Robust output regulation for Timoshenko beam equation with two inputs and two outputs

Bao-Zhu Guo1,2 | Tingting Meng2,3

1School of Mathematics and Physics, North China Electric Power University, Beijing, China
2School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China
3Institute of Artificial Intelligence, University of Science and Technology Beijing, Beijing, China

Correspondence
Bao-Zhu Guo, School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China. Email: bzguo@iss.ac.cn

Funding information
National Natural Science Foundation of China, Grant/Award Number: 61873260; Chinese Postdoctoral Science Foundation, Grant/Award Number: 2020M680531

Abstract
In this article, we consider two-dimensional robust output tracking for a Timoshenko beam equation where the disturbances and references are produced by a finite-dimensional exosystem. This requires at least two controls from basic output regulation theory of linear systems. The observer-based error feedback control approach is adopted in investigation through the following steps: (a) for the plant with no systematic uncertainties, choose a set of disturbances and references to obtain a nominal system; (b) for the nominal system, design an observer to realize 2-copy of the exosystem; (c) find a feedforward control for nominal system; and (d) allocate proportionally the states in feedforward control by those of the observer to obtain an observer based error feedback control also for nominal system. Since the control contains a 2-copy of the exo-system and the states of the observer are proportionally allocated in control, the problem is significantly different from SISO cases. The error feedback control is then proved to be bounded and also robust to systematic uncertainties and all disturbances and references produced from the exo-system. The numerical simulations validate the theoretical results.

KEYWORDS
internal model principle, output tracking, robustness, Timoshenko beam equation

1 | INTRODUCTION

Output tracking is one of the major concerns in control theory and applications. In many situations, one is only concerned with the performance output to be tracking possibly unknown reference in pursuing that the tracking error is as small as possible. At the same time, all subsystems in the closed-loop should be kept to be bounded and the closed-loop is required to be internally asymptotically stable. In output tracking, the tracking error is most often the only measurement for control design but to achieve better performance, additional measurements are also necessary. In any case, the feedback control should guarantee that the control is robust to all possible external disturbances and small system uncertainties as well. Since from 1970s, the output regulation has been investigated in the name of the internal model principle.1,2 By the internal model principle, the robust output tracking can be guaranteed by a dynamic tracking error feedback control with p-copy of the exosystem, where \( p \in \mathbb{N} \) is the dimension of the performance output.3 The internal model principle has been applied to nonlinear lumped parameter systems4 and even distributed parameter systems,3 where in latter case, the unboundedness of control and observation operators increases difficulties in solving the related operator equations. A systematic generalization of the internal model principle to infinite-dimensional systems was made in Reference 3 with
unbounded control and observation operators, and the disturbance related operators were also supposed to be unbounded. However, the tracking convergence is only limited to weak convergence due to unboundedness of the operators involved.

On the other hand, there is still a huge gap between abstract theory and PDEs, because abstract theory gives little insight in control design for PDEs, and in many cases, the involved unbounded operators should be extended to meet the regularity of the abstract framework but this is very difficult task for PDEs. In the last few years, several researchers have developed some direct PDE approaches under the guidance of the internal model principle. In a recent paper, the basic idea is straightforward: (1) specially freeze coefficients of the disturbances and references to get a nominal system; (2) for the nominal PDE-ODE system, design an extended state observer and a feedforward control; and (3) design an observer based control for nominal system through substitution of the states in feedforward control with the extended state observer. In this article, we consider a Timoshenko beam equation with two regulated outputs.

The system that we consider in this article is described by the following Timoshenko beam with disturbances:

\[
\begin{align*}
\rho w_t(x, t) - K(w_{xx}(x, t) - \varphi_x(x, t)) + \Theta_1(x)w_t(x, t) &+ \Theta_2(x)w_t(x, t) = d_1^T(x)v(t), \quad t > 0, \quad x \in (0, L), \\
I_\varphi w_t(x, t) - EI \varphi_{xx}(x, t) - K(w_{xx}(x, t) - \varphi(x, t)) + \Theta_3(x)\varphi_t(x, t) &+ \Theta_4(x)\varphi_t(x, t) = d_2^T(x)v(t), \quad t \geq 0, \quad x \in (0, L), \\
w(0, t) = d_1^T v(t), \quad \varphi(0, t) = d_2^T v(t), \quad t \geq 0, \\
K(w_{xx}(L, t) - \varphi(L, t)) = u_1(t) + d_1^T v(t), \quad t \geq 0, \\
EI \varphi_{xx}(L, t) = u_2(t) + d_2^T v(t), \quad t \geq 0, \\
w(x, 0) = w_0(x), \quad \varphi(x, 0) = \varphi_0(x), \quad x \in [0, L], \\
y_t(t) = (w(L, t), \varphi(L, t)),
\end{align*}
\]

where \(\Theta_k(x) \in L^\infty(0, L), k = 1, 2, 3, 4\) denotes systematic uncertainties, and the unknown coefficients \(d_k \in \mathbb{C}^n, k = 1, 2, 3, 4\) and unknown in-domain distributions \(d_k(x) = [d_{kj}(x)]\) and \(d_{kj} \in L^2(0, L), k = 1, 2, \quad j = 1 \ldots, n\). \((u_1(t), u_2(t))\) is the boundary control input which contains two components and \(y_t(t)\) is the regulated output which contains two components.

The external signal \(v(t) \in \mathbb{C}^{n \times 1}\) is produced from an exosystem governed by

\[
\dot{v}(t) = Sv(t), \quad v_0 = v(0),
\]
where \( v_0 \) is unknown and \( S \in \mathbb{C}^{n \times n} \) is known, which makes the disturbances unknown. Let the reference trajectory be represented by \( r(t) = (d_5^T v(t), d_6^T v(t)) \) with unknown \( d_5, d_6 \in \mathbb{C}^n \). Denote the tracking errors by

\[
e_1(t) = w(L, t) - d_5^T v(t), \quad e_2(t) = \varphi(L, t) - d_6^T v(t).
\]

The control objective is to achieve convergence of \((e_1(t), e_2(t))\) for any \( d_5^T, d_6^T \), namely,

\[
\lim_{t \to \infty} |e_1(t)| = 0, \quad \lim_{t \to \infty} |e_2(t)| = 0,
\]

regardless of the disturbances.

We consider system (1) in the state space \( H = (H^1_1(0, L) \times L^2(0, L))^2 \) where \( H^1_1(0, L) = \{ f \in H^1(0, L) | f(0) = 0 \} \) with the inner produce given by

\[
\langle (f_1, f_2, f_3, f_4), (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4) \rangle_H = \rho \int_0^L f_2(x) \hat{f}_2(x) dx + I_p \int_0^L f_4(x) \hat{f}_4(x) dx + EI \int_0^L f_3(x) \hat{f}_3(x) dx
\]

\[
+ K \int_0^L [f_3(x) - f_1(x)][\hat{f}_3(x) - \hat{f}_1(x)] dx,
\]

for any \((f_1, f_2, f_3, f_4), (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4) \in H \).

By general theory of output regulation for linear systems, a necessary condition must be presupposed that the eigenvalues of the exosystem cannot be the transmission zeros of the control plant. This condition is even necessary for existence of feedforward control for output regulation with all possible disturbances.\(^\text{10}\)

**Lemma 1.** Define \( \mu_1 \) and \( \mu_2 \) as the solutions to [11, theorem 2.1]

\[
\begin{align*}
\mu^2 - (a + b + c)\mu + ab &= 0, \\
a &= \frac{\rho}{K}, \quad b = \frac{I_p}{E s^2} + \frac{K}{EI}, \quad c = -\frac{K}{EI}.
\end{align*}
\]

There exist transmission zeros for the system with no uncertainties located on the imaginary axis, which satisfy

\[
\bigcup_{k=1,2} \{ s \in \mathbb{C} | c \hat{A}_k^2(L) - \hat{B}_k(L) \hat{C}_k(L) = 0 \},
\]

where \( \hat{A}_k(L), \hat{B}_k(L), \) and \( \hat{C}_k(L), k = 1, 2 \) are expressed by

If \( \Delta_1 = (a + b + c)^2 - 4ab \neq 0, \Delta_2 = ab \neq 0 \):

\[
\begin{align*}
\hat{A}_1(L) &= \frac{1}{\mu_1 - \mu_2} [\cosh \sqrt{\mu_1} L - \cosh \sqrt{\mu_2} L], \\
\hat{B}_1(L) &= \frac{1}{\mu_1 - \mu_2} (\mu_1 - b) \sqrt{\mu_1} \sinh \sqrt{\mu_1} L - \frac{1}{\mu_1 - \mu_2} (\mu_2 - b) \sqrt{\mu_2} \sinh \sqrt{\mu_2} L, \\
\hat{C}_1(L) &= \frac{1}{\mu_1 - \mu_2} (\mu_1 - a) \sqrt{\mu_1} \sinh \sqrt{\mu_1} L - \frac{1}{\mu_1 - \mu_2} (\mu_2 - a) \sqrt{\mu_2} \sinh \sqrt{\mu_2} L,
\end{align*}
\]

If \( \Delta_1 = (a + b + c)^2 - 4ab \neq 0, \Delta_2 = ab = 0 \):

\[
\begin{align*}
\hat{A}_2(L) &= \frac{1}{\mu_2} [\cosh \sqrt{\mu_2} L - 1], \\
\hat{B}_2(L) &= \sqrt{\mu_2} \sinh \sqrt{\mu_2} L, \quad \mu_2 \neq 0, \\
\hat{C}_2(L) &= \frac{a}{\mu_2} x + \frac{1}{\mu_2} (\mu_2 - a) \sqrt{\mu_2} \sinh(\sqrt{\mu_2} L).
\end{align*}
\]

**Assumption 1.** The eigenvalues of the matrix \( S \) are algebraically simple, located on the imaginary axis, and satisfy \( \sigma(S) \cap \bigcup_{k=1,2} \{ s \in \mathbb{C} | c \hat{A}_k^2(L) - \hat{B}_k(L) \hat{C}_k(L) = 0 \} = \emptyset \), where \( c = -\frac{K}{EI} \) and \( \hat{A}_k(L), \hat{B}_k(L) \) and \( \hat{C}_k(L), k = 1, 2 \) are defined in Lemma 1.
where $\mu_1, \mu_2 > 0$, is exponentially stable in the Hilbert space $(H^1(0, L) \times L^2(0, L))^2$, namely, the associated system operator $A$ defined by

$$
\begin{align*}
A(\phi_1, \dot{\phi}_1, \psi_1, \dot{\psi}_1) &= \left( \dot{\phi}_1, \frac{K}{\rho} (\phi_1'' - \psi_1'), \dot{\psi}_1, \frac{L}{\phi_1'} \psi_1'' + \frac{K}{\gamma} (\phi_1' - \psi_1) \right), \quad \forall (\phi_1, \dot{\phi}_1, \psi_1, \dot{\psi}_1) \in D(A), \\
D(A) &= \{ (\phi_1, \dot{\phi}_1, \psi_1) \in H^1(0, L) \times L^2(0, L) \times H^1(0, L) \times L^2(0, L) | \psi_1(0) = 0 \}, \\
K[\phi_1'(L) - \psi_1(L)] &= -\mu_1 \dot{\phi}_1(L), \quad E \dot{\psi}_1'(L) = -\mu_2 \dot{\psi}_1(L),
\end{align*}
$$

generates an exponentially stable $C_0$-semigroup on.$^{13}$

### 3. Observer-Based Robust Control Design

In this section, we design an observer-based error feedback control for system (1) with specially selected frozen disturbances and reference trajectories, which is called a nominal system. This constitutes three steps of designing an extended state observer, feedforward control, and finally an observer-based error feedback control. All three steps are done for the nominal system only. To enhance readability, we separate into three subsections according to these three steps.

To begin with, we select some frozen coefficients of the disturbances and reference trajectories in system (1) as follows ($D \in \mathbb{C}^n$):

$$
\begin{align*}
\Theta_k^0 &\equiv 0, \quad k = 1, 2, 3, 4, \quad \hat{d}_1^0(x) \equiv 0, \quad \hat{d}_2^0(x) \equiv 0, \quad d_1^{0T} = 0, \quad d_2^{0T} = 0, \\
D^T \phi_j &\neq 0, j = 1, 2, \ldots, n, \quad d_3^2 = -k_1 D^T S, \quad d_4^0 = -k_2 D^T S, \quad d_5^0 = D^T, \quad d_6^0 = D^T,
\end{align*}
$$

where $k_1, k_2 > 0$. Define $\phi_j$ as an eigenvector of $S$ corresponding to the eigenvalue $\lambda_j$ ($j = 1, \ldots, n$). Furthermore, define $J_s = [\phi_{s1}, \ldots, \phi_{sn}]$ and $S = J_s^{-1} J_s = \text{diag} [\lambda_{s1}, \ldots, \lambda_{sn}]$, which are frequently used thereafter. This produces a nominal system of (1) and (2):

$$
\begin{align*}
\rho w_3(x, t) - K(w_{ss}(x, t) - \dot{\phi}_s(x, t)) &= 0, \\
I_s \dot{\phi}_s(x, t) - E \dot{\psi}_s(x, t) - K(w_s(x, t) - \varphi(x, t)) &= 0, \\
w(0, t) = \varphi(0, t) = 0, \\
K(w_s(L, t) - \varphi(L, t)) &= u_1(t) - k_1 d_3^0 S \psi(t) = u_1(t) + k_1 [\hat{d}_3(t) - w_s(L, t)], \\
E \dot{\psi}_s(L, t) &= u_2(t) - k_2 d_5^0 S \psi(t) = u_2(t) + k_2 [\hat{d}_5(t) - \varphi_s(L, t)], \\
\dot{\psi}(t) &= S \psi(t), \\
\hat{e}_1(t) &= w(L, t) - D^T \psi(t), \\
\hat{e}_2(t) &= \varphi(L, t) - D^T \psi(t),
\end{align*}
$$

which is a coupled PDE+ODE system with two inputs and two outputs. The principle of choice of frozen coefficients (9) is that the coupled system (10) is detectable by the tracking errors.
### 3.1 Extended state observer for nominal system

In this subsection, we design an observer for nominal system (10) in terms of \((e_1(t), e_2(t), \dot{e}_1(t), \dot{e}_2(t))\), which produces automatically a 2-copy of the exosystem in dynamic feedback. Motivated from Lemma 2, an observer for nominal system (10) is designed as

\[
\begin{align*}
\rho \dot{\hat{w}}_i(x, t) &- K(\hat{w}_xx(x, t) - \hat{\phi}_x(x, t)) = 0, \\
I_\rho \hat{\phi}_i(x, t) - EI \hat{\phi}_xx(x, t) - K(\hat{w}_x(x, t) - \hat{\phi}(x, t)) = 0, \\
\dot{\hat{w}}(0, t) &- \hat{\phi}(0, t) = 0, \\
K(\hat{w}(L, t) - \hat{\phi}(L, t)) &- u_1(t) + k_1[\hat{e}_1(t) - \hat{w}_1(L, t)]. \\
\end{align*}
\]

\[(11)\]

where \(\hat{w}(t) = (\hat{w}_1(t), \hat{w}_2(t)), \hat{\phi}_k(t) \in \mathbb{C}^n, k = 1, 2, Q \in \mathbb{C}^n\) is determined to make \(S + QD^T\) Hurwitz. This is always possible since \((S, D^T)\) is detectable if and only if \((J^{-1}S^1, D^TJ_2)\) is detectable and the latter is guaranteed by Assumption 1 and \(D\hat{\phi}_j \neq 0\) for all \(j = 1, 2, \ldots, n\) in (9). In the designed observer (11), the exo-system is estimated by two observers \(\hat{w}_1(t)\) and \(\hat{w}_2(t)\), which aims to guarantee the 2-copy of the exo-system. It is noted that since the observation operator in (1) is compact, we use derivatives of the errors in observer design to enhance the convergence of the observer from asymptotically convergence to exponentially convergence. Because of this, the derivatives of the tracking errors are required to satisfy

\[
\int_0^\infty |e^{\alpha t}\hat{e}_1(t)|^2 dt < \infty, \quad \int_0^\infty |e^{\alpha t}\hat{e}_2(t)|^2 dt < \infty.
\]

\[(12)\]

for some \(\alpha > 0\), in order to guarantee the boundedness of the control. This makes our output regulation problem not the standard form in the abstract theory yet similar to [4, remark 1.29, p. 24] for lumped parameter systems.

Define the observer errors as

\[
\hat{\hat{w}}(x, t) = w(x, t) - \hat{\hat{w}}(x, t), \quad \hat{\phi}(x, t) = \hat{\phi}(x, t) - \hat{\phi}(x, t), \quad \hat{\hat{v}}(t) = \hat{\hat{v}}(t) - v(t), \quad k = 1, 2,
\]

which, from (10) and (11), are governed by

\[
\begin{align*}
\rho \hat{\hat{w}}_i(x, t) &- K(\hat{\hat{w}}_xx(x, t) - \hat{\phi}_x(x, t)) = 0, \\
I_\rho \hat{\phi}_i(x, t) - EI \hat{\phi}_xx(x, t) - K(\hat{\hat{w}}_x(x, t) - \hat{\phi}(x, t)) = 0, \\
\hat{\hat{w}}(0, t) &- \hat{\phi}(0, t) = 0, \\
K(\hat{\hat{w}}(L, t) - \hat{\phi}(L, t)) &- u_1(t) + k_1[\hat{e}_1(t) - \hat{w}_1(L, t)]. \\
\end{align*}
\]

\[(13)\]

By Lemma 2, the \((\hat{\hat{w}}, \hat{\hat{\phi}})\)-subsystem in (13) is exponentially stable in \(\mathcal{H}\), with \(\mu_1\) and \(\mu_2\) replaced by \(k_1\) and \(k_2\). The \(\hat{\hat{v}}\)-subsystem in (13) is exponentially stable in \(\mathbb{C}^{2n}\) since \(S + QD^T\) is Hurwitz and \((\hat{\hat{w}}(L, t), \hat{\phi}(L, t))\) converges to zero exponentially as \(t \to \infty\).

### 3.2 Feedforward control for nominal system

In order to obtain an observer based error feedback control for nominal system (10), we need first to design a feedforward control. To this purpose, introduce

\[
(\theta(x, t), \theta(x, t)) = (w(x, t), \phi(x, t)) - (f^T_0w(x)v(t), f^T_0\phi(x)v(t)),
\]

\[(14)\]
with \( f_{0w}(x) \in \mathbb{C}^n \) and \( f_{0p}(x) \in \mathbb{C}^n \) being defined by the regulator equation:

\[
\begin{align*}
\rho f_{0w}^T(x)S^2 - K(f_{0w}^T(x) - f_{0p}^T(x)) &= 0, \\
I f_{0p}^T(x)S^2 - EIf_{0p}^T(x) - K(f_{0w}^T(x) - f_{0p}^T(x)) &= 0, \\
\int_0^t 0 = 0, \quad f_{0p}(0) = 0, \\
\int_0^t (L) = D^T, \quad f_{0p}^T(L) = D^T.
\end{align*}
\]

(15)

The aim of the transformation (14) is to convert the output regulation of system (10) into a stabilization problem of the \((\theta, \theta)\) system.

**Lemma 3.** Under Assumption 1, the BVP (15) admits a unique solution.

In this way, \((\theta(x, t), \vartheta(x, t))\) is governed by

\[
\begin{align*}
\rho \theta_t(x, t) - K(\theta_{xx}(x, t) - \vartheta(x, t)) &= 0, \\
I \vartheta_t(x, t) - E\vartheta_{xx}(x, t) - K(\theta(x, t) - \vartheta(x, t)) &= 0, \\
\theta(0, t) = 0, \quad \vartheta(0, t) = 0, \\
K(\theta(L, t) - \vartheta(L, t)) &= u_1(t) - k_1 \vartheta^T S v(t) - K(f_{0w}^T(L) - f_{0p}^T(L))v(t), \\
E\vartheta(L, t) &= u_2(t) - k_2 \vartheta^T S v(t) - EIf_{0p}^T(L)v(t), \\
e_1(t) &= \theta(L, t), \\
e_2(t) &= \vartheta(L, t).
\end{align*}
\]

(16)

The output regulation of (10) is then transformed into stabilization for system (16), for which a feedforward control is naturally designed as

\[
\begin{align*}
u_1(t) &= -k_3 w_1(L, t) + [K(f_{0w}^T(L) - f_{0p}^T(L)) + (k_1 + k_3) f_{0w}^T(L)]v(t), \\
u_2(t) &= -k_4 \varphi_1(L, t) + [EIf_{0p}^T(L) + (k_2 + k_4) f_{0p}^T(L)]v(t).
\end{align*}
\]

(17)

with \( k_3, k_4 > 0 \). This makes the closed-loop of system (16) exponentially stable in \( H \), exactly the same as the \((\tilde{w}, \tilde{\varphi})\)-subsystem in system (13).

Now, an important issue is how to estimate, using \( \hat{v}_1(t) \) and \( \hat{v}_2(t) \), the disturbance terms in (17). It is a proportionally allocation process, which is significantly different from SISO cases. Define \( \zeta_k \in \mathbb{C}^{n \times n} \), \( k = 1, 2, 3, 4 \) with

\[
\begin{align*}
\begin{cases}
f_{0w}^T(L)\zeta_1 + f_{0p}^T(L)\zeta_2 = K(f_{0w}^T(L) - f_{0p}^T(L)) + (k_1 + k_3) f_{0w}^T(L)S, \\
f_{0w}^T(L)\zeta_3 + f_{0p}^T(L)\zeta_4 = EIf_{0p}^T(L) + (k_2 + k_4) f_{0p}^T(L)S.
\end{cases}
\end{align*}
\]

(18)

where \( f_{0w}^T(L) \) and \( f_{0p}^T(L) \) are coefficients of (14) for reference signal, which explains the ratio of the allocation of the \( \hat{v}_1(t) \) and \( \hat{v}_2(t) \).

**Lemma 4.** Under Assumption 1, (18) admits a unique solution \( (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) which is also independent of \( D \in \mathbb{C}^n \) in (9), that is, if \( (f_{0w}^T(L), f_{0p}^T(L)) \) meets the first three identities of (15), it also meets (18).

Then the feedback control can be updated as

\[
\begin{align*}
u_1(t) &= -k_3 w_1(L, t) + f_{0w}^T(L)\zeta_1 v_1(t) + f_{0p}^T(L)\zeta_2 v_2(t), \\
u_2(t) &= -k_4 \varphi_1(L, t) + f_{0w}^T(L)\zeta_3 v_1(t) + f_{0p}^T(L)\zeta_4 v_2(t).
\end{align*}
\]

(19)

where \( v_1(t) \) and \( v_2(t) \) are actually the same \( v(t) \), but \( v_1(t) \) and \( v_2(t) \) are estimated by \( \hat{v}_1(t) \) and \( \hat{v}_2(t) \), respectively.
3.3 Error feedback for nominal system

An observer based error feedback control is obtained by replacing the states in feedforward control (19) by the states of the observer (11):

\[
\begin{align*}
\dot{u}_1(t) &= -k_3 \hat{w}_1(L, t) + f_{0w}^T(L) \zeta_1 \hat{v}_1(t) + f_{0p}^T(L) \zeta_2 \hat{v}_2(t), \\
\dot{u}_2(t) &= -k_4 \hat{w}_2(L, t) + f_{0w}^T(L) \zeta_3 \hat{v}_1(t) + f_{0p}^T(L) \zeta_4 \hat{v}_2(t), \\
\rho \dot{w}_n(x, t) - K(\hat{w}_{cc}(x, t) - \hat{w}_n(x, t)) &= 0, \\
I_p \dot{\phi}_n(x, t) - EI \dot{\phi}_{cc}(x, t) - K(\hat{w}_n(x, t) - \hat{\phi}(x, t)) &= 0, \\
\dot{\hat{w}}(0, t) &= 0, \quad \dot{\phi}(0, t) = 0, \\
K(\hat{w}_n(L, t) - \hat{\phi}(L, t)) &= u_1(t) + k_1 [\hat{e}_1(t) - \hat{w}_1(L, t)], \\
EI \dot{\phi}_n(L, t) &= u_2(t) + k_2 [\hat{e}_2(t) - \hat{\phi}(L, t)], \\
\dot{\hat{v}}(t) &= \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \hat{v}(t) + \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} e_1(t) - \hat{w}(L, t) + D^T \hat{v}_1(t) \\ e_2(t) - \hat{\phi}(L, t) + D^T \hat{v}_2(t) \end{pmatrix},
\end{align*}
\]

which will be shown to be robust to all disturbances and systematic uncertainties in original system (1). This is the most advantage of the observer based approach. The control design is almost straightforward once we have chosen an appropriate nominal system.

4 ROBUSTNESS ANALYSIS

In this section, we show that the error feedback control (20) designed for nominal system is robust to all possible disturbances in all channels and system uncertainties. To do this, we write the closed-loop of system (1) under error feedback control (20) as follows:

\[
\begin{align*}
\rho \dot{w}(x, t) - K(w_{cc}(x, t) - \varphi_3(x, t)) + \Theta_1(x)w(x, t) + \Theta_2(x)w(x, t) &= \tilde{d}_1^T(x)v(t), \\
I_p \dot{\varphi}(x, t) - EI \varphi_{cc}(x, t) - K(w_{cc}(x, t) - \varphi(x, t)) + \Theta_3(x)\varphi_1(x, t) + \Theta_4(x)\varphi_2(x, t) &= \tilde{d}_2^T(x)v(t), \\
w(0, t) &= d_1^T v(t), \quad \varphi(0, t) = d_2^T v(t), \\
K(w_x(L, t) - \varphi(L, t)) &= -k_3 \hat{w}_2(L, t) + f_{0w}^T(L) \zeta_1 \hat{v}_1(t) + f_{0p}^T(L) \zeta_2 \hat{v}_2(t) + d_3^T v(t), \\
EI \varphi_1(L, t) &= -k_4 \hat{w}_2(L, t) + f_{0w}^T(L) \zeta_3 \hat{v}_1(t) + f_{0p}^T(L) \zeta_4 \hat{v}_2(t) + d_4^T v(t), \\
\rho \dot{w}_n(x, t) - K(\hat{w}_{cc}(x, t) - \hat{\varphi}(x, t)) &= 0, \\
I_p \dot{\varphi}_n(x, t) - EI \dot{\varphi}_{cc}(x, t) - K(\hat{w}_n(x, t) - \hat{\varphi}(x, t)) &= 0, \\
\dot{\hat{w}}(0, t) &= 0, \quad \dot{\varphi}(0, t) = 0, \\
K(\hat{w}_n(L, t) - \hat{\varphi}(L, t)) &= -k_3 \hat{w}_2(L, t) + f_{0w}^T(L) \zeta_1 \hat{v}_1(t) + f_{0p}^T(L) \zeta_2 \hat{v}_2(t) + d_3^T v(t), \\
EI \dot{\varphi}_n(L, t) &= -k_4 \hat{w}_2(L, t) + f_{0w}^T(L) \zeta_3 \hat{v}_1(t) + f_{0p}^T(L) \zeta_4 \hat{v}_2(t) + d_4^T v(t), \\
\dot{\hat{v}}(t) &= \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \hat{v}(t) + \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} e_1(t) - \hat{w}(L, t) + D^T \hat{v}_1(t) \\ e_2(t) - \hat{\phi}(L, t) + D^T \hat{v}_2(t) \end{pmatrix},
\end{align*}
\]

Define \( f_{1w}^T(x), f_{1p}^T(x) \in \mathbb{C}^{1 \times n} \) as

\[
\begin{align*}
\rho f_{1w}^T(x) &- K(f_{1w}^T(x) - f_{1p}^T(x)) + \Theta_1(x)f_{1w}^T(x)S + \Theta_2(x)f_{1p}^T(x)S = \tilde{d}_1^T(x), \\
I_p f_{1p}^T(x) &- EI f_{1p}^T(x) - K(f_{1w}^T(x) - f_{1p}^T(x)) + \Theta_3(x)f_{1w}^T(x)S + \Theta_4(x)f_{1p}^T(x)S = \tilde{d}_2^T(x), \\
f_{1w}^T(0) &= d_1^T, \quad f_{1p}^T(0) = d_2^T, \\
f_{1w}^T(L) &= d_3^T, \quad f_{1p}^T(L) = d_4^T.
\end{align*}
\]
This is the regulator equation for \((w, \varphi)\)-subsystem in (21) aiming at converting the output regulation into a stabilization problem.

We first introduce the transformation

\[
\begin{align*}
zw(x, t) &= w(x, t) + g_w^T(x)v(t), \quad z_w(x, t) = \varphi(x, t) + g_w^T(x)v(t), \\
(23)
\end{align*}
\]

to cluster disturbances in all channels into two channels, where \(g_w^T(x) \in \mathbb{C}^{1 \times r}\) and \(g_{w}^T(x) \in \mathbb{C}^{1 \times r}\) are defined as

\[
\begin{align*}
\rho \ddot{z}_w(x, t) &= -K(z_w(x, t) - z_w(x, t)) + \Theta_1(x)[w(x, t) - f_w^T(x)Sv(t)] \\
&\quad + \Theta_2(x)[w(x, t) - f_w^T(x)v(t)] = 0, \\
I_{\rho w}(x, t) &= Elz_w(x, t) - K(z_w(x, t) - z_w(x, t)) + \Theta_3(x)[\varphi(x, t) - f_{\varphi}(x)Sv(t)] \\
&\quad + \Theta_4(x)[\varphi(x, t) - f_{\varphi}(x)v(t)] = 0, \\
z_w(0, t) &= 0, \quad z_w(0, t) = 0, \\
K(z_w(L, t) - z_w(L, t)) &= -k_3 \ddot{w}_1(L, t) + f_{\varphi}^T(L) \zeta_1 \dot{v}_1(t) + f_{\varphi}^T(L) \zeta_2 \dot{v}_2(t) \\
&\quad - k_1 [d_1^T S + g_w(L)S]v(t), \\
Elw^T(L, t) &= -k_4 \dot{\varphi}_1(L, t) + f_{\varphi}^T(L) \zeta_1 \dot{v}_1(t) + f_{\varphi}^T(L) \zeta_2 \dot{v}_2(t) - k_1 [d_1^T S + g_w(L)S]v(t), \\
\rho \ddot{w}_1(x, t) &= -K(\dot{w}_1(x, t) - \varphi(x, t)) = 0, \\
\dot{w}_1(0, t) &= 0, \quad \varphi(0, t) = 0, \\
K(\dot{w}_1(L, t) - \varphi(L, t)) &= -k_3 \ddot{w}_1(L, t) + f_{\varphi}^T(L) \zeta_1 \dot{v}_1(t) + f_{\varphi}^T(L) \zeta_2 \dot{v}_2(t) \\
&\quad + k_1 [e_1(t) - \dot{w}_1(L, t)], \\
Elw^T(L, t) &= -k_4 \dot{\varphi}_1(L, t) + f_{\varphi}^T(L) \zeta_1 \dot{v}_1(t) + f_{\varphi}^T(L) \zeta_2 \dot{v}_2(t) + k_2[\dot{e}_2(t) - \varphi(L, t)], \\
\dot{\varphi}_1(t) &= w(L, t) - d_1^T v(t) = z_w(L, t) - (d_1^T + g_w^T(L))v(t), \\
e_1(t) &= w(L, t) - d_1^T v(t) = z_w(L, t) - (d_1^T + g_w^T(L))v(t).
\end{align*}
\]  

(25)

It is seen that the disturbances are appeared only in two channels in (25) instead of all channels in (21). Next, introduce the second transformation

\[
\begin{align*}
\begin{bmatrix}
w^c(x, t) \\
\varphi^c(x, t) \\
\dot{w}^c(x, t) \\
\dot{\varphi}^c(x, t) \\
\ddot{w}^c(t) \\
\ddot{\varphi}^c(t)
\end{bmatrix}
&= 
\begin{bmatrix}
zw(x, t) \\
z_w(x, t) \\
\dot{w}(x, t) \\
\dot{\varphi}(x, t) \\
\ddot{w}(t) \\
\ddot{\varphi}(t)
\end{bmatrix}
- 
\begin{bmatrix}
h_w^T(x)v(t) \\
h_{w}^T(x)v(t) \\
h_w^T(x)v(t) \\
h_{w}^T(x)v(t) \\
h_{w}^T(x)v(t) \\
h_{w}^T(x)v(t)
\end{bmatrix}.
\end{align*}
\]  

(26)
where \( h^w_{\omega}(x), h^\rho_{\varphi}(x) \in \mathbb{C}^{1 \times n} \) are defined as

\[
\begin{align*}
\rho h^w_{\omega}(x)S^2_y - K(h^w_{\omega}(x) - h^\rho_{\varphi}(x)) &= 0, \\
I \rho h^w_{\omega}(x)S^2_y - EL h^\varphi_{\varphi}(x) - K(h^w_{\omega}(x) - h^\rho_{\varphi}(x)) &= 0, \\
h^w_{\omega}(0) = 0, \quad h^\rho_{\varphi}(0) &= 0, \\
h^w_{\omega}(L) &= d^w_\omega + g^w_{\omega}(L), \quad h^\rho_{\varphi}(L) = d^\rho_\varphi + g^\rho_{\varphi}(L),
\end{align*}
\] (27)

which satisfy \( h^w_{\omega}(x) = f^w_{\omega}(x) + g^w_{\omega}(x) \) and \( h^\rho_{\varphi}(x) = f^\rho_{\varphi}(x) + g^\rho_{\varphi}(x) \). Similar to (15), (27) also aims to transform the output regulation for \((z_w, z_{\varphi})\)-subsystem in (25) into a stability problem of the \((w^c, \varphi^c)\) system.

By Lemma 4, \((h^w_{\omega}(x), h^\rho_{\varphi}(x))\) satisfies similarly

\[
\begin{align*}
h^w_{\omega}(L)\zeta_1 + h^\rho_{\varphi}(L)\zeta_2 &= K(h^w_{\omega}(L) - h^\rho_{\varphi}(L))S, \\
h^w_{\omega}(L)\zeta_3 + h^\rho_{\varphi}(L)\zeta_4 &= ELh^\varphi_{\varphi}(L) + (k_2 + k_4)h^\rho_{\varphi}(L)S, \\
\end{align*}
\] (28)

where \( \zeta_k, k = 1, 2, 3, 4 \) are the known solutions to (18). The (28) will be used to prove the existence of \( h_{\omega 1}, h_{\omega 2} \in \mathbb{C}^{n \times n} \), satisfying

\[
\begin{align*}
[S + Qd^w_{\omega}S^T]h_{\omega 1} - h_{\omega 2}S = Qh^w_{\omega}(L), \\
[S + Qd^\varphi_{\varphi}S^T]h_{\omega 2} - h_{\omega 2}S = Qh^\rho_{\varphi}(L), \\
K(h^w_{\omega}(L) - h^\rho_{\varphi}(L))S = f^w_{\omega}h^w_{\omega}(L)\zeta_1h_{\omega 1} + f^\rho_{\varphi}h^\rho_{\varphi}(L)\zeta_2h_{\omega 2}, \\
ELh^\varphi_{\varphi}(L) + (k_2 + k_4)h^\rho_{\varphi}(L)S = f^\varphi_{\varphi}h^\varphi_{\varphi}(L)\zeta_3h_{\omega 1} + f^\varphi_{\varphi}f^\rho_{\varphi}(L)\zeta_4h_{\omega 2}.
\end{align*}
\] (29)

which, together with (28), is used to transform (25) into a stability problem.

Through the transformation (26), the transformed system is described by

\[
\begin{align*}
\rho w^c_{\omega}(x, t) - K(w^c_{\omega}(x, t) - \varphi^c_{\omega}(x, t)) + \Theta_1(x)w^{c \varphi}_{\omega}(x, t) + \Theta_2(x)w^{c \varphi}_{\omega}(x, t) &= 0, \\
I \rho \varphi^c_{\omega}(x, t) - EL \varphi^c_{\omega}(x, t) - K(w^c_{\omega}(x, t) - \varphi^c_{\omega}(x, t)) &+ \Theta_3(x)\varphi^{c \varphi}_{\omega}(x, t) + \Theta_4(x)\varphi^{c \varphi}_{\omega}(x, t) &= 0, \\
w^c(0, t) = 0, \quad \varphi^c(0, t) &= 0, \\
K(w^c_{\omega}(L, t) - \varphi^c_{\omega}(L, t)) &= -k_3w^c_{\omega}(L, t) + f^w_{\omega 1}(L)\zeta_1\tilde{v}^c_1(t) + f^w_{\omega 2}(L)\zeta_2\tilde{v}^c_2(t), \\
EL \varphi^c_{\omega}(L, t) &= -k_4\varphi^c_{\omega}(L, t) + f^\varphi_{\varphi 1}(L)\zeta_3\tilde{v}^c_1(t) + f^\varphi_{\varphi 2}(L)\zeta_4\tilde{v}^c_2(t), \\
\rho \tilde{w}^c_{\omega}(x, t) - K(\tilde{w}^c_{\omega}(x, t) - \tilde{\varphi}^c_{\omega}(x, t)) &= 0, \\
I \rho \tilde{\varphi}^c_{\omega}(x, t) - EL \tilde{\varphi}^c_{\omega}(x, t) - K(\tilde{w}^c_{\omega}(x, t) - \tilde{\varphi}^c_{\omega}(x, t)) &= 0, \\
\tilde{w}^c(0, t) = 0, \quad \tilde{\varphi}^c(0, t) &= 0, \\
K(\tilde{w}^c_{\omega}(L, t) - \tilde{\varphi}^c_{\omega}(L, t)) &= -k_3\tilde{w}^c_{\omega}(L, t) + f^w_{\omega 1}(L)\zeta_1\tilde{v}^c_1(t) + f^w_{\omega 2}(L)\zeta_2\tilde{v}^c_2(t) \quad + k_1[w^c_{\omega}(L, t) - \tilde{w}^c_{\omega}(L, t)], \\
EL \tilde{\varphi}^c_{\omega}(L, t) &= -k_4\tilde{\varphi}^c_{\omega}(L, t) + f^\varphi_{\varphi 1}(L)\zeta_3\tilde{v}^c_1(t) + f^\varphi_{\varphi 2}(L)\zeta_4\tilde{v}^c_2(t) + k_2[\varphi^c_{\omega}(L, t) - \tilde{\varphi}^c_{\omega}(L, t)], \\
\tilde{v}^c(t) &= \begin{pmatrix} S + QD^T & 0 \\ 0 & S + QD^T \end{pmatrix} \begin{pmatrix} \tilde{v}^c(t) \\ \tilde{\varphi}^c(t) \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} w^c(L, t) - \tilde{w}^c(L, t) \\ \varphi^c(L, t) - \tilde{\varphi}^c(L, t) \end{pmatrix}, \\
e^c_1(t) &= z_{\omega 1}(L, t) - h^w_{\omega}(L)v(t) = w^c(L, t), \\
e^c_2(t) &= z_{\varphi}(L, t) - h^\rho_{\varphi}(L)v(t) = \varphi^c(L, t),
\end{align*}
\] (30)

which is just the system (21) with \( v(t) \equiv 0 \). Since the tracking errors are boundary values of the states of (30), it suffices to prove the stability of the system. This is the purpose we made two transformations (23) and (26).

**Lemma 5.** Under Assumption 1, the Sylvester equation (27) and (29) admit solution \((h^w_{\omega}, h^\rho_{\varphi}, h_{\omega 1}, h_{\omega 2})\). The BVPs (22) and (24) also admit unique solutions.
4.1 Robust regulation of closed-loop system without system uncertainty

In this subsection, we suppose that (30) with all $\mathcal{X}_k \equiv 0$, $k = 1, 2, 3, 4$, that is, system has only external disturbances. In such case, introduce

$$\ddot{w}^e(x, t) = w^e(x, t) - \ddot{w}^c(x, t), \quad \ddot{\varphi}^e(x, t) = \varphi^e(x, t) - \ddot{\varphi}^c(x, t).$$

(31)

In such case, (30) is transformed into the following system:

$$\begin{align*}
\rho \ddot{w}_n^c(x, t) - K(\ddot{w}_n^c(x, t) - \ddot{\varphi}_n^c(x, t)) &= 0, \\
I_p \ddot{\varphi}_n^c(x, t) - K(\ddot{w}_n^c(x, t) - \ddot{\varphi}_n^c(x, t)) &= 0, \\
\ddot{w}(0, t) &= 0, \quad \ddot{\varphi}(0, t) = 0, \\
K(\ddot{w}_n(L, t) - \ddot{\varphi}(L, t)) &= -k_1 \ddot{w}_n(L, t), \\
EI \ddot{\varphi}_n(L, t) &= -k_2 \ddot{\varphi}(L, t), \\
\rho \ddot{w}_n(L, t) - K(\ddot{w}_n(x, t) - \ddot{\varphi}_n(x, t)) &= 0, \\
I_p \ddot{\varphi}_n(L, t) - K(\ddot{w}_n(x, t) - \ddot{\varphi}_n(x, t)) &= 0, \\
\ddot{w}(0, t) &= 0, \quad \ddot{\varphi}(0, t) = 0, \\
K(\ddot{w}_n(L, t) - \ddot{\varphi}(L, t)) &= -k_3 \ddot{w}_n(L, t) + f_{0w}(L) \zeta_1^c \ddot{v}_1^c(t) + f_{0p}(L) \zeta_2^c \ddot{v}_2^c(t) + k_1 \ddot{w}_n(L, t), \\
EI \ddot{\varphi}_n(L, t) &= -k_4 \ddot{\varphi}(L, t) + f_{0w}(L) \zeta_1^c \ddot{v}_1^c(t) + f_{0p}(L) \zeta_2^c \ddot{v}_2^c(t) + k_2 \ddot{\varphi}(L, t), \\
\dot{v}_1^c(t) &= [S + QD^T] \dot{v}_1^c(t) + Q \ddot{\varphi}(L, t), \\
\dot{v}_2^c(t) &= [S + QD^T] \dot{v}_2^c(t) + Q \ddot{\varphi}(L, t), \\
e_1(t) &= \ddot{w}(L, t) + \ddot{w}^c(L, t), \quad e_2(t) = \ddot{\varphi}(L, t) + \ddot{\varphi}^c(L, t).
\end{align*}$$

(32)

It is seen from (32) that $(\ddot{w}^c, \ddot{\varphi}^c, \dot{v}_1^c, \dot{v}_2^c)$-subsystem is independent of the remaining subsystems and same as (13), they are exponentially stable in $H \times C^{2\eta}$. The $(\ddot{w}^e, \ddot{\varphi}^e)$-subsystem in (31) is rewritten as

$$\begin{cases}
\frac{d}{dt}(\ddot{w}^e, \ddot{\varphi}^e, \dot{v}_1^c, \dot{v}_2^c) = A(\ddot{w}^e, \ddot{\varphi}^e, \dot{v}_1^c, \dot{v}_2^c) + \Delta_{lw} f_{0w}(L) \zeta_1^c \ddot{v}_1^c(t) + f_{0p}(L) \zeta_2^c \ddot{v}_2^c(t) + k_1 \ddot{w}_n(L, t), \\
+ \Delta_{lp} f_{0w}(L) \zeta_1^c \ddot{v}_1^c(t) + f_{0p}(L) \zeta_2^c \ddot{v}_2^c(t) + k_2 \ddot{\varphi}(L, t), \\
\Delta_{lw} = (0, \delta(x - L), 0, 0), \quad \Delta_{lp} = (0, 0, 0, \delta(x - L)),
\end{cases}$$

(33)

where $\delta(\cdot)$ is the Dirac distribution, and the operator $A$ is defined in (8) with $\mu_1$ and $\mu_2$ being replaced by $k_3$ and $k_4$, respectively and hence generates an exponentially stable $C_0$-semigroup on $H$ that $\|e^{\mathcal{A}t}\| \leq M_A e^{-\omega_3 t}$, $M_A, \omega_3 > 0$. Before proving the exponentially stability of (30), we first prove the following lemma about the admissibility of $\Delta_{lw}$ and $\Delta_{lp}$ in case of no systematic uncertainties.

**Lemma 6.** The operators $\Delta_{lw}$ and $\Delta_{lp}$ are admissible to $e^{\mathcal{A}t}$.

**Theorem 1.** Suppose that $k_j > 0$, $j = 1, 2, 3, 4$ and $S + QD^T$ is Hurwitz. For any initial state $(w(\cdot, 0), w_1(\cdot, 0), \varphi(\cdot, 0), \varphi_1(\cdot, 0), \ddot{w}(\cdot, 0), \ddot{w}_1(\cdot, 0), \ddot{\varphi}(\cdot, 0), \ddot{\varphi}_1(\cdot, 0), \ddot{v}(0)) \in H^2 \times C^{2\eta}$ satisfying the compatibility conditions $w(0, 0) = d_1^T p(0)$ and $\varphi(0, 0) = d_2^T p(0)$, the closed-loop system (21) admits a unique bounded solution for $\nu(t) \neq 0$

$$\sup_{t \geq 0} \left\{\| (w(\cdot, t), w_1(\cdot, t), \varphi(\cdot, t), \varphi_1(\cdot, t)) \|_{\mathcal{H}}^2 + \| (\ddot{w}(\cdot, t), \ddot{w}_1(\cdot, t), \ddot{\varphi}(\cdot, t), \ddot{\varphi}_1(\cdot, t)) \|_{\mathcal{H}}^2 \right\} < +\infty;$$

(34)

and when $\nu(t) \equiv 0$, the system is internally exponentially stable.

$$\| (w(\cdot, t), w_1(\cdot, t), \varphi(\cdot, t), \varphi_1(\cdot, t)) \|_{\mathcal{H}}^2 + \| (\ddot{w}(\cdot, t), \ddot{w}_1(\cdot, t), \ddot{\varphi}(\cdot, t), \ddot{\varphi}_1(\cdot, t)) \|_{\mathcal{H}}^2 + \| \ddot{v}(t) \|_{C^{2\eta}}^2 \leq C e^{-\omega_1 t} E_0(0),$$

(35)
where \( C, \omega_1 > 0 \) and \( E_0(0) \geq 0 \) depends only on initial state. The convergence of tracking errors is guaranteed that

\[
\lim_{t \to \infty} |e_k(t)| = 0, \quad \int_0^\infty |e^t \dot{e}_k(t)|^2 dt < \infty, \quad k = 1, 2, \tag{36}
\]

for some \( \alpha > 0 \) and the first convergence above is uniformly exponentially.

Remark 1. The second convergence of (36) is crucial for the boundedness of the control (20). Actually, by (36) and (20), we have

\[
u_k(t) = I_k(t) + I_k^2(t), \quad k = 1, 2, \]

where \( \sup_{t \geq 0} |I_k(t)| < \infty \) is uniformly bounded and \( \int_0^\infty e^{\alpha t} |I_k(t)|^2 dt < \infty \).

### 4.2 Robust regulation of closed-loop system with system uncertainty

The closed-loop system (30) can be written as

\[
\frac{d}{dt} \begin{pmatrix}
  w^c(x, t) \\
  w^c_t(x, t) \\
  w^c_{tt}(x, t) \\
  \phi^c(x, t) \\
  \phi^c_t(x, t) \\
  \phi^c_{tt}(x, t) \\
  \hat{\phi}^c(x, t) \\
  \hat{\phi}^c_t(x, t) \\
  \hat{\phi}^c_{tt}(x, t) \\
  \bar{v}^c(t) \\
  \bar{v}^c_t(t) \\
  \bar{v}^c_{tt}(t)
\end{pmatrix}
= A_0
\begin{pmatrix}
  w^c(x, t) \\
  w^c_t(x, t) \\
  w^c_{tt}(x, t) \\
  \phi^c(x, t) \\
  \phi^c_t(x, t) \\
  \phi^c_{tt}(x, t) \\
  \hat{\phi}^c(x, t) \\
  \hat{\phi}^c_t(x, t) \\
  \hat{\phi}^c_{tt}(x, t) \\
  \bar{v}^c(t) \\
  \bar{v}^c_t(t) \\
  \bar{v}^c_{tt}(t)
\end{pmatrix}
+ \begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3 \\
  \phi_4 \\
  \phi_5 \\
  \phi_6
\end{pmatrix}, \tag{37}
\]

where \( A_0 \) is defined by

\[
A_0 = \begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3 \\
  \phi_4 \\
  \phi_5 \\
  \phi_6
\end{pmatrix}
\begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3 \\
  \phi_4 \\
  \phi_5 \\
  \phi_6
\end{pmatrix}^{-1}
\begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3 \\
  \phi_4 \\
  \phi_5 \\
  \phi_6
\end{pmatrix} \in D(A_0)
\]

\[
D(A_0) = \{(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in H \times H \times (C^\infty)^5 | \phi_1(0) = \phi_3(0) = 0, \phi_4(0) = \phi_5(0) = 0, K(\phi_1^T(L) - \phi_3^T(L)) = -k_3 \phi_5(L) + f^T_{0u}(L) \zeta_1 \phi_5 + f^T_{0p}(L) \zeta_2 \phi_6,
\]

\[
EI(\phi_4^T(L) - \phi_5^T(L)) = -k_4 \phi_4(L) + f^T_{0u}(L) \zeta_1 \phi_4 + f^T_{0p}(L) \zeta_1 \phi_5 + k_1[\phi_2(L) - \phi_2(L)],
\]

\[
K(\phi_4^T(L) - \phi_5^T(L)) = -k_4 \phi_4(L) + f^T_{0u}(L) \zeta_1 \phi_5 + f^T_{0u}(L) \zeta_2 \phi_6 + k_1[\phi_2(L) - \phi_2(L)].
\]
and $B$ is defined as
\[
BX = \left( 0, \frac{\Theta_1}{\rho} \phi_2 - \frac{\Theta_2}{\rho} \phi_1, 0, -\frac{\Theta_3}{I_p} \phi_4 - \frac{\Theta_4}{I_p} \phi_3, 0, 0, 0, 0, 0 \right),
\]

where $X = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ and obviously, $B$ is bounded. A simple computation shows that
\[
\|B\| \leq \max \left\{ \frac{2\|\Theta_1\|_2^2}{\rho^2}, \frac{2\|\Theta_3\|_2^2}{I_p^2}, \frac{4\|\Theta_2\|_2^2 L^2}{\rho K}, \left[ \frac{4\|\Theta_3\|_2^2 L^2}{\rho K} + \frac{2\|\Theta_4\|_2^2}{I_p} \right] L^2 / EI \right\},
\]

where $\|\Theta_i\|/(i = 1, 2, 3, 4)$ is the norm in $L^\infty(0, L)$. The following Lemma 7 is straightforward by virtue of Lemma 2 and Theorem 1.

**Lemma 7.** Suppose that $k_1, k_2, k_3, k_4$ are positive constants and the exponentially stable $C_0$-semigroup $e^{\lambda t}$ satisfies
\[
\|e^{\lambda t}\| \leq Ce^{-\omega_2 t},
\]

where $C, \omega_2 > 0$. For any
\[
\|B\| \leq \max \left\{ \frac{2\|\Theta_1\|_2^2}{\rho^2}, \frac{2\|\Theta_3\|_2^2}{I_p^2}, \frac{4\|\Theta_2\|_2^2 L^2}{\rho K}, \left[ \frac{4\|\Theta_3\|_2^2 L^2}{\rho K} + \frac{2\|\Theta_4\|_2^2}{I_p} \right] L^2 / EI \right\} < \frac{\omega_2}{C}.
\]

$A_0 + B$ also generates an exponentially stable $C_0$ semigroup.

**Theorem 2.** Under condition of Lemma 7, the tracking errors $e_k(t), k = 1, 2$ converge to zero exponentially as time goes to infinity and the weak convergence (36) of $\hat{e}_k(t), k = 1, 2$ is also guaranteed.

## 5 | NUMERICAL SIMULATIONS

In this section, we present numerical simulations for robust output regulation of the closed-loop system (21) under different sets of system uncertainties, disturbances, and references. The system parameters are chosen as $\rho = 1$, $EI = 35, I_p = 2, K = 1.5, L = 1$, and the initial states are chosen as
\[
w(x, 0) = \frac{7x}{10L}, \quad \varphi(x, 0) = 0, \quad w_r(x, 0) = 0, \quad \varphi_t(x, 0) = 0.
\]

The exosystem is defined with $S = \left[ 0, -\frac{7}{6}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right]$ and $v(0) = (1, 0, 1, 0)$. The error feedback control in (20) is defined with
\[
\begin{align*}
D^T &= (0.01, 0.01, -0.01, -0.01), \quad d_5^T = D^T, \quad d_6^T = D^T, \quad Q = \frac{\delta}{56}(-96, -92, -25, 135)^T, \\
k_1 &= 1.7, \quad k_2 = 3, \quad k_3 = 1.7, \quad k_4 = 6, \quad \dot{w}(x, 0) = \frac{x}{2L}, \quad \dot{\varphi}(x, 0) = \frac{3x}{10L}, \\
\dot{w}_1(x, 0) &= 0, \quad \dot{\varphi}_1(x, 0) = 0, \quad \dot{v}_1(0) = (1, 2, 3, 4)^T, \quad \dot{v}_2(0) = (4, 3, 2, 1)^T.
\end{align*}
\]

The robust output regulation is presented for two sets of external disturbances and references:
\[
\begin{align*}
\dot{d}_1^T(x) &= \left( \frac{x}{30}, \frac{x^2}{30}, \frac{x^2}{30}, \frac{x^2}{30} \right), \quad \dot{d}_2^T(x) = \left( \frac{x^2}{60}, \frac{x^2}{30}, \frac{x^2}{30}, \frac{x}{30} \right), \quad d_1^T = (0.1, 0.3, 0.2, 0.4), \\
\dot{d}_2^T &= (0.04, 0.02, 0.03, 0.01), \quad d_3^T = (0.1, 0.2, 0.3, 0.4), \quad d_4^T = (0.4, 0.3, 0.2, 0.1), \\
\Theta_1(x) &= 0.01, \quad \Theta_2(x) = 0.02, \quad \Theta_3(x) = 0.02, \quad \Theta_4(x) = 0.01.
\end{align*}
\]
and

\[
\begin{aligned}
\hat{d}_1^T(x) &= \left( \frac{x}{80}, \frac{3x^2}{70}, \frac{x^3}{20}, \frac{x^4}{70} \right), \\
\hat{d}_2^T(x) &= \left( \frac{x^2}{50}, \frac{x}{20}, \frac{3x^2}{70}, \frac{x}{80} \right), \\
d_1^T &= (0.1, 0.5, 0.3, 0.7), \\
d_2^T &= (0.07, 0.05, 0.03, 0.01), \\
d_3^T &= (0.1, 0.3, 0.5, 0.7), \\
d_4^T &= (0.2, 0.4, 0.6, 0.8), \\
d_5^T &= (0.07, 0.5, 0.3, 0.01), \\
d_6^T &= (0.3, 0.06, 0.04, 0.2), \\
\Theta_1(x) &= 0.02, \\
\Theta_2(x) &= 0.01, \\
\Theta_3(x) &= 0.01, \\
\Theta_4(x) &= 0.02.
\end{aligned}
\] (40)

By (20), (26), and (29), the proposed controllers are expressed by

\[
\begin{aligned}
u_1(t) &= -k_3 \hat{w}_1(L, t) + f_{0w}(L) \tilde{r}_1(t) + f_{0v}(L) \tilde{r}_2(t) \\
&\quad + [K(h^T_{wv}(L) - h^T_{vL}(L)) + k_1 h^T_{wL}(L)]v(t), \\
u_2(t) &= -k_4 \hat{w}_2(L, t) + f_{0w}(L) \tilde{r}_1(t) + f_{0v}(L) \tilde{r}_2(t) + [Eh^T_{\phi}(L) + k_2 h^T_{vL}(L)]v(t).
\end{aligned}
\] (41)

By Theorem 1, the controls are uniformly bounded.

Figure 1 depicts the closed-loop system with the signals (39). In Figure 1(A,B), the boundedness of \(w(x, t)\) and \(\varphi(x, t)\) are observed. As shown in Figure 1(C,D), the outputs \(w(L, t)\) and \(\varphi(L, t)\) track the references \(d_5^T v(t)\) and \(d_6^T v(t)\), respectively, after 10 seconds. The tracking errors \(\hat{e}_1(t) = w(L, t) - d_5^T v(t)\) and \(\hat{e}_2(t) = \varphi(L, t) - d_6^T v(t)\) are guaranteed to be bounded from Figure 1(E,F). Figure 1(G,H) presents the boundedness of the controller which coincides with the theoretical result (41). By comparison, Figure 2 displays the output regulation under other set of signals in (40). Similarly, the uniform boundedness of the displacements, the convergence of \(e_1(t)\) and \(e_2(t)\), and the boundedness of \(\hat{e}_1(t), \hat{e}_2(t), u_1(t),\) and \(u_2(t)\) are also observed.

6 PROOF FOR THE MAIN RESULTS

Proof of Lemma 1. For the homogeneous form of (1), its Laplace transform is found to be

\[
\begin{aligned}
\rho s^2 \tilde{y}_w(x, s) - K(y^w_{w}(x, s) - \tilde{y}_w(x, s)) &= 0, \\
I_m s^2 \tilde{y}_\varphi(x, s) - E\tilde{y}_\varphi(x, s) - K(y^\varphi_{w}(x, s) - \tilde{y}_\varphi(x, s)) &= 0, \\
\tilde{y}_w(0, s) &= 0, \\
\tilde{y}_{\varphi}(0, s) &= 0, \\
K(y^w_{w}(L, s) - \tilde{y}_w(L, s)) &= \tilde{u}_1(s), \\
E\tilde{y}_\varphi(L, s) &= \tilde{u}_2(s), \\
Y_w(s) &= (\tilde{y}_w(L, s), \tilde{y}_\varphi(L, s)),
\end{aligned}
\] (42)
The solutions to (42) can be analyzed in four cases in terms of symbols \( \Delta_1 = (a + b + c)^2 - 4ab \) and \( \Delta_2 = ab \), which read [11, theorem 2.1]

\[
\begin{align*}
\{ & \begin{array}{ll}
\hat{y}_w(x, s) = \frac{a}{3}x^3 + \frac{b}{2}x^2 - \frac{2EI}{K}\dot{x}x, & \text{when } \Delta_1 = \Delta_2 = 0, \\
\hat{y}_w(x, s) = \ddot{a}x^2 + b, & \text{otherwise,}
\end{array} \\
\hat{y}_w(x, s) = \hat{y}_w(0, s)\hat{A}_k(x) + \hat{y}_w(0, s)\hat{B}_k(x),
\end{align*}
\]

\[
\begin{align*}
\{ & \begin{array}{ll}
\hat{y}_w(x, s) = \hat{y}_w(0, s)\hat{C}_k(x) - \hat{y}_w(0, s)\frac{K}{EI}\hat{A}_k(x), & \text{otherwise},
\end{array}
\end{align*}
\]

where \( \hat{A}_k(x) \), \( \hat{B}_k(x) \), \( \hat{C}_k(x) \), \( k = 1, 2, 3 \), are defined as

\[
\begin{align*}
if \Delta_1 \neq 0 \text{ and } \Delta_2 \neq 0,
\hat{A}_1(x) &= \frac{1}{\mu_1 - \mu_2} [\cosh(\sqrt{\mu_1}x) - \cosh(\sqrt{\mu_2}x)], \\
\hat{B}_1(x) &= \frac{1}{\mu_1 - \mu_2}(\mu_1 - b)\sqrt{\mu_1}\sinh(\sqrt{\mu_1}x) - \frac{1}{\mu_1 - \mu_2}(\mu_2 - b)\sqrt{\mu_2}\sinh(\sqrt{\mu_2}x), \\
\hat{C}_1(x) &= \frac{1}{\mu_1 - \mu_2}(\mu_1 - a)\sqrt{\mu_1}\sinh(\sqrt{\mu_1}x) - \frac{1}{\mu_1 - \mu_2}(\mu_2 - a)\sqrt{\mu_2}\sinh(\sqrt{\mu_2}x),
\end{align*}
\]

\[
\begin{align*}
if \Delta_1 \neq 0 \text{ and } \Delta_2 = 0 \Leftrightarrow \mu_1 = 0, \mu_2 = a + c \neq 0,
\hat{A}_2(x) &= \frac{1}{\mu_2}[\cosh(\sqrt{\mu_2}x) - 1], \\
\hat{B}_2(x) &= \sqrt{\mu_2}\sinh(\sqrt{\mu_2}x), \\
\hat{C}_2(x) &= \frac{a}{\mu_2}x + \frac{1}{\mu_2}(\mu_2 - a)\sqrt{\mu_2}\sinh(\sqrt{\mu_2}x),
\end{align*}
\]

\[
\begin{align*}
if \Delta_1 = 0 \text{ and } \Delta_2 \neq 0 \Leftrightarrow \mu_1 = \mu_2 = \frac{a + b + c}{2} \neq 0,
\hat{A}_3(x) &= x\sqrt{\mu_1}\sinh(\sqrt{\mu_1}x), \\
\hat{B}_3(x) &= [\sqrt{\mu_1} + b\sqrt{\mu_1}^{-3}]\sinh(\sqrt{\mu_1}x) + (\mu_1 - b)\mu_1^{-1}x\cosh(\sqrt{\mu_1}x), \\
\hat{C}_3(x) &= [\sqrt{\mu_1} + a\sqrt{\mu_1}^{-3}]\sinh(\sqrt{\mu_1}x) + (\mu_1 - a)\mu_1^{-1}x\cosh(\sqrt{\mu_1}x).
\end{align*}
\]
Denote the transfer function from \((\hat{u}_1(s), \hat{u}_2(s))\) to \(Y(s) = (\hat{y}_w(L, s), \hat{y}_\sigma(L, s))\) by

\[
G(s) = \begin{pmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s)
\end{pmatrix}.
\]

By Reference 10, a necessary condition to achieve the robust output regulation is that \(\det G(i\omega) \neq 0\) for any \(i\omega \in \sigma(S)\), where \(i\) is the imaginary unit and \(\omega \in \mathbb{R}\). Now we discuss this condition in different cases.

**Step I:** When \(\Delta_1 = \Delta_2 = 0\), it follows from (42) that

\[
\begin{align*}
\dot{a} &= -\frac{\hat{a}_1}{2EI} + \frac{\hat{a}_1 L}{EI}, \\
\dot{\hat{b}} &= \frac{\hat{a}_2 L}{2EI} + \frac{\hat{a}_1 L}{EI}, \\
G_{11} &= \frac{L^3}{3EI} + \frac{L}{K}, \\
G_{12} &= \frac{L^2}{2EI}, \\
G_{21} &= \frac{L^2}{2EI}, \\
G_{22} &= \frac{L}{EI},
\end{align*}
\]

and hence

\[
\det(G) = \frac{L^2}{EI} \left( \frac{1}{K} + \frac{L^2}{12EI} \right) \neq 0.
\]

**Step II:** In other cases, by (42), it has

\[
\begin{align*}
\frac{\hat{a}_1(s)}{K} &= \hat{y}'_w(0, s)[\hat{A}_k(L) - \hat{C}_k(L)] + \hat{y}'_w(0, s)[\hat{B}_k(L) - cA_k(L)], \\
\frac{\hat{a}_2(s)}{L} &= \hat{y}'_w(0, s)\hat{C}_k(L) + \hat{y}'_w(0, s)c\hat{A}_k(L),
\end{align*}
\]

and hence

\[
\begin{align*}
G_{11}(s) &= \frac{c\hat{A}_k(L)s\hat{C}_k(L) - \hat{C}_k(L)s\hat{A}_k(L)}{E[\hat{A}_k(L)s\hat{C}_k(L) - \hat{C}_k(L)s\hat{A}_k(L)]}, \\
G_{12}(s) &= \frac{c\hat{A}_k(L)s\hat{C}_k(L) - \hat{C}_k(L)s\hat{A}_k(L)}{E[\hat{A}_k(L)s\hat{C}_k(L) - \hat{C}_k(L)s\hat{A}_k(L)]}, \\
G_{21}(s) &= \frac{-\hat{B}_k(L)s\hat{A}_k(L) + \hat{A}_k(L)s\hat{B}_k(L)}{E[\hat{B}_k(L)s\hat{A}_k(L) - \hat{A}_k(L)s\hat{B}_k(L)]}, \\
G_{22}(s) &= \frac{-\hat{B}_k(L)s\hat{A}_k(L) + \hat{A}_k(L)s\hat{B}_k(L)}{E[\hat{B}_k(L)s\hat{A}_k(L) - \hat{A}_k(L)s\hat{B}_k(L)]},
\end{align*}
\]

for any \(k = 1, 2, 3\). Therefore,

\[
\det(G) = \frac{c\hat{A}_k^2(L) - \hat{B}_k(L)s\hat{C}_k(L)}{KEI}.
\]

When \(\Delta_1 = 0\) and \(\Delta_2 \neq 0\), it must have \(s = \pm \frac{2\sqrt{\pi}}{\sqrt{L^2 - \frac{a^2}{K}}}\) which are not located on the imaginary axis.

**Step III:** For \(\{\Delta_1 \neq 0 \text{ and } \Delta_2 \neq 0\}\), there holds

\[
c\hat{A}_1^2(L) - \hat{B}_1(L)s\hat{C}_1(L)
= \frac{2c}{(\mu_1 - \mu_2)^2} \left[ 1 - \cosh(\sqrt{\mu_1} - \sqrt{\mu_2})L \right] + \frac{(a - b)^2 + (\sqrt{a} - \sqrt{b})^2c}{\sqrt{ab}} \frac{\sinh \mu_1 L}{\sinh \mu_2 L} \right].
\]

(48)

We claim that \(\{\Delta_1 \neq 0, \Delta_2 \neq 0\} \cap \{\det G(i\omega) = 0\}\) may have solutions on the imaginary axis. Actually, setting \(a = b\) and \(L = \frac{a}{K} \neq 0\) produces \(s = \pm i\sqrt{\frac{\pi}{L^2 - \frac{a^2}{K}}}\). \(\Delta_1 = (2a + c)^2 - 4a^2 = c(4a + c) > 0\), \(\mu_1 \neq \mu_2\) and \(\mu_1, \mu_2 < 0\). By (48), \(\det G(i\omega) = 0\) implies that

\[
c\hat{A}_1^2(L) - \hat{B}_1(L)s\hat{C}_1(L)
= -\frac{2K}{EI(\mu_1 - \mu_2)^2} \left[ 1 - \cosh(\sqrt{\mu_1} - \sqrt{\mu_2})L \right]
= -\frac{2K}{EI(\mu_1 - \mu_2)^2} \left[ 1 - \cos \left( \sqrt{\mu_1} - \sqrt{\mu_2} \right) L \right] = 0,
\]

which is satisfied. Therefore, \(\{\Delta_1 \neq 0, \Delta_2 \neq 0\}\) \(\cap \{\det G(i\omega) = 0\}\) does not exist.
which advises \( (\sqrt{|\mu_1|} - \sqrt{|\mu_2|})L = 2k\pi, \ k \in \mathbb{N}, \) and further
\[
\frac{4k^2\pi^2}{L^2} = |\mu_1| + |\mu_2| - 2\sqrt{|\mu_1\mu_2|} = -(\mu_1 + \mu_2) - 2\sqrt{\mu_1\mu_2} = \frac{K}{EI}.
\]
This is met when \( L^2 = \frac{4k^2\pi^2 EI}{K}. \)

**Step IV:** For \( \{\Delta_1 \neq 0 \text{ and } \Delta_2 = 0\} \), it must have \( s = \pm \sqrt{\frac{E}{I}} i \), and \( \det(G) = 0 \) implies that
\[
c\lambda_2^2(L) - B_2(L)C_2(L)
\]
\[
= \frac{1}{\mu_2^2} \left[ c \left( 2 - 2 \cos \sqrt{|\mu_2|L} \right) + aL \sqrt{|\mu_2|} \sin \sqrt{|\mu_2|L} \right]
\]
\[
= \frac{1}{\mu_2^2} \sin \frac{\sqrt{|\mu_2|L}}{2} \left[ 4c \sin \frac{\sqrt{|\mu_2|L}}{2} + 2aL \sqrt{|\mu_2|} \cos \frac{\sqrt{|\mu_2|L}}{2} \right] = 0,
\]
which holds true when
\[
\frac{\sqrt{|\mu_2|L}}{2} = \frac{\sqrt{|a + c|L}}{2} = (2k + 1)\pi \Leftrightarrow L = \frac{2(2k + 1)\pi}{\sqrt{|a + c|L}}.
\]

**Proof of Lemma 3.** Introduce \( F_{0e}^T(x) = F_{0e}^T(x)I_4 = [F_{0e1}, \ldots, F_{0en}] \) and \( F_{0p}^T(x) = F_{0p}^T(x)I_4 = [F_{0p1}, \ldots, F_{0pn}] \), which satisfy
\[
\begin{align*}
\rho \lambda_j^2 F_{0e,j}(x) - KE_{0e,j}(x) - F_{0e,j}'(x)) &= 0, \\
I_{\rho} \lambda_j^2 F_{0p,j}(x) - EI_{0p,j} F_{0p,j}(x) - F_{0p,j}(x)) &= 0,
\end{align*}
\]
(49)
\[
F_{0e,j}(0) = 0, \quad F_{0p,j}(0) = 0, \quad F_{0e,j}(L) = D^T \phi_j, \quad F_{0p,j}(L) = D^T \phi_j.
\]

Define \( \mu_{ij} \) and \( \mu_{2j} \) as the solutions to
\[
\mu_{ij}^2 - \left( \frac{\rho}{K} + \frac{I_4}{EI} \right) \lambda_j^2 \mu_j + \frac{\rho}{K} \lambda_j^2 \left[ \frac{I_4}{EI} \lambda_j^2 + \frac{K}{EI} \right] = 0,
\]
(50)
for \( \lambda_j \in \sigma(S) \) and \( j = 1, 2, \ldots, n \). The proof is accomplished by splitting into several cases.

**Case I:** \( \lambda_j \neq 0, \pm \sqrt{\frac{K}{I_4}} i \). In this case, \( \left( \frac{\rho}{K} + \frac{I_4}{EI} \right) \lambda_j^2 - \frac{4\mu_{ij}^2}{K} \lambda_j^2 \left[ \frac{I_4}{EI} \lambda_j^2 + \frac{K}{EI} \right] \neq 0 \) and thus \( \mu_1 \neq \mu_2 \) and \( \mu_k \neq 0, k = 1, 2. \) The solution to (49) can be expressed by [11, theorem 2.1]
\[
\begin{align*}
F_{0e,j}(x) &= F_{0e,j}'(0)A_{ij}(x) + F_{0e,j}'(0)B_{ij}(x), \\
F_{0p,j}(x) &= F_{0p,j}'(0)C_{ij}(x) + F_{0p,j}'(0)\frac{K}{EI}A_{ij}(x),
\end{align*}
\]
(51)
where \( A_{ij}(x), B_{ij}(x), \) and \( C_{ij}(x) \) are defined as
\[
A_{ij}(x) = \frac{1}{\mu_{ij} - \mu_{2j}} \left[ \cosh \sqrt{\mu_{ij}x} - \cosh \sqrt{\mu_{2j}x} \right],
\]
\[
B_{ij}(x) = \frac{1}{\mu_{ij} - \mu_{2j}} \left( \mu_{ij} - \frac{I_4}{EI} \lambda_j^2 - \frac{K}{EI} \right) \sqrt{\mu_{ij}^{-1}} \sinh \sqrt{\mu_{ij}x} - \frac{1}{\mu_{ij} - \mu_{2j}} \times \left( \mu_{2j} - \frac{I_4}{EI} \lambda_j^2 - \frac{K}{EI} \right) \sqrt{\mu_{2j}^{-1}} \sinh \sqrt{\mu_{2j}x},
\]
(52)
\[
C_{ij}(x) = \frac{1}{\mu_{ij} - \mu_{2j}} \left( \mu_{ij} - \frac{I_4}{EI} \lambda_j^2 \right) \sqrt{\mu_{ij}^{-1}} \sinh \sqrt{\mu_{ij}x} - \frac{1}{\mu_{ij} - \mu_{2j}} \times \left( \mu_{2j} - \frac{I_4}{EI} \lambda_j^2 \right) \sqrt{\mu_{2j}^{-1}} \sinh \sqrt{\mu_{2j}x}.
\]

By the boundary conditions at \( x = L \) in (15), the coefficients satisfy
\[
\begin{align*}
F_{0e,j}'(0)A_{ij}(L) + F_{0e,j}'(0)B_{ij}(L) &= D^T \phi_j, \\
F_{0p,j}'(0)C_{ij}(L) - F_{0p,j}'(0)\frac{K}{EI}A_{ij}(L) &= D^T \phi_j,
\end{align*}
\]
(53)
which admits a unique solution due to \(-\frac{K}{EI}A_{xy}^2(L) - B_{xy}(L)C_{xy}(L) \neq 0\) guaranteed by Assumption 1.

**Case II:** \(\lambda_{ij} = \pm \sqrt{\frac{K}{EI}}\). In this case, \(\mu_{ij} = 0\) and \(\mu_{ij} = -\left(\frac{L}{L} + \frac{K}{EI}\right)\). The solution to (49) is found to be

\[
\begin{align*}
F_{0w_j}(x) &= F_{0w_j}(0)A_{2j}(x) + F'_{0w_j}(0)B_{2j}(x), \\
F_{0p_j}(x) &= F'_{0p_j}(0)C_{2j}(x) - F'_{0w_j}(0)\frac{K}{EI}A_{2j}(x),
\end{align*}
\]  

(54)

where \(A_{2j}(x)\), \(B_{2j}(x)\), and \(C_{2j}(x)\) are given by

\[
\begin{align*}
A_{2j}(x) &= \frac{\cosh \sqrt{\mu_2}x}{\mu_2}, \\
B_{2j}(x) &= \frac{\sinh \sqrt{\mu_2}x}{\sqrt{\mu_2}}, \\
C_{2j}(x) &= \left[-\frac{\rho}{L}x + \frac{1}{\mu_2} \left(\mu_{ij} + \frac{\rho}{L}\right) \sqrt{\mu_2}^{-1} \sinh(\sqrt{\mu_2}x)\right].
\end{align*}
\]  

(55)

The coefficients which satisfy (53) with \(A_{xy}(L), B_{xy}(L), C_{xy}(L)\) being replaced by \(A_{2j}(x), B_{2j}(x), C_{2j}(x)\) admit solution due to \(-\frac{K}{EI}A_{2j}^2(L) - B_{2j}(L)C_{2j}(L) \neq 0\) guaranteed by Assumption 1.

**Case III:** \(\lambda_{ij} = 0\). In this case, \(\mu_{ij} = \mu_{ij} = 0\). The solution to (49) can be found as

\[
\begin{align*}
F_{0w_j}(x) &= \frac{a_{0w_j}}{3}x^3 + \frac{b_{0w_j}}{2}x^2 + \frac{2EI}{K}d_{0w_j}x, \\
F_{0p_j}(x) &= \frac{a_{0p_j}}{2}x^2 + \frac{b_{0p_j}}{2}x.
\end{align*}
\]  

(56)

where the coefficients satisfy

\[
\begin{align*}
d_{0F_j} &\left[\frac{L_j^2}{3} - \frac{2EI}{K}\right] + b_{0F_j}L_j^2 = D^T \phi_{ij}, \\
d_{0F_j}L_j^2 + b_{0F_j}L_j = D^T \phi_{ij},
\end{align*}
\]  

(57)

which admits a unique solution since the determinant of the coefficients is \(-\frac{L_j^0}{6} - \frac{2EI}{K} \neq 0\). This completes the proof of the lemma.

**Proof of Lemma 4.** Suppose \(L_j^{-1}\xi_kJ_0 = \xi_k = \text{diag}(\xi_{1,k}, \ldots, \xi_{k,k})\) is a diagonal matrix and then we have

\[
\begin{align*}
F_{0w_j}(L)\xi_{1,j} + F_{0p_j}(L)\xi_{2,j} &= K(F'_{0w_j}(L) - F_{0p_j}(L)) + (k_1 + k_3)\lambda_{ij}F_{0w_j}(L), \\
F_{0w_j}(L)\xi_{3,j} + F_{0p_j}(L)\xi_{4,j} &= EI F'_{0p_j}(L) + (k_2 + k_4)\lambda_{ij}F_{0p_j}(L),
\end{align*}
\]  

(58)

which is equivalent to

\[
\begin{align*}
&\begin{cases}
\text{if } \lambda_{ij} = 0, \\
\left[\xi_{1,j}A_{1j}(L) + \xi_{2,j}C_{1j}(L)\right]F'_{0w_j}(0) + \left[\xi_{1,j}B_{1j}(L) - \xi_{2,j}E_{1j}(L)\right]F_{0p_j}(0) = a_{1j}F'_{0w_j}(0) + b_{1j}F_{0w_j}(0), \\
\left[\xi_{3,j}A_{1j}(L) + \xi_{4,j}C_{1j}(L)\right]F'_{0p_j}(0) + \left[\xi_{3,j}B_{1j}(L) - \xi_{4,j}E_{1j}(L)\right]F_{0p_j}(0) = c_{1j}F_{0w_j}(0) + d_{1j}F_{0w_j}(0),
\end{cases}
\end{align*}
\]
By Lemma 3, the coefficients \( \hat{a}_{j} \) with \( \gamma_{ij} \) defined by (52) and (55), and \( a_{mj}, b_{mj}, c_{mj}, d_{mj}, m = 1, 2 \), being expressed by

\[
\begin{align*}
 a_{mj} &= KA_{mj}(L) - KC_{mj}(L) + (k_{1} + k_{3}) \lambda_{ij} A_{mj}(L), \\
b_{mj} &= KB_{mj}(L) + K_{11} A_{mj}(L) + (k_{1} + k_{3}) \lambda_{ij} B_{mj}(L), \\
c_{mj} &= EIC_{mj}(L) + (k_{2} + k_{4}) \lambda_{ij} C_{mj}(L), \\
d_{mj} &= -KA_{mj}(L) - (k_{2} + k_{4}) \lambda_{ij} K_{11} A_{mj}(L).
\end{align*}
\]

Therefore, the coefficients \( \hat{b}_{mj}, \hat{F}_{0mj}(0), \) and \( F''_{0mj}(0) \) are determined by the boundary conditions at \( x = L \), and thus are depending on \( D^{T} \) in (9). In order to find the solutions independent of \( D^{T} \), the \( \hat{c}_{k,j}, k = 1, 2, 3, 4, \) should satisfy

\[
\begin{align*}
 \hat{c}_{1,j}(L) + \frac{L}{2} + \frac{2EI}{K} \lambda_{ij} L^{2}, \quad & \text{if } \lambda_{ij} = 0, \\
 \hat{c}_{1,j}(L) + \frac{L}{2} + \frac{2EI}{K} \lambda_{ij} L^{2}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{2,j}(L) - \frac{\hat{c}_{2,j}}{EIL} A_{1j}(L) = a_{1j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{2,j}(L) - \frac{\hat{c}_{2,j}}{EIL} A_{1j}(L) = a_{1j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{3,j}(L) - \frac{\hat{c}_{3,j}}{EIL} A_{2j}(L) = b_{2j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{3,j}(L) - \frac{\hat{c}_{3,j}}{EIL} A_{2j}(L) = b_{2j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{4,j}(L) - \frac{\hat{c}_{4,j}}{EIL} A_{3j}(L) = c_{1j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{4,j}(L) - \frac{\hat{c}_{4,j}}{EIL} A_{3j}(L) = c_{1j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}}, \\
 \hat{c}_{4,j}(L) - \frac{\hat{c}_{4,j}}{EIL} A_{3j}(L) = c_{1j}, \quad & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{K}{I_{p}}},
\end{align*}
\]

The uniqueness of the solutions to (59)-(61) is guaranteed because by Assumption 1, the determinants of the coefficients of \( \hat{c}_{1,j}, \hat{c}_{2,j} \) and \( \hat{c}_{3,j}, \hat{c}_{4,j} \), are non-zero, namely, \( \frac{L}{6} - \frac{2EI}{K} \neq 0 \) and \( \frac{K}{EIL} A_{mj}(L) + B_{mj}(L)C_{mj}(L) \neq 0 \). Therefore,

\[
\tilde{c}_{k} = J_{k} \hat{c}_{k}^{-1}, \quad k = 1, 2, 3, 4
\]

is the solution to (68), which is independent of \( D^{T} \) in (9).

**Proof of Lemma 5.** We split the proof into four steps.

**Step I:** Define

\[
F_{w}^{T}(x) = f_{w}^{T}(x)J_{s} = [F_{w,1}(x), \ldots, F_{w,n}(x)], \quad F_{\varphi}^{T}(x) = f_{\varphi}^{T}(x)J_{s} = [F_{\varphi,1}(x), \ldots, F_{\varphi,n}(x)],
\]

where \( F_{w,j}(x) \) and \( F_{\varphi,j}(x) \) satisfy

\[
\begin{align*}
 [p \lambda_{ij}^{2} + \Theta_{1}(x) \lambda_{ij} + \Theta_{2}(x)]F_{w,j}(x) - K(F_{w}^{T}(x) - F_{\varphi}^{T}(x)) = d_{1}^{T}(x) \phi_{yj}, \\
 [I_{p} \lambda_{ij}^{2} + \Theta_{3}(x) \lambda_{ij} + \Theta_{4}(x)]F_{w,j}(x) - EIF_{w}^{T}(x) - K(F_{w}^{T}(x) - F_{\varphi}^{T}(x)) = d_{2}^{T}(x) \phi_{yj}, \\
 F_{w,j}(0) = d_{1}^{T} \phi_{yj}, \quad F_{w,j}(L) = d_{2}^{T} \phi_{yj}, \\
 F_{w,j}(L) = d_{3}^{T} \phi_{yj}, \quad F_{\varphi,j}(L) = d_{4}^{T} \phi_{yj},
\end{align*}
\]
which can be rewritten as

\[
\begin{align*}
\frac{d}{dx} [F_{\psi,j}, F_{\psi,j}', F''_{\psi,j}, F'''_{\psi,j}] &= [F_{\psi,j}, F_{\psi,j}', F''_{\psi,j}, F'''_{\psi,j}] M(x) + \begin{bmatrix} 0, 0, -\frac{d_1^T \phi_j}{K}, -\frac{d_1^T \phi_j}{E I} \end{bmatrix} \\
M(x) &= \begin{bmatrix} 0 & 0 & \rho \mu_j^2 + \Theta_0 \lambda_0 + \Theta_0 \lambda_0 \xi_j & 0 \\
0 & 0 & 0 & \frac{1}{E I} \lambda_0 \Theta_0 \lambda_0 + \Theta_0 \lambda_0 \xi_j \\
1 & 0 & 0 & -\frac{K}{E I} \\
0 & 1 & 1 & 0 \end{bmatrix}.
\end{align*}
\]

The solution has the form:

\[
[F_{\psi,j}, F_{\psi,j}', F''_{\psi,j}, F'''_{\psi,j}] = [d_1^T \phi_j, d_2^T \phi_j, F''_{\psi,j}(0), F'''_{\psi,j}(0)] e^{\int_0^x M(s)ds} + \int_0^x \begin{bmatrix} 0, 0, \frac{-d_1^T \phi_j}{K}, -\frac{d_1^T \phi_j}{E I} \end{bmatrix} e^{-\int_0^s M(t)ds} ds e^{\int_0^s M(t)ds},
\]

where \(F''_{\psi,j}(0)\) and \(F'''_{\psi,j}(0)\) are determined by the boundary conditions at \(x = L\). Since the systematic uncertainty \(\Theta_k \in C[0, L], k = 1, 2, 3, 4\) are too small that the eigenvalues of \(S\) are still not the transmission zeros of the uncertain plant, it then suffices to prove the existence of \(F''_{\psi,j}(0)\) and \(F'''_{\psi,j}(0)\) with \(\Theta_k \equiv 0, k = 1, 2, 3, 4\).

In such case that \(\Theta_k \equiv 0, k = 1, 2, 3, 4\), the general solution can be expressed, by [11, theorem 2.2], as

\[
\begin{align*}
&\text{if } \lambda_j = 0,
F_{\psi,j}(x) = \begin{bmatrix} x^2 \left( -\frac{2E I}{x} \right) + b_{F,j} x^2 + d_1^T \phi_j + \frac{1}{2EI} \int_0^x \left( \frac{(x-s)^3}{3} - \frac{2E I}{K} (x-s) \right) \frac{d_1^T \phi_j}{ds} ds \\
-\frac{1}{E I} \int_0^x (x-s) \frac{d_2^T \phi_j}{ds} ds,
F_{\psi,j}'(x) = \frac{d_1^T \phi_j}{\mu_j^2 - \mu_j^2} \left( \mu_j - \frac{\rho \mu_j^2}{K} \right) \cosh \sqrt{\mu_j^2 x} - \left( \mu_j - \frac{\rho \mu_j^2}{K} \right) \cosh \sqrt{\mu_j^2 x}.
\end{align*}
\]

Otherwise,

\[
\begin{align*}
F_{\psi,j}(x) &= F_{\psi,j}'(0) A_{\psi,j}(x) + F_{\psi,j}'(0) B_{\psi,j}(x) - \frac{1}{E I} \int_0^x B_{\psi,j}(x-s) \frac{d_1^T \phi_j}{ds} ds \\
&- \frac{1}{E I} \int_0^x A_{\psi,j}(x-s) \frac{d_2^T \phi_j}{ds} ds + \frac{d_1^T \phi_j}{\mu_j^2 - \mu_j^2} \left( \mu_j - \frac{\rho \mu_j^2}{K} \right) \cosh \sqrt{\mu_j^2 x} - \left( \mu_j - \frac{\rho \mu_j^2}{K} \right) \cosh \sqrt{\mu_j^2 x}.
\end{align*}
\]

\[
\begin{align*}
F_{\psi,j}'(x) &= F_{\psi,j}'(0) C_{\psi,j}(x) - F_{\psi,j}'(0) E_{\psi,j}(x) + \frac{1}{E I} \int_0^x A_{\psi,j}(x-s) \frac{d_2^T \phi_j}{ds} ds \\
&- \frac{1}{E I} \int_0^x C_{\psi,j}(x-s) \frac{d_2^T \phi_j}{ds} ds + \frac{d_1^T \phi_j}{\mu_j^2 - \mu_j^2} \left( \mu_j - \frac{\rho \mu_j^2}{K} \right) \cosh \sqrt{\mu_j^2 x} - \left( \mu_j - \frac{\rho \mu_j^2}{K} \right) \cosh \sqrt{\mu_j^2 x},
\end{align*}
\]

where \(A_{\psi,j}(x), B_{\psi,j}(x), C_{\psi,j}(x),\) and \(D_{\psi,j}(x), m = 1, 2,\) are given by (52) and (55). The \(F''_{\psi,j}(0), F'''_{\psi,j}(0), \frac{d_1^T \phi_j}{K},\) and \(\frac{d_2^T \phi_j}{K}, k = 1, 2, j = 1, \ldots, n,\) are uniquely obtained by Assumption 1. This proves the existence of the solution to (22).

**Step II:** By \(H_{\psi,j}^L = h_{\psi,j}^L = [H_{\psi,j,1}(x), \ldots, H_{\psi,j,n}(x)]\) and \(H_{\psi,j}' = h_{\psi,j}' = [H_{\psi,j,1}(x), \ldots, H_{\psi,j,n}(x)]\), (27) is then transformed into

\[
\begin{align*}
\begin{cases}
\rho \mu_j^2 H_{\psi,j}(x) - K(H_{\psi,j}'(x) - H_{\psi,j}'(x)) = 0, \\
I \rho \mu_j^2 H_{\psi,j}(x) - EIH_{\psi,j}(x) - K(H_{\psi,j}'(x) - H_{\psi,j}'(x)) = 0, \\
H_{\psi,j}(0) = 0, \quad H_{\psi,j}'(0) = 0, \\
K(H_{\psi,j}'(L) - H_{\psi,j}'(L)) = K(F_{\psi,j}'(L) - F_{\psi,j}'(L)) - d_3^T \phi_j - k_1 \lambda_j H_{\psi,j}(L), \\
EIH_{\psi,j}'(L) = EIH_{\psi,j}'(L) - d_4^T \phi_j - k_2 \lambda_j H_{\psi,j}(L),
\end{cases}
\end{align*}
\]
where the boundary conditions at \( x = L \) are obtained from \( h_w^\top(x) = f_w^\top(x) + g_w^\top(x) \), \( h_\varphi^\top(x) = f_\varphi^\top(x) + g_\varphi^\top(x) \).

Similar to Lemma 3, the solution of (63) can be found to be

\[
\begin{cases}
H_{w,j}(x) = H'_{\varphi,j}(0)A_{j1}(x) + H'_{w,j}(0)B_{j1}(x), & \text{if } \lambda_{ij} \neq 0, \pm \sqrt{\frac{\psi_j}{\rho_j}} \\
H_{\varphi,j}(x) = H'_{\varphi,j}(0)C_{j1}(x) - H'_{w,j}(0)\frac{\psi_j}{\rho_j}A_{j1}(x), & \text{if } \lambda_{ij} = \pm \sqrt{\frac{\psi_j}{\rho_j}} \\
H_{w,j}(x) = H'_{\varphi,j}(0)A_{j2}(x) + H'_{w,j}(0)B_{j2}(x), & \text{if } \lambda_{ij} = \pm \sqrt{\frac{\psi_j}{\rho_j}} \\
H_{\varphi,j}(x) = H'_{\varphi,j}(0)C_{j2}(x) - H'_{w,j}(0)\frac{\psi_j}{\rho_j}A_{j2}(x), & \text{if } \lambda_{ij} = 0, \\
\end{cases}
\]

(64)

where \( A_{mj}(x), B_{mj}(x), \) and \( C_{mj}(x), m = 1, 2, \) are given by (52) and (55). The \( (H'_{w,j}(0), H'_{\varphi,j}(0)) \) and \( (\hat{a}_{H,j}, \hat{b}_{H,j}) \) are determined by the boundary conditions at \( x = L \). This concludes the existence of the solution to (27).

**Step II:** The existence of the solutions to (22) and (27) further results into the bounded solution to (24), since \( h_w^\top(x) = f_w^\top(x) + g_w^\top(x) \) and \( h_\varphi^\top(x) = f_\varphi^\top(x) + g_\varphi^\top(x) \).

**Step IV:** In order to solve (29), we first find the solution to the first two equations of (29). Set

\[
J_s^{-1}h_{0k}J = \text{diag}[H_{0k,1}, \ldots, H_{0k,n}] = H_{0k}, \quad k = 1, 2.
\]

(65)

Then, (29) is equivalent to

\[
F_{0w,j}(L)H_{0j} = H_{w,j}(L), \quad F_{0\varphi,j}(L)H_{02,j} = H_{\varphi,j}(L).
\]

(66)

from which we obtain

\[
h_{01} = J_s\text{diag} \left[ \frac{H_{w,j}(L)}{F_{0w,j}(L)} \right] J_s^{-1}, \quad h_{02} = J_s\text{diag} \left[ \frac{H_{\varphi,j}(L)}{F_{0\varphi,j}(L)} \right] J_s^{-1}.
\]

(67)

We next show that the solution (67) makes the last two equations of (29) hold automatically. Actually, by (66), the last two equations in (29) can be transformed into

\[
\begin{align*}
K(H'_{w,j}(L) - H_{\varphi,j}(L)) + (k_1 + k_3)H_{w,j}(L)\lambda_{ij} &= F_{0w,j}(L)\hat{\chi}_{1j}H_{01,j} + F_{0\varphi,j}(L)\hat{\theta}_{1j}H_{02,j} \\
= H_{0w,j}(L)\hat{\chi}_{1j} + H_{0\varphi,j}(L)\hat{\theta}_{1j}, \\
E(H'_{\varphi,j}(L) + (k_2 + k_4)H_{\varphi,j}(L))\lambda_{ij} &= F_{0w,j}(L)\hat{\chi}_{3j}H_{01,j} + F_{0\varphi,j}(L)\hat{\theta}_{3j}H_{02,j} \\
= H_{0w,j}(L)\hat{\chi}_{3j} + H_{0\varphi,j}(L)\hat{\theta}_{3j},
\end{align*}
\]

(68)

which is exactly (28) and thus holds naturally.

**Proof of Lemma 6.** By Reference 14, the admissibility of \( \Delta_{Lw} \) and \( \Delta_{L\varphi} \) is equivalent to: (a) \( \Delta_{Lw}A^{*-1} \) and \( \Delta_{L\varphi}A^{*-1} \) are bounded from \( H \) to \( C \), and (b) for every \( T > 0 \), there exists \( M_{T_1} > 0 \) such that the adjoint system of the following

\[
\begin{align*}
\rho \hat{u}_0^c(x, t) - K(\hat{u}_0^c(x, t) - \hat{\phi}_x^c(x, t)) &= 0, \\
L_p\hat{\phi}_0^c(x, t) - E(\hat{\phi}_0^c(x, t) - K(\hat{u}_0^c(x, t) - \hat{\phi}_x^c(x, t))) &= 0, \\
\hat{u}_0^c(0, t) &= 0, \quad \hat{\phi}_0^c(0, t) = 0, \\
K(\hat{u}_0^c(L, t) - \hat{\phi}_x^c(L, t)) &= -k_3\hat{u}_0^c(L, t), \\
E(\hat{\phi}_0^c(L, t) - \hat{\phi}_x^c(L, t)) &= -k_4\hat{\phi}_0^c(L, t),
\end{align*}
\]

(69)

satisfies

\[
\int_0^T [||\hat{u}_0^c(L, t)||^2 + ||\hat{\phi}_0^c(L, t)||^2] dt \leq M_{T_1} [||\hat{u}_0^c(., 0)||^2, ||\hat{\phi}_0^c(., 0)||^2].
\]
Since $A$ generates an exponentially stable $C_0$-semigroup on $H$, so does for $A^*$ on $H$ and hence system (69) admits a unique solution. In what follows, for simplicity, we only consider the real solutions which do not bring big difference to complex solutions. Define

$$E^*(t) = \frac{\rho}{2} \int_0^L \left[ \hat{\omega}^r_c(x, t) \right]^2 \, dx + \frac{I_c}{2} \int_0^L \left[ \hat{\omega}^f_c(x, t) \right]^2 \, dx + \frac{EI}{2} \int_0^L \left[ \hat{\phi}^c_c(x, t) \right]^2 \, dx$$

$$+ \frac{K}{2} \int_0^L \left[ \hat{\phi}^c_c(x, t) - \hat{\omega}^f_c(x, t) \right]^2 \, dx,$$

which decays exponentially that $E^*(t) \leq C_e e^{-\omega_s t} E^*(0)$ for some $C_e, \omega_s > 0$. Since

$$E^*(t) = -k_3[\hat{\omega}^r_c(L, t)]^2 - k_4[\hat{\phi}^c_c(L, t)]^2,$$

we have

$$\int_0^\infty \{k_3[\hat{\omega}^r_c(L, t)]^2 + k_4[\hat{\phi}^c_c(L, t)]^2\} \, dt \leq E^*(0).$$

For any $(\phi_1, \phi_2, \phi_3, \phi_4) \in D(A^*)$ and $(f_1, f_2, f_3, f_4) \in H$, solve $A^*(\phi_1, \phi_2, \phi_3, \phi_4) = (f_1, f_2, f_3, f_4)$ to obtain

$$\begin{cases} 
\phi_2(x) = -f_1(x), & \phi_4(x) = -f_3(x), \\
\phi_1(x) = \dot{a}_1 \left[ t^3 - \frac{3EI}{K} x \right] + \hat{b}_1 x^2 + \frac{\rho}{2EI} \int_0^x \left[ \frac{(x-s)^3}{3} - \frac{3EI}{K} (x-s) \right] f_2(s) \, ds \\
\phi_3(x) = \dot{a}_1 x^2 + \hat{b}_1 x + \frac{\rho}{2EI} \int_0^x (x-s)^2 f_4(s) \, ds - \frac{\rho}{2EI} \int_0^x (x-s) f_4(s) \, ds,
\end{cases}$$

where $\dot{a}_1$ and $\hat{b}_1$ are uniquely determined as

$$\begin{cases} 
\dot{a}_1 = \frac{k_3}{2EI} f_1(L) - \frac{\rho}{2EI} \int_0^L f_2(x) \, dx, \\
\hat{b}_1 = -\frac{k_2}{EI} f_3(L) + \frac{I_c}{EI} \int_0^L f_4(x) \, dx - \frac{\rho}{2EI} \int_0^L 2(L-x) f_2(x) \, dx - \frac{k_1}{EI} f_1(L) + \frac{\rho L}{EI} \int_0^L f_2(x) \, dx.
\end{cases}$$

Now,

$$\Delta_{Lw} A^{*-1}(f_1, f_2, f_3, f_4) = -f_1(L), \quad \Delta_{L\varphi} A^{*-1}(f_1, f_2, f_3, f_4) = -f_3(L),$$

which advises that both $\Delta_{Lw} A^{*-1}$ and $\Delta_{L\varphi} A^{*-1}$ are bounded from $H$ to $\mathbb{C}$. This proves the (infinite) admissibility of $\Delta_{Lw}$ and $\Delta_{L\varphi}$ to $e^{At}$.

**Proof of Theorem 1.** By (23), (26), and (31), the closed-loop system (21) is transformed into (32). It then suffices to prove the exponential stability of (32). Same to (13), the $(\tilde{\omega}^r, \tilde{\phi}^c, \tilde{\nu}^r)$-subsystem in (32) is exponentially stable that

$$\| (\tilde{\omega}^r(x, t), \tilde{\omega}^f_c(x, t), \tilde{\phi}^c_r(x, t), \tilde{\phi}^c_c(x, t), \tilde{\nu}^r(t)) \|_{H^1 C^2\mathbb{R}} \leq C e^{-\omega_2 t} \| (\tilde{\omega}^r(x, 0), \tilde{\omega}^f_c(x, 0), \tilde{\phi}^c_r(x, 0), \tilde{\phi}^c_c(x, 0), \tilde{\nu}^r(0)) \|_{H^1 C^2\mathbb{R}},$$

with $C, \omega_2 > 0$. We then have

$$\begin{cases} 
|\tilde{\phi}^c_r(L, t)|^2 \leq \| \tilde{\phi}^c_r(\cdot, t) \|^2_{L^2} \leq C e^{-\omega_2 t} \| (\tilde{\omega}^r(x, 0), \tilde{\omega}^f_c(x, 0), \tilde{\phi}^c_r(x, 0), \tilde{\phi}^c_c(x, 0), \tilde{\nu}^r(0)) \|_{H^1 C^2\mathbb{R}}, \\
|\tilde{\nu}^r(L, t)|^2 \leq \| \tilde{\nu}^r(\cdot, t) \|^2_{L^2} \leq \max \{ EI, K \} \left[ \| \tilde{\phi}^c_r(\cdot, t) \|^2 + \| \tilde{\phi}^c_r(\cdot, t) - \tilde{\omega}^f_c(\cdot, t) \|^2 \right] \leq C e^{-\omega_2 t} \| (\tilde{\omega}^r(x, 0), \tilde{\omega}^f_c(x, 0), \tilde{\phi}^c_r(x, 0), \tilde{\phi}^c_c(x, 0), \tilde{\nu}^r(0)) \|_{H^1 C^2\mathbb{R}}.
\end{cases}$$

(71)

Define

$$\beta < \min \left\{ \frac{\omega_2}{M_A}, \frac{\omega_2}{2} \right\},$$

(72)
and then $\hat{Y}(t) = e^{\beta t}(\hat{w}, \hat{w}_t, \hat{\psi}, \hat{\phi}_c)$ satisfies (only real part of the solution needs to be discussed)

$$
\frac{d}{dt} \hat{Y}(t) = \left[ A + \beta \right] \hat{Y}(t) + \Delta_{tLw} \left[ \eta_1^T e^{\beta t} \hat{\psi}(t) + k_1 e^{\beta t} \hat{w}_c(t) \right] + \Delta_{t\phi} \left[ \eta_2^T e^{\beta t} \hat{\phi}_c(t) + k_2 e^{\beta t} \hat{\phi}_c(t) \right],
$$

where $\eta_1^T = \left[ \int_{0}^{\infty} \langle L \xi_1, f_{0\phi}^T(L) \xi_2 \rangle, \int_{0}^{\infty} \langle L \xi_2, f_{0\phi}^T(L) \xi_2 \rangle \right]$ and $A$, $\Delta_{tLw}$, and $\Delta_{t\phi}$ are defined as (33). By (70), define (as before, we also consider only real solution for simplicity)

$$
E_1(t) = \frac{\rho}{2} \int_{0}^{L} [\hat{w}_c(x, t)]^2 dx + \frac{I_p}{2} \int_{0}^{L} [\hat{\psi}(x, t)]^2 dx + \frac{E_l}{2} \int_{0}^{L} [\hat{\phi}_c(x, t)]^2 dx + \frac{K}{2} \int_{0}^{L} [\hat{\phi}_c(x, t) - \hat{w}_c(x, t)]^2 dx,
$$

which satisfies $E_1(t) \leq Ce^{-\alpha t}$. By (72), we have $E_1(t) = -k_1 [\hat{w}_c(L, t)]^2 - k_2 [\hat{\phi}_c(L, t)]^2$ and further

$$
k_1 \int_{0}^{\infty} e^{2\beta t} [\hat{w}_c(L, t)]^2 dt + k_2 \int_{0}^{\infty} e^{2\beta t} [\hat{\phi}_c(L, t)]^2 dt = E_1(0) + \int_{0}^{\infty} 2\beta e^{2\beta t} E_1(t) dt < \infty.
$$

As a result

$$
\int_{0}^{\infty} e^{2\beta t} [\hat{w}_c(L, t)]^2 dt + \int_{0}^{\infty} e^{2\beta t} [\hat{\phi}_c(L, t)]^2 dt + \int_{0}^{\infty} e^{2\beta t} ||\hat{\psi}(t)||^2 dt < \infty.
$$

Since $A + \beta$ generates an exponentially stable $C_0$-semigroup on $H$, and by Lemma 6, $\Delta_{tLw}$ and $\Delta_{t\phi}$ are admissible to $e^{(A + \beta)t}$, $\hat{Y}(t)$ is asymptotically stable. For any $2\alpha < 2\beta = \omega_1$, define

$$
E_2(t) = \frac{\rho}{2} \int_{0}^{L} \hat{w}_c(x, t) dx + \frac{I_p}{2} \int_{0}^{L} [\hat{\psi}(x, t)]^2 dx + \frac{E_l}{2} \int_{0}^{L} [\hat{\phi}_c(x, t)]^2 dx + \frac{K}{2} \int_{0}^{L} [\hat{\phi}_c(x, t) - \hat{w}_c(x, t)]^2 dx,
$$

which satisfies $E_2(t) \leq Ce^{-2\beta t}$. This further advises

$$
|\hat{\phi}_c(L, t)|^2 \leq Ce^{-2\beta t}, \quad |\hat{w}_c(L, t)|^2 \leq Ce^{-2\beta t},
$$

which, together with (71), implies that

$$
\lim_{t \to \infty} |e_k(t)| = 0, \quad k = 1, 2,
$$

exponentially. Finally, for any $\alpha < \beta$, there holds

$$
\frac{d}{dt} \left[ e^{2\alpha t} E_2(t) \right] = 2\alpha e^{2\alpha t} E_2(t) - k_3 e^{2\alpha t} [\hat{w}_c(L, t)]^2 + \eta_1^T \hat{\psi}(t) e^{2\alpha t} \hat{w}_c(L, t) + k_1 e^{2\alpha t} \hat{w}_c(L, t) \hat{w}_c(L, t) - k_4 e^{2\alpha t} [\hat{\phi}_c(L, t)]^2 + \eta_2^T \hat{\psi}(t) e^{2\alpha t} \hat{\phi}_c(L, t) \hat{\phi}_c(L, t)
$$

$$
\leq 2\alpha e^{2\alpha t} E_2(t) - \left[ k_3 - \frac{\|\eta_1^T\|}{\delta_1} - \frac{k_1}{\delta_2} \right] e^{2\alpha t} [\hat{w}_c(L, t)]^2 + \left[ \|\eta_1^T\| \delta_1 + \|\eta_2^T\| \delta_3 \right] e^{2\alpha t} ||\hat{\psi}(t)||^2 + k_1 \delta_2 e^{2\alpha t} [\hat{w}_c(L, t)]^2 - \left[ k_4 - \frac{\|\eta_1^T\|}{\delta_3} - \frac{k_2}{\delta_4} \right] e^{2\alpha t} [\hat{\phi}_c(L, t)]^2 + k_2 \delta_4 e^{2\alpha t} [\hat{\phi}_c(L, t)]^2,
$$

where we choose $\delta_i, i = 1, 2, 3, 4$ sufficiently large so that

$$
k_3 - \frac{\|\eta_1^T\|}{\delta_1} - \frac{k_1}{\delta_2} > 0, \quad k_4 - \frac{\|\eta_1^T\|}{\delta_3} - \frac{k_2}{\delta_4} > 0.
$$

We then have

$$
\left[ k_3 - \frac{\|\eta_1^T\|}{\delta_1} - \frac{k_1}{\delta_2} \right] \int_{0}^{\infty} e^{2\alpha t} [\hat{w}_c(L, t)]^2 dt + \left[ k_4 - \frac{\|\eta_1^T\|}{\delta_3} - \frac{k_2}{\delta_4} \right] \int_{0}^{\infty} e^{2\alpha t} [\hat{\phi}_c(L, t)]^2 dt
$$
\[ \leq 2a \int_0^\infty e^{2at}E_2(t)dt + E_2(0) + \left[ \| \theta \|_1 + \| \theta \|_2 \right] \int_0^\infty e^{2at}[\dot{\psi}(t)]^2dt + k_1\delta_2 \int_0^\infty e^{2at}[\tilde{\psi}_1(L, t)]^2dt + k_2\delta_3 \int_0^\infty e^{2at}[\tilde{\psi}_2(L, t)]^2dt < \infty, \]

which, together with (74), implies that

\[ \int_0^\infty e^{2at}[\dot{e}_k(t)]^2dt < \infty, \quad k = 1, 2. \]

**Proof of Theorem 2.** Introduce the invertible transformation (31), the system of \((\tilde{u}, \tilde{w}, \tilde{v})\) is described by

\[
\begin{align*}
\rho \tilde{w}_n^c(x, t) - K(w_n^c(x, t) - \varphi_n^c(x, t)) + \Theta_1(x)w_n^c(x, t) + \Theta_2(x)w_n^c(x, t) &= 0, \\
I_p\tilde{\varphi}_n^c(x, t) - E\tilde{\varphi}_n^c(x, t) - K(w_n^c(x, t) - \varphi_n^c(x, t)) + \Theta_3(x)\varphi_n^c(x, t) + \Theta_4(x)\varphi_n^c(x, t) &= 0, \\
w_n^c(0, t) = 0, \quad \varphi_n^c(0, t) &= 0,
\end{align*}
\]

\[
K(w_n^c(L, t) - \varphi_n^c(L, t)) = -k_3\tilde{w}_n^c(L, t),
\]

\[
E\tilde{\varphi}_n^c(L, t) = -k_4\tilde{\varphi}_n^c(L, t) + f_0^T(L)\xi_1\tilde{v}_1(t) + f_0^T(L)\xi_2\tilde{v}_2(t),
\]

\[
\rho \tilde{w}_n^c(x, t) - K(\tilde{w}_n^c(x, t) - \tilde{\varphi}_n^c(x, t)) + \Theta_1(x)w_n^c(x, t) + \Theta_2(x)w_n^c(x, t) = 0,
\]

\[
I_p\tilde{\varphi}_n^c(x, t) - E\tilde{\varphi}_n^c(x, t) - K(\tilde{w}_n^c(x, t) - \tilde{\varphi}_n^c(x, t)) + \Theta_3(x)\varphi_n^c(x, t) + \Theta_4(x)\varphi_n^c(x, t) = 0,
\]

\[
\tilde{w}_n^c(0, t) = 0, \quad \tilde{\varphi}_n^c(0, t) = 0,
\]

\[
K(\tilde{w}_n^c(L, t) - \tilde{\varphi}_n^c(L, t)) = -k_1\tilde{w}_n^c(L, t),
\]

\[
E\tilde{\varphi}_n^c(L, t) = -k_2\tilde{\varphi}_n^c(L, t),
\]

\[
\dot{\tilde{v}}^c(t) = \begin{pmatrix} S + QD^T & 0 \\ 0 & S + QD^T \end{pmatrix} \tilde{v}^c(t) + \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \tilde{w}(L, t) \\ \tilde{\varphi}(L, t) \end{pmatrix},
\]

\[
e_1(t) = w^c(L, t),
\]

\[
e_2(t) = \varphi^c(L, t),
\]

which is exponentially stable because of the exponential stability of the closed-loop system (32) from Lemma 7. Define the energy function (also for simplicity, we discuss only the real solution)

\[
E_3(t) = \frac{\rho}{2} \int_0^L \left[ \tilde{w}_n^c(x, t) \right]^2dx + \frac{I_p}{2} \int_0^L \left[ \tilde{\varphi}_n^c(x, t) \right]^2dx + \frac{EI}{2} \int_0^L \left[ \tilde{\varphi}_n^c(x, t) \right]^2dx
\]

\[
+ \frac{K}{2} \int_0^L \left[ \tilde{w}_n^c(x, t) - w_n^c(x, t) \right]^2dx + \frac{\gamma\rho}{2} \int_0^L \left[ \tilde{\varphi}_n^c(x, t) \right]^2dx + \frac{\gamma I_p}{2} \int_0^L \left[ \tilde{\varphi}_n^c(x, t) \right]^2dx
\]

\[
+ \frac{\gamma EI}{2} \int_0^L \left[ \tilde{\varphi}_n^c(x, t) \right]^2dx + \frac{\gamma K}{2} \int_0^L \left[ \tilde{\varphi}_n^c(x, t) - \tilde{w}_n^c(x, t) \right]^2dx + \|\tilde{v}_1\|^2 + \|\tilde{v}_2\|^2,
\]

with \( \gamma > 0 \), and suppose that

\[
E_3(t) \leq Ce^{-\omega_3 t},
\]

for some \( C, \omega_3 > 0 \) by Lemma 7. This implies that \( e_1(t) = w^c(L, t) \) and \( e_2(t) = \varphi^c(L, t) \) converge exponentially to zero as \( t \to \infty \). Finally, the time derivative of \( E_3(t) \) is computed to satisfy

\[
\dot{E}_3(t) \leq C_1E_3(t) - \left[ \gamma k_1 - k_3\delta_5 - \| Q \| \delta_{11} \right] \left[ \tilde{w}_n^c(L, t) \right]^2 - \left[ \gamma k_2 - k_4\delta_8 - \| Q \| \delta_{12} \right] \left[ \tilde{\varphi}_n^c(L, t) \right]^2
\]

\[
- \left( k_3 - \frac{\| f_0^T(L) \| \delta_1 \| \delta_6 \} }{\delta_6} - \frac{\| f_0^T(L) \| \delta_2 \| \delta_6 \} }{\delta_6} \right) \left[ w_n^c(L, t) \right]^2
\]

\[
- \left( k_4 - \frac{\| f_0^T(L) \| \delta_3 \| \delta_8 \} }{\delta_8} - \frac{\| f_0^T(L) \| \delta_4 \| \delta_8 \} }{\delta_8} \right) \left[ \varphi_n^c(L, t) \right]^2,
\]

(78)
where $\delta_5 - \delta_{12}$ are positive constants, and $C_1 > 0$ is related to $\delta_6, \delta_7, \delta_8, \delta_{10}, \delta_{11},$ and $\delta_{12}$. Choose $\delta_5 - \delta_{10}$ and $\gamma$ large enough so that

$$\gamma k_1 - k_3 \delta_6 - ||Q|| \delta_{11} > 0, \quad \gamma k_2 - k_4 \delta_8 - ||Q|| \delta_{12} > 0,$$

$$k_3 - \frac{k_3}{\delta_5} - \frac{||f_{0v}^T(L)\zeta_1||}{\delta_6} - \frac{||f_{0v}^T(L)\zeta_2||}{\delta_7} > 0, \quad k_4 - \frac{k_4}{\delta_8} - \frac{||f_{0v}^T(L)\zeta_3||}{\delta_9} - \frac{||f_{0v}^T(L)\zeta_4||}{\delta_{10}} > 0,$$

(79)

from which and (78), we obtain

$$\dot{E}_3(t) \leq C_1 E_3(t) - C^* \{ |w(L, t)|^2 + |\varphi(L, t)|^2 \},$$

(80)

for $C^*_1 > 0$. For any $0 < \alpha < \omega_3$ where $\omega_3$ is defined in (77), we obtain from (80) that

$$\frac{d}{dt} \{ e^{\alpha t} \dot{E}_3(t) \} \leq (\alpha + C_1) e^{\alpha t} E_3(t) - C^*_1 e^{\alpha t} \{ |w(L, t)|^2 + |\varphi(L, t)|^2 \}. \quad (81)$$

Integrating both sides of (81) over $[0, \infty)$ with respect to $t$ gives

$$\int_0^\infty e^{\alpha t} \dot{e}_k^2(t) dt < \infty, \quad k = 1, 2,$$

by (77) and the fact $\dot{e}_1(t) = w(L, t), \quad \dot{e}_2(t) = \varphi(L, t)$.

7 | CONCLUDING REMARKS

In this article, we apply observer based approach to deal with two inputs, two outputs robust regulation problem for a Timoshenko beam equation. This is an continuation of our recent works on SISO PDEs by the internal model principle. The key idea is to make a 2-copy of the exosystem in the observer and to allocate proportionally the estimates in error feedback control. The others are similar to SISO PDEs. We first build a nominal system which is a coupled system consisting of the original PDE and exosystem, with some specially selected coefficients of the disturbances and references. For the nominal system, we design an extended state observer in terms of the tracking errors and their derivatives. This produces naturally a dynamic error feedback with 2-copy of the exosystem yet it is not coincident with the general abstract theory in literature because we used their derivatives in addition to the tracking errors. We next find a feedforward control for nominal system, which leads finally to an error feedback control by simply substitution of the state and disturbances in the feedforward control with their estimates obtained from the extended state observer. The robustness analysis is performed by two state transformations through solving related regulator equations. Since our closed-loop is exponentially stable without disturbances, the controller is also shown to be robust to low order system uncertainties as we showed in Reference 9 for SISO PDEs. Finally, the feedback control is shown to be bounded, which is crucial in output tracking, and our approach is genetic even for MIMO PDEs.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (No. 61873260) and Chinese Postdoctoral Science Foundation (No. 2020M680531).

<table>
<thead>
<tr>
<th>Nomenclature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
</tr>
<tr>
<td>$L$</td>
</tr>
<tr>
<td>$EI$</td>
</tr>
<tr>
<td>$I_p$</td>
</tr>
<tr>
<td>$K$</td>
</tr>
</tbody>
</table>
REFERENCES


How to cite this article: Guo B-Z, Meng T. Robust output regulation for Timoshenko beam equation with two inputs and two outputs. *Int J Robust Nonlinear Control*. 2021;31:1245–1269. https://doi.org/10.1002/rnc.5345