Robust output regulation of 1-d wave equation

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A B S T R A C T
In the last a few years, there have been a couple of works addressing output regulation for 1-d linear wave equations without robustness. The aim of this paper is to provide, under the guidance of the internal model principle, a unified way to achieve more profound results including fast convergence, no man-made assumption, simple control design and particularly robustness. Different from existing works where disturbance appears only in one channel, we allow disturbances in all possible channels. Our approach is an observer based control approach which is designed only for a nominal system yet also has the power to guarantee output regulation for original uncertain PDE system. Simulation examples are presented to show the effectiveness of the proposed control.

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1. Introduction

In engineering, the most important task is to achieve performance output tracking regardless of external disturbances and system uncertainties, and at the same time to keep the system internally asymptotically stable, which is frequently referred to as output regulation problem in control theory (Lamaré, Auriol, Meglio, & Aarsnes, 2018). For linear time invariant systems with disturbances from an exo-system, the internal model principle which was proposed in 1970s (Davison, 1976; Francis & Wonham, 1976) has a perfect mathematical solution to solve the output regulation problems. By the internal model principle, the robust output tracking is simplified to designing a dynamic tracking error feedback control containing a \( p \)-copy of the exosystem, where \( p \in \mathbb{N} \) is the dimension of regulated output (Paunonen, 2016). This powerful method has also been applied to nonlinear lumped parameter systems (Huang, 2004) and distributed parameter systems (Natarajan, Gilliam, & Weiss, 2014; Paunonen, 2017). However, in order to apply the abstract theory to the systems described by partial differential equations (PDEs), one has to verify many abstract conditions in casting into abstract form and solving operator equations which also involve difficult unbounded operator extensions. For example, in the large framework of abstract well-posed and regular infinite-dimensional systems (Natarajan et al., 2014; Paunonen, 2017), due to unboundedness of control and observation operators, the convergence claimed from an abstract point of view is usually very weak. Specifically, for internal asymptotic stability of closed-loop system in Paunonen (2017), the tracking error convergence is \( \int_{t}^{t+1} \| e(s) \|^{2} ds \to 0 (t \to \infty) \), which actually can be improved by PDE techniques. Most importantly, the abstract theory of the internal model principle for PDEs, not like for finite-dimensional counterparts, gives rarely insight into the error feedback control design. This can be seen from the PDE examples presented in these aforementioned papers.

On the other hand, for given PDEs, it is likely to develop a direct solution. Robust state (feedforward) feedback controls were addressed for hyperbolic PDEs in Deutscher (2016b), Deutscher and Jakob (2018). In Deutscher (2016a), a robust output feedback control was designed based on an observer of an extended PDE–ODE system, where different to present paper, the reference signal was supposed to be known. In Xu and Dubljevic (2017), an output regulation problem for a class of first-order hyperbolic PID equation systems was investigated by both state and output feedbacks but the robustness was also not discussed.

In the last a few years, some control methods have been proposed directly for distributed parameter systems. A recent paper (Feng, Guo, & Wu, 2020) proposed an observer based control by the trajectory planning approach for a wave equation where one channel disturbance and control were non-collocated. Although the observer design is skillful yet the robustness of the controller was not addressed. In paper Jin and Guo (2019), an unstable wave equation with disturbances in different channels was discussed but the robustness has not been touched either.

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Papers Guo, Zhou, and Krstic (2018) and Guo, Shao, and Krstic (2017) proposed an adaptive control for a 1-d wave equation with harmonic disturbances of the form \( d(t) = a \cos \omega t + b \sin \omega t \) being in one channel. The authors estimated the unknown constants \( a, b \) in terms of the tracking error and its derivative. This brings some limitations to the results: (a) There is a man-made assumption (for instance in theorem 3.2 of Guo et al. (2017) or theorem 2 of Guo et al. (2018)) that \( \omega \neq n \pi + \pi / 2 \) which is not necessary by our approach; (b) The internal stability and output tracking error convergence is asymptotically not exponentially; (c) The convergence of the derivative of the tracking error was not mentioned although it was used in the control design; (d) Although the method can be possibly used to deal with disturbances in other channels, the order of the control will then increase significantly because it needs to estimate all unknown constants in disturbances; (e) The robustness was not addressed and is very likely to have no robustness to disturbances in other channels because of its slow (asymptotically) convergence. This motivates the present paper to deal with this problem in a united way under the guidance of the internal model principle.

This approach was recently proposed in our work (Guo & Meng, 2020a) which covers the recent paper (Guo & Jin, 2020) and Meng, 2020a) which coverstherecentpaper(Guo&Jin,2020) under the guidance of the internal model principle.

The wave equation that we consider in this paper is described by

\[
\begin{align*}
 y_t(x,t) &= y_{ss}(x,t) + d_1(x)p(t), \\
y_s(0,t) &= d_2p(t), \\
y_s(1,t) &= u(t) + d_3p(t), \\
Y(t) &= y(0,t),
\end{align*}
\]

where \( u(t) \) is the control and \( Y(t) \) is the regulated output which is non-collocated with control. The unknown disturbances \( d_1(x)p(t), d_2p(t), k = 2, 3 \) are generated by the following system

\[
\dot{p}(t) = 5p(t), \quad p(0) = p_0 \in C^{n \times 1},
\]

where \( d_1(x) \in C^{1 \times n} \) and \( d_2 \in C^{n \times 1} \), \( k = 2, 3 \) are unknown, and \( S \in C^{n \times n} \) is supposed to be known but the initial value \( p_0 \) of the exosystem is unknown. In addition, we assume that \( d_1 \in L^2(0, 1) \). The following assumption simply means that the disturbances are sum of finite harmonic disturbances discussed in Guo et al. (2017, 2018).

**Assumption 1.1.** The eigenvalues of the matrix \( S \) are algebraically simple, and are located on the imaginary axis.

The disturbance covered by Assumption 1.1 contains sinusoid signals with unknown phase. For instance \( \sin(\alpha t + \phi) \) where \( \phi \) is unknown phase can be written as \( \sin(\alpha t + \phi) = \cos \phi \sin \alpha t + \sin \phi \cos \alpha t \). When \( S \) can be diagonalized, the multi-eigenvalues do not produce more disturbances than simple eigenvalues. We are therefore concerned only the algebraic simply eigenvalues, which was also assumed in other papers like Deutscher (2016a).

The reference signal is \( r(t) = d_4 p(t) \) where \( d_4 \in C^{1 \times n} \) is also unknown. The tracking error is denoted by

\[
e(t) = Y(t) - d_4 p(t).
\]

The objective of the regulation in Section 2 is to design an error feedback control so that

\[
limit_{t \rightarrow \infty} |e(t)| = 0,
\]

regardless of disturbances and at the same time to keep the closed-loop system internally asymptotically stable. Section 3 aims to guarantee the exponential convergence by additionally using derivative of the tracking error.

For output regulation problem of SISO system, it is also necessary that the eigenvalues of \( S \) cannot contain zeros of the transfer function. Take Laplace transform for system (1) without disturbance to get

\[
\begin{align*}
 s^2 r(x, s) &= r_{xx}(x, s), \\
r_x(0,s) &= 0, \\
r_x(1, s) &= \hat{u}(s), \\
Y(s) &= r(0, s),
\end{align*}
\]

where \( r(x, s), \hat{u}(s) \) and \( Y(s) \) are the Laplace transforms of \( y(x, t), u(t) \) and \( Y(t) \) respectively. The transfer function from the input to the output is then obtained as

\[
H(s) = \frac{1}{s \sinh(s)}.
\]

which has no zeros. We therefore do not need any additional conditions for system (1), which is significantly different from Guo et al. (2017, 2018) where some man-made conditions were imposed.

We proceed as follows. In the next section, Section 2, in terms of the tracking error only, we design a robust error feedback to guarantee the asymptotic internal stability and tracking error convergence. In Section 3, we use additionally the derivative of the tracking error to achieve robust output tracking with exponential internal stability and tracking error convergence. Some numerical simulations are presented in each section to verify the proposed controls. Section 4 is devoted to the proofs of the main results, followed up concluding remarks in Section 5. For the sake of notation simplicity, we postulate that \( k + S \) denotes \( kI_n + S \) where \( I_n \) is the \( n \)-dimensional unit matrix. We denote \( \mathcal{H} = H^1(0, 1) \times L^2(0, 1) \times C \).

## 2. Tracking error feedback robust control design

Since one of the requirements for output regulation is that the closed-loop system should be internally asymptotically stable, we first need to know how to asymptotically stabilize system (1) without disturbance. Now suppose that \( p(t) \equiv 0 \) in (1). Since the output operator is compact, we need to design a dynamic feedback control

\[
\begin{align*}
u(t) &= (k_1 + k_2)Y(t) - k_1 z(t), k_1, k_2 > 0, \\
z(t) &= -k_1 z(t) - Y(t),
\end{align*}
\]

to stabilize asymptotically system (1) with \( p(t) \equiv 0 \). This is motivated from Li, Jia, and Liu (2015) for beam equation. In other words, the system of the following:

\[
\begin{align*}
y_t(x,t) &= y_{xx}(x,t), \\
y_s(0,t) &= (k_1 + k_2)y(0,t) - k_1 z(t), k_1, k_2 > 0, \\
y_s(1,t) &= 0, \\
z(t) &= -k_1 z(t) - y(0,t),
\end{align*}
\]

is asymptotically stable, which is claimed by the following Lemma 2.1.
Lemma 2.1. Suppose that $k_1, k_2 > 0$. Then, for any initial value $(y(\cdot, 0), y_0(\cdot, 0), z(0)) \in \mathcal{H}$, system (6) admits a unique solution and is asymptotically stable.

Proof. See “Proof of Lemma 2.1” in Section 4.

When we discuss the output tracking, the output becomes $e(t)$ instead of $y(0, t)$ and the dynamic part $\dot{z}(t) = -k_1[z(t) - y(0, t)]$ becomes

$$\dot{\theta}(t) = -k_1 \theta(t) + k_1 e(t), \quad k_1 > 0,$$  
(7)

which is completely determined by the tracking error $e(t)$ and will turn out to be same useful as $z$-subsystem in (6).

First of all, we choose frozen coefficients:

$$d_0^1(x) \equiv 0, \quad d_0^2 = 0, \quad d_0^3 \neq 0, \quad j = 1, 2, \ldots, n, \quad d_0^4 = (k_1 + k_2) d_0^3 - k_1^2 d_0^2 [S + k_1]^{-1}, \quad k_1, k_2 > 0,$$  
(8)

where $\phi_j, j = 1, \ldots, n$ are the eigenvectors of $S$. Such $d_0^3$ always exists like $d_0^4$. The principle of choice of the frozen coefficients is that the ODE (11) is satisfied by $\dot{\theta}(t)$, $\dot{e}(t)$ and $\dot{z}(t)$.

Lemma 2.2. Suppose that $k_1, k_2 > 0$ and $S + Qd_0^4$ is Hurwitz. Then, for any initial value $\{y(\cdot, 0), y_0(\cdot, 0), z(0), \dot{z}(0)\} \in \mathcal{H} \times \mathbb{C}^n$, system (13) admits a unique solution and is asymptotically stable.

Proof. See “Proof of Lemma 2.2” in Section 4.

Step 2: Feedforward control design. To find feedforward control, the output regulation problem is usually transformed into a stabilization problem. Make a transformation for the $y$-subsystem in (9): $\varepsilon(x, t) = y(x, t) - f_0(x)p(t)$. Then, $\varepsilon(x, t)$ satisfies

$$\varepsilon_t(x, t) = \varepsilon_x(x, t),$$  
$$\varepsilon_x(0, t) = 0,$$  
$$\varepsilon_t(1, 0) = p(t) - f_0(1)p(t),$$  
(14)

where $f_0(x) \in \mathbb{C}^1 \times \mathbb{C}^n$ satisfies the regulator equation:

$$f_0^t(x) = f_0(x)[S],$$  
$$f_0^x(0) = (k_1 + k_2) d_0^3 - k_1^2 d_0^2 [S + k_1]^{-1},$$  
$$f_0(0) = d_0^4,$$  
(15)

which is an initial value problem of ODE and hence admits a unique solution. A feedforward control is therefore designed as

$$u(t) = f_0^t(1)p(t) - k_3 \varepsilon_t(1, 1) - k_3 \varepsilon_t(1, 1) + k_3 \varepsilon_t(1, 1),$$  
$$+ f_0^t(1)[S + k_3][f_0^x(1) + k_3][f_0(1)]p(t),$$  
(16)

under which the closed-loop system of (14) is exponentially stable in $H^1[0, 1] \times L^2(0, 1)$ and hence the output is regulated that

$$\lim_{t \to \infty} |e(t)| = 0, \quad |e(0, t)| = 0.$$  

By observer (12) and feedforward control (16), we obtain immediately an observer based tracking error feedback control by replacing the state in feedforward control with the state of the observer as follows:

$$\dot{u}(t) = -k_2 \dot{y}_x(1, 1) - k_3 \dot{y}_x(1, 1) + k_3 \varepsilon_t(1, 1),$$  
$$+ f_0^t(1)[S + k_3][f_0^x(1) + k_3][f_0(1)]p(t),$$  
$$\dot{y}_x(x, t) = y_x(x, t),$$  
$$\dot{y}_x(0, t) = (k_1 + k_2)[\dot{y}(0, t) - e(t)] - k_1 \dot{z}(0, t) - \theta(t),$$  
$$\dot{x}(t) = -k_1 [\dot{z}(0, t) - \dot{y}(0, t)],$$  
$$\dot{x}(1, t) = u(t).$$  
(17)
which is designed only for nominal system (9) yet also works for original system (1) shown in next subsection because it contains 1-copy of the exosystem that \( \hat{p}(t) \to p(t) | t \to \infty \).

### 2.2. Robustness analysis

The closed-loop of system (1) under control (17) is written as

\[
\begin{align*}
\begin{cases}
y_r(x, t) &= y_w(x, t) + d_1(x)p(t), \\
y_\delta(0, t) &= d_2 p(t), \\
y_\delta(1, t) &= u(t) + d_3 p(t), \\
y_\delta(x, t) &= y_w(x, t), \\
y_\delta(0, t) &= (k_1 + k_2)[y_\delta(0, t) - e(t)] - k_3[\tilde{z}(t) - \theta(t)], \\
y_\delta(1, t) &= u(t), \\
\hat{z}(t) &= -k_1[\hat{z}(t) - \tilde{y}(0, t)], \\
\hat{p}(t) &= \hat{p}(t) + Q[e(t) - \tilde{y}(0, t) + d_4^T p(t)], \\
\tilde{t}(t) &= -k_1[\tilde{t}(t) - k_4 \hat{y}(1, t)] + \hat{g}_1^T(0, 1)S + k_4 \hat{d}_1 \hat{p}(t), \\
e(t) &= \tilde{y}(0, t) - d_4 p(t), \\
\end{cases}
\end{align*}
\]

which is considered in the state space \( \mathbb{H} = \mathbb{R} \times \mathbb{C}^m \). Introduce the transformations

\[
\begin{align}
w(x, t) &= y(x, t) + g_5(x)p(t), \\
z_1(t) &= z(t) + k_5 g_5(0) S + k_1^{-1} p(t) - \theta(t), \\
\end{align}
\]

where \( g_5(x) \in \mathbb{C}^{1 \times n} \) is defined as

\[
\begin{align}
g_5(0) &= -d_2 + [k_1 + k_2][d_4 + g_6(0)] \\
g_5^T(1) &= -d_3,
\end{align}
\]

and similarly to (10) and (11), \( z_1(t) \) satisfies

\[
\begin{align}
z_1'(t) &= -k_1 z_1(t) - w(0, t).
\end{align}
\]

With this transformation, the system is governed now by

\[
\begin{align}
\begin{cases}
w_\delta(x, t) &= w_\delta(x, t), \\
w_\delta(0, t) &= (k_1 + k_2)[d_4 + g_6(0)]p(t) - k_1^2[d_4 + g_6(0)]S + k_1^{-1} p(t), \\
\tilde{t}(t) &= -k_1[z_1(t) - \theta(t)], \\
\end{cases}
\end{align}
\]

\[
\begin{align}
w_\delta(1, t) &= u(t), \\
\hat{p}(t) &= \hat{p}(0) + f_0^T \hat{p}(t) - k_4 \hat{y}(1, t) - \hat{f}_0(0, 1)S \hat{p}(t), \\
e(t) &= w(0, t) - g_6(0) + d_4 p(t).
\end{align}
\]

It is seen that the disturbances being all channels of \( y(x, t) \) in closed-loop (18) are now in one channel of \( w(x, t) \) in (22). Moreover, the transformation (19) makes \( (w, z_1) \) in (21)–(22) very similar to \( (\hat{y}, \hat{z}) \)-subsystem in observer (12) so that the second transformation becomes much straightforward.

We next introduce the second transformation

\[
\begin{align}
\begin{cases}
\hat{y}_c(x, t) &= y_c(x, t), \\
\hat{y}_c(0, t) &= (f_0^T(0, 1) + k_5 \hat{d}_1(0) S + k_4 \hat{d}_1(0)) \hat{p}(t), \\
\hat{y}_c(1, t) &= (k_1 + k_2) [\hat{y}_c(0, t) - \hat{y}(0, t)] - k_4 \hat{y}(1, t), \\
\hat{y}_c(1, t) &= \hat{y}_c(0, t), \\
\hat{y}_c(1, t) &= (f_0^T(0, 1) + k_5 \hat{d}_1(0) S + k_4 \hat{d}_1(0)) \hat{p}(t) + f_0^T \hat{p}(t), \\
\tilde{t}(t) &= -k_1[\hat{z}(t) - (\hat{y}(0, t) - \hat{y}(0, t))], \\
\hat{p}(t) &= (S + Q d_4^T) \hat{p}(t) + Q[\hat{y}(0, t) - \hat{y}(0, t)], \\
e(t) &= \hat{y}(0, t), \\
\end{cases}
\end{align}
\]

where all channels have no disturbance at all. This is the reason we make twice transformations (19) and (23), which make the closed-loop system (18) free of disturbance in (26).

**Lemma 2.3.** The boundary value problems (20) and (25) admit bounded solutions.

**Proof.** See “Proof of Lemma 2.3” in Section 4.

Finally, define

\[
\hat{y}^\ast(x, t) = y_c^\ast(x, t) - \tilde{y}(x, t).
\]

Then, system (26) is further reduced to

\[
\begin{align}
\begin{cases}
\hat{y}^\ast(x, t) &= y_c^\ast(x, t), \\
\hat{y}^\ast(0, t) &= (k_1 + k_2) \hat{y}(0, t) - k_4 \hat{y}(1, t), \\
\hat{y}^\ast(1, t) &= 0, \\
\hat{z}^\ast(t) &= -k_1 \hat{z}^\ast(t) + k_4 \hat{y}(1, t), \\
\hat{p}(t) &= (S + Q d_4^T) \hat{p}(t) + Q[\hat{y}(0, t) - \hat{y}(0, t)], \\
e(t) &= \hat{y}(0, t) + \hat{y}^\ast(t), \\
\end{cases}
\end{align}
\]

which is our starting point to prove the following **Theorem 2.1.**

**Theorem 2.1.** Suppose that \( k_j > 0, j = 1, 2, 3, 4 \) and \( S + Q d_4^T \) is Hurwitz. For any initial state \( (y(0, t), y_c(0, t), \hat{y}(1, t), \hat{z}(0), \hat{p}(0)) \in \mathbb{H} \) with compatible condition \( y(0, 0) - d_4 p(0) = 0 \), the closed-loop system (18) admits a unique solution and

(i). If \( p(t) = 0 \), the system is bounded: \( \| y(t), y_c(t), \hat{y}(1, t), \hat{z}(t), \hat{p}(t) \|_H < \infty \);

(ii). When \( p(t) \to 0 \), the system is internally asymptotically stable: \( \lim_{t \to \infty} \| y(t), y_c(t), \hat{y}(1, t), \hat{z}(t), \hat{p}(t) \|_H = 0 \);

(iii). The tracking error is convergent asymptotically that \( \lim_{t \to \infty} e(t) = 0 \).
The control is set as 

\[
\begin{align*}
\dot{x}(0) &= 0.1, \quad \ddot{x}(0) = (0.1, 0.2, 0.3, 0.4)^T, \quad \dddot{x}(0) = \frac{x}{2}, \\
\dot{y}(0) &= 0, \quad k_1 = 1.8, \quad k_2 = 15, \quad k_3 = 0.5, \quad k_4 = 4.8. \\
Q &= \frac{\pi}{36}(-96, -92, -25, 135)^T, \quad d_4 = (1, 1, 1, 1).
\end{align*}
\]

The initial states are chosen as \(y(x, 0) = \frac{7x}{10}\) and \(z(x, 0) = 0\). The following disturbances and reference trajectory are firstly considered 

\[
\begin{align*}
\dot{d}_1(x) &= \left(\cos(x), \sin(x), \frac{x^2}{2}, x\right), \quad d_1 = (0, 2, 0, 4), \\
\dot{d}_3 &= (1, 0, 3, 0), \quad d_3 = (0, 3, 0, 2).
\end{align*}
\]

From Fig. 1, the system states and control input are bounded and the output is regulated to track the reference trajectory.

3. Robust control design using additionally derivative of tracking error 

In the last section, we only claim that the closed-loop system is internally asymptotically stable and the tracking error is asymptotically convergent. This is because the measured signal, corresponding to a compact output operator, is too weak to make closed-loop exponentially stable (Guo, 1998). In this section, we explore further tracking error and its derivative (PD control) to enhance the internal stability and convergence, i.e., we shall use both 

\[
e(t) = y(0, t) - d_4 p(t), \quad \dot{e}(t) = y_t(0, t) - d_4 S p(t)
\]

in the control design. This is exactly the same case performed in Guo et al. (2017). The objective of the output regulation now becomes 

\[
\lim_{t \to \infty} \|e(t)\|^2 = 0, \quad \int_0^\infty e^{\alpha t} |e(t)|^2 dt < \infty,
\]

for some \(\alpha > 0\). The first convergence is exponentially convergence and the second weak convergence was not addressed in Guo et al. (2017). Since usually \(\dot{e}(t)\) is not continuous, the weak convergence might be the best result we are expected (Natarajan et al., 2014). 

Once again, first of all, we choose a set of frozen coefficients of the disturbances as follows (slightly different to (8)) 

\[
\begin{align*}
d^0_1 &= 0, \quad d^1_1 = 0, \quad d^2_1[\phi_1, \ldots, \phi_m] \neq 0, \\
d^0_3 &= k_1 d^1_4 + k_2 d^2_4 s, \quad k_1, k_2 > 0.
\end{align*}
\]

by which we obtain a nominal system: 

\[
\begin{align*}
y_{\hat{u}}(x, t) &= y_{\hat{u}}(x, t), \\
y_{\hat{u}}(0, t) &= k_1[y_0(0, t) - e(t)] + k_2[y_t(0, t) - \dot{e}(t)], \\
y_{\hat{u}}(1, t) &= u(t), \\
\dot{p}(t) &= S \hat{p}(t), \\
e(t) &= y(0, t) - \hat{y}(0, t) - d_4 \hat{p}(t), \\
\dot{e}(t) &= y_t(0, t) - d_4 \hat{p}(t).
\end{align*}
\]

which is also slightly simpler than (9) because we have more measured output signals. In addition, since additional signal is used, the problem is not covered in abstract setting by tracking error robust control (Paunonen, 2017).

3.1. Robust control 

Once again, we use two steps to design a robust control for nominal system (34).

Step 1: Observer design. In terms of \((e(t), \dot{e}(t))\), an observer for nominal system (34) is designed as 

\[
\begin{align*}
\hat{y}_{\hat{u}}(x, t) &= \hat{y}_{\hat{u}}(x, t), \\
\hat{y}_{\hat{u}}(0, t) &= k_1[y_0(0, t) - e(t)] + k_2[\hat{y}_t(0, t) - \dot{e}(t)], \\
\hat{y}_{\hat{u}}(1, t) &= u(t), \\
\ddot{p}(t) &= S \hat{p}(t) + Q[e(t) - \hat{y}(0, t) + d_4 \hat{p}(t)].
\end{align*}
\]

Then, the observer errors \(\hat{y}(x, t) = y(x, t) - \hat{y}(x, t)\) and \(\dot{p}(t) = p(t) - \ddot{p}(t)\) are governed by 

\[
\begin{align*}
\hat{y}_{\hat{u}}(x, t) &= y_{\hat{u}}(x, t), \\
\hat{y}_{\hat{u}}(0, t) &= k_1 y_0(0, t) - k_2 \hat{y}_t(0, t), \\
\hat{y}_{\hat{u}}(1, t) &= 0, \\
\ddot{p}(t) &= (S + Q d_4) \ddot{p}(t) + Q\hat{e}(0, t),
\end{align*}
\]

which is obviously exponentially stable in the space \(H^1(0, 1) \times L^2(0, 1) \times C^m\).

Step 2: Feedforward control design. Once again, we introduce \(e(x, t) = y(x, t) - f_0(x)p(t)\) to transfer the output regulation into a stabilization problem, where \(e(x, t)\) satisfies (14) and \(f_0(x) \in C^{1+n}\) is updated as 

\[
\begin{align*}
f_0'(x) &= f_0(x) S^2, \\
f_0(0) &= d_4, \\
f_0(0) &= d_4,
\end{align*}
\]

which is an initial value problem and hence admits a unique solution. Same to (14), a feedforward control is designed as 

\[
\begin{align*}
u(t) &= -k_3 \hat{y}_t(1, t) - k_4 \hat{y}(1, t), \\
+ \left[ I_{f_0}(1) + k_3 f_0(1) S + k_4 f_0(1) \right] \hat{p}(t).
\end{align*}
\]

Combining observer (35) and feedforward control (38), by replacing the states in feedforward control with the states of the observer, we immediately obtain an error feedback control as 

\[
\begin{align*}
u(t) &= -k_3 \hat{y}_t(1, t) - k_4 \hat{y}(1, t), \\
+ \left[ I_{f_0}(1) + k_3 f_0(1) S + k_4 f_0(1) \right] \ddot{p}(t), \\
\hat{y}_{\hat{u}}(x, t) &= y_{\hat{u}}(x, t), \\
\hat{y}_{\hat{u}}(0, t) &= k_1[y_0(0, t) - e(t)] + k_2[\hat{y}_t(0, t) - \dot{e}(t)], \\
\hat{y}_{\hat{u}}(1, t) &= u(t), \\
\ddot{p}(t) &= S \hat{p}(t) + Q[e(t) - \hat{y}(0, t) + d_4 \hat{p}(t)].
\end{align*}
\]

Once again the control (39) is designed only for nominal system (34) yet is also shown to be working for original system (1) in next subsection.
3.2. Robust analysis

The closed-loop system (1) under control (39) is written as

\[\begin{align*}
y_t(x, t) &= y_u(x, t) + d_1(x)p(t), \\
y_0(0, t) &= d_2p(t), \\
y_1(1, t) &= u(t) + d_3p(t), \\
y_u(0, t) &= y_0(x, t), \\
y_0(0, t) &= k_1[y_0(0, t) - e(t)] + k_2[y_1(0, t) - e(t)], \\
y_1(1, t) &= u(t), \\
\hat{p}(t) &= S\hat{p}(t) + Qe(t) - \hat{y}(0, t) + d_4p(t), \\
u(t) &= -k_0\hat{y}(0, t) - k_1\hat{y}(1, t), \\
e(t) &= y(0, t) - d_5p(t), \\
\hat{e}(t) &= y(0, t) - d_6p(t). \\
\end{align*}\]  

(40)

Certainly, disturbances appear in all channels. What we need to do next is to find two transformations to make closed-loop free from disturbance. System (40) will be discussed in the state space \(\mathbb{H}_e = (H^2(0, 1) \times L^2(0, 1))^2 \times \mathbb{C}^n\).

The first transformation is again

\[\begin{align*}
w(x, t) &= y(x, t) + g_0(x)p(t), \\
\end{align*}\]  

(41)

where \(g_0(x) \in \mathbb{C}^{1 \times n}\), parallel to (20), is updated to

\[\begin{align*}
g_0(x) &= g_0(x) - d_1(x), \\
g_0'(0) &= -d_2 + k_1[d_4 + g_0(0)] + k_2[d_4 + g_0(0)]S, \\
g_0'(1) &= -d_3. \\
\end{align*}\]  

(42)

Then, \(w(x, t)\) satisfies

\[\begin{align*}
w_0(x, t) &= w(x, t), \\
w_0(0, t) &= k_1[d_4 + g_0(0)]p(t) + k_2[d_4 + g_0(0)]Sp(t), \\
w_0(1, t) &= u(t), \\
w_0 = \hat{f}_0[1]\hat{p}(t) - k_1\hat{y}(1, t) - f_0(1)S\hat{p}(t) \\
e(t) &= w(0, t) - [d_4 + g_0(0)]p(t), \\
\hat{e}(t) &= w(0, t) - [d_4 + g_0(0)]Sp(t). \\
\end{align*}\]  

(43)

Again, only one channel of \(w(x, t)\) has disturbance and the \(w\)-subsystem in (21) is similar to the \(y\)-subsystem in observer (35), which makes our second transformation much easy. The second transformation is

\[\begin{align*}
(y', y', \hat{y}', \hat{p}') &= (w(x, t), [y(x, t), \hat{p}(t)] - (h(x)p(t), h(x)p(t), h_2p(t)), \\
h(x)S &= h'(x), \\
h(0) &= k_1h(0) + k_2h(0)S, \\
h(0) &= g_0(0) + d_4, \\
\end{align*}\]  

(44)

where \(h_2\) satisfies (25) with \(f_0(x)\) defined in (37) and \(h(x) \in \mathbb{C}^{1 \times n}\), parallel to (24), being updated as

\[\begin{align*}
h(x) &= h'(x), \\
h(0) &= k_1h(0) + k_2h(0)S, \\
h(0) &= g_0(0) + d_4, \\
\end{align*}\]  

(45)

which admits, as an initial value problem of ODE, a unique solution provided (42) does.

After the second transformation, system (43) is transformed into

\[\begin{align*}
y'_0(x, t) &= y'_0(x, t), \\
y'_0(0, t) &= 0, \\
y'_0(1, t) &= (f_0(1) + k_1f_0(1)S + k_2f_0(1))\hat{p}'(t) - k_3\hat{y}'(1, t), \\
y'_1(x, t) &= y'_1(x, t), \\
y'_1(0, t) &= k_1[y'_0(0, t) - y^*(0, t)], \\
y'_1(1, t) &= (f_0(1) + k_1f_0(1)S + k_2f_0(1))\hat{p}'(t) - k_3\hat{y}'(1, t), \\
\hat{p}'(t) &= (S + Qd_4\hat{p}'(t) + Q\hat{y}'(0, t) - \hat{y}'(0, t)), \\
e(t) &= y'(0, t), \\
\hat{e}(t) &= y'(0, t), \\
\end{align*}\]  

(46)

where none of the channels has disturbance.

**Lemma 3.1.** The boundary value problem (42) admits bounded solutions.

**Proof.** See “Proof of Lemma 3.1” in Section 4.

Finally, define

\[\begin{align*}
\hat{y}'(x, t) &= y'(x, t) - \hat{y}'(x, t). \\
\end{align*}\]

(47)

Then, system (46) is further reduced to

\[\begin{align*}
\mathcal{L}_i[\hat{y}'_i(x, t)] &= k_i\hat{y}'(0, t) - k_2\hat{y}'(1, t), \\
\hat{y}'(0, t) &= 0, \\
\hat{p}'(t) &= (S + Qd_4\hat{p}'(t) + Q\hat{y}'(0, t) - \hat{y}'(0, t)), \\
\hat{y}'(x, t) &= y'_i(x, t), \\
\hat{y}'(x, t) &= -k_2\hat{y}'(1, t) + k_3\hat{y}'(0, t), \\
\hat{y}'(1, t) &= (f_0(1) + k_1f_0(1)S + k_2f_0(1))\hat{p}'(t) - k_3\hat{y}'(1, t), \\
e(t) &= y'(0, t) + \hat{y}'(0, t), \\
\hat{e}(t) &= \hat{y}'(0, t) + \hat{y}'(0, t), \\
\end{align*}\]  

(48)

which is our starting point to prove the following Theorem 3.1.

**Theorem 3.1.** Suppose \(k_i > 0, j = 1, 2, 3, 4\) and \(S + Qd_4p\) is Hurwitz. For any initial value \((y_0(0), y_1(0), \hat{y}_0(0), \hat{y}_1(0), p(0)) \in \mathbb{H}_e = (H^2(0, 1) \times L^2(0, 1))^2 \times \mathbb{C}^n\) with compatible condition \(y_0(0) - d_4p(0) = 0\), the closed-loop system (40) admits a unique bounded solution

\[\sup_{t \in [0, t]} \|y'(t), y_1(t), \hat{y}'(t), \hat{y}_1(t), p(t)\|_{\mathbb{H}_e} < +\infty; \]  

(49)

and when \(p(t) \equiv 0\), the closed-loop system is internally exponentially stable:

\[\|y'(t), y_1(t), \hat{y}'(t), \hat{y}_1(t), p(t)\|_{\mathbb{H}_e} \leq M_2e^{-\alpha_2t}\|y_0(0), y_1(0), \hat{y}_0(0), \hat{y}_1(0), p(0)\|_{\mathbb{H}_e}, \]  

(50)

where \(M_2, \alpha_2 > 0\). The tracking errors are regulated to zero such that

\[\lim_{t \to \infty} |e(t)| = 0, \]  

(51)

for some \(\alpha > 0\). In particular, the first convergence is exponentially convergence.

**Proof.** See “Proof of Theorem 3.1” in Section 4.

We point out that the result of Theorem 3.1 is much stronger than the results in existing literature where only asymptotic stability and convergence were claimed, yet with additional man-made assumption and no robustness (e.g., Guo et al., 2017, 2018). Compared with Feng et al. (2020), Jin and Guo (2019), we have not only the robustness but also more straightforward feedback control.

**Remark 3.1.** Through the transformations (41) and (44), the closed-loop system (40) is rewritten as an exponentially stable system (46), which is claimed by Theorem 3.1 and (47). The evolution equation of (46) in the state space \(\mathbb{H}_e = (H^2(0, 1) \times L^2(0, 1))^2 \times \mathbb{C} \times \mathbb{C}^n\) can be written as

\[\frac{d}{dt}(y', y', \hat{y}', \hat{y}', \hat{p}') = T_1(y', y', \hat{y}', \hat{y}', \hat{z}', \hat{p}'), \]  

where \(T_1\) generates an exponentially stable \(C_0\)-semigroup such that \(\|e^{\lambda t}\| \leq C_1e^{-\alpha_1t}\) for some \(C_1, \alpha_1 > 0\) and all \(t \geq 0\).

The internal model principle claims that the output is still regulated as long as the system is internally exponentially stable. However, there are too many such possibilities for system uncertainties and seems impossible to formulate the uncertainties in
3.3 Simulation

In this subsection, we present some numerical simulations for closed-loop system (40). For the exosystem, take \( p(0) = (1, 0, 1, 0) \) and
\[
S = \left[ 0, -\frac{\pi}{6}, 0, \frac{\pi}{6}, 0, 0, 0, 0, 0, 0, \frac{\pi}{3}, 0, 0, \frac{\pi}{3}, 0 \right].
\]
(53)
For the control, set
\[
\hat{p}(0) = (0.1, 0.2, 0.3, 0.4) \quad \text{and} \quad \hat{u}(x, 0) = \frac{x}{2},
\]
\[
\hat{y}_1(x, 0) = 0, \quad k_1 = 0.6, \quad k_2 = 0.8, \quad k_3 = 2, \quad k_4 = 1.8,
\]
\[
Q = \frac{\pi}{36}(-96, -92, -25, 135), \quad d_2^0 = (1, 1, 1, 1).
\] (54)
We choose initial state \( y(x, 0) = \frac{7x}{10}, y_1(x, 0) = 0 \). The disturbances and reference trajectory are chosen as
\[
d_1(x) = \left( \frac{\cos(x)}{4}, \frac{\sin(x)}{5}, \frac{x^2}{3}, \frac{x}{2} \right),
\]
\[
d_2 = (0, 0.2, 0.4), \quad d_3 = (0.1, 0, 0.3, 0), \quad d_4 = (0, 0.3, 0.2).
\] (55)
From Fig. 2(a), the closed-loop system is bounded and from Fig. 2(b), the output is regulated to track the reference trajectory. Furthermore, the derivative of the tracking error is guaranteed to be bounded, as well as the control input.

4. Proofs of the main results

Proof of Lemma 2.1. Define the operator \( A_1 \) by
\[
A_1(f, \hat{f}, g) = \left( \hat{f}(x), f''(x), -k_1 g + k_2 f(0) \right), 
\]
\[
\forall (f, \hat{f}, g) \in \mathcal{D}(A_1),
\]
\[
\mathcal{D}(A_1) = \{ (f, \hat{f}, g) \in H^2(0, 1) \times H^1(0, 1) \times C 
\]
\[
f'(0) = -k_1 g + (k_1 + k_2) f(0), \quad f'(1) = 0 \}.
\] (56)
Proof of Lemma 2.3. To solve (20), we define \( f_j(x) \in \mathbb{C}^{1 \times n} \) as:
\[
\begin{align*}
    f_j(x) & = x_j \frac{f_j'(x)}{f_j(0)}, \\
    f_j(0) & = d_2, f_j(0) = d_4,
\end{align*}
\]
which is an initial value problem and hence admits a unique solution. It is easy to see that the \( h(x) \) defined by (24) satisfies
\[
h(x) = g_0(x) + f_j(x).
\]
In order to solve (20), we only need to solve (24) with \( h(0) = g_0(0) + d_3d_4 \) being replaced by \( h(1) = f_j(1) - d_3 \), for which we have \( h(x) = H_j(x) \phi_{\lambda_j} \) by the eigenvector to the eigenvalue \( \lambda_j \in \sigma(S) \) for \( j = 1, \ldots, n \). It is found that
\[
H_j(x) = (H_j(0), H_j'(0), H_j''(1)) \phi_{\lambda_j} \neq 0,
\]
which has solution
\[
H_j(x) = \begin{cases} 
    k_2 \phi_{\lambda_j} x + c_{ij}, & \lambda_j = 0, \\
    \frac{\lambda_j^2 - \mathcal{L}_j^2}{\lambda_j^2 + 2 \mathcal{L}_j \lambda_j + k_2 \lambda_j} \mathcal{L}_j e^{-\mathcal{L}_j x}, & \lambda_j \neq 0.
\end{cases}
\]
(61)
where \( c_{ij} \) is uniquely determined by \( h(1) = f_j'(1) - d_3 \).

Last, since (25) involves \( f_0(x) \) and \( h(x) \), we need to solve (15). Now, \( f_0(x) = f_0(0) \phi_{\lambda_j} \) satisfies
\[
\begin{align*}
    F_0(0) & = 0, \\
    F_0'(0) & = (k_2 \phi_{\lambda_j} + k_2) d_4^2 \phi_{\lambda_j}, \\
    F_0''(0) & = d_2^2 \phi_{\lambda_j},
\end{align*}
\]
which has solution
\[
F_0(x) = \begin{cases} 
    k_2 d_2^2 \phi_{\lambda_j} x + d_4^2 \phi_{\lambda_j}, & \lambda_j = 0, \\
    \frac{\lambda_j^2 - k_2 \mathcal{L}_j}{\lambda_j^2 + 2 \mathcal{L}_j \lambda_j + k_2 \lambda_j} \mathcal{L}_j e^{-\mathcal{L}_j x}, & \lambda_j \neq 0.
\end{cases}
\]
(63)
with \( c_{ij} \) being determined by \( h(1) = f_j'(1) - d_3 \). This further advises the existence of solution to (42).
Proof of Theorem 3.1. By (41), (44) and (47), the closed-loop system (40) is transformed into (48). It then suffices to consider the exponential stability of (48). In addition, we only consider the real part of the solution since the imaginary part can be dealt with exactly the same way. The $(\tilde{y}, \tilde{p})$-subsystem of (48) is a well-known exponentially stable system in $H^1(0, 1) \times L^2(0, 1) \times C^n$. Hence, for any initial value $(\tilde{y}(\cdot, 0), \tilde{p}(\cdot, 0)) \in H^1(0, 1) \times L^2(0, 1) \times C^n$, there exists a unique solution in $H^1(0, 1) \times L^2(0, 1) \times C^n$ to $(\tilde{y}, \tilde{p})$-subsystem of (48) that

$$E_0(t) = \frac{1}{2} \int_0^1 [\tilde{y}_x(x, t)]^2 dx + \frac{1}{2} \int_0^1 [\tilde{y}_x(x, t)]^2 dx + \frac{k_1}{2} ||\tilde{y}(0, t)||^2 \leq M_1 e^{-\omega_1 t} ||\tilde{y}(\cdot, 0), \tilde{y}_x(\cdot, 0), \tilde{p}(0)||_{H^1(0, 1) \times L^2(0, 1) \times C^n},$$

(69)

where $M_1, \omega_1 > 0$.

Define $\rho(t) = -\int_0^1 (x - 1) \tilde{y}_x^2(x, t) dx$ which satisfies

$$|\rho(t)| \leq \int_0^1 \left[ |\tilde{y}_x(x, t)|^2 + |\tilde{y}_x(x, t)|^2 \right] dx \leq M_1 e^{-\omega_1 t} ||\tilde{y}(\cdot, 0), \tilde{y}_x(\cdot, 0), \tilde{p}(0)||_{H^1(0, 1) \times L^2(0, 1) \times C^n}.$$

Differentiating $\rho(t)$ with respect to time and integrating by parts, we have

$$\rho(t) = -2 \int_0^1 (x - 1) \tilde{y}_x^2(x, t) dx + \int_0^1 [\tilde{y}_x(x, t)]^2 dx - [\tilde{y}_x(0, t)]^2 + \int_0^1 [\tilde{y}_x(x, t)]^2 dx \leq M_1 e^{-\omega_1 t} ||\tilde{y}(\cdot, 0), \tilde{y}_x(\cdot, 0), \tilde{p}(0)||_{H^1(0, 1) \times L^2(0, 1) \times C^n}.$$

For any $0 < \beta < \omega_1^{-1}$, it has

$$\int_0^\infty e^{\beta t} ||\tilde{y}(\cdot, 0), t||^2 dt \leq \rho(0) + M_1 e^{-\omega_1 t} ||\tilde{y}(\cdot, 0), \tilde{y}_x(\cdot, 0), \tilde{p}(0)||_{H^1(0, 1) \times L^2(0, 1) \times C^n},$$

(70)

and

$$\lim_{t \to \infty} e^{\beta t} \hat{e}(t) = \lim_{t \to \infty} e^{\beta t} \hat{e}(t) = 0.$$

For the $\tilde{y}$-subsystem of (48), we write

$$\frac{d}{dt} \left[ e^{\beta t} \begin{bmatrix} \tilde{y}(\cdot, t) \\ \tilde{p}(\cdot, t) \end{bmatrix} \right] = (A_2 + \beta e^{\beta t}) \begin{bmatrix} \tilde{y}(\cdot, t) \\ \tilde{p}(\cdot, t) \end{bmatrix} + \begin{bmatrix} 0 \\ \delta(x - 1) \end{bmatrix} \cdot [\tilde{y}_x(0, 1) + k_1 \tilde{y}_x(0, 1) + k_2 \tilde{y}_x(0, 1)] e^{\beta t} \tilde{p}(t)$$

$$+ \begin{bmatrix} 0 \\ \delta(x) \end{bmatrix} [-k_1 e^{\beta t} \tilde{y}(0, t) - k_2 e^{\beta t} \tilde{y}_x(0, t)].$$

(71)

where $A_2$ is defined in (66), and $e^{\beta t}$ is an exponentially stable $C_0$-semigroup on $H^1(0, 1) \times L^2(0, 1)$ that

$$e^{\beta t} \leq M_2 e^{-\omega_2 t}, M_3, \omega_2 > 0.$$

For any

$$\beta = \min \left\{ \frac{\alpha_1}{M_3}, \frac{\alpha_2}{2} \right\},$$

(72)

we have

$$\int_0^\infty e^{2\beta t} ||\tilde{y}(\cdot, 0), t||^2 dt < \infty, \quad \int_0^\infty e^{2\beta t} ||\tilde{y}_x(\cdot, 0), t||^2 dt < \infty, \quad \int_0^\infty e^{2\beta t} ||\tilde{p}(\cdot, t)||^2 dt < \infty.$$

Since $A_2 + \beta$ generates an exponentially stable $C_0$-semigroup, and $(0, \delta(x - 1) \cdot [\tilde{y}_x(0, 1) + k_1 \tilde{y}_x(0, 1) + k_2 \tilde{y}_x(0, 1)] e^{\beta t} \tilde{p}(t)$ is admissible to $e^{\beta t} \tilde{y}(\cdot, t), e^{\beta t} \tilde{y}_x(\cdot, t)$ is asymptotically stable in $H^1(0, 1) \times L^2(0, 1)$. This further implies that the $\tilde{y}$-subsystem is exponentially stable [Zhou & Guo, 2018]

$$||\tilde{y}(\cdot, t), \tilde{y}_x(\cdot, t)||_{H^1(0, 1) \times L^2(0, 1)} \leq C e^{-\beta t} ||\tilde{y}(\cdot, 0), \tilde{y}_x(\cdot, 0), \tilde{y}_x(\cdot, t), \tilde{y}_x(\cdot, t), \tilde{p}(0)||_{\mathcal{B}_c},$$

where $C$ is a positive constant independent of the initial values.

Same to (67), $e(t)$ is exponentially convergent that

$$\lim_{t \to \infty} |e(t)| = \lim_{t \to \infty} |y(0, t) + y(0, t) = 0, (73)$$

exponentially.

Define $\rho(t) = -\int_0^1 (x - 1) \tilde{y}_x^2(x, t) dx$. Then,

$$|\rho(t)| \leq C e^{-\beta t} ||\tilde{y}(\cdot, 0), \tilde{y}_x(\cdot, 0), \tilde{y}_x(\cdot, t), \tilde{y}_x(\cdot, t), \tilde{p}(0)||_{\mathcal{B}_c}.$$

By

$$\rho(t) = -|\tilde{y}_x(0, t)|^2 - |\tilde{y}_x(0, t)|^2 + ||\tilde{y}_x(\cdot, t), \tilde{y}_x(\cdot, t), \tilde{p}(0)||_2^2,$$

and similarly to (70), we can obtain

$$\int_0^\infty e^{2\beta t} ||\tilde{y}_x(\cdot, 0), t||^2 dt < \infty,$$

for some $0 < \alpha < \beta$. This further implies that

$$\lim_{t \to \infty} \int_0^\infty |e(t)|^2 < \infty,$$

(74)

This completes the proof of the theorem.

5. Concluding remarks

This paper develops a unified way to deal with output regulation for wave equation with non-collocated control and observation, and with disturbances in all possible channels. When the performance output is weak, we can achieve internal asymptotic stability and asymptotic convergence for tracking error, the same effect to Guo et al. (2017, 2018) by adaptive control using additionally the derivative of the tracking error in control design and man-made assumption. If the derivative of the tracking error can be used in control design as existing literature (e.g., Guo et al., 2017, 2018), we can enhance to exponential stability and convergence. Very importantly, our control design is only done for a nominal system regardless how many and where the disturbances are. This reduces significantly the control order by comparison with the adaptive control where each unknown constant in disturbances must be estimated. The last but not the least is that our control is robust to all external disturbances described by exosystem and likely conditional robust to system uncertainty as explained in Remark 3.1 because our controller contains actually internal model, which is sharp contrast to the adaptive control method that there is likely no robustness. The results also enhance significantly the recent results in Feng et al. (2020), Jin and Guo (2019). Finally, the mathematics analysis is also straightforward and can be applicable to other PDEs (Guo & Meng, 2020).
One reviewer indicated an interesting question that how $k_3, k_4$ in controller (38) affect the performance of the closed-loop. This is hard to be answered because this relies on spectral analysis for system (48) which is not easy due to highly coupling of the system.

**CRediT authorship contribution statement**

Bao-Zhu Guo: Conceptualization, Methodology, Investigation, Writing - reviewing & editing. Tingting Meng: Investigation, Writing - original draft, Numerical simulation.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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