Robust Output Regulation of 1-d Wave Equation

Bao-Zhu Guo\textsuperscript{a}, Tingting Meng\textsuperscript{b}

\textsuperscript{a}School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China and Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, P.R. China

\textsuperscript{b}Institute of Artificial Intelligence, University of Science and Technology Beijing, Beijing 100083, China

Abstract

In the last a few years, there have been a couple of works addressing output regulation for 1-d linear wave equations without robustness. The aim of this paper is to provide, under the guidance of the internal model principle, a united way to achieve more profound results including fast convergence, no man-made assumption, simple control design and particularly robustness. Different from existing works where disturbance appears only in one channel, we allow disturbances in all possible channels. Our approach is an observer based control approach which is designed only for a nominal system yet also has the power to guarantee output regulation for original uncertain PDE system. Simulation examples are presented to show the effectiveness of the proposed control.

Key words: Wave equation, robust control, internal model principle

1 Introduction

In engineering, the most important task is to achieve performance output tracking regardless of external disturbances and system uncertainties, and at the same time to keep the system internally asymptotically stable, which is frequently referred to as output regulation problem in control theory ([17]). For linear time invariant systems with disturbances from an exo-system, the internal model principle which was proposed in 1970s ([1,5]) has a perfect mathematical solution to solve the output regulation problems. By the internal model principle, the robust output tracking is simplified to designing a dynamic tracking error feedback control containing a $p$-copy of the exosystem, where $p \in \mathbb{N}$ is the dimension of regulated output ([19]). This powerful method has also been applied to nonlinear lumped parameter systems ([14]) and distributed parameter systems [20,18]. However, in order to apply the abstract theory to the systems described by partial differential equations (PDEs), one has to verify many abstract conditions in casting into abstract form and solving operator equations which so involve difficult unbounded operator extensions. For example, in the large framework of abstract well-posed and regular infinite-dimensional systems ([20,18]), due to unboundedness of control and observation operators, the convergence claimed from an abstract point of view is usually very weak. Specifically, for internal asymptotic stability of closed-loop system in [20], the tracking error convergence is $\int_{t}^{t+1} \|e(s)\|^2 \, ds \to 0 (t \to \infty)$, which actually can be improved by PDE techniques. Most importantly, the abstract theory of the internal model principle for PDEs, not like for finite-dimensional counterparts, gives rarely insight into the error feedback control design. This can be seen from the PDE examples presented in these aforementioned papers.

On the other hand, for given PDEs, it is likely to develop a direct solution. Robust state (feedforward) feedback controls were addressed for hyperbolic PDEs in [3,4]. In [2], a robust output feedback control was designed based on an observer of an extended PDE-ODE system, where different to present paper, the reference signal was supposed to be known. In [21], an output regulation problem for a class of first-order hyperbolic PID equation systems was investigated by both state and output feedbacks but the robustness was also not discussed.

In the last a few years, some control methods have been proposed directly for distributed parameter systems. A
recent paper [6] proposed an observer based control by the trajectory planning approach for a wave equation where one channel disturbance and control were non-collocated. Although the observer design is skillful yet the robustness of the controller was not addressed. In paper [15], an unstable wave equation with disturbances in different channels was discussed but the robustness has not been touched either. Papers [9,10] proposed an adaptive control for a 1-d wave equation with harmonic disturbances of the form \( d(t) = a \cos \omega t + b \sin \omega t \) being in one channel. The authors estimated the unknown constants \( a, b \) in terms of the tracking error and its derivative. This brings some limitations to the results: a) There is a man-made assumption (for instance in theorem 3.2 of [10] or theorem 2 of [9]) that \( \omega \neq 0 + n\pi/2 \) which is not necessary by our approach; b) The internal stability and output tracking error convergence is asymptotically not exponentially; c) The convergence of the derivative of the tracking error was not mentioned although it was used in the control design; d) Although the method can be possibly used to deal with disturbances in other channels, the order of the control will then increase significantly because it needs to estimate all unknown constants in disturbances; e) The robustness was not addressed and is very likely to have no robustness to disturbing forces in other channels because of its slow (asymptotically) convergence. This motivates the present paper to deal with this problem in a unified way under the guidance of the internal model principle.

This approach was recently proposed in our work [13] which covers the recent paper [11] by adaptive control for a heat equation with the same drawbacks aforementioned. The contributions of the present paper are therefore as follows: 1) When the tracking error is the only measurable output, we can achieve asymptotic convergence, which was obtained in [9] but with additionally man-made condition and tracking error derivative; 2) Using the same outputs as [10], i.e., tracking error and its derivative, we can achieve internal exponential stability, exponential convergence of the tracking error and weak convergence of the derivative of the tracking error; 3) The control design is almost straightforward without increasing the order of the controller no matter where the disturbances are entering the system; 4) The controller is shown to be robust to all disturbances and is likely conditionally robust to system uncertainties, because it contains the internal model of the exosystem.

The wave equation that we consider in this paper is described by

\[
\begin{align*}
y_{xx}(x,t) &= y_{tt}(x,t) + d_1(x)p(t), \\
y_x(0,t) &= d_2p(t), \\
y_x(1,t) &= u(t) + d_3p(t), \\
Y(t) &= y(0,t),
\end{align*}
\]

(1)

where \( u(t) \) is the control and \( Y(t) \) is the regulated output which is non-collocated with control. The unknown disturbances \( d_1(x)p(t), d_k p(t), k = 2,3 \) are generated by the following exosystem

\[
\dot{p}(t) = Sp(t), \quad p(0) = p_0 \in \mathbb{C}^{n \times 1},
\]

(2)

where \( d_1(x) \in \mathbb{C}^{1 \times n} \) and \( d_k \in \mathbb{C}^{1 \times n} \), \( k = 2,3 \) are unknown, and \( S \in \mathbb{C}^{n \times n} \) is supposed to be known yet the initial value \( p_0 \) of the exosystem is unknown. In addition, we assume that \( d_1 \in L^2(0,1) \). The following assumption simply means that the disturbances are sum of finite harmonic disturbances discussed in [9,10].

**Assumption 1.1** The eigenvalues of the matrix \( S \) are algebraically simple, and are located on the imaginary axis.

The disturbance covered by Assumption 1.1 contains sinusoidal signals with unknown phase. For instance \( \sin(\omega t + \phi) \) where \( \phi \) is an unknown phase can be written as \( \sin(\omega t + \phi) = \cos \phi \sin \omega t + \sin \phi \cos \omega t \). When \( S \) can be diagonalized, the multi-eigenvalues do not produce more disturbances than simple eigenvalues. We are therefore concerned only the algebraic simply eigenvalues, which was also assumed in other papers like [2].

The reference signal is \( r(t) = d_4 \dot{p}(t) \) where \( d_4 \in \mathbb{C}^{1 \times n} \) is also unknown. The tracking error is denoted by

\[
e(t) = Y(t) - d_4 \dot{p}(t).
\]

(3)

The objective of the regulation in Section 2 is to design an error feedback control so that

\[
\lim_{t \to \infty} |e(t)| = 0,
\]

regardless of disturbances and at the same time to keep the closed-loop system internally asymptotically stable. Section 3 aims to guarantee the exponential convergence by additionally using derivative of the tracking error.

For output regulation problem of SISO system, it is also necessary that the eigenvalues of \( S \) cannot contain zeros of the transfer function. Take Laplace transform for system (1) without disturbance to get

\[
\begin{align*}
s^2r(x,s) &= r_{xx}(x,s), \\
r_x(0,s) &= 0, \\
r_x(1,s) &= \dot{u}(s), \\
Y(s) &= r(0,s),
\end{align*}
\]

(4)

where \( r(x,s), \dot{u}(s) \) and \( Y(s) \) are the Laplace transforms of \( y(x,t), \dot{u}(t) \) and \( Y(t) \) respectively. The transfer func-
tion from the input to the output is then obtained as
\[ H(s) = \frac{1}{s \sinh(s)}. \] (5)
which has no zeros. We therefore do not need any additional conditions for system (1), which is significantly different from [9,10] where some man-made conditions were imposed.

We proceed as follows. In the next section, section 2, in terms of the tracking error only, we design a robust error feedback to guarantee the asymptotically internally stability and tracking error convergence. In section 3, we use additionally the derivative of the tracking error to achieve robust output tracking with exponentially internally stability and tracking error convergence. Some numerical simulations are presented in each section to verify the proposed controls. Section 4 is devoted to the proofs of the main results, followed up concluding remarks in section 5. For the sake of notation simplicity, we postulate that \( k + S \) denotes \( kI_n + S \) where \( I_n \) is the \( n \)-dimensional unit matrix. We denote \( \mathcal{H} = \mathcal{H}_1(0,1) \times L^2(0,1) \times \mathbb{C} \).

### 2 Tracking error feedback robust control design

Since one of the requirements for output regulation is that the closed-loop system should be internally asymptotically stable, we first need to know how to asymptotically stabilize system (1) without disturbance. Now suppose that \( p(t) \equiv 0 \) in (1). Since the output operator is compact, we need to design a dynamic feedback control
\[
\begin{align*}
\hat{u}(t) &= (k_1 + k_2)Y(t) - k_1z(t), k_1, k_2 > 0, \\
\hat{z}(t) &= -k_1[z(t) - Y(t)],
\end{align*}
\]
to stabilize asymptotically system (1) with \( p(t) \equiv 0 \). This is motivated from [16] for beam equation. In other words, the system of the following:
\[
\begin{align*}
y_{\text{tr}}(x,t) &= y_{xx}(x,t), \\
y_x(0,t) &= (k_1 + k_2)y(0,t) - k_1z(t), k_1, k_2 > 0, \\
y_x(1,t) &= 0, \\
\hat{z}(t) &= -k_1[z(t) - y(0,t)],
\end{align*}
\] (6)
is asymptotically stable, which is claimed by the following lemma 2.1.

**Lemma 2.1** Suppose that \( k_1, k_2 > 0 \). Then, for any initial value \((y(\cdot,0), y_x(\cdot,0), z(0)) \in \mathcal{H}\), system (6) admits a unique solution and is asymptotically stable.

**Proof:** See “Proof of Lemma 2.1” in Section 4. When we discuss the output tracking, the output becomes \( e(t) \) instead of \( y(0,t) \) and the dynamic part \( \dot{z}(t) = -k_1[z(t) - y(0,t)] \) becomes
\[
\dot{\theta}(t) = -k_1\theta(t) + k_1e(t), k_1 > 0,
\] (7)
which is completely determined by the tracking error \( e(t) \) and will turns out to be same useful as \( z \)-subsystem in (6).

First of all, we choose frozen coefficients:
\[
\begin{align*}
d_1^0(x) &\equiv 0, \\
d_2^0 = (k_1 + k_2)\phi_j \neq 0, j = 1, \ldots, n, \\
d_2^2 = (k_1 + k_2)d_1^0 - k_1^2d_1^2[S + k_1]^{-1}, k_1, k_2 > 0,
\end{align*}
\] (8)
where \( \phi_j, j = 1, \ldots, n \) are the eigenvectors of \( S \). Such \( d_2^0 \) always exists like \( d_1^0 = [1, \ldots, 1][\phi_{s_1}, \ldots, \phi_{s_n}]^{-1} \). For those frozen coefficients, system (1)-(2) is reduced to a nominal system of the following:
\[
\begin{align*}
y_{\text{tr}}(x,t) &= y_{xx}(x,t), \\
y_x(0,t) &= (k_1 + k_2)[y(0,t) - e(t)] - k_1[z(t) - \theta(t)], \\
y_x(1,t) &= u(t), \\
\hat{p}(t) &= Sp(t), \\
\dot{\theta}(t) &= -k_1\theta(t) + k_1e(t), \\
e(t) &= y(0,t) - d_2^0p(t),
\end{align*}
\] (9)
where by (8), the \( z(t) \) is defined as
\[
z(t) = \theta(t) + k_1d_1^0[S + k_1]^{-1}p(t),
\] (10)
which satisfies
\[
\dot{z}(t) = -k_1[z(t) - Y(t)].
\] (11)
In what follows, by \( z(t) \) which is unknown, we mean (10) but the ODE (11) satisfied by \( z(t) \) is used to motivate the observer design for \( z(t) \).

The principle of choice of the frozen coefficients is that the outputs \( e(t) \) makes coupled PDE-ODEs system (9) detectable. The control design is only for system (9), which is the most advantage of the observer based approach.
2.1 Robust control design

**Step 1: Observer design.** For coupled system (9), an observer is designed as

\[
\begin{align*}
\dot{y}_t(x,t) &= \dot{y}_{xx}(x,t), \\
\dot{y}_x(0,t) &= (k_1 + k_2)[\dot{y}(0,t) - c(t)] - k_1[\dot{z}(t) - \theta(t)], \\
\dot{y}_x(1,t) &= u(t), \\
\dot{z}(t) &= -k_1[\dot{z}(t) - \dot{y}(0,t)], \\
\hat{p}(t) &= S\hat{p}(t) + Q[e(t) - \dot{y}(0,t) + d_0^p\hat{p}(t)],
\end{align*}
\]

(12)

where \( Q \in \mathbb{C}^{n \times 1}. \) By (7), it is seen that the observer is determined by the tracking error \( e(t) \) only. Denote the observer errors by

\[
\begin{align*}
\dot{\tilde{y}}(x,t) &= y(x,t) - \dot{y}(x,t), \\
\tilde{z}(t) &= z(t) - \hat{z}(t), \\
\hat{p}(t) &= \hat{p}(t) - p(t).
\end{align*}
\]

By (7),

\[
\tilde{z}(t) = \theta(t) + k_1d_4^p[S + k_1^{-1}Sp(t) + k_1[\dot{z}(t) - \dot{y}(0,t)]
\]

\[
= -k_1\theta(t) + k_1e(t) + k_1d_4^p[S + k_1^{-1}Sp(t)
\]

\[
+ k_1[\dot{z}(t) - \dot{y}(0,t)]
\]

\[
= -k_1[\theta(t) + k_1d_4^p(t)[S + k_1^{-1}p(t) - \tilde{z}(t)] + k_1\dot{y}(0,t)
\]

\[
= -k_1\tilde{z}(t) + k_1\dot{y}(0,t),
\]

where the last equality was obtained in view of (10).

The observer errors are then governed by

\[
\begin{align*}
\dot{\tilde{y}}_t(x,t) &= \dot{y}_{xx}(x,t), \\
\dot{\tilde{y}}_x(0,t) &= (k_1 + k_2)\dot{y}(0,t) - k_1\dot{z}(t), \\
\dot{\tilde{y}}_x(1,t) &= 0, \\
\dot{\tilde{z}}(t) &= -k_1[\dot{z}(t) - \dot{y}(0,t)], \\
\hat{\tilde{p}}(t) &= [S + Qd_4^p]\hat{p}(t) + Q\dot{y}(0,t),
\end{align*}
\]

(13)

where the coupled (\( \tilde{y}, \tilde{z} \))-subsystem is just the form of (6). The \( Q \) is chosen to make \( S + Qd_4^p \) be Hurwitz. This can be always guaranteed since \( (S, d_4^p) \) is detectable under the choice of \( d_4^p \). Actually, \( (S, d_4^p) \) is detectable if and only if \( (\phi_{s1}, \cdots, \phi_{sn})^{-1}S[\phi_{s1}, \cdots, \phi_{sn}]d_4^p[\phi_{s1}, \cdots, \phi_{sn}] \) is detectable which is obviously true by Assumption 1.1 that

\[
[\phi_{s1}, \cdots, \phi_{sn}]^{-1}S[\phi_{s1}, \cdots, \phi_{sn}] = \text{diag}[]_{s1, \cdots, sn},
\]

and

\[
d_4^p\phi_{sj} \neq 0, j = 1, 2, \cdots, n,
\]

by (8).

The following Lemma 2.2 is a direct consequence of Lemma 2.1 because system (13) is asymptotically stable if and only if (\( \tilde{y}, \tilde{z} \))-subsystem is asymptotically stable and the latter is claimed by Lemma 2.1.

**Lemma 2.2** Suppose that \( k_1, k_2 > 0 \) and \( S + Qd_4^p \) is Hurwitz. Then, for any initial value \( (\tilde{y}(\cdot, 0), \dot{\tilde{y}}(\cdot, 0), \tilde{z}(0), \hat{p}(0)) \) \( \in \mathcal{H} \times \mathbb{C}^n \), system (13) admits a unique solution and is asymptotically stable.

**Proof:** See “Proof of Lemma 2.2” in Section 4.

**Step 2: Feedforward control design.** To find feedforward control, the output regulation problem is usually transformed into a stabilization problem. Make a transformation for the \( y \)-subsystem in (9): \( \varepsilon(x,t) = y(x,t) - f_0(x)p(t) \). Then, \( \varepsilon(x,t) \) satisfies

\[
\begin{align*}
\varepsilon_t(x,t) &= \varepsilon_{xx}(x,t), \\
\varepsilon_x(0,t) &= 0, \\
\varepsilon_x(1,t) &= u(t) - f_0(1)p(t),
\end{align*}
\]

(14)

where \( f_0(x) \in \mathbb{C}^1 \times n \) satisfies the regulator equation:

\[
\begin{align*}
f_0'(x) &= f_0(x)S^2, \\
f_0(0) &= (k_1 + k_2)d_4^p - k_1d_4^p(S + k_1)^{-1}, \\
f_0(0) &= d_4^p,
\end{align*}
\]

(15)

which is an initial value problem of ODE and hence admits a unique solution. A feedforward control is therefore designed as

\[
\begin{align*}
u(t) &= f_0(1)p(t) - k_3\varepsilon(1,t) - k_4\varepsilon(1,t)
\]

\[
= -k_3y_h(1,t) - k_4y(1,t)
\]

(16)

under which the closed-loop system of (14) is exponentially stable in \( H^1(0,1) \times L^2(0,1) \) and hence the output is regulated that

\[
\lim_{t \to \infty} |\varepsilon(t)| = \lim_{t \to \infty} |\varepsilon(0,t)| = 0.
\]

By observer (12) and feedforward control (16), we obtain immediately an observer based tracking error feedback control by replacing the state in feedforward control with
the state of the observer as follows:

\[
\begin{align*}
    u(t) &= -k_3 \hat{y}_t(t) - k_4 \hat{y}(1, t) \\
    &\quad + [f'_0(1) + k_3 f_0(1) S + k_4 f_0(1)] \hat{p}(t), \\
    \hat{y}_t(x, t) &= \hat{y}_{xx}(x, t), \\
    \hat{y}_x(0, t) &= (k_1 + k_2) [\hat{y}(0, t) - e(t)] - k_1 \hat{z}(t) - \theta(t)], \\
    \hat{y}_x(1, t) &= u(t), \\
    \hat{z}(t) &= -k_1 \hat{z}(t) - \hat{y}(0, t), \\
    \hat{p}(t) &= S \hat{p}(t) + Q [e(t) - \hat{y}(0, t) + d_0^p \hat{p}(t)], \\
    e(t) &= y(0, t) - d_4 p(t),
\end{align*}
\]

which is designed only for nominal system (9) yet also works for original system (1) shown in next subsection because it contains 1-copy of the exosystem that \( \hat{p}(t) \to p(t)(t \to \infty) \).

### 2.2 Robustness analysis

The closed-loop of system (1) under control (17) is written as

\[
\begin{align*}
    y_t(x, t) &= y_{xx}(x, t) + d_1(x) p(t), \\
    y_0(t) &= d_2 p(t), \\
    y_x(1, t) &= u(t) + d_3 p(t), \\
    \hat{y}_t(x, t) &= \hat{y}_{xx}(x, t), \\
    \hat{y}_x(0, t) &= (k_1 + k_2) [\hat{y}(0, t) - e(t)] - k_1 \hat{z}(t) - \theta(t)], \\
    \hat{y}_x(1, t) &= u(t), \\
    \hat{z}(t) &= -k_1 \hat{z}(t) - \hat{y}(0, t), \\
    \hat{p}(t) &= S \hat{p}(t) + Q [e(t) - \hat{y}(0, t) + d_0^p \hat{p}(t)], \\
    e(t) &= y(0, t) - d_4 p(t),
\end{align*}
\]

which is considered in the state space \( \mathbb{H} = \mathcal{H} \times \mathbb{C}^n \). Introduce the transformations

\[
\begin{align*}
    w(x, t) &= y(x, t) + g_y(x) p(t), \\
    z^1(t) &= z(t) + k_1 g_y(0)[S + k_1]^{-1} p(t) \\
    &= \theta(t) + k_1 [g_y(0) + d_0^l][S + k_1]^{-1} p(t),
\end{align*}
\]

where \( g_y(x) \in \mathbb{C}^{1 \times n} \) is defined as

\[
\begin{align*}
    g_y(x) S^2 &= g_y'(x) - d_1(x), \\
    g_y'(0) &= -d_2 + (k_1 + k_2) [d_4 + g_y(0)] \\
    &\quad - k_1^2 [d_4 + g_y(0)][S + k_1]^{-1}, \\
    g_y'(1) &= -d_3,
\end{align*}
\]

and similarly to (10) and (11), \( z^1(t) \) satisfies

\[
\hat{z}^1(t) = -k_1 [\hat{z}^1(t) - w(0, t)].
\]

With this transformations, \( w(x, t) \) is governed now by

\[
\begin{align*}
    w_t(x, t) &= w_{xx}(x, t), \\
    w_x(0, t) &= (k_1 + k_2) [d_4 + g_y(0)] p(t) \\
    &\quad - k_1^2 [d_4 + g_y(0)][S + k_1]^{-1} p(t), \\
    &= (k_1 + k_2) [w(0, t) - e(t)] \\
    &\quad - k_1 [z^1(t) - \theta(t)],
\end{align*}
\]

\[
\begin{align*}
    w_x(1, t) &= u(t), \\
    u(t) &= f'_0(1) \hat{p}(t) - k_3 \hat{y}_t(1, t) - f_0(1) S \hat{p}(t), \\
    e(t) &= w(0, t) - [g_y(0) + d_4] p(t).
\end{align*}
\]

It is seen that the disturbances being all channels of \( y(x, t) \) in closed-loop (18) are now in one channel of \( w(x, t) \) in (22). Moreover, the transformation (19) makes \( (w, z^1) \) in (21)-(22) very similar to \( (\hat{y}, \hat{z}) \)-subsystem in observer (12) so that the second transformation becomes much straightforward.

We next introduce the second transformation

\[
\begin{align*}
    (y^c(x, t), \hat{y}^c(x, t), \hat{p}^c(t), \tilde{z}^c(t)) \\
    &= (w(x, t), \hat{y}(x, t), \hat{p}(t), z^1(t)) \\
    &\quad - (h(x)p(t), h(x)p(t), h_{\hat{p}}p(t), \tilde{z}(t)),
\end{align*}
\]

where \( h(x) \in \mathbb{C}^{1 \times n} \) satisfies

\[
\begin{align*}
    h(x) S^2 &= h''(x), \\
    h'(0) &= (k_1 + k_2) h(0) - k_1^2 h(0)[S + k_1]^{-1}, \\
    h(0) &= g_y(0) + d_4,
\end{align*}
\]

which is an initial value ODE and hence admits a solution provided (20) does, and \( h_{\hat{p}} \in \mathbb{C}^{n \times n} \) satisfies

\[
\begin{align*}
    h'(1) + k_3 h(1) S + k_4 h(1) \\
    &= [f'_0(1) + k_3 f_0(1) S + k_4 f_0(1)] h_{\hat{p}}, \\
    \{S + Q d_0^l\} h_{\hat{p}} - h_{\hat{p}} S &= Q h(0),
\end{align*}
\]
Then, the system of $(y^c, \hat{y}^c, \hat{z}^c, \hat{p}^c)$ is transformed into

$$
\begin{cases}
y^c_{\Delta}(x, t) = y^c(x, t), \\
y^c(0, t) = 0, \\
y^c(1, t) = (f_0(1) + k_3 f_0(1) S + k_4 f_0(1)) \hat{p}^c(t) - k_3 \hat{y}^c(1, t) - k_4 \hat{y}^c(1, t), \\
\hat{y}^c_{\Delta}(x, t) = \hat{y}^c_{xx}(x, t), \\
\hat{y}^c(0, t) = (k_1 + k_2) \hat{y}^c(0, t) - \hat{y}^c(0, t)] + k_1 \hat{z}^c(t), \\
\hat{y}^c(1, t) = (f_0(1) + k_3 f_0(1) S + k_4 f_0(1)) \hat{p}^c(t) - k_3 \hat{y}^c(1, t) - k_4 \hat{y}^c(1, t), \\
\hat{z}^c(t) = -k_1 [\hat{z}^c(t) - (y^c(0, t) - \hat{y}^c(0, t))], \\
\hat{p}^c(t) = (S + Qd_1^c) \hat{p}^c(t) + Q [y^c(0, t) - \hat{y}^c(0, t)], \\
e(t) = y^c(0, t),
\end{cases}
$$

\[(26)\]

where all channels have no disturbance at all. This is the reason we make twice transformations (19) and (23), which make the closed-loop system (18) free of disturbances in (26).

**Lemma 2.3** The boundary value problems (20) and (25) admit bounded solutions.

**Proof:** See “Proof of Lemma 2.3” in Section 4.

Finally, define

$$
\tilde{y}^c(x, t) = y^c(x, t) - \hat{y}^c(x, t).
$$

Then, system (26) is further reduced to

$$
\begin{cases}
\tilde{y}^c_{\Delta}(x, t) = \tilde{y}^c_{xx}(x, t), \\
\tilde{y}^c(0, t) = (k_1 + k_2) \tilde{y}^c(0, t) - k_1 \hat{z}^c(t), \\
\tilde{y}^c(1, t) = 0, \\
\hat{z}^c(t) = -k_1 \hat{z}^c(t) + k_1 \hat{y}^c(0, t), \\
\hat{p}^c(t) = (S + Qd_1^c) \hat{p}^c(t) + Q \tilde{y}^c(0, t), \\
\hat{y}^c_{\Delta}(x, t) = \hat{y}^c_{xx}(x, t), \\
\hat{y}^c(0, t) = -(k_1 + k_2) \hat{y}^c(0, t) + k_1 \hat{z}^c(t), \\
\hat{y}^c(1, t) = [f_0(1) + k_3 f_0(1) S + k_4 f_0(1)] \hat{p}^c(t) - k_3 \hat{y}^c(1, t) - k_4 \hat{y}^c(1, t), \\
e(t) = \tilde{y}^c(0, t) + \hat{y}^c(0, t),
\end{cases}
$$

\[(28)\]

which is our starting point to prove the following Theorem 2.1.

**Theorem 2.1** Suppose that $k_j > 0, j = 1, 2, 3, 4$ and $S + Qd_1^c$ is Hurwitz. For any initial state $(y^c(\cdot, 0), \hat{y}^c(\cdot, 0), \hat{z}(\cdot, 0), \hat{p}(\cdot, 0)) \in \mathbb{R}$ with the compatible condition $y(0, 0) - d_2p(0) = 0$, the closed-loop system (18) admits a unique solution and

(i). If $p(t) \neq 0$, the system is bounded:

$$
\|y^c(\cdot, t), y^c(\cdot, t), \hat{y}^c(\cdot, t), \hat{z}(\cdot, t), \hat{p}(\cdot, t)\| \leq \infty;
$$

(ii). When $p(t) \equiv 0$, the system is internally asymptotically stable:

$$
\lim_{t \to \infty} \|y^c(\cdot, t), y^c(\cdot, t), \hat{y}^c(\cdot, t), \hat{z}(\cdot, t), \hat{p}(\cdot, t)\| = 0;
$$

(iii). The tracking error is convergent asymptotically that $\lim_{t \to \infty} |e(t)| = 0$.

**Proof:** See “Proof of Theorem 2.1” in Section 4.

2.3 Numerical simulations

This subsection presents some numerical simulations for closed-loop (18). For the exosystem, choose $p(0) = (1, 0, 1, 0)^T$ and

$$
S = \left[0, -\pi 6, 0, 0; \pi 6, 0, 0, 0, 0, 0, 0, 0, 0, -\pi 3, 0, 0, \pi 3, 0\right].
$$

\[(29)\]

The control is set as

$$
\begin{cases}
\dot{z}(0) = 0.1, \dot{\hat{p}}(0) = (0.1, 0.2, 0.3, 0.4)^T, \dot{\hat{y}}(x, 0) = \frac{x}{2}, \\
\hat{y}(x, 0) = 0, k_1 = 1.8, k_2 = 15, k_3 = 0.5, k_4 = 4.8, \\
Q = \frac{\pi}{36} (-96, -92, -25, 135)^T, d_4^c = (1, 1, 1, 1).
\end{cases}
$$

\[(30)\]

The initial states are chosen as $y(x, 0) = \frac{7x}{10}$ and $y(x, 0) = 0$. The following disturbances and reference trajectory are firstly considered

$$
\begin{cases}
d_1(x) = \left(\frac{\cos(x)}{4}, \frac{\sin(x)}{5}, \frac{x^2}{3}, \frac{x}{2}\right), d_2 = (0, 2, 0, 4), \\
d_3 = (1, 0, 3, 0), d_4 = (0, 3, 0, 2).
\end{cases}
$$

\[(31)\]

From Figure 1, the system states and control input are bounded and the output is regulated to track the reference trajectory.
3 Robust control design using additionally derivative of tracking error

In the last section, we only claim that the closed-loop system is internally asymptotically stable and the tracking error is asymptotically convergent. This is because the measured signal, corresponding to a compact output operator, is too weak to make closed-loop exponentially stable ([17]). In this section, we explore further tracking error and its derivative (PD control) to enhance the internal stability and convergence, i.e., we shall use both

\[ e(t) = y(0, t) - d_4 p(t), \quad \dot{e}(t) = y_t(0, t) - d_4 S p(t) \]

in the control design. This is exactly the same case performed in [10]. The objective of the output regulation now becomes

\[ \lim_{t \to \infty} |e(t)| = 0, \quad \int_0^\infty e^{\alpha t} |\dot{e}(t)|^2 dt < \infty, \quad (32) \]

for some \( \alpha > 0 \). The first convergence is exponentially convergence and the second weak convergence was not addressed in [10]. Since usually \( \dot{e}(t) \) is not continuous, the weak convergence might be the best result we are expected ([18]).

Once again, first of all, we choose a set of frozen coefficients of the disturbances as follows (slightly different to (8))

\[
\begin{align*}
&d_1^0(x) \equiv 0, \quad d_0^0 = 0, \quad d_0^\alpha[\phi_{a1}, \ldots, \phi_{an}] \neq 0, \\
&d_2^2 = k_1 d_1^0 + k_2 d_0^0 S, \quad k_1, k_2 > 0, \quad (33)
\end{align*}
\]

by which we obtain a nominal system:

\[
\begin{align*}
y_t(x, t) &= y_{xx}(x, t), \\
y_x(0, t) &= k_1[y(0, t) - e(t)] + k_2[y_t(0, t) - \dot{e}(t)], \\
y_x(1, t) &= u(t), \\
\dot{p}(t) &= S p(t), \\
e(t) &= y(0, t) - d_4^0 p(t), \\
\dot{e}(t) &= y_t(0, t) - d_4^0 S p(t),
\end{align*}
\]

which is also slightly simpler than (9) because we have more measured output signals. In addition, since additional signal is used, the problem is not covered in abstract setting by tracking error robust control ([20]).

3.1 Robust control

Once again, we use two steps to design a robust control for nominal system (34).

Step 1: Observer design. In terms of \((e(t), \dot{e}(t))\), an observer for nominal system (34) is designed as

\[
\begin{align*}
\dot{y}_t(x, t) &= \dot{y}_{xx}(x, t), \\
\dot{y}_x(0, t) &= k_1[y(0, t) - e(t)] + k_2[\dot{y}_t(0, t) - \dot{e}(t)], \\
\dot{y}_x(1, t) &= u(t), \\
\dot{p}(t) &= S \dot{p}(t) + Q[e(t) - \dot{y}(0, t) + d_4^0 \dot{p}(t)].
\end{align*}
\]

Then, the observer errors \(\tilde{y}(x, t) = y(x, t) - \hat{y}(x, t)\) and \(\tilde{p}(t) = p(t) - \hat{p}(t)\) are governed by

\[
\begin{align*}
\dot{\tilde{y}}_t(x, t) &= \dot{\tilde{y}}_{xx}(x, t), \\
\dot{\tilde{y}}_x(0, t) &= k_1[\tilde{y}(0, t) - e(t)] - k_2 \tilde{y}_t(0, t), \\
\dot{\tilde{y}}_x(1, t) &= 0, \\
\dot{\tilde{p}}(t) &= (S + Q d_4^0) \dot{\tilde{p}}(t) + Q \tilde{y}(0, t),
\end{align*}
\]

which is obviously exponentially stable in the space \(H^1(0, 1) \times L^2(0, 1) \times \mathbb{C}^n\).

Step 2: Feedforward control design. Once again, we introduce \(\varepsilon(x, t) = y(x, t) - f_0(x) p(t)\) to transfer the output regulation problem into a stabilization problem, where \(\varepsilon(x, t)\) satisfies (14) and \(f_0(x) \in \mathbb{C}^{1 \times n}\) is updated as

\[
\begin{align*}
f^\varepsilon_0(x) &= f_0(x) S^2, \\
f^0_0(0) &= k_1 d_1^0 + k_2 d_0^0 S, \\
f^0_0(0) &= d_0^0,
\end{align*}
\]

which is an initial value problem and hence admits a unique solution. Same to (14), a feedforward control is
The first transformation is again
designed as
\[
    u(t) = -k_3y_t(1,t) - k_4y(1,t) + [f_0'(1) + k_3 f_0(1)S + k_4 f_0(1)]p(t).
\] (38)

Combining observer (35) and feedforward control (38), by replacing the states in feedforward control with the states of the observer, we immediately obtain an error feedback control as
\[
    \begin{align*}
    y(t) = y_{xx}(x, t) + d_1(x)p(t), \\
    y_x(0, t) = d_2 p(t), \\
    y_x(1, t) = u(t) + d_3 p(t), \\
    \dot{y}_x(t, x) = \ddot{y}_{xx}(x, t), \\
    \dot{y}_x(0, t) = k_1[\dot{y}(0, t) - c(t)] + k_2[\ddot{y}(0, t) - \dot{c}(t)], \\
    \dot{y}_x(1, t) = u(t), \\
    \ddot{p}(t) = S \dot{p}(t) + Q[e(t) - \dot{y}(0, t) + d_4 \dot{p}(t)], \\
    u(t) = -k_3\dot{y}_t(1, t) - k_4 \ddot{y}(1, t) + [f_0'(1) + k_3 f_0(1)S + k_4 f_0(1)]\ddot{p}(t), \\
    e(t) = y(0, t) - d_4 \dot{p}(t), \\
    \dot{c}(t) = y_t(0, t) - d_4 S \dot{p}(t).
    \end{align*}
\] (39)

Once again the control (39) is designed only for nominal system (34) yet is also shown to be working for original system (1) in next subsection.

3.2 Robust analysis

The closed-loop of system (1) under control (39) is written as
\[
    \begin{align*}
    y_{tt}(x, t) = y_{xx}(x, t) + d_1(x)p(t), \\
    y_x(0, t) = d_2 p(t), \\
    y_x(1, t) = u(t) + d_3 p(t), \\
    \dot{y}_x(t, x) = \ddot{y}_{xx}(x, t), \\
    \dot{y}_x(0, t) = k_1[\dot{y}(0, t) - c(t)] + k_2[\ddot{y}(0, t) - \dot{c}(t)], \\
    \dot{y}_x(1, t) = u(t), \\
    \ddot{p}(t) = S \dot{p}(t) + Q[e(t) - \dot{y}(0, t) + d_4 \dot{p}(t)], \\
    u(t) = -k_3\dot{y}_t(1, t) - k_4 \ddot{y}(1, t) + [f_0'(1) + k_3 f_0(1)S + k_4 f_0(1)]\ddot{p}(t), \\
    e(t) = y(0, t) - d_4 \dot{p}(t), \\
    \dot{c}(t) = y_t(0, t) - d_4 S \dot{p}(t).
    \end{align*}
\] (40)

Certainly, disturbances appear in all channels. What we need to do next is to find two transformations to make closed-loop free from disturbance. System (40) will be discussed in the state space \( \mathbb{H}_e = (H^2(0, 1) \times L^2(0, 1))^2 \times \mathbb{C}^n \).

The first transformation is again
\[
    w(x, t) = y(x, t) + g_y(x)p(t),
\] (41)

where \( g_y(x) \in \mathbb{C}^{1 \times n} \), parallel to (20), is updated to
\[
    \begin{align*}
    g_y(x)S^2 &= g_y''(x) - d_1(x), \\
    g_y'(0) &= -d_2 + k_1[d_4 + g_y(0)] + k_2[d_4 + g_y(0)]S, \\
    g_y''(1) &= -d_3.
    \end{align*}
\] (42)

Then, \( w(x, t) \) satisfies
\[
    \begin{align*}
    w_{tt}(x, t) &= w_{xx}(x, t), \\
    w_x(0, t) &= k_1[d_4 + g_y(0)]p(t) + k_2[d_4 + g_y(0)]Sp(t) \\
    &= k_1[w(0, t) - c(t)] + k_2[w_t(0, t) - \dot{c}(t)], \\
    w_x(1, t) &= u(t), \\
    w(t) &= f_0'(1)p(t) - k_3[\dot{y}(1, t) - f_0(1)\dot{p}(t)] \\
    &- k_4[\ddot{y}(1, t) - f_0(1)\ddot{p}(t)], \\
    e(t) &= w(0, t) - [d_4 + g_y(0)]p(t), \\
    \dot{c}(t) &= w_t(0, t) - [d_4 + g_y(0)]Sp(t),
    \end{align*}
\] (43)

Again, only one channel of \( w(x, t) \) has disturbance and the \( w \)-subsystem in (21) is similar to the \( \dot{y} \)-subsystem in observer (35), which makes our second transformation much easy. The second transformation is
\[
    (y^c(x, t), \dot{y}^c(x, t), \ddot{y}^c(t)) = (w(x, t), \dot{y}(x, t), \ddot{y}(t)) - (h(x)p(t), h(x)p(t), h \dot{p}(t)),
\] (44)

where \( h \) satisfies (25) with \( f_0(x) \) defined in (37) and \( h(x) \in \mathbb{C}^{1 \times n} \), parallel to (24), being updated as
\[
    \begin{align*}
    h(x)S^2 &= h''(x), \\
    h'(0) &= k_1 h(0) + k_2 h(0)S, \\
    h(0) &= g_y(0) + d_4,
    \end{align*}
\] (45)

which admits, as an initial value problem of ODE, a unique solution provided (42) does.
formed into

\[
\begin{align*}
&\frac{d\tilde{y}}{dt}(x,t) = \tilde{g}_x(x,t), \\
&\tilde{g}_x(0,t) = 0, \\
&\tilde{g}_x(1,t) = (f'_0(1) + k_3f_0(1)+k_4f_0(1))\tilde{p}(t) \\
&\quad -k_3\tilde{g}_x(1,t) - k_4\tilde{y}_c(1,t), \\
&\tilde{y}_c(x,t) = \tilde{g}_x(x,t), \\
&\tilde{g}_x(0,t) = k_1[\tilde{y}_c(0,t) - \tilde{y}(0,t)] \\
&\quad + k_2[\tilde{y}(0,t) - \tilde{y}_c(0,t)], \\
&\tilde{g}_x(1,t) = (f'_0(1) + k_3f_0(1)+k_4f_0(1))\tilde{p}(t) \\
&\quad - k_3\tilde{y}_c(1,t) - k_4\tilde{y}_c(1,t), \\
&\tilde{p}(t) = (S + Qd_1^2)\tilde{p}(t) + Q[\dot{\tilde{y}}_c(0,t) - \tilde{y}(0,t)], \\
&e(t) = \tilde{y}(0,t), \quad \dot{e}(t) = \tilde{y}_c(0,t),
\end{align*}
\]

where none of the channels has disturbance.

**Lemma 3.1** The boundary value problem (42) admits bounded solutions.

**Proof:** See “Proof of Lemma 3.1” in Section 4.

Finally, define

\[
\tilde{y}_c(x,t) = y_c(x,t) - \tilde{y}_c(x,t).
\]

Then, system (46) is further reduced to

\[
\begin{align*}
&\frac{d\tilde{y}_c}{dt}(x,t) = \tilde{g}_x(x,t), \\
&\tilde{g}_x(0,t) = k_1\tilde{y}_c(0,t) + k_2\tilde{g}_x(0,t), \\
&\tilde{y}_c(1,t) = 0, \\
&\tilde{p}(t) = (S + Qd_1^2)\tilde{p}(t) + Q[\dot{\tilde{y}}_c(0,t) - \tilde{y}(0,t)], \\
&e(t) = \tilde{y}_c(0,t), \quad \dot{e}(t) = \tilde{y}_c(0,t),
\end{align*}
\]

which is our starting point to prove the following Theorem 3.1.

**Theorem 3.1** Suppose \(k_j > 0, j = 1, 2, 3, 4\) and \(S + Qd_1^2\) is Hurwitz. For any initial value \((y(\cdot, 0), y_c(\cdot, 0), \tilde{y}_c(\cdot, 0), \tilde{y}(0, 0), \dot{p}(0)) \in \mathbb{H}_c = (H^2((0, 1) \times L^2((0, 1)))^2 \times \mathbb{C}^n\) with compatible condition \(y_0(0) - d_2p_0 = 0\), the closed-loop system (40) admits a unique bounded solution

\[
\sup_{t \geq 0} \|y(\cdot, t), y_c(\cdot, t), \tilde{y}_c(\cdot, t), \tilde{y}(\cdot, t), \dot{p}(t)\|_{\mathbb{H}_c} < +\infty;
\]

and when \(p(t) \equiv 0\), the closed-loop system is internally exponentially stable:

\[
\|y(\cdot, t), y_c(\cdot, t), \tilde{y}_c(\cdot, t), \tilde{y}(\cdot, t), \dot{p}(t)\|_{\mathbb{H}_c} \leq M_2 e^{-\omega_2 t} \|y(0, 0), y_c(0, 0), \tilde{y}_c(0, 0), \tilde{y}(0, 0), \dot{p}(0)\|_{\mathbb{H}_c},
\]

where \(M_2, \omega_2 > 0\). The tracking errors are regulated to zero such that

\[
\lim_{t \to \infty} |e(t)| = 0, \quad \int_0^\infty |e^\alpha \dot{e}(t)|^2 < \infty,
\]

for some \(\alpha > 0\). In particular, the first convergence is exponentially convergence.

**Proof:** See “Proof of Theorem 3.1” in Section 4.

We point out that the result of Theorem 3.1 is much stronger than the results in existing literature where only asymptotic stability and convergence were claimed, yet with additional man-made assumption and no robustness (e.g., [9,10]). Compared with [6,15], we have not only the robustness but also more straightforward feedback control.

**Remark 3.1** Through the transformations (41) and (44), the closed-loop system (40) is rewritten as an exponentially stable system (46), which is claimed by Theorem 3.1 and (47). The evolution equation of (46) in the state space \(\mathbb{H}_c = (H^2(0, 1) \times L^2(0, 1))^2 \times \mathbb{C} \times \mathbb{C}^n\) can be written as

\[
\frac{d}{dt}(y_c^2, y_c^1, \tilde{y}_c^1, \tilde{y}_c^2, \tilde{y}_c^3, \tilde{p}^2) = T_1(y_c^2, y_c^1, \tilde{y}_c^1, \tilde{y}_c^2, \tilde{y}_c^3, \tilde{p}^2)
\]

where \(T_1\) generates an exponentially stable \(C_0\)-semigroup such that \(\|e^{T_1 t}\| \leq C_1 e^{-\omega_1 t}\) for some \(C_1, \omega_1 > 0\) and all \(t \geq 0\).

The internal model principle claims that the output is still regulated as long as the system is internally exponentially stable. However, there are so many too much possibilities for system uncertainties and seems impossible to formulate the uncertainties in a united PDE form. Here we consider only a simple case that for \((y_c^2, y_c^1)\)-subsystem of (46), there exist an uncertain perturbation

\[
T_p(y_c^2, y_c^1, \tilde{y}_c^1, \tilde{y}_c^2, \tilde{y}_c^3, \tilde{p}^2) = (0, \Theta_1(x))y_c^2 + \Theta_2(x)y_c^1, 0, 0, 0, 0
\]

such that \(\|T_p\|_{\mathbb{H}_c} \leq C_2\) where \(C_2 > 0\) depends on unknown functions \(\Theta_k \in L^2(0, 1), k = 1, 2\).
The corresponding perturbed closed-loop system then becomes
\[
\frac{d}{dt}(y^c, \dot{y}^c, \ddot{y}^c, \dddot{y}^c, \dot{p}^c) = (T_1 + T_p)(y^c, \dot{y}^c, \ddot{y}^c, \dddot{y}^c, \dot{p}^c)
\]
where \( C_2 < \frac{1}{\rho_2} \) which makes \( T_1 + T_p \) generate an exponentially stable \( C_0 \)-semigroup on \( \mathbb{R}^n \) (internal exponential stability). The perturbed system (52) corresponds the following PDE:

\[
\begin{align*}
\dot{y}_t(x, t) &= y_{xx}(x, t) + \Theta_1 y_t(x, t) + \Theta_2 y(x, t) + d_1(x)p(t), \\
y_x(0, t) &= d_2 p(t), \\
y_e(1, t) &= u(t) + d_3 p(t), \\
y_t(x, t) &= \dot{y}_x(x, t), \\
y_e(0, t) &= k_1 [\hat{y}(0, t) - c(t)] + k_2 [\hat{y}_t(0, t) - \dot{c}(t)], \\
\dot{y}_e(1, t) &= u(t), \\
\dot{e}(t) &= S \dot{p}(t) + Q[e(t) - \hat{y}(0, t) + d_3 \dot{p}(t)], \\
u(t) &= -k_3 \hat{y}_t(1, t) - k_4 \hat{y}(1, t) + [f_0(1) + k_3 f_0(1) + k_4 f_0(1)] \dot{p}(t), \\
c(t) &= y(0, t) - d_4 p(t), \quad \dot{c}(t) = y_t(0, t) - d_4 S \dot{p}(t).
\end{align*}
\]

We can still prove that the tracking error \( e(t) = y^c(0, t) \) is convergent exponentially to zero as time \( t \to \infty \), which implies that the controller is conditionally robust to system uncertainties. Since this step is almost the same as that for heat equation discussed in our previous work [13], we omit the details in this paper.

3.3 Simulation

In this subsection, we present some numerical simulations for closed-loop system (40). For the exosystem, take \( p(0) = (1, 0, 1, 0)^T \) and
\[
S = \left[0, -\pi/6; 0, 0; 0, 0; 0, 0, 0, 0, -\pi/3; 0, 0, 0\right].
\]
For the control, set
\[
\begin{align*}
\dot{p}(0) &= (0.1, 0.2, 0.3, 0.4)^T, \\
\dot{w}(x, 0) &= x/2, \\
\dot{w}_t(x, 0) &= 0, k_1 = 0.6, k_2 = 0.8, k_3 = 2, k_4 = 1.8, \\
Q &= \frac{\pi}{36}(-96, -92, -25, 135)^T, \\
d_4 &= (1, 1, 1, 1).
\end{align*}
\]
We choose initial state \( y(x, 0) = \frac{7x}{10}, y_t(x, 0) = 0 \).

The disturbances and reference trajectory are chosen as
\[
\begin{align*}
d_1(x) &= \left(\cos(x), \sin(x), \frac{x^2}{3}, \frac{x}{2}\right), \\
d_2 &= (0, 0.2, 0, 0.4), \\
d_3 &= (0, 1, 0, 0.3, 0), \\
d_4 &= (0, 0, 3, 0, 0.2).
\end{align*}
\]
From Figure 2(a), the closed-loop system is bounded and from Figure 2(b), the output is regulated to track the reference trajectory. Furthermore, the derivative of the tracking error is guaranteed to be bounded, as well as the control input.

Fig. 2. The closed-loop system under (55).

4 Proofs of the main results

Proof of Lemma 2.1. Define the operator \( A_1 \) by
\[
\begin{align*}
A_1(f, \dot{f}, g) &= \left(\dot{f}, f''(x), -k_1 g + k_1 f(0)\right), \\
\forall (f, \dot{f}, g) &\in \mathcal{D}(A_1), \\
\mathcal{D}(A_1) &= \{(f, \dot{f}, g) \in H^2(0, 1) \times H^1(0, 1) \times \mathbb{C} : f'(0) = -k_1 g + (k_1 + k_2) f(0), \quad f'(1) = 0\}. 
\end{align*}
\]
For any \((f_2, f_2, z_2) \in \mathcal{H}\) and any \((f_1, \dot{f}_1, z_1) \in \mathcal{D}(A_1)\), \( A_1(f_1, \dot{f}_1, z_1) = (f_2, \dot{f}_2, z_2) \) admits a unique solution that
\[
\begin{align*}
f_1(x) &= x \int_1^x \dot{f}_2(\xi) d\xi - \int_0^x \xi f_2(\xi) d\xi + \frac{-\int_0^x f_2(\xi) d\xi - z_2}{k_2}, \\
\dot{f}_1(x) &= f_2(x), \\
z_1 &= \frac{-\int_0^x f_2(\xi) d\xi - z_2}{k_2} - \frac{1}{k_1} z_2.
\end{align*}
\]
By the Sobolev embedding theorem, $A_1^{-1}$ is compact on $\mathcal{H}$, and thus $\sigma(A_1)$, the spectrum of $A_1$, consists of isolated eigenvalues only. Solve the eigenvalue problem $A_1(f, g) = \mu(f, g) \neq 0$, that is,

\[
\begin{align*}
\lambda g &= -k_1 g + k_1 f(0), \\
J''(x) - \lambda^2 J(x) &= 0, \\
J'(0) &= -k_1 g + (k_1 + k_2) f(0), \quad J'(1) = 0.
\end{align*}
\]

For $\lambda \neq -k_1$, $J(x)$ satisfies

\[
\begin{align}
\lambda^2 h &= -k_1 h + k_1 f(0), \\
h(0) &= k_1 \lambda f(0) + k_2 h(0), \\
h'(1) &= 0.
\end{align}
\]

Taking the inner product with $J(x)$ on both sides of the first equation of (57) gives

\[
0 = \int_0^1 [\lambda^2 f(x) - f''(x)] J(x) dx
= \frac{k_1 \lambda}{k_1 + \lambda + i\lambda^2} [f(0)]^2 + k_2 [f(0)]^2 + \int_0^1 |f'(x)|^2 dx \\
+ \lambda^2 \int_0^1 |J(x)|^2 dx.
\]

If $\lambda$ is real, the above identity implies that $\lambda < 0$. Otherwise, let $\lambda = \lambda_1 + i\lambda_2, \lambda_2 \neq 0$ where $\lambda_i \in \mathbb{R}, i = 1, 2$. From the imaginary part of the above identity, we find

\[
\frac{k_1^2 \lambda_2}{k_1 + \lambda_1 + i\lambda_2} |f(0)|^2 + 2\lambda_1 \lambda_2 \int_0^1 |f(x)|^2 dx = 0,
\]

which advises that $\lambda_1 = \text{Re}(\lambda) \leq 0$. If $\lambda_1 = \text{Re}(\lambda) = 0$, the above identity implies that $f(0) = 0$. In this case, (57) becomes

\[
\begin{align}
\lambda^2 f(x) &= 0, \\
f(0) &= 0, \\
f'(x) &= 0.
\end{align}
\]

which admits only zero solution. This implies that the eigenvalues of $A_1$ have negative real parts. Solve the eigenvalue problem (57) to obtain

\[
e^{2\lambda} = \frac{\lambda - \frac{k_1 \lambda}{k_1 + \lambda} - k_2}{\lambda + \frac{k_1 \lambda}{k_1 + \lambda} + k_2} = 1 + O(\lambda^{-1}).
\]

Solving above eigenvalue problem asymptotically ([12]), we find eigenpairs $(\lambda_n, f_n(x))$ of $A_1$ as follows:

\[
\lambda_n = n\pi i + O(n^{-1}), f_n(x) = \cos n\pi x + O(n^{-1}),
\]

where $n$’s are integers. Since \{$(\cos n\pi x/n\pi i, \cos n\pi x)$\}$_{n \in \mathbb{Z}}$ forms an orthonormal basis for $H^1(0, 1) \times L^2(0, 1)$ and \{$\Phi_n(x)$\}$_{n \in \mathbb{Z}}$ \{\{\Psi(x)\}$

\[
\left\{ \begin{array}{l}
\Phi_n(x) = (\cos n\pi x/n\pi i, \cos n\pi x, 0, n \in \mathbb{Z}, \\
\Psi(x) = (0, 0, 1).
\end{array} \right.
\]

By theorem A1 of [12] and (58), there is a sequence of generalized eigenfunction of $A_1$, which forms a Riesz basis for $\mathcal{H}$. Moreover, all eigenvalues of $A_1$ are algebraically simple with large modulus. As a result, $A_1$ generates a $C_0$-semigroup on $\mathcal{H}$, which is asymptotically stable. $\blacksquare$

**Proof of Lemma 2.3.** To solve (20), we define $f_y(x) \in C^{1 \times n}$ as

\[
\begin{align}
0 &= \int_0^1 \lambda^2 f(x) - f''(x) f(x) dx \\
&= \frac{k_1 \lambda}{k_1 + \lambda + i\lambda^2} |f(0)|^2 + k_2 [f(0)]^2 + \int_0^1 |f'(x)|^2 dx \\
&+ \lambda^2 \int_0^1 |f(x)|^2 dx.
\end{align}
\]

In order to solve (20), we only need to solve (24) with $h(0) = g(0) + d_4$ being replaced by $h'(1) = f_y'(1)$, for which we have $h(x) = H(x)[\phi_{s_1}, \cdots, \phi_{s_n}]^{-1}$ by defining $\phi_{s_j}$ as the eigenvector of the eigenvalue $\lambda_{s_j} \in \sigma(S)$ for $j = 1, \cdots, n$. It is found that $H_j(x) = H(x)\phi_{s_j}$ satisfies

\[
\begin{align}
H''_j(x) &= H_j(x)\lambda_{s_j}^2, \\
H'_j(0) &= \left( \frac{k_1 \lambda_{s_j}}{\lambda_{s_j} + k_1} + k_2 \right) H_j(0), \\
H''_j(1) &= f_y''(1) \phi_{s_j} - d_3 \phi_{s_j},
\end{align}
\]

which has solution

\[
H_j(x) = e^{C_{H_j} x} \left[ \begin{array}{c}
\frac{\lambda_{s_j}^2 - k_2 (k_1 + \lambda_{s_j})}{\lambda_{s_j}^2 + 2k_1 \lambda_{s_j} + k_2 (k_1 + \lambda_{s_j})} e^{-\lambda_{s_j} x} \\
+ e^{\lambda_{s_j} x}
\end{array} \right], \quad \lambda_{s_j} \neq 0,
\]

where $C_{H_j}$ is uniquely determined by $h'(1) = f_y'(1) - d_3$.

Last, since (25) involves $f_0(x)$ and $h(x)$, we need to solve (15). Now, $F_{0j}(x) = f_0(x)\phi_{s_j}$ satisfies

\[
\begin{align}
F''_{0j}(x) &= F_{0j}(x)\lambda_{s_j}^2, \\
F'_{0j}(0) &= \left( \frac{k_1 \lambda_{s_j}}{\lambda_{s_j} + k_1} + k_2 \right) d_{0j} \phi_{s_j}, \\
F_{0j}(0) &= d_{0j} \phi_{s_j},
\end{align}
\]
which has solution

$$F_{0j}(x) = \begin{cases} k_2 d_4^j \phi_{sj} x + d_3^j \phi_{sj}, \lambda_{sj} = 0, \\
\frac{\lambda_{sj}^2}{2 \lambda_{sj}^2 + 2 k_1 \lambda_{sj} + 2 k_1} x, \lambda_{sj} \neq 0,
\end{cases}$$

(63)

where $c_{F0j} = \frac{\lambda_{sj}^2 + 2 k_1 \lambda_{sj} + 2 k_1}{2 \lambda_{sj} + 2 k_1 \lambda_{sj}} d_4^j \phi_{sj}$. Finally, (25) is equivalent to

$$H_j(1) + k_3 \lambda_{sj} H_j(1) + k_4 H_j(1)$$

$$= [F'_{0j}(1) + k_3 \lambda_{sj} F_{0j}(1) + k_4 F_{0j}(1)] H_{\bar{p}j},$$

(64)

$$F_{0j}(0) H_{\bar{p}} = H_j(0), j = 1, 2, \ldots, n,$$

where $\text{diag}(H_{\bar{p}j}) = H_{\bar{p}} = [\phi_{s1}, \ldots, \phi_{sn}]^{-1} H_p [\phi_{s1}, \ldots, \phi_{sn}]$. In this way, we find a solution to (25) as

$$H_{\bar{p}j} = \begin{cases} \frac{c_{Hj}}{\lambda_{sj}}, \lambda_{sj} = 0, \\
\frac{c_{Hj}}{\lambda_{sj}}, \lambda_{sj} \neq 0.
\end{cases}$$

**Proof of Theorem 2.1.** By (19), (23) and (27), the closed-loop system (18) is transformed into (28). It is therefore sufficiently considering asymptotic stability and convergence of system (28).

First, it is seen that $(\vec{y}^e, \vec{z}^e, \vec{p}^e)$-subsystem in (28) is exactly the same as system (13). By Lemma 2.2, for any initial value $(\vec{y}^e(0), \vec{y}^f(0), \vec{z}^e(0), \vec{p}^f(0)) \in H \times \mathbb{C}^n$, there exists a unique solution to $(\vec{y}^e, \vec{z}^e, \vec{p}^e)$-subsystem in (28) that

$$\lim_{t \to \infty} \left\| (\vec{y}^e(t), t), \vec{y}^f(t), \vec{z}^e(t), \vec{p}^e(t)) \right\|_{H \times \mathbb{C}^n} = 0,$$

$$\lim_{t \to \infty} \| \vec{y}^e(0, t) \| = 0.$$  

(65)

The $\vec{y}^e$-subsystem in (28) is rewritten as

$$\frac{d}{dt} \left( \vec{y}^e(t), \vec{y}^f(t) \right) = A_2 \left( \vec{y}^e(t), \vec{y}^f(t) \right)$$

$$+ \left( \begin{array}{c} 0 \\
\delta(x - 1)
\end{array} \right) \left( f'_0(1) + k_3 f_0(1) S + k_4 f_0(1) \right) \vec{p}^e(t)$$

+ \left( \begin{array}{c} 0 \\
\delta(x)
\end{array} \right) \left[ -(k_1 + k_2) \vec{y}^e(0, t) + k_4 \vec{z}^e(t) \right],$$

where the operator $A_2$ is defined as

$$A_2(f(x), f'(x)) = (\tilde{f}(x), \tilde{f}'(x)), \forall (f, \tilde{f}) \in D(A_2),$$

$$D(A_2) = \{ (f, \tilde{f}) | H^2(0, 1) \times H^2(0, 1) \},$$

$$\tilde{f}'(1) = -k_3 \tilde{f}(1) + k_4 f(1)$$

and $e^{A_2 t}$ is a well-known exponentially stable $C_0$-semigroup on $H^1(0, 1) \times L^2(0, 1)$. Since the operators $A_0$ and $A_0$ are admissible to $e^{A_2 t}$ (122), and by (65), $(\vec{y}^e(0), \vec{y}^f(0), \vec{z}^e(0)) \in H^1(0, 1) \times L^2(0, 1)$, the $\vec{y}^e$-subsystem is asymptotically stable. Therefore, for any initial value $(\vec{y}^e(0), \vec{y}^f(0), \vec{z}^e(0)) \in H^1(0, 1) \times L^2(0, 1)$, there exists a unique solution in $H^1(0, 1) \times L^2(0, 1)$ to $\vec{y}^e$-subsystem in (28) and (122).

$$\lim_{t \to \infty} \| \vec{y}^e(0, t) \| \leq \lim_{t \to \infty} \| (\vec{y}^e(t), \vec{y}^f(t)) \|_{H^1(0, 1) \times L^2(0, 1)} = 0.$$  

(66)

This, together with (65), gives

$$\lim_{t \to \infty} [\vec{y}^e(0, t) + \vec{y}^f(0, t)] = 0.$$  

(67)

**Proof of Lemma 3.1.** Same to Lemma 2.3, apply $f_0(x)$ defined by (59) to write $h(x) = f_0(x) + g_0(x)$ where $g_0(x)$ is defined by (42). The solution of (45), similar to Lemma 3.1, can be expressed by $h(x) = H(x) [\phi_0, \ldots, \phi_n]^{-1}$ where $H(x) = \{ H_j(x) \}$ satisfies

$$H_j(x) = \begin{cases} k_1 c_{Hj} x + c_{Hj}, \lambda_{sj} = 0, \\
0, \lambda_{sj} \neq 0.
\end{cases}$$

(68)

with $c_{Hj}$ being determined by $h'(1) = f'_0(1) - d_3$. This further advances the existence of solution to (42).

**Proof of Theorem 3.1.** By (41), (44) and (47), the closed-loop system (40) is transformed into (48). It then suffices to consider the exponential stability of (48). In addition, we only consider the real part of the solution since the imaginary part can be dealt with exactly the same way. The $(\vec{y}^e, \vec{p}^e)$-subsystem of (48) is a well-known exponentially stable system in $H^1(0, 1) \times L^2(0, 1) \times \mathbb{C}^n$. Hence, for any initial value $(\vec{y}^e(0), \vec{y}^f(0), \vec{p}^f(0)) \in H^1(0, 1) \times L^2(0, 1) \times \mathbb{C}^n$, there exists a unique solution in $H^1(0, 1) \times L^2(0, 1) \times \mathbb{C}^n$ to $(\vec{y}^e, \vec{p}^e)$-subsystem of (48)
Define $\rho(t) = -2 \int_0^1 (x-1) \tilde{g}_2(x,t) \tilde{g}_1(x,t) dx$, which satisfies
\[
|\rho(t)| \leq \int_0^1 \{[\tilde{y}_1(x,t)]^2 + [\tilde{y}_2(x,t)]^2\} dx \\
\leq M_1 e^{-\omega_1 t} \| \tilde{g}^\circ(\cdot,0), \tilde{g}^\circ_l(\cdot,0), \tilde{p}^\circ(0) \|_{H^2(0,1) \times L^2(0,1) \times \mathbb{C}^n}.
\] 

Differentiating $\rho(t)$ with respect to time and integrating by parts, we have
\[
\dot{\rho}(t) = -2 \int_0^1 (x-1) \tilde{g}_2(x,t) \tilde{g}_1(x,t) \\
+ (x-1) \tilde{g}_2(x,t) \tilde{g}_1(x,t) dx \\
= -[\tilde{g}_2(0,t)]^2 + \int_0^1 [\tilde{g}_1(x,t)]^2 dx - [\tilde{g}_2(0,t)]^2 \\
+ \int_0^1 [\tilde{g}_1(x,t)]^2 dx \\
\leq -[\tilde{g}_2(0,t)]^2 + \int_0^1 [\tilde{g}_1(x,t)]^2 dx + \int_0^1 [\tilde{g}_2(x,t)]^2 dx \\
\leq M_1 e^{-\omega_1 t} \| \tilde{g}^\circ(\cdot,0), \tilde{g}^\circ_l(\cdot,0), \tilde{p}^\circ(0) \|_{H^2(0,1) \times L^2(0,1) \times \mathbb{C}^n} \\
-|\tilde{g}_2(0,t)|^2,
\]
for any $0 < \beta < \frac{\omega_1}{2}$, it has
\[
\int_0^\infty e^{2\beta t} |\tilde{g}_2(0,t)|^2 dt \\
\leq \rho(0) + \frac{M_1}{\omega_1 - 2\beta} \| \tilde{g}^\circ(\cdot,0), \tilde{g}^\circ_l(\cdot,0), \tilde{p}^\circ(0) \|_{H^2(0,1) \times L^2(0,1) \times \mathbb{C}^n} \\
+ 2\beta \int_0^\infty e^{2\beta t} \rho(t) dt < \infty.
\] 

For the $\tilde{g}^\circ$-subsystem of (48), we write
\[
\frac{d}{dt} \left[ e^{\beta t} \begin{pmatrix} \tilde{g}^\circ(\cdot,t) \\ \tilde{g}^\circ_l(\cdot,t) \end{pmatrix} \right] \\
= (A_2 + \beta) e^{\beta t} \begin{pmatrix} \tilde{g}^\circ(\cdot,t) \\ \tilde{g}^\circ_l(\cdot,t) \end{pmatrix} + \begin{pmatrix} 0 \\ \delta(x-1) \end{pmatrix} \\
	imes [f'_0(1) + k_3f_0(1)S + k_4f_0(1)] e^{\beta t} \tilde{p}^\circ(t) \\
+ \begin{pmatrix} 0 \\ \delta(x) \end{pmatrix} [-k_1 e^{\beta t} \tilde{g}^\circ(0,t) - k_2 e^{\beta t} \tilde{g}^\circ_l(0,t)],
\]
where $A_2$ is defined in (66), and $e^{A_2 t}$ is an exponentially stable $C_0$-semigroup on $H^1(0,1) \times L^2(0,1)$ that
\[
\| e^{A_2 t} \| \leq M_3 e^{-\omega_3 t}, M_3, \omega_3 > 0.
\]
For any
\[
\beta = \min \left\{ \frac{\omega_1}{M_3}, \frac{\omega_1}{2} \right\},
\]
we have
\[
\int_0^\infty e^{2\beta t} |\tilde{g}^\circ(0,t)|^2 dt < \infty, \int_0^\infty e^{2\beta t} |\tilde{g}^\circ_l(0,t)|^2 dt < \infty, \\
\int_0^\infty e^{2\beta t} |\tilde{p}^\circ(t)|^2 dt < \infty.
\]
Since $A_2 + \beta$ generates an exponentially stable $C_0$-semigroup, and $(0, \delta(x-1)) \mathbb{T}$ and $(0, \delta(x)) \mathbb{T}$ are admissible to $e^{(A_2+\beta)t}$ ([22]), we have that $(e^{\beta \tilde{g}^\circ(\cdot,t)}, e^{\beta \tilde{g}^\circ_l(\cdot,t)})$ is asymptotically stable in $H^1(0,1) \times L^2(0,1)$. This further implies that the $\tilde{g}^\circ$-subsystem is exponentially stable that ([22])
\[
\| (\tilde{g}^\circ(\cdot,t), \tilde{g}^\circ_l(\cdot,t)) \|_{H^1(0,1) \times L^2(0,1)} \\
\leq C e^{-2\beta t} \| (\tilde{g}^\circ(\cdot,0), \tilde{g}^\circ_l(\cdot,0), \tilde{g}^\circ(\cdot,t), \tilde{g}^\circ_l(\cdot,t), \tilde{p}^\circ(0)) \|_{H_r},
\]
where $C$ is a positive constant independent of the initial values.

Same to (67), $e(t)$ is exponentially convergent that
\[
\lim_{t \to \infty} |e(t)| = \lim_{t \to \infty} |\tilde{g}^\circ(0,t) + \tilde{g}^\circ(0,t)| = 0,
\] 
extponentially.

Define $\rho_1(t) = -2 \int_0^1 (x-1) \tilde{g}^\circ_l(x,t) \tilde{g}^\circ(0,t) dx$. Then,
\[
|\rho_1(t)| \leq C e^{-2\beta t} \| (\tilde{g}^\circ(\cdot,0), \tilde{g}^\circ_l(\cdot,0), \tilde{g}^\circ(\cdot,t), \tilde{g}^\circ_l(\cdot,t), \tilde{p}^\circ(0)) \|_{H_r}.
\]
By
\[\dot{\rho}_1(t) = -|\bar{g}_x^c(0,t)|^2 - |\bar{g}_x(0,t)|^2 + \| \langle \bar{g}_x^c(\cdot, t), \bar{g}_x^c(\cdot, t) \rangle \|_{L^2(0,1)}^2,\]
and similarly to (70), we can obtain
\[\int_0^\infty e^{2\alpha t} |\bar{g}_x^c(0,t)|^2 dt < \infty,
\]
for some $0 < \alpha < \beta$. This further implies
\[\lim_{t \to \infty} \int_0^\infty |e^{\alpha t} \bar{e}(t)|^2 dt \leq \lim_{t \to \infty} \int_0^\infty |e^{\alpha t} \bar{g}_x^c(0,t)|^2 dt + \int_0^\infty |e^{\alpha t} \bar{g}_x(0,t)|^2 dt < \infty.
\]
This completes the proof of the theorem.

5 Concluding remarks

This paper develops a unified way to deal with output regulation for wave equation with non-collocated control and observation, and with disturbances in all possible channels. When the performance output is weak, we can achieve internal asymptotic stability and asymptotic convergence for tracking error, the same effect to [9,10] by adaptive control using additionally the derivative of the tracking error in control design and man-made assumption. If the derivative of the tracking error can be used in control design as existing literature (e.g.,[9,10]), we can enhance to exponential stability and convergence. Very importantly, our control design is only done for a nominal system regardless how many and where the disturbances are. This significantly reduces the control order by comparison with the adaptive control where each unknown constant in disturbances must be estimated. The last but not the least is that our control is robust to all external disturbances described by exosystem and likely conditional robust to system uncertainty as explained in Remark 3.1 because our controller contains actually internal model, which is sharp contrast to the adaptive control method that there is likely no robustness. The results also enhance significantly the recent results in [6,15]. Finally, the mathematics analysis is also straightforward and can be applicable to other PDEs [8].

One reviewer indicated an interesting question that how $k_3,k_4$ in controller (38) affect the performance of the closed-loop. This is hard to be answered because this relies on spectral analysis for system (48) which is not easy due to highly coupling of the system.

References
