Automatica 114 (2020) 108818

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Robust error based non-collocated output tracking control for a heat equation[☆]

Bao-Zhu Guo[a,b,c,*], Tingting Meng[b,c]

[a] School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
[b] Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China
[c] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

ARTICLE INFO

Article history:
Received 11 April 2019
Received in revised form 19 December 2019
Accepted 20 December 2019
Available online xxxx

Keywords:
Internal model principle
Observer design
Output tracking
Disturbance rejection

ABSTRACT

In this paper, an output tracking problem for a heat equation is considered, where all possible disturbances produced from an exosystem and a systematic uncertainty are considered and the performance output is non-collocated with control. The objective is twofold: to look at how the internal model principle works for the output tracking of PDEs; and to see how to design a robust tracking error feedback control for PDEs. To this purpose, we first select a frozen case with specially selected frozen coefficients of the disturbances. For this frozen system, we design a feedforward control by solving simply regulator equation and an infinite-dimensional extended state observer in terms of tracking error only which gives an estimation of both states of the frozen plant and exosystem. An observer-based tracking error feedback control is then designed for the frozen system, which is shown to be in line with the internal model principle. As a result, the system is shown to be robust to system uncertainty and disturbances in all channels. The numerical simulations validate the results.

1. Introduction

Output regulation or output tracking is one of the major concerns in control theory. In many engineering applications, we are only concerned with the partial states of the system, which are called performance outputs, to be regulated and others are kept to be bounded (Meng & Wei, 2017; Wang & Wu, 2014). At the same time, the disturbances from both systems and measurements must be taken into account to guarantee the robustness of the control so that the closed-loop system can operate normally in the presence of the disturbances (Christofides, 2001; Deutscher, 2016). In this regard, a systematic research has been carried out since from 1970s in the name of the internal model principle (Davison, 1976; Francis & Wonham, 1976). By the internal model principle, the robust output tracking is largely simplified to a dynamic tracking error feedback control which contains a p-copy of the exosystem, where $p \in \mathbb{N}$ is the dimension of the performance output (Paunonen, 2016). The exosystem is supposed to generate both reference signals and disturbances. This powerful method has been applied to nonlinear lumped parameter systems (Huang, 2004) and even distributed parameter systems (Deutscher, 2015; Paunonen & Pohjolainen, 2010), where in latter case, the unboundedness of control and observation operators increases difficulties in solving the related Sylvester equations. A systematic generalization of the internal model principle to infinite-dimensional systems was made in Paunonen and Pohjolainen (2014) with unbounded control and observation operators. However, the disturbance related operators and the input operator in dynamic tracking error feedback control are still assumed to be bounded. Unbounded case was investigated in Paunonen (2017) but the output convergence is limited to weak convergence due to unboundedness of the output operator (see also Natarajan, Gilliam, & Weiss, 2014).

On the other hand, progresses on output tracking from PDE point of view have also been made over the years but the complexity is quite different depending on the locations of the control, disturbance and performance output. A non-collocated output tracking problem was considered in Guo, Shao, and Krstic (2017), with the harmonic disturbance rejected via adaptive control method. An early effort along the same line can also be found in Guo and Guo (2011). Some other interesting regulation problems can also be solved by constructing special solutions for regulator equations from which the controllers can be designed

☆ This work was supported by the National Natural Science Foundation of China (61873260), and the Project of Department of Education of Guangdong Province, China (No.2017KZDXM087). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Joachim Deutscher under the direction of Editor Miroslav Krstic.

* Corresponding author at: School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China.

E-mail addresses: bzguo@iss.ac.cn (B.-Z. Guo), mengting@amss.ac.cn (T. Meng).

https://doi.org/10.1016/j.automatica.2020.108818

in terms of kernel equations (Deutscher, 2015, 2016). In particular, paper Deutscher (2016) obtained a robust output feedback control based on an observer of an extended PDE–ODE system, where different to present paper, the reference signal is supposed to be known although in principle it can use error feedback only. With a combination of the internal model principle and the backstepping approach, the regulation problem via state feedback control has been solved for some PDEs in Deutscher and Kerschbaum (2016) and Gabriel and Deutscher. The paper (Deutscher & Kerschbaum, 2016) considered second-order hyperbolic PDEs and Gabriel and Deutscher addressed an n-coupled wave equation, both with spatially varying coefficients, and in-domain and boundary disturbances. An interesting situation of Gabriel and Deutscher is that the performance output can be pointwise, namely, distributed in-domain or defined on a boundary. In Jin and Guo (2018), an extended state observer was constructed to estimate the state and the very general external disturbance, but the control and performance output there are collocated. Other methods directly for PDEs can also be found in He, Ge, How, Choo, and Hong (2011) and He, He, and Ge (2015). The state feedback regulation problem was investigated systematically in Aulisa and Gilliam (2016).

However, from the applications of the internal model principle for abstract infinite-dimensional systems to PDEs like those presented in Natarajan et al. (2014), Paunonen (2017) and Paunonen and Pohjolainen (2014), there is still a huge gap. Given a PDE, how to check all assumptions and conditions required in the abstract form is never an easy task. In this paper, under the guideline of the internal model principle presented in Paunonen and Pohjolainen (2014), we attempt to design a robust tracking error feedback control for a PDE. The difference from Paunonen and Pohjolainen (2014) is that our disturbance related operator and the input operator in dynamic tracking error feedback control are also unbounded, and compared with Paunonen (2017), we seek pointwise convergence of the output and do not need to check some assumptions. The contributions of this paper are: (a) It demonstrates how to solve robust output tracking for PDEs by the internal model principle through this example; (b) It reveals the tricks in observer design behind the general framework which has been started recently in Paunonen (2020); (c) The method proposed in this paper is quite general to be applied to other PDEs, for which an output regulation for a non wellposed Euler–Bernoulli beam can be similarly developed in Guo and Meng (2019) during review of the present paper. In addition, another contribution of the paper is the design of observer for non-collocated PDEs. All contributions focus indeed on the design of a dynamic error feedback control. If we look at PDE examples presented in aforementioned papers, the control design following abstract results is very complicated. Indeed, there is an observer-based design theory generally for finite-dimensional systems, see, e.g., Deutscher (2017) and Huang (2004, Theorem 1.14, p.14). The core idea of Paunonen (2020) is to generalize observer-based design for infinite-dimensional systems, where the observer design must be Lunberger type without much option and the convergence output must be in weak sense due to unboundedness of the output operator. The present paper is to exemplify the idea of Paunonen (2020) to a PDE, where differently from (Paunonen, 2020), an observer-based error feedback robust control is proposed directly from PDE point of view without an additional assumption and the pointwise convergence is guaranteed for the output tracking error. Roughly speaking, our approach says that if one can design an extended state observer for any specially selected frozen coefficients of a coupled PDE with exosystem, then, an observer based error feedback control contains 1-copy of the exosystem. This approach is genetic even for MIMO PDEs by simply expanding the exosystem as $\dot{v} = \text{diag}(G, G, \ldots, G)v$ from the single exosystem $\dot{p} = Gp(t)$ (Paunonen, 2020).

The system that we consider in this paper is described by the following heat equation with disturbances and system uncertainty:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + F(x)p(t) + \varDelta(x)w(x,t), \\ \qquad\qquad x \in (0,1),\ t > 0, \\ w_x(0,t) = Np(t), \quad t \geq 0, \\ w_x(1,t) = u(t) + Dp(t), \quad t \geq 0, \\ w(x,0) = w_0(x), \quad 0 \leq x \leq 1,\ t \geq 0, \\ y_c(t) = w(0,t), \quad t \geq 0, \end{cases} \tag{1}$$

where the real function $\varDelta \in C^1[0,1]$ represents the system uncertainty, $F(x) \in \mathbb{C}^{1\times n}$, $N \in \mathbb{C}^{1\times n}$ and $D \in \mathbb{C}^{1\times n}$ are unknown coefficients of the in-domain and boundary disturbances, $u(t)$ is the control, $w_0(x)$ is the initial state and $y_c(t)$ is the performance output to be regulated. We consider system (1) in the state space $H = L^2(0,1)$.

The finite-dimensional exosystem is described by

$$\begin{cases} \dot{p}(t) = Gp(t), \quad t > 0, \\ p(0) = p_0, \end{cases} \tag{2}$$

where the unknown $p \in \mathbb{C}^{n\times 1}$. It is assumed that the matrix $G \in \mathbb{C}^{n\times n}$ is known and the initial value p_0 is unknown.

Denote the reference trajectory by

$$y_{ref}(t) = Mp(t), \tag{3}$$

and the tracking error by $y_e(t) = y_c(t) - y_{ref}(t)$, where $M \in \mathbb{C}^{1\times n}$ is also unknown. The control objective is to design a tracking error feedback control so that

$$\lim_{t\to\infty} |y_e(t)| = \lim_{t\to\infty} |y_c(t) - y_{ref}(t)| = 0, \tag{4}$$

regardless of the in-domain, input, and non-collocated boundary disturbances, and systematic uncertainty. It is seen that this is a typical output tracking problem for infinite-dimensional system discussed abstractly in Paunonen (2017) and Paunonen and Pohjolainen (2014), where the weak convergence of the output was pursued due to unbounded output operator but here we seek pointwise convergence. In the literature of PDE systems, the specialty of system (1) lies in that the performance output is non-collocated with control, which represents a difficult case in output tracking for PDEs.

In order to guarantee the robust output regulation, a necessary condition that the spectrum of the exosystem cannot be the transmission zeros of the control plant must be assumed. For our system without disturbances and uncertainty, by taking Laplace transform, there holds

$$\begin{cases} sy_w(x,s) = y_w''(x,s), \\ y_w'(0,s) = 0,\ y_w'(1,s) = \hat{u}(s), \\ Y_c(s) = w(0,s), \end{cases}$$

where $y_w(x,s)$, $\hat{u}(s)$ and $Y_c(s)$ are the Laplace transforms of $w(x,t)$, $u(t)$ and $y_c(t)$, respectively. The transfer function is then found to be

$$T(s) = \frac{2}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}$$

which has no zero and therefore we do not need any necessary condition like Assumption 2.2 of Paunonen (2017). Instead, the following Assumption 1.1 is assumed throughout the paper.

Assumption 1.1. The matrix G is diagonalizable and all eigenvalues of G are located on the imaginary axis and are distinct.

Assumption 1.1 simply means that all the disturbances and reference signals are produced from finite sum of the harmonic signals, which has been discussed in Guo and Guo (2011) by adaptive control method. For notation simplicity, all obvious domains for both time and spatial variables are omitted in equations hereafter.

Lemma 1.1. *For any $\alpha_2 > 0$, the following system*

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = -\alpha_2\varepsilon(1,t). \end{cases} \tag{5}$$

is exponentially stable and

$$|\varepsilon(0,t)| \le \tilde{M}_\varepsilon e^{-\tilde{\omega}_\varepsilon t}\|\varepsilon(\cdot,0)\|, \forall\, t \ge 0,$$

for some constants $\tilde{M}_\varepsilon, \tilde{\omega}_\varepsilon > 0$.

Proof. The proof of this lemma is simple but we need some facts for the proof of this result in later sections. Define the system operator $A : (D(A) \subset H) \to H$ as follows

$$\begin{cases} A\phi(x) = \phi''(x), \\ D(A) = \{\phi(x) \in H^2(0,1) | \phi'(0) = 0, \\ \qquad\qquad\qquad\qquad \phi'(1) = -\alpha_2\phi(1)\}, \end{cases} \tag{6}$$

which generates an analytic exponentially stable semigroup that

$$\|\varepsilon(\cdot,t)\| \le M_\varepsilon e^{-\omega_\varepsilon t}\|\varepsilon(\cdot,0)\|, \tag{7}$$

where $M_\varepsilon, \omega_\varepsilon > 0$ are independent on initial conditions. Next, it is a simple exercise that A is self-adjoint in H, with compact resolvent. By theory of functional analysis, there is a sequence of eigenfunctions of A, which forms an orthonormal basis for H. Solve the eigenvalue problem $A\phi = \lambda\phi = \rho^2\phi$, i.e.,

$$\begin{cases} \phi''(x) = \rho^2\phi(x), \\ \phi'(0) = 0, \phi'(1) = -\alpha_2\phi(1), \end{cases} \tag{8}$$

from which we see obviously that each eigenvalue must be geometrically simple. Since A is self-adjoint, all λ must be real and it is easy to check from (8) that $\lambda < 0$. Solve (8) to obtain

$$\phi(x) = e^{\rho x} + e^{-\rho x}, e^{2\rho} = 1 - \tfrac{2\alpha_2}{\rho+\alpha_2} = 1 - \tfrac{2\alpha_2}{\rho} + \mathcal{O}(\rho^{-2}).$$

From the second identity, we first solve $e^{2\rho} = 1 - \frac{2\alpha_2}{\rho}$ by Rouché's theorem in complex analysis to obtain $\rho_n = n\pi i + \mathcal{O}(n^{-1})$. Substituting it into $e^{2\rho} = 1 - \frac{2\alpha_2}{\rho} + \mathcal{O}(\rho^{-2})$, we then have $\mathcal{O}(n^{-1}) = \frac{2\alpha_2 i}{n\pi} + \mathcal{O}(n^{-2})$. Since $\lambda = \rho^2$, we finally obtain

$$\begin{cases} \lambda = \lambda_n = \left[n\pi i + \frac{2\alpha_2 i}{n\pi} + \mathcal{O}(n^{-2})\right]^2 \\ \qquad = -4\alpha_2 - (n\pi)^2 + \mathcal{O}(n^{-1}), \\ \phi(x) = \phi_n(x) = 2\cos\left(n\pi + \frac{2\alpha_2}{n\pi}\right)x + \mathcal{O}(n^{-2}). \end{cases} \tag{9}$$

The solution of (5) can be written in $H = L^2(0,1)$ as

$$\varepsilon(x,t) = \sum_{n=0}^{\infty} a_n e^{\lambda_n t}\phi_n(x),\ \ \|\varepsilon(\cdot,0)\|^2 = \sum_{n=0}^{\infty}|a_n|^2, \tag{10}$$

where $\lambda_n < 0$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ forms an orthonormal basis for H. Therefore, for any $t \ge 1$, it follows from (10) that

$$\begin{aligned} |\varepsilon(0,t)|^2 &= \left|\sum_{n=0}^{\infty} a_n e^{\lambda_n t}\phi_n(0)\right|^2 \\ &\le \sum_{n=0}^{\infty}|a_n\phi_n(0)|^2 \sum_{n=0}^{\infty} e^{2\lambda_n t} \\ &\le C_1 \sum_{n=0}^{\infty}|a_n|^2 \sum_{n=0}^{\infty} e^{2\lambda_n t}, \end{aligned} \tag{11}$$

where $C_1 > 0$ is a constant independent of the initial value. We may suppose without loss of generality that $\{\lambda_n\}$ is decreasing with respect to n. From the asymptotic expression of λ_n in (9), there exists an integer $N > 0$ such that $2(\lambda_n - \lambda_0) \le -n\pi$ as $n > N$, which results in

$$\begin{aligned} &\lim_{t\to\infty}\sum_{n=0}^{\infty} e^{2(\lambda_n-\lambda_0)t} \\ =\ &\lim_{t\to\infty}\sum_{n=0}^{N} e^{2(\lambda_n-\lambda_0)t} + \lim_{t\to\infty}\sum_{n=N+1}^{\infty} e^{2(\lambda_n-\lambda_0)t} \\ \le\ &\lim_{t\to\infty}\sum_{n=N+1}^{\infty} e^{-n\pi t} = \lim_{t\to\infty}\frac{e^{-(N+1)\pi t}}{1-e^{-\pi t}} = 0. \end{aligned}$$

This, together with (11), shows that

$$\lim_{t\to\infty}|\varepsilon(0,t)|^2 \le \lim_{t\to\infty} C_2 e^{2\lambda_0 t}\|\varepsilon(\cdot,0)\|^2 = 0, \tag{12}$$

where $C_2 > 0$ is a constant independent of the initial value. ■

We proceed as follows. In Section 2, we first design a feedforward control for the system (1) with specially selected disturbances and system uncertainty, and then an extended state observer in terms of tracking error for the coupled frozen system and exosystem is designed. As a result, we obtain an observer-based error feedback control for frozen system and 1-copy property is briefly discussed. In Section 3, we show the robust output tracking for system (1) with the observer-based error feedback control designed for the frozen system. Numerical simulations are presented in Section 4 for illustration, followed by concluding remarks in Section 5.

2. Observer-based error feedback control design

In this section, we design an observer-based error feedback control for the system (1) with specially selected disturbances and system uncertainty. To this purpose, we need three steps. Firstly, we need to select the special frozen coefficients so that the coupled system (w,p) is detectable with respect to the tracking error and design a feedforward control for the frozen system. Secondly, we design an extended state observer for the coupled frozen system and exosystem. By these three steps we produce an observer-based error feedback control by simply substitution of the states in feedforward control with its estimations obtained from the extended state observer.

First, the frozen coefficients are chosen as

$$F(x) \equiv 0,\ \ \varDelta(x) \equiv 0,\ \ D = 0, M = M_0,\ \ N = N_0, \tag{13}$$

where M_0 and N_0 are chosen to guarantee the existence of the observer for coupled system (1) and (2) with those frozen coefficients selected from (13). It is noted that the observer design for this case has its independent significance itself because the output and control are non-collocated.

With frozen coefficients (13), systems (1) and (2) can be written as

$$\begin{cases} \dot{p}(t) = Gp(t), \\ w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = N_0p(t),\ \ w_x(1,t) = u(t), \\ y_e(t) = w(0,t) - M_0p(t). \end{cases} \tag{14}$$

We design an observer-based error feedback control for frozen system (14) by three steps.

Step 1: Feedforward control for frozen system. We design a feedforward control for subsystem $w(x,t)$ in (14). Since the disturbance $N_0p(t)$ is on the left end and control is on the right end, we introduce standardly the variable

$$\varepsilon(x,t) = w(x,t) - f_0(x)p(t) \text{ with } f_0(x) \in \mathbb{C}^{1\times n}, \tag{15}$$

in which $f_0(x)p(t)$ is the particular steady state to achieve output regulation. A simple calculation gives

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = 0,\ \ \varepsilon_x(1,t) = u(t) - f_0'(1)p(t), \\ y_e(t) = \varepsilon(0,t), \end{cases} \tag{16}$$

provided that

$$\begin{cases} f_0''(x) = f_0(x)G, \\ f_0'(0) = N_0, \ f_0(0) = M_0, \end{cases} \tag{17}$$

which has the obvious solution:

$$(f_0(x), f_0'(x)) = (M_0, N_0)e^{\begin{pmatrix} 0 & G \\ I & 0 \end{pmatrix}x}. \tag{18}$$

It is seen from (16) that the disturbance has been moved to the control side. This is the role played by the transformation (15). Compared with (5), we naturally have a feedforward control of the following

$$u(t) \quad = -\alpha_2\varepsilon(1,t) + f_0'(1)p(t), \alpha_2 > 0. \tag{19}$$

The closed-loop ε-subsystem of (16) therefore becomes

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = 0, \\ \varepsilon_x(1,t) = -\alpha_2\varepsilon(1,t), \end{cases}$$

which by Lemma 1.1 is exponentially stable and $|\varepsilon(0,t)|$ converges to zero uniformly exponentially.

Step 2: Observer design for frozen system. Now we are in a position to design an observer for system (14). To this purpose, we introduce a transformation

$$z(x,t) = w(x,t) + g_0(x)p(t), \tag{20}$$

where $g_0(x) \in \mathbb{C}^{1\times n}$. By (14), the extended system of $(z(x,t), p(t))$ is governed by

$$\begin{cases} z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = \alpha_1 z(0,t) - \alpha_1 y_e(t), \\ z_x(1,t) = u(t), \\ \dot{p}(t) = Gp(t), \\ y_e(t) = z(0,t) - (f_0(0) + g_0(0))p(t), \end{cases} \tag{21}$$

provided that $g_0(x)$ is chosen to satisfy

$$\begin{cases} g_0''(x) = g_0(x)G, \\ g_0'(0) = \alpha_1[g_0(0) + M_0] - N_0, \quad \alpha_1 > 0, \quad \alpha_1 \in \mathbb{R}, \\ g_0'(1) = 0. \end{cases} \tag{22}$$

It is seen from (21) that a damping for z-part is introduced at boundary $x = 0$, and meanwhile the term $f_0(0) + g_0(0)$ in $y_e(t)$ turns out to play an important role in stabilizing the p-part. The existence of the solution to (22) is proved by the following lemma.

Lemma 2.1. *The boundary value problem (22) admits a unique solution.*

Proof. Let w_j be an eigenvector of G corresponding to the eigenvalue $i\omega_j$, $\omega_j \in \mathbb{R}$, which will be used throughout the paper. Right multiply by w_j in (22) to obtain

$$\begin{cases} g_{0j}''(x) = i\omega_j g_{0j}(x), \\ g_{0j}'(0) = \alpha_1 g_{0j}(0) - N_{0j} + \alpha_1 M_{0j}, \ g_{0j}'(1) = 0, \end{cases} \tag{23}$$

where $j = 1, 2, \ldots, n$, $g_{0j}(x) = g_0(x)w_j$, $M_{0j} = M_0 w_j$ and $N_{0j} = N_0 w_j$. There are two cases:

Case 1: $\omega_j = 0$. In this case, the solution of (23) is found to be

$$g_{0j}(x) = \frac{N_{0j} - \alpha_1 M_{0j}}{\alpha_1}. \tag{24}$$

Case 2: $\omega_j \neq 0$. We may suppose without loss of generality that $\omega_j > 0$. The general solution of (23) is

$$g_{0j}(x) = c\left(e^{\sqrt{i\omega_j}x} + e^{2\sqrt{i\omega_j}}e^{-\sqrt{i\omega_j}x}\right), \tag{25}$$

where c is found to be

$$c = \frac{\alpha_1 M_{0j} - N_{0j}}{\sqrt{i\omega_j}\left(1 - e^{2\sqrt{i\omega_j}}\right) - \alpha_1\left(1 + e^{2\sqrt{i\omega_j}}\right)}, \tag{26}$$

if the denominator above is non-zero which is shown to be always true. Indeed, if $\sqrt{i\omega_j}\left(1 - e^{2\sqrt{i\omega_j}}\right) - \alpha_1\left(1 + e^{2\sqrt{i\omega_j}}\right) = 0$, then

$$\alpha_1 \quad = k_j \frac{1 - e^{2k_j} + 2e^{k_j}\sin k_j + i\left(1 - e^{2k_j} - 2e^{k_j}\sin k_j\right)}{2(1 + e^{k_j}\cos k_j)^2 + (e^{k_j}\sin k_j)^2},$$

where $k_j = \sqrt{2\omega_j}$. Since $\alpha_1 \in \mathbb{R}$, we must have $1 - e^{2k_j} - 2e^{k_j}\sin k_j = 0$, which however turns out to be wrong. Let $y(x) = e^{2x} + 2e^x \sin x - 1$, $x > 0$. Then, for any $x > 0$ $y'(x) \quad = 4e^x\left[1 + x + \sum_{m=1}^{\infty}\left(\frac{x^{4m}}{(4m)!} + \frac{x^{(4m+1)}}{(4m+1)!}\right)\right] > 0$, which together with $y(0) = 0$, shows that $y(x) > 0$ for all $x > 0$. Negative ω_j can be treated similarly. Therefore, the solution of (22) always exists for any $\alpha_1 > 0$ and is found to be

$$g_0(x) = (g_{01}(x), g_{02}(x), \ldots, g_{0n}(x))[w_1, w_2, \ldots, w_n]^{-1}. \quad \blacksquare \tag{27}$$

Let $h_0(x) = f_0(x) + g_0(x)$ where $f_0(x)$ and $g_0(x)$ satisfy (17) and (22) respectively. Then, $h_0(x)$ satisfies

$$\begin{cases} h_0''(x) = h_0(x)G, \\ h_0'(0) = \alpha_1 h_0(0), \ h_0'(1) = f_0'(1), \end{cases} \tag{28}$$

which admits a unique solution by (17) and Lemma 2.1.

We next design an observer for (21) as follows:

$$\begin{cases} \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1\hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = u(t), \\ \dot{\hat{p}}(t) = G\hat{p}(t) + Q[y_e(t) - \hat{z}(0,t) + h_0(0)\hat{p}(t)], \end{cases} \tag{29}$$

where $Q \in \mathbb{C}^{n\times 1}$ will be determined later. Once again, we see the role played by the term $h_0(0)$ in $y_e(t)$. By (20), we have produced an observer for system (14) with $\hat{w}(x,t) = \hat{z}(x,t) - g_0(x)\hat{p}(t)$ as an approximation of $w(x,t)$.

Set the observer errors to be $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$, $\tilde{p}(t) = p(t) - \hat{p}(t)$ and then

$$\begin{cases} \tilde{z}_t(x,t) = \tilde{z}_{xx}(x,t), \\ \tilde{z}_x(0,t) = \alpha_1\tilde{z}(0,t), \ \tilde{z}_x(1,t) = 0, \\ \dot{\tilde{p}}(t) = [G + Qh_0(0)]\tilde{p}(t) - Q\tilde{z}(0,t). \end{cases} \tag{30}$$

A sufficient condition to ensure $G + Qh_0(0)$ to be Hurwitz is that $\Sigma(G, h_0(0))$ is detectable for which we have the following Lemma 2.2.

Lemma 2.2. *The pair $(G, h_0(0))$ is detectable if and only if $H_{0j}(0) \neq 0$ for all $j = 1, 2, \ldots, n$, which is equivalent to*

$$\begin{cases} N_{0j} \neq 0, \ \text{if } \omega_j = 0, \\ M_{0j}\sqrt{i\omega_j}(1 - e^{2\sqrt{i\omega_j}}) - N_{0j}(1 + e^{2\sqrt{i\omega_j}}) \neq 0, \ \text{if } \omega_j \neq 0, \end{cases} \tag{31}$$

where $M_{0j} = M_0 w_j$, $N_{0j} = N_0 w_j$ and w_j is the eigenvector corresponding to $i\omega_j$.

Proof. It is known that $(G, h_0(0))$ is detectable if and only if $(G_0, H_0(0))$ is detectable, where $G_0 = J^{-1}GJ = \text{diag}\{i\omega_j\}$, $H_0(x) = h_0(x)J$ and J is the eigenvector matrix of G. For any $i\omega_j \in \sigma(G)$,

$j = 1, 2, \ldots, n$, it is seen that

$$\operatorname{rank}\begin{pmatrix} G_0 - i\omega_j \\ H_0(0) \end{pmatrix} = n$$
$$\Updownarrow$$
$$\operatorname{rank}\begin{pmatrix} i\omega_1 - i\omega_j & 0 & 0 & \cdots & 0 \\ 0 & i\omega_2 - i\omega_j & 0 & \cdots & 0 \\ & & & & \\ 0 & 0 & \ddots & \cdots & 0 \\ 0 & \cdots & 0 & i\omega_{n-1} - i\omega_j & 0 \\ 0 & \cdots & 0 & 0 & i\omega_n - i\omega_j \\ H_{01}(0) & H_{02}(0) & \cdots & \cdots & H_{0n}(0) \end{pmatrix} = n,$$

which is equivalent to $H_{0j}(0) \neq 0$ for $j = 1, 2, \ldots, n$.

Case 1: $\omega_j = 0$. By (24), it has

$$H_{0j}(0) = M_{0j} + g_{0j}(0) = \frac{N_{0j}}{\alpha_1} \neq 0. \tag{32}$$

Case 2: $\omega_j \neq 0$. By (25) and (26), it has

$$\begin{aligned} H_{0j}(0) &= \frac{M_{0j}\sqrt{i\omega_j}\left(1 - e^{2\sqrt{i\omega_j}}\right)}{\sqrt{i\omega_j}\left(1 - e^{2\sqrt{i\omega_j}}\right) - \alpha_1\left(1 + e^{2\sqrt{i\omega_j}}\right)} \\ &\quad - \frac{\left(1 + e^{2\sqrt{i\omega_j}}\right) N_{0j}}{\sqrt{i\omega_j}\left(1 - e^{2\sqrt{i\omega_j}}\right) - \alpha_1\left(1 + e^{2\sqrt{i\omega_j}}\right)} \neq 0, \end{aligned} \tag{33}$$

where the denominator above is non-zero by the proof of Lemma 2.1. ■

Remark 2.1. With the notation of Lemma 2.2, since

$$J^{-1}[G + Qh_0(0)]J = G_0 + J^{-1}QH_0(0),\ G_0^* = -G_0,$$

we can take $Q = -JH_0^*(0) = -JJ^*h_0^*(0)$, where $*$ denotes the conjugate transpose of the matrix.

Lemma 2.3. *Suppose that M_0 and N_0 satisfy Lemma 2.2, which makes $G + Qh_0(0)$ Hurwitz. Then, the observer (29) is convergent exponentially to system (21).*

Proof. It is seen that the introduction of damping at $x = 0$ makes $\tilde{z}$-subsystem in (30) exponentially stable, same to Lemma 1.1:

$$\|\tilde{z}(\cdot, t)\| \leq \kappa_{\tilde{z}} e^{-\mu_{\tilde{z}} t} \|\tilde{z}(\cdot, 0)\|, \tag{34}$$

where $\kappa_{\tilde{z}}, \mu_{\tilde{z}} > 0$. Similarly to (10), we can find the solution of $\tilde{z}$-subsystem of (30) as

$$\begin{cases} \tilde{z}(x, t) = \displaystyle\sum_{n=0}^{\infty} b_n e^{\lambda_n t} \phi_n(x), \\ \|\tilde{z}(\cdot, 0)\|^2 = \displaystyle\sum_{n=0}^{\infty} |b_n|^2 < \infty, \end{cases}$$

where (λ_n, ϕ_n) may be different to those in (10) since generally, $\alpha_1 \neq \alpha_2$, but they have the same asymptotic expressions of (9) and so we use, by abuse of the notation, the same notation of (λ_n, ϕ_n). Same as (12), there exists a $C_3 > 0$ independent of initial value such that

$$|\tilde{z}(0, t)|^2 \leq C_3 \|\tilde{z}(\cdot, 0)\|^2 e^{2\lambda_0 t},\ \forall\ t \geq 1, \tag{35}$$

which implies particularly that $\tilde{z}(0, t)$ is convergent to zero exponentially as $t \to \infty$.

For $\tilde{p}$-subsystem in (30), by (35) and whenever $G + Qh_0(0)$ is Hurwitz

$$\begin{aligned} \lim_{t\to\infty} \|\tilde{p}(t)\|^2 &= \lim_{t\to\infty} \|e^{[G+Qh_0(0)]t}\tilde{p}(0)\|^2 \\ &\quad + \lim_{t\to\infty} C_4 \|\tilde{z}(\cdot, 0)\|^2 e^{2\lambda_0 t} = 0, \end{aligned} \tag{36}$$

where C_4 is independent on initial conditions. ■

Step 3: Observer-based feedback for frozen system. By Lemma 2.2, we can further simplify M_0 and N_0 as

$$N_0 = \alpha_1 M_0,\ M_0 w_j \neq 0, \tag{37}$$

where w_j are eigenvectors of S appeared in Lemma 2.2. It is seen that such (M_0, N_0) always exists because $M_0 = [1, \ldots, 1][w_1, \ldots, w_n]^{-1}$ solves $M_0[w_1, \ldots, w_n] = [1, 1, \ldots, 1]$. The choice of (37) is based on the condition (31).

This makes (17) and (28) be updated as

$$\begin{cases} f_0''(x) = f_0(x)G, \\ f_0'(0) = N_0 = \alpha_1 M_0, \\ f(0) = M_0, \end{cases} \quad \begin{cases} h_0''(x) = h_0(x)G, \\ h_0'(0) = \alpha_1 h_0(0), \\ h_0'(1) = f_0'(1), \end{cases} \tag{38}$$

which further advise $f_0(x) \equiv h_0(x)$ and hence $g_0(x) \equiv 0$. In what follows, all $h_0(x)$ are therefore replaced by $f_0(x)$.

By the choice of (37), the condition (31) is always satisfied, that is, $\Sigma(G, h_0(0)) = \Sigma(G, M_0)$ is detectable and hence from Lemma 2.3 that the observer (29) with (M_0, N_0) in (37) is also convergent exponentially to system (21) by selecting Q so that $G + QM_0$ is Hurwitz.

Since $w(x, t) = z(x, t)$ by the choice of (M_0, N_0) in (37), by (15), (29) and (19), a tracking error feedback observer-based control is designed as

$$\begin{cases} u(t) = -\alpha_2 \hat{z}(1, t) + [\alpha_2 f_0(1) + f_0'(1)]\hat{p}(t), \\ \hat{z}_t(x, t) = \hat{z}_{xx}(x, t), \\ \hat{z}_x(0, t) = \alpha_1 \hat{z}(0, t) - \alpha_1 y_e(t), \\ \hat{z}_x(1, t) = u(t), \\ \dot{\hat{p}}(t) = [G + QM_0]\hat{p}(t) + Qy_e(t) - Q\hat{z}(0, t). \end{cases} \tag{39}$$

Make the transformation

$$\zeta(x, t) = \hat{z}(x, t) - f_0(x)\hat{p}(t). \tag{40}$$

Then, the controller in (39) becomes

$$\begin{cases} u(t) = -\alpha_2 \zeta(1, t) + f_0'(1)\hat{p}(t), \\ \zeta_t(x, t) = \zeta_{xx}(x, t) + f_0(x)Q\zeta(0, t) - f_0(x)Qy_e(t), \\ \zeta_x(0, t) = \alpha_1 \zeta(0, t) - \alpha_1 y_e(t), \\ \zeta_x(1, t) = -\alpha_2 \zeta(1, t), \\ \dot{\hat{p}}(t) = G\hat{p}(t) - Q\zeta(0, t) + Qy_e(t). \end{cases} \tag{41}$$

Clearly, the controller (41) contains 1-copy of the exosystem. This is just the internal model that we need for robustness.

Remark 2.2. The controller (39) can be written into the standard form of dynamic error feedback control in internal model principle:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \hat{z}(\cdot, t) \\ \hat{p}(t) \end{pmatrix} = \mathcal{G}_1 \begin{pmatrix} \hat{z}(\cdot, t) \\ \hat{p}(t) \end{pmatrix} + \mathcal{G}_2 y_e(t), \\ u(t) = K \begin{pmatrix} \hat{z}(\cdot, t) \\ \hat{p}(t) \end{pmatrix}, \end{cases} \tag{42}$$

where $\mathcal{G}_2$ is unbounded as in Paunonen (2017).

To end this section, we explain more about the control design (39). First, for frozen system (14), design feedforward control (19); Second, for frozen system (14), design an observer (29); Replace the state in feedback of (19) with the state of the observer (29) to obtain the error feedback control (39). The control (39) contains naturally 1-copy of the exosystem, which is confirmed by (41). This is in line with the internal model principle for finite-dimensional systems in Deutscher (2017) and Huang (2004, Theorem 1.14, p.14), and Paunonen (2020) for infinite-dimensional systems. However, it is hard to know if our observer is Lunberger type presented in Paunonen (2020). In addition, the results of Paunonen (2020) are based on previously established results of Paunonen (2017) where there are a series of abstract assumptions which are not easy to check. Therefore, we need to give a robustness analysis in next section. Another point is that before (37), we design an observer for system (14) with general (M_0, N_0) to show the design principle of the observer, which is of independent significance itself.

3. Robustness

In this section, we always choose the frozen coefficients (M_0, N_0) as that in (37). The closed-loop system with the dynamic control (39) then becomes

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + F(x)p(t) + \Delta(x)w(x,t), \\ w_x(0,t) = Np(t), \\ w_x(1,t) = -\alpha_2\hat{z}(1,t) + [\alpha_2 f_0(1) + f_0'(1)]\hat{p}(t) + Dp(t), \\ \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1\hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = -\alpha_2\hat{z}(1,t) + [\alpha_2 f_0(1) + f_0'(1)]\hat{p}(t), \\ \dot{\hat{p}}(t) = [G + QM_0]\hat{p}(t) + Qy_e(t) - Q\hat{z}(0,t), \\ y_e(t) = w(0,t) - Mp(t). \end{cases} \tag{43}$$

For $w(x,t)$ in (43), introduce the following transformation

$$z^c(x,t) = w(x,t) - \hat{f}(x)p(t)$$

with $\hat{f}(x) \in \mathbb{C}^{1\times n}$ satisfying

$$\begin{cases} \hat{f}''(x) = \hat{f}(x)[G - \Delta(x)] - F(x), \\ \hat{f}'(0) = N,\ \hat{f}(0) = M, \end{cases} \tag{44}$$

whose solution, same as (18), is uniquely expressed by

$$\begin{aligned} (\hat{f}(x), \hat{f}'(x)) = &\ (M, N)e^{A(x)} \\ &+ \int_0^x [0_{1\times n}, -F(s)]e^{A(x)-A(s)}ds, \\ A(x) = &\int_0^x \begin{pmatrix} 0 & G - \Delta(s) \\ I & 0 \end{pmatrix} ds. \end{aligned} \tag{45}$$

Since the disturbances appear in all channels of the $w(\cdot,t)$ system in (43), we first make the following transformation to cluster all external disturbances in all channels into one channel:

$$z^1(x,t) = w(x,t) + \hat{g}(x)p(t), \tag{46}$$

where $\hat{g}(x) \in \mathbb{C}^{1\times n}$ is defined by

$$\begin{cases} \hat{g}''(x) = \hat{g}(x)G + \Delta(x)\hat{f}(x) + F(x), \\ \hat{g}'(0) = \alpha_1[\hat{g}(0) + M] - N,\ \hat{g}'(1) = -D, \end{cases} \tag{47}$$

which admits a solution by virtue of Lemma 2.1. Then, $\hat{h}(x) = \hat{f}(x) + \hat{g}(x)$ satisfies

$$\begin{cases} \hat{h}''(x) = \hat{h}(x)G, \\ \hat{h}'(0) = \alpha_1\hat{h}(0),\ \hat{h}'(1) = \hat{f}'(1) - D, \end{cases} \tag{48}$$

which similarly to (17), admits a solution

$$(\hat{h}(x), \hat{h}'(x)) = (\hat{h}(0), \alpha_1\hat{h}(0))e^{\begin{pmatrix} 0 & G \\ I & 0 \end{pmatrix}x} \tag{49}$$

with $\hat{h}(0)$ being determined by $\hat{h}'(1) = \hat{f}'(1) - D$.

In this way, $(z^1(x,t), \hat{z}(x,t), \hat{p}(t))$ is governed by

$$\begin{cases} z_t^1(x,t) = z_{xx}^1(x,t) + \Delta(x)[w(x,t) - \hat{f}(x)p(t)], \\ z_x^1(0,t) = \alpha_1\hat{h}(0)p(t) = \alpha_1 z^1(0,t) - \alpha_1 y_e(t), \\ z_x^1(1,t) = -\alpha_2\hat{z}(1,t) + [\alpha_2 f_0(1) + f_0'(1)]\hat{p}(t), \\ \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \alpha_1\hat{z}(0,t) - \alpha_1 y_e(t), \\ \hat{z}_x(1,t) = -\alpha_2\hat{z}(1,t) + [\alpha_2 f_0(1) + f_0'(1)]\hat{p}(t), \\ \dot{\hat{p}}(t) = [G + QM_0]\hat{p}(t) + Qy_e(t) - Q\hat{z}(0,t), \\ y_e(t) = z^1(0,t) - \hat{h}(0)p(t). \end{cases} \tag{50}$$

It is seen that only one channel and output contain explicitly the external disturbance except the one caused by the system uncertainty $\Delta(x)$, which is the advantage of the introduction of (46).

Now, we make a transformation:

$$\begin{aligned} \begin{pmatrix} z^c(x,t) \\ \hat{z}^c(x,t) \\ \hat{p}^c(t) \end{pmatrix} &= \begin{pmatrix} z^1(x,t) \\ \hat{z}(x,t) \\ \hat{p}(t) \end{pmatrix} - \begin{pmatrix} \hat{h}(x) \\ \hat{h}(x) \\ h_{\hat{p}} \end{pmatrix} p(t) \\ &= \begin{pmatrix} w(x,t) \\ \hat{z}(x,t) \\ \hat{p}(t) \end{pmatrix} - \begin{pmatrix} \hat{f}(x) \\ \hat{h}(x) \\ h_{\hat{p}} \end{pmatrix} p(t), \end{aligned} \tag{51}$$

where $h_{\hat{p}} \in \mathbb{C}^{n\times n}$ satisfies

$$\begin{cases} \hat{h}'(1) = -\alpha_2\hat{h}(1) + [\alpha_2 f_0(1) + f_0'(1)]h_{\hat{p}}, \\ h_{\hat{p}}G - [G + QM_0]h_{\hat{p}} = -Q\hat{h}(0), \end{cases} \tag{52}$$

and $(z^c(x,t), \hat{z}^c(x,t), \hat{p}^c(t))$ is governed by

$$\begin{cases} z_t^c(x,t) = z_{xx}^c(x,t) + \Delta(x)z^c(x,t), \\ z_x^c(0,t) = 0, \\ z_x^c(1,t) = -\alpha_2\hat{z}^c(1,t) + [f_0'(1) + \alpha_2 f_0(1)]\hat{p}^c(t), \\ \hat{z}_t^c(x,t) = \hat{z}_{xx}^c(x,t), \\ \hat{z}_x^c(0,t) = \alpha_1\hat{z}^c(0,t) - \alpha_1 z^c(0,t), \\ \hat{z}_x^c(1,t) = -\alpha_2\hat{z}^c(1,t) + [f_0'(1) + \alpha_2 f_0(1)]\hat{p}^c(t), \\ \dot{\hat{p}}^c(t) = [G + QM_0]\hat{p}^c(t) + Q[z^c(0,t) - \hat{z}^c(0,t)], \\ y_e(t) = z^c(0,t). \end{cases} \tag{53}$$

It is observed that in (51), we choose the same $\hat{h}(x)$ for both $z^1(x,t)$ and $\hat{z}(x,t)$ because they have the similar structure. This is just an observation not the general principle. The system (53) is just the closed-loop system (43) with $p(t) \equiv 0$, which is the purpose of twice transformations first by (46) and second by (51).

Lemma 3.1. *For every set of disturbance coefficients and systematic uncertainty, (52) always admits a solution.*

Proof. Suppose that J is the matrix such that

$$\begin{cases} J^{-1}GJ = \mathrm{diag}\{i\omega_j\} = G_0, \\ X = J^{-1}h_{\hat{p}}J, \quad Q_0 = J^{-1}Q, \\ \hat{H}(x) = \hat{h}(x)J = (\hat{H}_1(x), \hat{H}_2(x), \dots, \hat{H}_n(x)), \\ F_0(x) = f_0(x)J = (F_{01}(x), F_{02}(x), \dots, F_{0n}(x)). \end{cases} \tag{54}$$

Then, (52) is equivalent to

$$\begin{cases} \hat{H}'(1) + \alpha_2\hat{H}(1) = [\alpha_2 F_0(1) + F_0'(1)]X, \\ XG_0 - [G_0 + Q_0F_0(0)]X = -Q_0\hat{H}(0). \end{cases} \tag{55}$$

Suppose that the solution of (55) is of the form $X = \mathrm{diag}\{x_j\},\ j = 1, 2, \dots, n$. Then,

$$\begin{cases} \hat{H}_j'(1) + \alpha_2\hat{H}_j(1) = [\alpha_2 F_{0j}(1) + F_{0j}'(1)]x_j, \\ F_{0j}(0)x_j = \hat{H}_j(0), \end{cases} \tag{56}$$

whose solution, by (49) and (18), is then expressed by

$$X = \mathrm{diag}\{x_j\},\ x_j = \frac{\hat{H}_j(0)}{F_{0j}(0)}, j = 1, 2, \dots, n \tag{57}$$

and hence

$$h_{\hat{p}} = JXJ^{-1}.$$

This completes the proof of the lemma. ■

We consider (53) in the Hilbert space $\mathbb{H} \doteq (L^2(0,1))^2 \times \mathbb{C}^n$ with the following inner product

$$\begin{aligned} &\langle(\phi_1, \phi_2, \phi_3), (\varphi_1, \varphi_2, \varphi_3)\rangle \\ &= \int_0^1 \phi_1(x)\overline{\varphi_1(x)}dx + \nu\int_0^1 \phi_2(x)\overline{\varphi_2(x)}dx + \phi_3^\top\overline{\varphi_3}, \end{aligned}$$

for any $\nu > 0$, and $(\phi_1, \phi_2, \phi_3), (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{H}$.

The system (53) is then written as

$$\frac{d}{dt}\begin{pmatrix} z^c(\cdot,t) \\ \hat{z}^c(\cdot,t) \\ \hat{p}^c(t) \end{pmatrix} = (A_e + \boldsymbol{\Delta})\begin{pmatrix} z^c(\cdot,t) \\ \hat{z}^c(\cdot,t) \\ \hat{p}^c(t) \end{pmatrix}, \tag{58}$$

where A_e is defined by

$$\begin{cases} A_e(\phi_1(x),\phi_2(x),\phi_3) = (\phi_1''(x),\phi_2''(x),[G+QM_0]\phi_3 \\ +Q[\phi_1(0)-\phi_2(0)]),\ \forall(\phi_1,\phi_2,\phi_3)\in\mathcal{D}(A_e); \\ \mathcal{D}(A_e) = \{\forall(\phi_1,\phi_2,\phi_3)\in H^2(0,1)\times H^2(0,1)\times\mathbb{C}^n| \\ \phi_1'(0)=0,\ \phi_1'(1)=\phi_2'(1),\ \phi_2'(0)=\alpha_1\phi_2(0)-\alpha_1\phi_1(0), \\ \phi_2'(1) = -\alpha_2\phi_2(1)+[f_0'(1)+\alpha_2 f_0(1)]\phi_3\} \end{cases} \tag{59}$$

and $\boldsymbol{\Delta}$ is a bounded operator defined by

$$\boldsymbol{\Delta}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \Delta(\cdot)\phi_1 \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

and $\|\boldsymbol{\Delta}\| \le \|\Delta(\cdot)\|_{L^\infty(0,1)}$.

Lemma 3.2. *For any $\alpha_1 > 0$ and $\alpha_2 > 0$, the operator A_e generates a C_0-semigroup on $\mathbb{H} = (L^2(0,1))^2 \times \mathbb{C}^n$.*

Proof. Introduce the invertible transformation

$$\Gamma\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_1-\phi_2 \\ \phi_3 \end{pmatrix}.$$

Define $\mathbb{A} = \Gamma A_e \Gamma^{-1}$ which is found to be

$$\begin{cases} \mathbb{A}(\phi_1(x),\phi_2(x),\phi_3) = (\phi_1''(x),\phi_2''(x),[G+QM_0]\phi_3 \\ +Q\phi_2(0)),\ \forall(\phi_1,\phi_2,\phi_3)\in\mathcal{D}(\mathbb{A}); \\ \mathcal{D}(\mathbb{A}) = \{\forall(\phi_1,\phi_2,\phi_3)\in H^2(0,1)\times H^2(0,1)\times\mathbb{C}^n| \\ \phi_1'(0)=0,\ \phi_2'(0)=\alpha_1\phi_2(0),\ \phi_2'(1)=0, \\ \phi_1'(1) = -\alpha_2\phi_1(1)+\alpha_2\phi_2(1)+[f_0'(1)+\alpha_2 f_0(1)]\phi_3\}. \end{cases} \tag{61}$$

For any $(\phi_1,\phi_2,\phi_3)\in\mathcal{D}(\mathbb{A})$, solving $\mathbb{A}(\phi_1,\phi_2,\phi_3) = (\varphi_1,\varphi_2,\varphi_3)$ leads to

$$\begin{cases} \phi_1(x) = (x-1)\displaystyle\int_0^x \varphi_1(s)ds - \int_1^x (s-1)\varphi_1(s)ds \\ -\dfrac{1}{\alpha_2}\displaystyle\int_0^1 \varphi_1(s)ds - \int_0^1 s\varphi_2(s)ds - \dfrac{1}{\alpha_1}\int_0^1 \varphi_2(s)ds \\ +\dfrac{1}{\alpha_2}[f_0'(1)+\alpha_2 f_0(1)][G+QM_0]^{-1} \\ \times\left[\varphi_3 + \dfrac{1}{\alpha_1}Q\displaystyle\int_0^1 \varphi_2(s)ds\right], \\ \phi_2(x) = x\displaystyle\int_1^x \varphi_2(s)ds - \int_0^x s\varphi_2(s)ds - \dfrac{1}{\alpha_1}\int_0^1 \varphi_2(s)ds, \\ \phi_3 = [G+QM_0]^{-1}\left[\varphi_3 + \dfrac{1}{\alpha_1}Q\displaystyle\int_0^1 \varphi_2(s)ds\right], \end{cases}$$

which advises that $\mathbb{A}^{-1}$ is compact on $\mathbb{H}$. For any $(\phi_1,\phi_2,\phi_3)\in\mathcal{D}(\mathbb{A})$, we have

$$\begin{aligned} &\mathrm{Re}\langle\mathbb{A}(\phi_1,\phi_2,\phi_3),(\phi_1,\phi_2,\phi_3)\rangle \\ &= \mathrm{Re}(-\alpha_2|\phi_1(1)|^2 + \alpha_2\phi_2(1)\overline{\phi_1(1)} + [f_0'(1)+\alpha_2 f_0(1)] \\ &\times\phi_3\overline{\phi_1(1)} - \|\phi_1'\|_{L^2(0,1)} - \nu\alpha_1|\phi_2(0)|^2 - \nu\|\phi_2'\|_{L^2(0,1)} \\ &+\phi_3^\top[G+QM_0]^\top\phi_3 + \phi_2(0)Q^\top\phi_3) \\ &\le \alpha_2(1-\delta_1-\|f_0'(1)+\alpha_2 f_0(1)\|\delta_2)|\phi_1(1)|^2 \\ &-(\delta_3-\frac{\alpha_2}{\delta_1})|\phi_2(1)|^2 - \|\phi_1'\|_{L^2(0,1)} - (\nu\alpha_1 - 2\delta_3 - \|Q\|\delta_4) \\ &\times|\phi_2(0)|^2 - (\nu-2\delta_3)\|\phi_2'\|_{L^2(0,1)} \\ &+\left(\|G+QM_0\| + \frac{\alpha_2\|f_0'(1)+\alpha_2 f_0(1)\|}{\delta_2} + \frac{\|Q\|}{\delta_4}\right)\|\phi_3\|^2, \end{aligned}$$

where $\delta_k > 0$, $k = 1,2,3,4$. Choose δ_1 and δ_2 small enough to make $1-\delta_1-\|f_0'(1)+\alpha_2 f_0(1)\|\delta_2 > 0$ and choose δ_3 and ν largely enough to make $\delta_3 - \frac{\alpha_2}{\delta_1} > 0$, $\nu\alpha_1 - 2\delta_3 - \|Q\|\delta_4 > 0$ and $\nu - 2\delta_3 > 0$. Then, for any constant $M_{\mathbb{A}} > 0$, there exists

$$\mathrm{Re}\langle(\mathbb{A}-M_{\mathbb{A}})(\phi_1,\phi_2,\phi_3),(\phi_1,\phi_2,\phi_3)\rangle \le 0,$$

which advises that $\mathbb{A}-M_{\mathbb{A}}$ is dissipative. According to the Lumer–Phillips theorem (Pazy, 1983, theorem 4.3, ch.1), $\mathbb{A}-M_{\mathbb{A}}$ generates a C_0-semigroup of contractions on $\mathbb{H}$. Therefore, $\mathbb{A}$ generates a C_0-semigroup on $\mathbb{H}$ and so does A_e by $A_e = \Gamma^{-1}\mathbb{A}\Gamma$. ∎

Theorem 3.1. *Suppose $\alpha_1,\alpha_2 > 0$ and $G+QM_0$ is Hurwitz which always holds from Lemma 2.3. Then, the semigroup generated by A_e is exponentially stable:*

$$\|e^{A_e t}\| \le M_{A_e} e^{-\omega_{A_e} t}, \tag{62}$$

where $M_{A_e}, \omega_{A_e} > 0$ are constants independent of the initial values.

For any $\|\Delta(\cdot)\|_{L^\infty(0,1)} < \frac{\omega_{A_e}}{M_{A_e}}$ and any initial state $(w(\cdot,0),\hat{z}(\cdot,0),\hat{p}(0)) \in (L^2(0,1))^2\times\mathbb{C}^n$, the closed-loop system (43) is uniformly bounded:

$$\sup_{t\ge0}[\|w(\cdot,t)\| + \|\hat{z}(\cdot,t)\| + \|\hat{p}(t)\|] < \infty, \tag{63}$$

and internally exponentially stable whence $p(t) = 0$:

$$\lim_{t\to\infty}[\|w(\cdot,t)\| + \|\hat{z}(\cdot,t)\| + \|\hat{p}(t)\|] = 0 \tag{64}$$

exponentially.

Proof. By Lemma 3.2, we first show that $e^{\mathbb{A}t}$ is exponentially stable. By $\tilde{z}^c(x,t) = z^c(x,t) - \hat{z}^c(x,t)$, this is equivalent to the exponential stability of the following system:

$$\begin{cases} z_t^c(x,t) = z_{xx}^c(x,t), \\ z_x^c(0,t) = 0, \\ z_x^c(1,t) = -\alpha_2 z^c(1,t) + \alpha_2\tilde{z}^c(1,t) \\ \qquad\qquad + [f_0'(1)+\alpha_2 f_0(1)]\hat{p}^c(t), \\ \tilde{z}_t^c(x,t) = \tilde{z}_{xx}^c(x,t), \\ \tilde{z}_x^c(0,t) = \alpha_1\tilde{z}^c(0,t),\ \hat{z}_x^c(1,t) = 0, \\ \dot{\hat{p}}^c(t) = [G+QM_0]\hat{p}^c(t) + Q\tilde{z}^c(0,t), \\ y_e(t) = z^c(0,t). \end{cases} \tag{65}$$

First, by Lemma 2.3 (see (34) and (36)), it does have

$$\lim_{t\to\infty}[\|\tilde{z}^c(\cdot,t)\| + \|\hat{p}^c(t)\|] = 0 \tag{66}$$

exponentially and so we only need to show that

$$\lim_{t\to\infty}\|z^c(\cdot,t)\| = 0$$

exponentially. However, this is a trivial fact because we can write

$$\dot{z}^c(\cdot,t) = Az^c(\cdot,t) - \delta(x-1)\xi(t),$$

where A is defined by (6), and

$$\xi(t) = \left[\alpha_2 f_0(1) + f_0'(1)\right]\hat{p}^c(t) + \alpha_2\tilde{z}^c(1,t). \tag{67}$$

Same as (35), $\tilde{z}^c(1,t)$ decays exponentially as long as $t\to\infty$. This, together with (66), shows that

$$\lim_{t\to\infty}\xi(t) = 0 \tag{68}$$

exponentially. Finally, since by (7), A generates an exponentially stable C_0-semigroup on H, and $\delta(x-1)$ is admissible for e^{At} (Jin & Guo, 2018), we conclude immediately from Lemma 1.1 of Zhou and Guo (2018) (see also Feng & Guo, 2017) that

$$\lim_{t\to\infty}\|z^c(\cdot,t)\| = 0$$

exponentially.

We then prove (63) and (64) for any $\|\varDelta(\cdot)\|_{L^\infty(0,1)} < \frac{\omega_{A_e}}{M_{A_e}}$, which by (46) and (51), are equivalent to

$$\lim_{t\to\infty}[\|z^c(\cdot,t)\| + \|\hat{z}^c(\cdot,t)\| + \|\hat{p}^c(t)\|] = 0$$

exponentially. This, however, follows from

$$\|e^{(A_e+\boldsymbol{\varDelta})t}\| \le M_{A_e}e^{-(\omega_{A_e}-M_{A_e}\|\boldsymbol{\varDelta}\|)t}. \quad \blacksquare \tag{69}$$

Remark 3.1. If we write the closed-loop (43) with $\varDelta(x) = 0$ as

$$\frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ \hat{z}(\cdot,t) \\ \hat{p}(t) \end{pmatrix} = A_e \begin{pmatrix} w(\cdot,t) \\ \hat{z}(\cdot,t) \\ \hat{p}(t) \end{pmatrix} + B_e p(t), \tag{70}$$

where B_e is unbounded. Then, the Sylvester equation

$$\varSigma G = A_e \varSigma + B_e \tag{71}$$

admits a solution $\varSigma$, which is the precondition required in Theorem 6.2 of Paunonen and Pohjolainen (2014) and Theorem 3.3 of Paunonen (2017). It is noted that (71) is equivalent to the stability of system (53) under the transformations (46) and (51). This explains the motivation of the introduction the transformations of (46) and (51). Although for our problem, (71) always admits solution (Phong, 1991) $\varSigma \in \mathcal{L}(\mathbb{C}^n, H^2_{-1}\times\mathbb{C}^n)$ where H_{-1} is the completion of $H = L^2(0,1)$ with respect to the norm $\|A_e^{-1}\|$, we need actually the solution $\varSigma \in \mathcal{L}(\mathbb{C}^n, H^2\times\mathbb{C}^n)$ (Paunonen, 2017) because we need to solve the closed-loop system (53) in $H^2\times\mathbb{C}^n$. In addition, even if we have the solution of (71), we still do not know the expression of the tracking error. Therefore, we need series previous transformations. The first effort for unbounded B_e was investigated in Paunonen (2017) but we are not able to apply directly the results there because of series of abstract assumptions which seem not easy to check.

Now, we come to the robust output regulation, which is stated as the following Theorem 3.2.

Theorem 3.2. *For any $F(x)$, M, N, D and any $\|\varDelta(\cdot)\|_{L^\infty(0,1)} < \frac{\omega_{A_e}}{M_{A_e}}$ with M_{A_e} and ω_{A_e} being used in (62), the output tracking of the closed-loop system (43) is guaranteed such that*

$$\lim_{t\to\infty}|y_e(t)| = 0 \tag{72}$$

exponentially.

Proof. We first consider the case of $\varDelta(x) \equiv 0$. For the closed-loop system (53) by $y_e(t) = z^c(0,t)$, set

$$\tilde{\varepsilon}(x,t) = z^c(x,t) - \frac{1}{\alpha_2}\xi(t)$$

with $\xi(t)$ defined by (67), which satisfies

$$\begin{cases} \tilde{\varepsilon}_t(x,t) = \tilde{\varepsilon}_{xx}(x,t) - \frac{1}{\alpha_2}\dot{\xi}(t), \\ \tilde{\varepsilon}_x(0,t) = 0, \\ \tilde{\varepsilon}_x(1,t) = -\alpha_2\tilde{\varepsilon}(1,t), \end{cases} \tag{73}$$

which is written as

$$\frac{d}{dt}\tilde{\varepsilon}(\cdot,t) = A\tilde{\varepsilon}(\cdot,t) - \frac{1}{\alpha_2}\dot{\xi}(t).$$

Let $\{(\lambda_n,\phi_n)\}$ be defined by (9) and let 1 be expressed by the orthonormal basis $\{\phi_n(x)\}$ in $H = L^2(0,1)$, namely

$$1 = \sum_{n=0}^{\infty} c_n\phi_n(x),$$

where by (9), its Fourier coefficient is defined as

$$c_n = \int_0^1 \phi_n(x)dx = 2\frac{\cos(n\pi)\sin(\frac{2\alpha_2}{n\pi})}{n\pi} + \mathcal{O}(n^{-2}). \tag{74}$$

Then, likewise (10), we can write the solution of (73) as

$$\begin{aligned}\tilde{\varepsilon}(x,t) &= \sum_{n=0}^{\infty} a_n e^{\lambda_n t}\phi_n(x) \\ &\quad -\frac{1}{\alpha_2}\int_0^t \sum_{n=0}^{\infty} c_n\phi_n(x)e^{\lambda_n(t-s)}\dot{\xi}(s)ds,\end{aligned}$$

where

$$\sum_{n=0}^{\infty} a_n^2 = \|\tilde{\varepsilon}(\cdot,0)\|^2. \tag{75}$$

Hence,

$$\begin{aligned}\tilde{\varepsilon}(0,t) &= \sum_{n=0}^{\infty} a_n e^{\lambda_n t}\phi_n(0) \\ &\quad - \frac{1}{\alpha_2}\sum_{n=0}^{\infty} c_n\phi_n(0)\left[\xi(t) - \xi(0)e^{\lambda_n t}\right] \\ &\quad - \frac{1}{\alpha_2}\sum_{n=0}^{\infty} c_n\phi_n(0)\int_0^t \lambda_n e^{\lambda_n(t-s)}\xi(s)ds.\end{aligned} \tag{76}$$

By (74) and (75), the first and the second terms on the right side of (76), respectively as (12) and (68), tend to zero exponentially as $t\to\infty$. As for the third term, suppose

$$|\xi(t)| \le Ce^{-\mu t},$$

where $C > 0$ and $0 < \mu < \min|\lambda_n|$. Then,

$$\begin{aligned}\left|\int_0^t \lambda_n e^{\lambda_n(t-s)}\xi(s)ds\right| &\le \tfrac{C\lambda_n}{-\lambda_n-\mu}\left[e^{-\mu t} - e^{\lambda_n t}\right] \\ &\le C_0e^{-\mu_0 t},\ \forall\, t \ge 1,\end{aligned}$$

where $C_0, \mu_0 > 0$ are independent of n. This, together with (74), shows that the third term on the right side of (76) also tends to zero exponentially as $t\to\infty$. Therefore we have

$$\lim_{t\to\infty}|z^c(0,t)| = \lim_{t\to\infty}|\tilde{\varepsilon}(0,t) + \frac{1}{\alpha_2}\xi(t)| = 0.$$

We next consider the case of $\varDelta(x) \neq 0$. In this case, (73) is then written as

$$\begin{cases} \tilde{\varepsilon}_t(x,t) = \tilde{\varepsilon}_{xx}(x,t) + \varDelta(x)\tilde{\varepsilon}(x,t) + \frac{\varDelta(x)}{\alpha_2}\xi(t) - \frac{1}{\alpha_2}\dot{\xi}(t), \\ \tilde{\varepsilon}_x(0,t) = 0, \\ \tilde{\varepsilon}_x(1,t) = -\alpha_2\tilde{\varepsilon}(1,t), \end{cases} \tag{77}$$

which can be rewritten as

$$\frac{d}{dt}\tilde{\varepsilon}(\cdot,t) = \tilde{A}\tilde{\varepsilon}(\cdot,t) + \left[\frac{\varDelta(x)}{\alpha_2}\xi(t) - \frac{1}{\alpha_2}\dot{\xi}(t)\right],$$

where $\tilde{A} = A + \varDelta(\cdot)I$. Since $\varDelta(\cdot)$ is real, $\tilde{A}$ is still an adjoint operator with compact resolvent and hence there is a sequence of eigenfunctions of $\tilde{A}$, which forms an orthonormal basis for $H = L^2(0,1)$. Now, suppose $\tilde{A}\phi = \lambda\phi$. Then, (λ,ϕ) satisfies the eigenvalue problem:

$$\begin{cases} \phi''(x) = \lambda\phi(x) - \varDelta(x)\phi(x), \\ \phi'(0) = 0,\ \phi'(1) = -\alpha_2\phi(1). \end{cases} \tag{78}$$

Let $\lambda = \rho^2$. The general solutions of (78) are

$$\phi(x) = C_1\phi_1(x) + C_2\phi_2(x),$$

where

$$\phi_1(x) = e^{\rho x} - \frac{1}{2\rho}\int_0^x e^{\rho(x-\xi)}\Delta(\xi)\phi_1(\xi)\,d\xi - \frac{1}{2\rho}\int_x^1 e^{-\rho(x-\xi)}\Delta(\xi)\phi_1(\xi)\,d\xi,$$
$$\phi_2(x) = e^{-\rho x} - \frac{1}{2\rho}\int_0^x \left(e^{\rho(x-\xi)} - e^{-\rho(x-\xi)}\right)\Delta(\xi)\phi_2(\xi)\,d\xi.$$

After one iteration for $\phi_1(x)$, one has

$$\phi_1(x) = e^{\rho x}\left(1 - \frac{1}{2\rho}\int_0^x \Delta(\xi)d\xi - \frac{1}{2\rho}\int_x^1 e^{-2\rho(x-\xi)}\Delta(\xi)d\xi + \mathcal{O}(\rho^{-2})\right).$$

Since $\Delta \in C^1[0,1]$, we have

$$\int_x^1 e^{-2\rho(x-\xi)}\Delta(\xi)\,d\xi = \frac{e^{-2\rho x}}{2\rho}\left(e^{2\rho\xi}\Delta(\xi)|_x^1 - \int_x^1 e^{2\rho\xi}\Delta'(\xi)\,d\xi\right) = \mathcal{O}(\rho^{-1}), \tag{79}$$

and hence

$$\phi_1(x) = e^{\rho x}\left(1 - \frac{1}{2\rho}\int_0^x \Delta(\xi)d\xi + \mathcal{O}(\rho^{-2})\right). \tag{80}$$

By(79) and (80), we have (Guo & Wang, 2019, Theorem 3.4, p.218)

$$\begin{aligned}\phi_1'(x) &= \rho e^{\rho x} - \frac{1}{2}\int_0^x e^{\rho x}\Delta(\xi)\left(1 - \frac{1}{2\rho}\int_0^\xi \Delta(s)ds + \mathcal{O}(\rho^{-2})\right)d\xi + \frac{1}{2}\int_x^1 e^{-\rho(x-2\xi)}\Delta(\xi) \\ &\quad\times\left(1 - \frac{1}{2\rho}\int_0^\xi \Delta(s)ds + \mathcal{O}(\rho^{-2})\right)d\xi, \\ &= e^{\rho x}\left(\rho - \frac{1}{2}\int_0^x \Delta(\xi)d\xi + \frac{1}{4\rho}\int_0^x \Delta(\xi)\int_0^\xi \Delta(s)dsd\xi + \frac{1}{2}\int_x^1 e^{-2\rho(x-\xi)}\Delta(\xi)d\xi + \mathcal{O}(\rho^{-2})\right) \\ &= e^{\rho x}\left(\rho - \frac{1}{2}\int_0^x \Delta(\xi)d\xi + \frac{1}{8\rho}\left[\int_0^x \Delta(\xi)d\xi\right]^2 + \frac{1}{2}\int_x^1 e^{-2\rho(x-\xi)}\Delta(\xi)d\xi + \mathcal{O}(\rho^{-2})\right)\end{aligned} \tag{81}$$

Similarly,

$$\begin{aligned}\phi_2(x) &= e^{-\rho x}\left(1 + \frac{1}{2\rho}\int_0^x \Delta(\xi)d\xi + \mathcal{O}(\rho^{-2})\right), \\ \phi_2'(x) &= e^{-\rho x}\left(-\rho - \frac{1}{2}\int_0^x \Delta(\xi)d\xi - \frac{1}{8\rho}\left[\int_0^x \Delta(\xi)d\xi\right]^2 - \frac{1}{2}\int_0^x e^{2\rho(x-\xi)}\Delta(\xi)d\xi + \mathcal{O}(\rho^{-2})\right).\end{aligned} \tag{82}$$

Substitute into the two boundary conditions to have that $\lambda = \rho^2$ is an eigenvalue if and only if $\det(B) = 0$, where

$$\begin{aligned}B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4\end{pmatrix},\ a_1 = \rho + \frac{1}{2}\int_0^1 e^{2\rho x}\Delta(x)dx + \mathcal{O}(\rho^{-2}), \\ a_2 &= -\rho + \mathcal{O}(\rho^{-2}), \\ a_3 &= e^{\rho}\left(\alpha_2 + \rho - \frac{\int_0^1 \Delta(x)dx}{2} + \frac{\left[\int_0^1 \Delta(x)dx\right]^2}{8\rho} - \frac{\alpha_2\int_0^1 \Delta(x)dx}{2\rho} + \mathcal{O}(\rho^{-2})\right), \\ a_4 &= e^{-\rho}\left(\alpha_2 - \rho - \frac{\int_0^1 \Delta(x)dx}{2} - \frac{\left[\int_0^1 \Delta(x)dx\right]^2}{8\rho} - \frac{1}{2}\int_0^1 e^{2\rho(1-x)}\Delta(x)dx + \frac{\alpha_2\int_0^1 \Delta(x)dx}{2\rho} + \mathcal{O}(\rho^{-2})\right),\end{aligned}$$

and same to (79), $\frac{1}{2}\int_0^1 e^{2\rho x}\Delta(x)dx = \mathcal{O}(\rho^{-1})$ and $\frac{1}{2}\int_0^1 e^{2\rho(1-x)}\Delta(x)dx = \mathcal{O}(\rho^{-1})$.

It is easy to verify that $\det(B) = 0$ if and only if

$$\begin{aligned}e^{2\rho} &= \frac{-\alpha_2 + \rho + \frac{\int_0^1 \Delta(x)dx}{2} + \mathcal{O}(\rho^{-1})}{\alpha_2 + \rho - \frac{\int_0^1 \Delta(\xi)d\xi}{2} + \mathcal{O}(\rho^{-1})} \\ &= 1 - 2\frac{\alpha_2 - \frac{1}{2}\int_0^1 \Delta(x)dx}{\alpha_2 + \rho - \frac{\int_0^1 \Delta(\xi)d\xi}{2} + \mathcal{O}(\rho^{-1})} \\ &= 1 + \frac{1}{\rho}\left(\int_0^1 \Delta(x)dx - 2\alpha_2\right) + \mathcal{O}(\rho^{-2}).\end{aligned}$$

Hence,

$$\rho = n\pi i + \frac{i\left(2\alpha_2 - \int_0^1 \Delta(x)\,d\xi\right)}{n\pi} + \mathcal{O}(n^{-2}), \tag{83}$$

for large integer n. Therefore,

$$\lambda = \rho^2 = -(n\pi)^2 - 4\alpha_2 + 2\int_0^1 \Delta(x)dx + \mathcal{O}(n^{-1}), \tag{84}$$

which is similar to (9). By assumption $\|\Delta(\cdot)\|_{L^\infty(0,1)} < \frac{\omega_{A_e}}{M_{A_e}}$, we have all $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(\tilde{A})$. The associate eigenfunction is

$$\begin{aligned}\phi(x) &= \frac{1}{\rho}\begin{vmatrix} a_1 & a_2 \\ \phi_1(x) & \phi_2(x)\end{vmatrix} = \phi_2(x) + \phi_1(x) + \mathcal{O}(\rho^{-2}) \\ &= e^{\rho x} + e^{-\rho x} + \frac{[e^{-\rho x} - e^{\rho x}]}{2\rho}\int_0^x \Delta(\xi)d\xi + \mathcal{O}(\rho^{-2}) \\ &= 2\cos\left(n\pi x + \frac{2\alpha_2 - \int_0^1 \Delta(\xi)d\xi}{n\pi}x\right) - \frac{1}{n\pi}\int_0^x \Delta(\xi)d\xi \\ &\quad\times\sin\left(n\pi x + \frac{2\alpha_2 - \int_0^1 \Delta(\xi)d\xi}{n\pi}x\right) + \mathcal{O}(n^{-2}).\end{aligned} \tag{85}$$

This results in

$$\int_0^1 \phi(x)dx = \mathcal{O}(n^{-2}), \tag{86}$$

which is the same as (74). Following the same procedure for $\Delta(x) = 0$, we can still obtain that $\lim_{t\to\infty}|\tilde{\varepsilon}(0,t)| = 0$ exponentially for system (77). This completes the proof of the theorem. ■

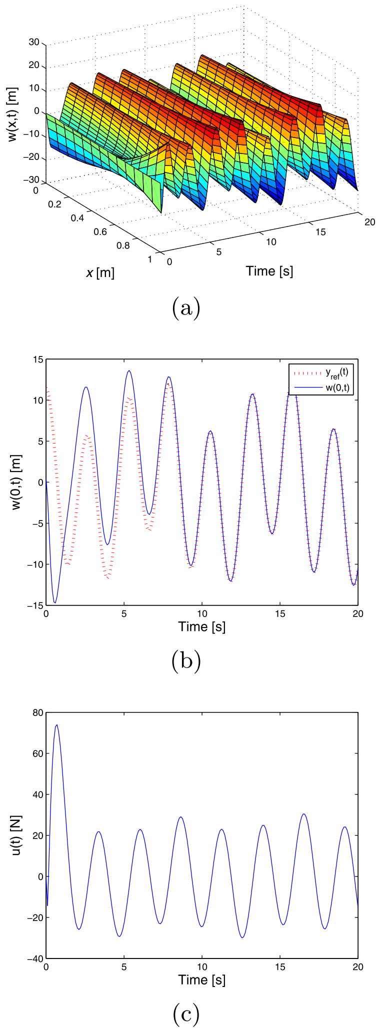

Fig. 1. The closed-loop system under (89).

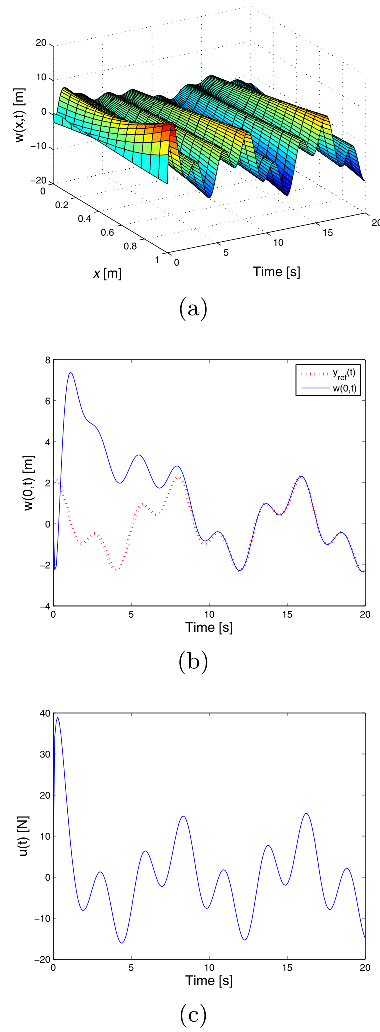

Fig. 2. The closed-loop system under (90).

4. Numerical simulations

In this section, we present two examples with different disturbances and system uncertainty to demonstrate the robustness of the designed controller. The initial states are taken as

$$\begin{cases} p(0) = (0.3, 0.1, 0.2, 0.3)^\top, \ \hat{p}(0) = (0.4, 0.3, 0.2, 0.1)^\top, \\ w(x, 0) = \dfrac{\cos(\pi x)}{2} + \dfrac{x}{2}, \ \hat{z}(x, 0) = \sin \pi x, \end{cases}$$

and the matrix G is

$$G = \begin{pmatrix} 0 & 0.8 & 0 & 0 \\ -0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.4 \\ 0 & 0 & -2.4 & 0 \end{pmatrix}. \tag{87}$$

The control gains are chosen as $\alpha_1 = 1.7$ and $\alpha_2 = 1$. Set $M_0 = (1, 1, 1, 1)$ and $N_0 = \alpha_1 M_0$, where $M_0[w_1, \ldots, w_n] = \frac{1}{\sqrt{2}}(1+i, 1-i, 1-i, 1+i)$ with each component nonzero required by Lemma 2.2. Set $Q = \dfrac{1}{60}(-195, -270, 252, -339)^\top$ and then we have

$$[G + Qh_0(0)] \simeq \begin{pmatrix} -3.2 & 0 & 0 & 0 \\ 0 & -2.4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1.6 \end{pmatrix}. \tag{88}$$

The controller (39) is then determined by frozen (M_0, N_0) and the tracking error $y_e(t)$ only. Fig. 1 depicts the performance of the closed-loop associated system (43) with

$$\begin{cases} M = (5, 10, 15, 20), \ N = (3, 4, 5, 6), \\ D = (1, 2, 3, 4), F(x) = (\frac{x}{2}, x, \frac{x}{2}, x), \ \Delta(x) \equiv 0. \end{cases} \tag{89}$$

From Fig. 1(a), it is seen that $w(x, t)$ is bounded. Fig. 1(b) depicts the output regulation by the controller inputs in Fig. 1(c). For

$$\begin{cases} M = (4, 3, 2, 1), \ N = (10, 9, 8, 7), \\ D = (8, 7, 6, 5), F(x) = \left(\dfrac{x}{3}, \dfrac{x}{4}, x, \dfrac{x}{5}\right), \\ \Delta(x) \equiv 0.01, \end{cases} \tag{90}$$

the same controller can also stabilize the closed-loop system, as shown in Fig. 2 where $w(x, t)$ is bounded and furthermore $w(0, t)$ is regulated to track $y_{ref}(t)$ as time involves. In both figures, the control is seen to be bounded as time evolves. This can be clearly seen from (52)

$$\begin{aligned} u(t) &= -\alpha_2[\hat{z}(1, t) - \hat{h}(1)p(t)] + [\alpha_2 f_0(1) + f_0'(1)] \\ &\quad \times [\hat{p}(t) - h_{\hat{p}}p(t)] + \hat{h}'(1)p(t) \\ &= -\alpha_2 \hat{z}^c(1, t) + [\alpha_2 f_0(1) + f_0'(1)]\hat{p}^c(t) + \hat{h}'(1)p(t). \end{aligned} \tag{91}$$

5. Concluding remarks

In this paper, under the guidance of the internal model principle, we design an observer based dynamic tracking error feedback control to realize output tracking for a heat equation. Since the observer design is the key step towards the control design, we explain step by step the observer design process for slightly general frozen uncertainty. However, for the dynamic error feedback control, we only need an observer for some more specially frozen uncertainties, which turns out to be 1-copy of the exosystem. The error feedback control is shown to be robust to all disturbances from in-domain, non-collocated and input channels, and system uncertainty, from which we see clearly why the internal model is not simply an observer. The method is general and systematic in the spirit of internal model principle presented in a recent communication (Paunonen, 2020) and can be applied to other PDEs like recent work (Guo & Meng, 2019) after this paper. The Associate editor proposed input saturation problem considered in Deutscher (2017), which would be an interesting problem in future works. In addition, it seems that it is difficult to formulate all possible system uncertainties from PDE point of view, here we only consider a simple case although we believe that our controller is always conditionally robust as claimed by the internal model principle. Finally, our approach is genetic even for MIMO PDEs. For instance, for m-output, we simply expand the single exo-system $\dot{p}(t) = Gp(t)$ as $\dot{v}(t) = \mathrm{diag}(G, G, \ldots, G)v(t)$ (Paunonen, 2020) and follows the procedures of this paper to construct robust output error feedback control. The last point is that our system happens no zeros and therefore there is no necessary condition to be pre-assumed. However, for other PDEs like those considered in Guo and Meng (2019), this assumption about zeros must be pre-assumed.

References

Aulisa, E., & Gilliam, D. A. (2016). *A practical guide to geometric regulation for distributed parameter systems*. Boca Raton: CRC Press.

Christofides, P. D. (2001). *Robust control of hyperbolic PDE systems*. Berlin: Springer-Verlag.

Davison, E. J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control*, *21*, 25–34.

Deutscher, J. (2015). A backstepping approach to the output regulation of boundary controlled parabolic PDEs. *Automatica*, *57*, 56–64.

Deutscher, J. (2016). Backstepping design of robust output feedback regulators for boundary controlled parabolic PDEs. *IEEE Transactions on Automatic Control*, *61*, 2288–2294.

Deutscher, J. (2017). Robust output regulation by observer-based feedforward control. *International Journal of Systems Science*, *48*, 795–804.

Deutscher, J., & Kerschbaum, S. (2016). Backstepping design of robust state feedback regulators for second order hyperbolic PIDEs. *IFAC PapersOnLine*, 81–86.

Feng, H., & Guo, B. Z. (2017). New unknown input observer and output feedback stabilization for uncertain heat equation. *Automatica*, *86*, 1–10.

Francis, B. A., & Wonham, W. M. (1976). The internal model principle of control theory. *Automatica*, *12*, 457–465.

Gabriel, J., & Deutscher, J. State feedback regulator design for coupled linear wave equations. In *Proc. ECC 2018 in Limassol*. (pp. 3013–3018). Cyprus.

Guo, W., & Guo, B. Z. (2011). Parameter estimation and stabilisation for a one-dimensional wave equation with boundary output constant disturbance and non-collocated control. *International Journal of Control*, *84*, 381–395.

Guo, B. Z., & Meng, T. T. (2019). *Robust tracking error feedback control for beam equation, preprint*. China: Academy of Mathematics and Systems Science, Academia Sinica.

Guo, W., Shao, Z. C., & Krstic, M. (2017). Adaptive rejection of harmonic disturbance anticollocated with control in 1D wave equation. *Automatica*, *79*, 17–26.

Guo, B. Z., & Wang, J. M. (2019). *Control of wave and beam PDEs: The Riesz basis approach*. Cham: Springer-Verlag.

He, W., Ge, S. S., How, B. V. E., Choo, Y. S., & Hong, K. S. (2011). Robust adaptive boundary control of a flexible marine riser with vessel dynamics. *Automatica*, *47*, 722–732.

He, W., He, X., & Ge, S. S. (2015). Modeling and vibration control of a coupled vessel-mooring-riser system. *IEEE/ASME Transactions on Mechatronics*, *20*, 2832–2840.

Huang, J. (2004). *Nonlinear output regulation theory and application*. Philadelphia, PA: SIAM.

Jin, F. F., & Guo, B. Z. (2018). Performance boundary output tracking for one-dimensional heat equation with boundary unmatched disturbance. *Automatica*, *96*, 1–10.

Meng, T., & Wei, H. (2017). Iterative learning control of a robotic arm experiment platform with input constraint. *IEEE Transactions on Industrial Electronics*, *65*, 664–672.

Natarajan, V., Gilliam, D. S., & Weiss, G. (2014). The state feedback regulator problem for regular linear systems. *IEEE Transactions on Automatic Control*, *59*, 2708–2723.

Paunonen, L. (2016). Controller design for robust output regulation of regular linear systems. *IEEE Transactions on Automatic Control*, *61*, 2974–2986.

Paunonen, L. (2017). Robust controllers for regular linear systems with infinite-dimensional exosystems. *SIAM Journal on Control and Optimization*, *55*, 1567–1597.

Paunonen, L. (2020). *Robustness and internal model property of observer-based controllers* (Research manuscript).

Paunonen, L., & Pohjolainen, S. (2010). Internal model theory for distributed parameter systems. *SIAM Journal on Control and Optimization*, *48*, 4753–4775.

Paunonen, L., & Pohjolainen, S. (2014). The internal model principle for systems with unbounded control and observation. *SIAM Journal on Control and Optimization*, *52*, 3967–4000.

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations*. New York: Springer-Verlag.

Phong, V. Q. (1991). The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems. *Mathematische Zeitschrift*, *208*, 567–588.

Wang, J. W., & Wu, H. N. (2014). Fuzzy output tracking control of semi-linear first-order hyperbolic PDE systems with matched perturbations. *Fuzzy Sets and Systems*, *254*, 47–66.

Zhou, H. C., & Guo, B. Z. (2018). Performance output tracking for one-dimensional wave equation subject to unmatched general disturbance and non-collocated control. *European Journal of Control*, *39*, 39–52.

Bao-Zhu Guo received the Ph.D. degree from Chinese University of Hong Kong in applied mathematics in 1991. From 1985 to 1987, he was a Research Assistant at Beijing Institute of Information and Control, China. During the period 1993–2000, he was with the Beijing Institute of Technology, first as an associate professor (1993–1998) and subsequently a professor (1998–2000). Since 2000, he has been with the Academy of Mathematics and Systems Science, the Chinese Academy of Sciences, where he is a research professor in mathematical system theory.

Dr. Guo is the recipient of the One Hundred Talent Program from the Chinese Academy of Sciences (1999), and the National Science Fund for Distinguished Young Scholars (2003). He is the author or co-author of four books including the books: "Stability and Stabilization of Infinite Dimensional Systems with Applications" (Springer-Verlag, 1999), "Control of Wave and Beam PDEs: The Riesz Basis Approach (Springer-Verlag, 2019) and "Active Disturbance Rejection Control for Nonlinear Systems: An Introduction" (John Wiley \& Sons, 2016). His research interests include theory of control of infinite-dimensional systems

Tingting Meng received the B.Eng. degree in automation from Henan Polytechnic University (HPU), China, in 2014 and the M.E. degree in control engineering from University of Electronic Science and Technology of China (UESTC), China, in 2017.

She is currently a Ph.D. student in Academy of Mathematics and Systems Science, Academia Sinica, China. Her current research interests include active disturbance rejection control, iterative learning control, distributed paover 180 international peer/refereed journal papers, and rameter system control.