \tag{24}$$

where $K(x) = 1 - e^{\chi(\vartheta(x)-\ell-\delta)^2 - \chi(x-\ell-\delta)^2}\left(\dfrac{\ell - s_{n-1}}{\ell - s_n}\right)$, and with the boundary conditions:

$$\begin{aligned} \eta'_n(\ell) + (\beta_n + \beta_{n-1} + 2\chi\delta)\eta_n(\ell) &= \pm\left[\left[\frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n} - \beta_n\beta_{n-1}\right]v'_{\varepsilon n}(\ell)\right. \\ &\left.- \frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n}v'_{\varepsilon(n-1)}(\ell) + \frac{\beta_n - \beta_{n-1}}{\Delta t}(q - v_{\varepsilon n}(\ell))\right]e^{-\chi\delta^2} = K_2(\varepsilon), \\ \eta'_n(s_n) - &\left[2\chi(\ell - s_n - \delta) + \beta'_\varepsilon(\xi) - \beta_n - \frac{s_n - s_{n-1}}{\Delta t}\right]\eta_n(s_n) \\ &= \pm\left[\left[\frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n} + \beta'_\varepsilon(\xi)\left(\beta_n + \frac{s_n - s_{n-1}}{\Delta t}\right)\right]v'_{\varepsilon n}(s_n)\right. \\ &+ \left(\beta'_\varepsilon(\vartheta) + \frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n}\right)v'_{\varepsilon n}(s_n) - \frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n}v'_{\varepsilon(n-1)}(s_{n-1}) \\ &\left.+ \frac{Q(\beta_n) - Q(\beta_{n-1})}{\Delta t}\right]e^{-\chi(s_n-\ell-\delta)^2} = K_3(\varepsilon), \end{aligned}$$

where $-c_0\varepsilon \le \xi \le q$ and $\chi > 0$ is taken such that

$$2\chi\delta \ge 1, \quad 2\chi(\ell - s_n - \delta) - \beta_n - 2\,\frac{s_n - s_{n-1}}{\Delta t} \ge 1.$$

Suppose that $\eta_{n-1}(x) \le C_{n-1}$, then if η_n takes a maximum at the interior of (s_n, ℓ), by noticing the fact that $|K(x)| \le K_0\Delta t$,

$$\eta_n(x) \le C_{n-1} + K_{10}\Delta t C_{n-1}, \qquad \text{where } K_{10} > 0 \text{ is a constant,}$$

and

$$\eta_n(x) \le (1 + K_{10}\Delta t)^N C_0, \qquad \Delta t = \frac{T}{N}.$$

If $\eta_n(x)$ attains maximum at $x = \ell$ or $x = s_n$, then

$$\eta_n(x) \le K_2(\varepsilon) \le K_{20}, \quad \eta_n(x) \le K_3(\varepsilon) \le K_{30},$$

by (i) and (ii), which have just been proved. Since

$$\lim_{t\to\infty}(1 + K_{10}\Delta t)^N = K_{10}T,$$

we finally get $\eta_n(x) \le C$ or $|v''_{\varepsilon n}| \le C$ from (21). Since

$$\frac{v_{\varepsilon n}(x) - v_{\varepsilon(n-1)}(x)}{\Delta t} = \alpha^2 v''_{\varepsilon n}(x) + \left[\beta_n + \frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}\right] v'_{\varepsilon n}(x) - v'_{\varepsilon(n-1)}(\xi)\frac{x - \vartheta}{\Delta t},$$
$$\xi \in (\vartheta(x), x), \quad \frac{x - \vartheta}{\Delta t} = \frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n},$$

we have $\left|\dfrac{v_{\varepsilon n}(x) - v_{\varepsilon(n-1)}(x)}{\Delta t}\right| \le C.$ ∎

Since the constant C in Lemma 2 is independent of ε, by the Ascoli-Arzela Lemma, for any integer n, there exists a subsequence of $\{v_{\varepsilon n}(x), v'_{\varepsilon n}(x)\}$, still denoted by $v_{\varepsilon n}$, for simplification, such that

$$v_{\varepsilon n} \to v_n, \quad v'_{\varepsilon n} \to v'_n, \qquad \text{uniformly as } \varepsilon \to 0,$$

and from equation (21),

$$v''_{\varepsilon n} \to v''_n, \qquad \text{uniformly as } \varepsilon \to 0.$$

If $v_n(x_0) > 0$, then by uniform convergence, for small enough ε, there is a constant $c > 0$ such that $v_{\varepsilon n}(x_0) \ge c > 0$. Therefore, $-v'_{\varepsilon n}(s_n) + Q(\beta_n) = 0$ from (21) and $-v'_n(s_n) + Q(\beta_n) = 0$, i.e., $v_n(x)$ is the solution of (13). Since the solution of (13) is unique, the whole sequence of solution of (21) converges uniformly to the solution of (13) as ε goes to zero. We thus have proved the following lemma.

LEMMA 3. *The solution of equation (13),(14) always holds*

$$|v_n(x) - v_{(n-1)}(x)| \le C\Delta t, \quad \|v'_n\| \le C,$$

where C is a constant only depending on the C^1 norm of $\beta(t)$, $s(t)$, and $\ell - s(T)$, and the C^2 norm of v_0.

Define

$$\begin{cases} s_N(t) = \dfrac{1}{\Delta t}[(t - t_{n-1})s_n + (t_n - t)s_{n-1}], & \text{for } t \in [t_{n-1}, t_n), \\ v_N(x,t) = \dfrac{1}{\Delta t}[(t - t_{n-1})v_n(x) + (t_n - t)v_{n-1}(x)], & \text{for } t \in [t_{n-1}, t_n), \end{cases} \tag{25}$$

LEMMA 4.

(a) $0 \le s'_N(t) \le \|Q\|$;
(b) $0 \le v_N(x,t) \le q$;
(c) $|v_{Nx}(x,t)| \le \max\{\|v_0\|, q\beta_M, \|Q\|\}$;
(d) $v_{Nx}(x,t) \ge 0$, *if* $v_0'(x) \ge 0$;
(e) $|v_{Nxx}(x,t)| \le C$,
(f) $|v_{Nx}(x,t_1)| - |v_{Nx}(x,t_2)| \le C|t_1 - t_2|^{1/2}$, *for any* $t_1, t_2 \in [t_{n-1}, t_n)$, *where* C *has the same meaning as in Lemma 3.*

PROOF. (a)–(e) are direct consequences of Lemma 1 and Lemma 3 by the definition of $v_N(x,t)$. Now we only consider (f). Notice that

$$v_{Nxt}(x,t) = \frac{v_n'(x) - v_{n-1}'(x)}{\Delta t}$$

and

$$\begin{aligned} v_{Nxt}(x,t) = v_n'''(x) + \left[\beta_n + \frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}\right] v_n''(x) \frac{s_n - s_{n-1}}{\Delta t}\frac{1}{\ell - s_n} v_n'(x) \\ - v_{n-1}'(\xi)\frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n} = v_n'''(x) + G_n(x), \end{aligned}$$

where $|G_n(x)| \le C$, $\xi \in (\vartheta(x), x)$, $t \in [t_{n-1}, t_n)$. For any $t_{n-1} \le t_1 < t_2 < t_n$, $\tau = (t_2 - t_1)^{1/2}$,

$$\begin{aligned} \int_x^{x+\tau} [v_{Nx}(y,t_1) - v_{Nx}(y,t_2)]\,dy &= \int_x^{x+\tau}\int_{t_1}^{t_2} v_{Nxt}(y,t)\,dy\,dt \\ &= \left[v_n''(x+\tau) - v_n''(x) + \int_x^{x+\tau} G_n(y)\,dy\right](t_2 - t_1). \end{aligned}$$

Hence, there exists a point $\xi \in (x, x+\tau)$ such that

$$|v_{Nx}(\xi,t_1) - v_{Nx}(\xi,t_2)| \le C|t_1 - t_2|^{1/2}.$$

Thus,

$$\begin{aligned} |v_{Nx}(x,t_1) - v_{Nx}(x,t_2)| &\le |v_{Nx}(x,t_1) - v_{Nx}(\xi,t_1)| + |v_{Nx}(x,t_2) - v_{Nx}(\xi,t_2)| + C\,|t_1 - t_2|^{1/2} \\ &\le (C + 2\,\|v_{Nxx}\|\,)\,|t_1 - t_2|^{1/2}. \end{aligned}$$

This is (f). The proof is complete. ∎

By the Ascoli-Arzela Lemma, there exists a subsequence of $v_N(x,t)$, $s_N(t)$, still denoted by v_N, s_N for ease of notation, and a function $\phi(x,t)$, $s(t)$ such that

$$\begin{aligned} &s_N \to s, \quad v_N \to \phi, \quad v_{Nx} \to \phi_x, \qquad \text{uniformly as } N \to \infty, \\ &v_{Nt} \to \phi_t, \quad v_{Nxx} \to \phi_{xx}, \qquad \text{in } L^\infty \text{ weak star topology,} \end{aligned} \tag{26}$$

and

$$\begin{aligned} &0 \le s'(t) \le \|Q\|, \\ &0 \le \phi \le q, \quad |\phi_x(x,t)| \le C_0, \quad |\phi_t(x,t)| \le C, \quad |\phi_{xx}(x,t)| \le C, \\ &|\phi_x(x,t_1)| - |\phi_x(x,t_2)| \le C|t_1 - t_2|^{1/2}, \qquad \text{for any } t_1, t_2 \in [0,T], \end{aligned}$$

where C has same meaning as in Lemma 4. Now

$$s_N(T) = s_N = \sum_{n=1}^N (s_n - s_{n-1}) = \frac{k}{\rho L}\sum_{n=1}^N [-v_{n-1}'(s_{n-1}) + Q(\beta_{n-1})]\,\Delta t \le C_0 T.$$

If T is small enough, then $s_N <$ Const. $< \ell$,

$$\begin{aligned} s_N(T) &= \sum_{n=1}^{N}(s_n - s_{n-1}) = \sum_{n=1}^{N}[-v'_{n-1}(s_{n-1}) + Q(\beta_{n-1})]\,\Delta t \\ &= \sum_{n=1}^{N} -v_{Nx}(s_N(t_{n-1}), t_{n-1})\,\Delta t + \sum_{n=1}^{N} Q(\beta_{n-1})\,\Delta t \\ &= \sum_{n=1}^{N}[\phi_x(s_N(t_{n-1}), t_{n-1}) - v_{Nx}(s_N(t_{n-1}), t_{n-1})]\,\Delta t \\ &\quad + \sum_{n=1}^{N}[-\phi_x(s_N(t_{n-1}), t_{n-1}) + \phi_x(s(t_{n-1}), t_{n-1})]\,\Delta t \\ &\quad + \sum_{n=1}^{N}[-\phi_x(s(t_{n-1}), t_{n-1}) + Q(\beta_{n-1})]\,\Delta t. \end{aligned}$$

The last term of the above is just a Riemann sum. By the uniform convergence, we immediately have that

$$s(T) = \int_0^T [-\phi_x(s(t), t) + Q(\beta(t))\,]\,dt.$$

For any $t \in (0, T]$, we then have

$$s(t) = \int_0^t [-\phi_x(s(t), t) + Q(\beta(t))\,]\,dt. \tag{27}$$

Next, notice that

$$\begin{aligned} &-v'_N(s_N(t_n), t_n) + Q(\beta(t_n)) \geq 0, \\ &v_N(s_N(t_n), t_n)[-v'_N(s_N(t_n), t_n) + Q(\beta(t_n))\,] = 0, \\ &v'_N(\ell, t_n) + \beta(t_n)v_N(\ell, t_n) = q\beta(t_n), \end{aligned}$$

at each point of the partition $\{t_n\}$ associated with N. Since these points are dense in $[0, T]$ as $N \to \infty$ and the convergence of $v_{Nx}(x,t)$, $v_N(x,t)$, $s_N(t)$ is uniform, it follows immediately that

$$\begin{cases} -\phi_x(s(t), t) + Q(\beta(t)) \geq 0, \\ \phi(s(t), t)[-\phi'(s(t), t) + Q(\beta(t)] = 0, \\ \phi_x(\ell, t) + \beta(t)\phi(\ell, t) = q\beta(t). \end{cases} \tag{28}$$

We shall show that $\phi(x,t) = v(x,t)$, the solution of problem (8). Since

$$\begin{aligned} v_{Nt}(x,t) = v''_n(x) + \beta_n v''_n(x) + K_n(x) &= v_{Nxx}(x, t_n) + \beta(t_n)v_{Nx}(x, t_n) + K_n(x), \\ &s_N(t_n) < x < \ell, \quad t \in (t_{n-1}, t_n), \quad n \geq 1, \end{aligned}$$

where

$$\begin{aligned} K_n(x) &= [v'_n(x) - v'_{n-1}(\xi)]\frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n} = [v'_n(x) - v'_{n-1}(x)]\frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n} \\ &\quad + v''_{n-1}(\Theta)\frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}(x - \xi), \quad \xi, \Theta \in (\vartheta(x), x), \quad \frac{x - \vartheta}{\Delta t} = \frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}, \\ |K_n(x)| &\leq C\Delta t^{1/2}, \end{aligned}$$

for any $f(x,t) \in C^2(\Omega)$, $\Omega = \{(x,t),\, t \in [0,T],\, x \in [s(t),\ell]\}$,

$$\sum_{n=1}^{N} \Delta t \int_{s_N(t_n)}^{\ell} v_{Nt}(x,t) f(x,t_n) = \sum_{n=1}^{N} \Delta t \int_{s_N(t_n)}^{\ell} v_{Nxx}(x,t_n) f(x,t_n) + \sum_{n=1}^{N} \Delta t \int_{s_N(t_n)}^{\ell} \beta(t_n) v_{Nx}(x,t_n) f(x,t_n) + G_N, \quad (29)$$

where

$$|G_N| \le \left| \sum_{n=1}^{N} \Delta t \int_{s_N(t_n)}^{\ell} K_n(x) f(x,t_n) \right| \le CT(\ell - s(T)) \|f\| \Delta t^{1/2},$$
$$\lim_{N\to\infty} G_N = 0.$$

By using the uniform convergence (26) and (29), we can easily obtain

$$- \iint_\Omega f_t \phi \, dx\, dt + \int_0^T s'(t)[\phi f](s(t),t) + \int_{s(T)}^{\ell} [f\phi](x,T)\, dx - \int_0^{\ell} f(x,0) v_0(x)\, dx$$
$$= \int_0^T [\, [\phi_x f](\ell \cdot t) - [\phi_x f](s(t),t)]\, dt - \iint_\Omega \phi_x f_x \, dx\, dt + \iint_\Omega \beta(t) \phi_x f \, dx\, dt, \quad (30)$$

and (30) holds for any continuous function f with first order derivatives being bounded on Ω by smooth function approximation, if necessary.

From [7], we know that the solution v of equation (8) is the only function satisfying (30) for such class of function f, thus we have proved that $\phi = v$. Since v is unique by Theorem 1, the whole sequence defined by (25) converges to (s, v).

THEOREM 2. *Let $s_N(t)$ and $v_N(x,t)$ be defined by (25), where $v_n(x)$ is produced by Rothe approximation scheme (13),(14), then*

$$s_N \to s, \quad v_N \to v, \quad v_{Nx} \to v_x, \qquad \text{uniformly as } N \to \infty,$$
$$v_{Nt} \to v_t, \quad v_{Nxx} \to v_{xx}, \qquad \text{in } L^\infty \text{ weak star topology},$$

where (s, v) is the unique solution of equation (8).

Summarizing the above and returning to the original system (4), we have the following theorem.

THEOREM 3. *The solution of ablation-transpiration cooling equation (4) can be obtained as the uniform limit of Rothe approximation solutions $s_N(t)$ and $u_N(x,t)$ as N goes to infinity:*

$$\begin{cases} s_N(t) = \dfrac{1}{\Delta t}[(t - t_{n-1})s_n + (t_n - t)s_{n-1}], & \text{for } t \in [t_{n-1}, t_n), \\ u_N(x,t) = \dfrac{1}{\Delta t}[(t - t_{n-1})u_n(x) + (t_n - t)u_{n-1}(x)], & \text{for } t \in [t_{n-1}, t_n), \end{cases} \quad (31)$$

where s_n and v_n are produced by

$$\begin{cases} s_n = s_{n-1} + \dfrac{k}{\rho L}[u'_{n-1}(s_{n-1}) - a_1(\beta_{n-1}) v_{n-1}(s_{n-1}) + a_2(\beta_{n-1})]\Delta t \\ u'_n(x) = R_n(x) u_n(x) + z_n(x) \\ u_n(s_n) = \min\left\{ u_m, \dfrac{a_2(\beta_n) + z_n(s_n)}{-R_n(s_n) + a_1(\beta_n)} \right\}, \end{cases} \quad (32)$$

where $R_n(x) < 0$, $z_n(x) \ge 0$, $\vartheta(x) = \dfrac{\ell - s_{n-1}}{\ell - s_n} x - \dfrac{s_n - s_{n-1}}{\ell - s_n} \ell$

$$R'_n(x) = -R_n^2(x) - \frac{1}{\alpha^2}\left[\beta_n + \frac{s_n - s_{n-1}}{\Delta t} \frac{\ell - x}{\ell - s_n}\right] R_n(x) + \frac{1}{\alpha^2 \Delta t}, \qquad R_n(\ell) = -\frac{\beta_n}{\alpha^2},$$

$$z'_n(x) = -\left[R_n(x) + \frac{1}{\alpha^2}\left(\beta_n + \frac{s_n - s_{n-1}}{\Delta t}\frac{\ell - x}{\ell - s_n}\right)\right] z_n(x) - \frac{u_{n-1}(\vartheta(x))}{\alpha^2 \Delta t}, \qquad z_n(\ell) = \frac{q\beta_n}{\alpha^2}.$$

3. NUMERICAL SIMULATION

The following parameters are taken from an experimental example with a steel bar as the ablation material, the melting point of which is 1450°C, and helium as the coolant:

$$\ell = 0.015\,\text{m}, \quad c_\rho = 0.153\,\text{kcal/kg}\cdot{}^\circ\text{C}, \quad k = 0.0063\,\text{kcal}/{}^\circ\text{C}\cdot\text{s}\cdot\text{m}$$
$$c_{\rho L} = 1.24\,\text{kcal/kg}\cdot{}^\circ\text{C}, \quad \rho = 7850\,\text{kg/m}^3, \quad h_r = 3500\,\text{kcal/kg},$$
$$h_e = 1728\,\text{kcal/kg}, \quad q_0 = 5000\,\text{kcal/s}\cdot\text{m}^2, \quad L = 65\,\text{kcal/kg},$$
$$u_m = 1450^\circ\text{C}, \quad u_c = 315^\circ\text{C}, \quad u_0(x) = 516^\circ\text{C}.$$

The numerical results, shown in Tables 1 and 2 and Figure 2, are obtained by solving (32) through a Fortran program. The results demonstrate that if coolant is not input, the steel bar is melted in 8.7 seconds, but in the case of coolant being supplied, $m = 2.8$ (the maximum coolant value in our example is $m = 4.38$), the temperature at the outer surface never attains the melting point.

Table 1. $m = 0$, $T = 10$, $N = 30$.

t_n	$s_N(t_n)$	$u_N(s_N(t_n), t_n)$
0.0333333	$3.186269E-003$	
0.0666667	$3.186269E-003$	843.4351
1.0000000	$3.186269E-003$	1406.767
1.3333330	$3.913654E-003$	1450
1.6666670	$4.102717E-003$	1450
2.0000000	$4.738931E-003$	1450
2.3333330	$4.962004E-003$	1450
2.6666670	$5.557141E-003$	1450
3.0000000	$5.800145E-003$	1450
3.3333330	$6.375198E-003$	1450
3.6666660	$6.635741E-003$	1450
4.0000000	$7.181665E-003$	1450
4.3333330	$7.435005E-003$	1450
4.6666670	$7.990170E-003$	1450
5.0000000	$8.259417E-003$	1450
5.3333330	$8.818359E-003$	1450
5.6666670	$9.115268E-003$	1450
6.0000000	$9.678342E-003$	1450
6.3333340	$1.000987E-002$	1450
6.6666670	$1.058151E-002$	1450
7.0000010	$1.096510E-002$	1450
7.3333340	$1.155357E-002$	1450
7.6666680	$1.201060E-002$	1450
8.0000010	$1.277367E-002$	1450
8.3333340	$1.334426E-002$	1450
8.6666670	$1.449326E-002$	1450

Table 2. $m = 2.8$, $T = 10$, $N = 30$.

t_n	$s_N(t_n)$	$u_N(s_N(t_n), t_n)$
0.3333333	$1.341841E-003$	
0.6666667	$1.341841E-003$	706.2200
1.0000000	$1.341841E-003$	789.2970
1.3333330	$1.341841E-003$	799.4745
1.6666670	$1.341841E-003$	792.7454
2.0000000	$1.341841E-003$	782.6046
2.3333330	$1.341841E-003$	779.1092
2.6666670	$1.341841E-003$	769.8155
3.0000000	$1.341841E-003$	765.6064
3.3333330	$1.341841E-003$	764.4061
3.6666660	$1.341841E-003$	765.0331
4.0000000	$1.341841E-003$	766.6870
4.3333330	$1.341841E-003$	769.0807
4.6666670	$1.341841E-003$	771.1402
5.0000000	$1.341841E-003$	773.2373
5.3333330	$1.341841E-003$	774.7437
5.6666670	$1.341841E-003$	776.2341
6.0000000	$1.341841E-003$	784.6915
6.3333340	$1.341841E-003$	781.1058
6.6666670	$1.341841E-003$	779.8939
7.0000010	$1.341841E-003$	779.6635
7.3333340	$1.341841E-003$	779.7626
7.6666680	$1.341841E-003$	779.9884
8.0000010	$1.341841E-003$	780.2437
8.3333340	$1.341841E-003$	780.4897
8.6666670	$1.341841E-003$	780.7261
9.0000000	$1.341841E-003$	780.9316
9.3333330	$1.341841E-003$	781.1009
9.6666660	$1.341841E-003$	781.2509
9.9999990	$1.341841E-003$	781.3953

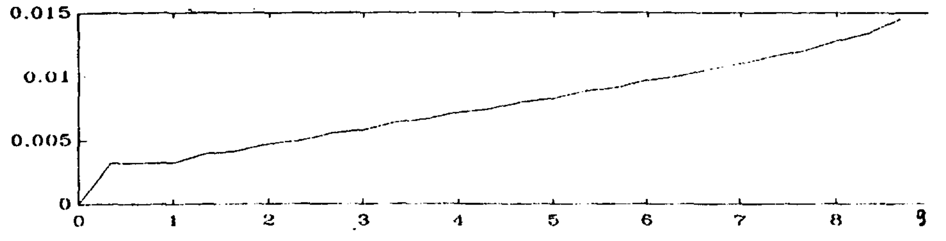

Figure 2. Numerical simulation of an ablation steel bar.

REFERENCES

1. G.H. Meyer, One-dimensional parabolic free boundary problems, *SIAM* **19**, 17–34 (1977).
2. X.S. Yang, Transpiration cooling control of thermal protection, *Acta Automatica Sinica* **11**, 345–350 (1985).
3. X.S. Yang, Transpiration cooling with water temperature field, *Acta Automatica Sinica* **17**, 385–394 (1991).
4. H.D. Fair, Electromagnetic earth-to-space launch, *IEEE Transactions on Magnetics* **25**, 9–16 (1989).
5. M.R. Palmer, Electromagnetic space launch: A re-evaluation in light of current technology and launch needs and feasibility of a near term demonstration, *IEEE Transactions on Magnetics* **25** (1) (1989).
6. L.S. Jiang, Remarks on the Stefan-Signorini problem, In *Free Boundary Problems: Vol. III, Res. Notes in Math.* **120**, 13–19, (1985).
7. B.Z. Guo and Y. Hao, Modelling and solution of a transpiration cooling control system, (to appear).
8. B.Z. Guo and Y. Hao, A control related problem of an ablation-transpiration cooling control system, (to appear).
9. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, (1984).