Non-fragile $H_{\infty}$ filtering for delayed Takagi–Sugeno fuzzy systems with randomly occurring gain variations

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Received 9 June 2016; received in revised form 25 October 2016; accepted 1 November 2016
Available online 22 November 2016

Abstract

In this study, we consider a non-fragile $H_{\infty}$ filter design for a class of continuous time delayed Takagi–Sugeno (T–S) fuzzy systems. The filter design is assumed to include randomly occurring gain variations according to the filter’s implementation. Two independent Bernoulli distributions are employed to describe these random phenomena. We focus mainly on the design of a non-fragile filter so the filtering error system is asymptotically mean-square stable with the prescribed $H_{\infty}$ performance in the presence of both randomly occurring gain variations and a time delay. A sufficient condition is developed by employing the Lyapunov functional approach. Based on this condition, an improved $H_{\infty}$ filter for delayed T–S fuzzy system is described, which is free of randomly occurring gain variations. The filter parameters are obtained by solving a set of linear matrix inequalities. The effectiveness and the advantages of the proposed techniques are demonstrated by two numerical examples.

Keywords: Interval time-varying delay; Non-fragile $H_{\infty}$ filter; Randomly occurring gain variation; Takagi–Sugeno fuzzy system

1. Introduction

In the last few decades, fuzzy systems based on the Takagi–Sugeno (T–S) model have attracted much attention from the control community because T–S fuzzy models provide an efficient method for representing complex nonlinear systems using simple local linear dynamics [1–7]. A time delay is encountered frequently in various practical systems, such as chemical systems, biological systems, and networked control systems, and the existence of a time delay may lead to instability or poor performance by the systems. Thus, the stability and control of delayed T–S fuzzy systems have attracted much attention, and numerous results have been reported [8–12].

The full state of a control system is not always available, so state estimation has also attracted considerable interest in recent decades due to its widespread applications in process control, space exploitation, and many other uses [13].

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http://dx.doi.org/10.1016/j.fss.2016.11.001
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In recent years, there has been increasing interest in $\mathcal{H}_\infty$ filtering for dynamical systems [14]. The advantage of using an $\mathcal{H}_\infty$ filter rather than the conventional Kalman filter is that no statistical assumptions are required regarding the exogenous signals. In addition, $\mathcal{H}_\infty$ filtering is known to be robust against unmodeled dynamics [15]. Recently, many studies have addressed $\mathcal{H}_\infty$ filtering for T–S fuzzy systems with a time-varying delay. In [16] and [17], the $\mathcal{H}_\infty$ filtering of T–S fuzzy systems with time-varying delays was investigated using a free-weighting matrix method. Improved results were obtained by [18] and [20] by constructing a new Lyapunov functional and a new free-weighing matrix. However, the time-varying delay varied from 0 to an upper bound in [16–20], whereas in practice, the delay may vary in a range where the lower bound is not restricted to zero, i.e., the time delay is interval time-varying. Typical real-world examples of dynamic systems with time-varying interval delays are networked control systems [21]. Thus, a $\mathcal{H}_\infty$ filtering design for continuous-time nonlinear systems with an interval time-varying delay was investigated based on T–S fuzzy models by [22]. Some less conservative results were derived by [23,24] using the delay partitioning method and new bound real lemma, respectively. However, these methods are affected by the following common drawbacks [16–24]: (1) the augmented Lyapunov functional method is not considered; (2) Jensen’s inequality, which neglects some terms, is utilized to estimate the upper bound of some derivative of the Lyapunov functional; (3) a reciprocally convex combination approach is not employed. Therefore, there is still the possibility of further reducing the conservatism of the filter design for delayed T–S fuzzy systems, which is one of the motivations for the present study.

It should be noted that all of the aforementioned results in [16–24] concerning $\mathcal{H}_\infty$ filtering problems are based on an implicit assumption that the filters can be implemented exactly. However, in practical situations, it is difficult for an exactly implemented filter to meet the actual requirements because inaccuracies or uncertainties may occur during filter implementation. These uncertainties can be attributed to unexpected errors during the implementation of the controller, such as analog-digital and digital-analog conversion, round-off errors in numerical computations, and aging of the components. Thus, a significant issue is how to design a filter or controller for a given plant such that the filter or controller is insensitive to some gain variations, which is known as the non-fragile control/filtering problem [25]. Recently, many studies have addressed non-fragile $\mathcal{H}_\infty$ filtering for T–S fuzzy systems. In [26–28], non-fragile $\mathcal{H}_\infty$ filtering for continuous-time T–S fuzzy systems was investigated. In [29,30], distributed non-fragile filtering was studied for discrete-time fuzzy systems. Furthermore, the filter gain variations may be subject to random changes because of environmental conditions during filter implementation among networks [31]. In this case, the gain variations may be presented in a probabilistic manner with specific types and intensities. Based on this observation, the phenomenon of randomly occurring gain variations should be consider in filter design. Indeed, a non-fragile $\mathcal{H}_\infty$ fuzzy filtering design for T–S fuzzy systems with randomly occurring gain variations and channel fading has been proposed [32] where the time delay is not considered (also see [26]), and randomly occurring gain variations are only employed in discrete-time fuzzy systems. When the randomly occurring gain variations phenomenon issue is coupled with a time delay, the non-fragile filtering problem is still open and it requires further intensive investigation, especially for continuous-time T–S fuzzy systems with $\mathcal{H}_\infty$ performance constraints.

Motivated by the studies mentioned above, we propose a non-fragile $\mathcal{H}_\infty$ filter design for delayed T–S fuzzy systems. The time delay is assumed to belong to a given interval and the designed filter includes additive gain variation, which is assumed to be random and it satisfies the Bernoulli distribution. Using the Lyapunov functional approach and improved integral inequality, a sufficient condition for designing the non-fragile $\mathcal{H}_\infty$ filter is developed such that the filtering error system is asymptotically mean-square stable with the prescribed $\mathcal{H}_\infty$ performance. Furthermore, an improved result for $\mathcal{H}_\infty$ filtering in a delayed T–S fuzzy system is presented, which is free of randomly occurring gain variations. We show that the filter parameters can be obtained by solving a set of linear matrix inequalities (LMIs). Finally, two numerical examples are used to demonstrate the effectiveness of the proposed design technique. The main contributions of this study are: a) for the first time, we consider randomly occurring gain variations where the gain variations appear in a random manner based on a certain type of probabilistic law in a non-fragile $\mathcal{H}_\infty$ filter design for continuous-time T–S fuzzy systems; b) compared with previous studies, we propose improved $\mathcal{H}_\infty$ filtering for continuous-time T–S fuzzy systems with a time-varying delay by employing an augmented Lyapunov functional approach, which is combined with an improved integral inequality and reciprocally convex combination technique.

**Notations.** The notations used in this study are fairly standard. $\top$ denotes matrix transposition, $I$ is the identity matrix with appropriate dimensions, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices, and $*$ are the elements below the main diagonal of a symmetric block matrix. For symmetric matrices $A$ and $B$, the notation $A > B$ (respectively, $A \geq B$) means that the matrix $A - B$ is positive definite (respectively,
nonnegative), and \( \text{diag} \{ \ldots \} \) denotes the block diagonal matrix. \( \| \cdot \| \) refers to the induced matrix 2-norm. \( L_2 \) is the space of square integral vector functions on \([0, \infty)\) with norm \( \| \cdot \|_2 \equiv \left( \int_0^\infty \| \cdot \|^2 \, dt \right)^{1/2} \). Moreover, \( \text{Prob}\{\cdot\} \) is the probability of an event occurring and \( \mathbb{E}\{\cdot\} \) denotes the mathematical expectation.

### 2. Problem statement

We consider the following nonlinear system with an interval time-varying delay, which can be represented by a delayed T–S fuzzy model with \( r \) plant rules.

**Plant rule:** IF \( \theta_i(t) \) is \( M_{i1} \), and \( \ldots \), and \( \theta_p(t) \) is \( M_{ip} \), THEN

\[
\begin{align*}
\dot{x}(t) &= A_i x(t) + A_{ri} x(t - \tau(t)) + B_i w(t), \\
y(t) &= C_i x(t) + C_{ri} x(t - \tau(t)) + D_i w(t), \\
z(t) &= L_i x(t) + L_{ri} x(t - \tau(t)) + F_i w(t), \\
x(t) &= \psi(t), t \in [-\tau_2, 0],
\end{align*}
\]

(2.1)

where \( r \) is the number of IF–THEN rules, and \( M_{ij} \) is the fuzzy set; \( x(t) \in \mathbb{R}^n \), \( y(t) \in \mathbb{R}^m \), and \( z(t) \in \mathbb{R}^p \) are the state vector, the output measurement, and the signal to be estimated, respectively. The external disturbance \( w(t) \in \mathbb{R}^q \) is assumed to belong to \( L_2[0, \infty) \). \( \tau(t) \) is a time-varying delay that satisfies the following inequality

\[ \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \mu, \]

where \( \tau_2 > \tau_1 \geq 0 \), and \( \mu \) are known constants. \( \theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)] \) is the premise variables vector. \( A_i, A_{ri}, B_i, C_i, C_{ri}, D_i, L_i, L_{ri}, F_i \) are known constant matrices with appropriate dimensions, and \( \psi(t) \) denotes the continuous initial vector function defined on \([-\tau_2, 0]\).

By employing the commonly used center-average defuzzifier product interference and singleton fuzzifier, the delayed T–S fuzzy system (2.1) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(\theta(t))[A_i x(t) + A_{ri} x(t - \tau(t)) + B_i w(t)], \\
y(t) &= \sum_{i=1}^{r} h_i(\theta(t))[C_i x(t) + C_{ri} x(t - \tau(t)) + D_i w(t)], \\
z(t) &= \sum_{i=1}^{r} h_i(\theta(t))[L_i x(t) + L_{ri} x(t - \tau(t)) + F_i w(t)], \\
x(t) &= \psi(t), t \in [-\tau_2, 0],
\end{align*}
\]

(2.2)

where

\[
h_i(\theta(t)) = \frac{\prod_{j=1}^{p} M_{ij}(\theta_j(t))}{\sum_{i=1}^{r} \prod_{j=1}^{p} M_{ij}(\theta_j(t))},
\]

(2.3)

and \( M_{ij}(\theta_j(t)) \) represents the grade of membership for \( \theta_j(t) \) in \( M_{ij} \).

It can be seen that

\[
h_i(\theta(t)) \geq 0, \quad \sum_{i=1}^{r} h_i(\theta(t)) = 1.
\]

(2.4)

The filter is designed by parallel distributed compensation while considering randomly occurring gain variations. The proposed filter is given as follows:
\[
\begin{align*}
\dot{x}_f (t) &= \sum_{i=1}^{r} h_i(\theta(t))[A_{fi}(t) + \alpha(t)\Delta A_{fi}(t)]x_f(t) + (B_{fi}(t) + \beta(t)\Delta B_{fi}(t))y(t), \\
\dot{z}_f (t) &= \sum_{i=1}^{r} h_i(\theta(t))[C_{fi}x_f(t) + D_{fi}y(t)], \\
x(t) &= \psi(t), t \in [-\tau_2, 0],
\end{align*}
\]

where \(A_{fi}, B_{fi}, C_{fi}, \) and \(D_{fi}\) are the filtering parameters that need to be determined, and \(\alpha(t)\) and \(\beta(t)\) are mutually independent Bernoulli-distributed white sequences. A natural assumption on \(\alpha(t)\) and \(\beta(t)\) is given as

\[
\begin{align*}
\text{Prob}[\alpha(t) = 1] &= \mathbb{E}[\alpha(t)] = \bar{\alpha}, \quad \text{Prob}[\alpha(t) = 0] = 1 - \bar{\alpha}, \\
\text{Prob}[\beta(t) = 1] &= \mathbb{E}[\beta(t)] = \bar{\beta}, \quad \text{Prob}[\beta(t) = 0] = 1 - \bar{\beta}.
\end{align*}
\]

The uncertain perturbation matrices \(\Delta A_{fi}\) and \(\Delta B_{fi}\) are defined as

\[
\Delta A_{fi}(t) = M_{ai} \Delta_1(t) N_{ai}, \quad \Delta B_{fi}(t) = M_{bi} \Delta_2(t) N_{bi},
\]

where \(M_{ai}, M_{bi}, N_{ai},\) and \(N_{bi}\) are known constant matrices with appropriate dimensions, and \(\Delta_1(t)\) and \(\Delta_2(t)\) are unknown matrix functions that satisfy

\[
\Delta_1^T(t) \Delta_1(t) \leq I, \ r = 1, 2.
\]

**Remark 1.** In real-world networked systems, inaccuracies or uncertainties occur during the implementation of a designed filter because of unexpected errors [26–29]. In addition, uncertainties may enter the system in random manner according to their individual Bernoulli distributions [31,32]. This description is more suitable to reflect the random nature of parameter variations, particularly during signal transmissions through a communication network. In contrast to [22–24], the uncertainties are taken into consideration, but two random variables \(\alpha(t)\) and \(\beta(t)\) are also introduced to model the randomly occurring gain variations of the filters, which have not been investigated previously for continuous-time T–S fuzzy systems.

We define an augmented state vector \(\eta(t) = [x^T(t), x_f^T(t)]^T\) and an error vector \(e(t) = z(t) - z_f(t)\), to obtain the following filtering error system:

\[
\begin{align*}
\dot{\eta}(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t))[(\hat{A}_{ij} + \bar{\alpha} \hat{A}_j(t) + \bar{\alpha}(t)\hat{A}_j(t))\eta(t) \\
&\quad + (\hat{A}_{rjj} + \bar{\beta} \hat{A}_{rjj}(t) + \bar{\beta}(t)\hat{A}_{rjj}(t))\eta(t - \tau(t)) \\
&\quad + (\hat{B}_{ij} + \bar{\beta} \hat{B}_{ij}(t) + \bar{\beta}(t)\hat{B}_{ij}(t))w(t)], \\
e(t) &= \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t))[\tilde{C}_{ij}\eta(t) + \tilde{C}_{rjj}\eta(t - \tau(t)) + \tilde{D}_{ij}w(t)],
\end{align*}
\]

where

\[
\begin{align*}
\hat{A}_{ij} &= \begin{bmatrix} A_i & 0 \\ B_{fj}C_i & A_{fj} \end{bmatrix}, \quad \hat{A}_j(t) &= \begin{bmatrix} 0 & 0 \\ 0 & \Delta A_{fj}(t) \end{bmatrix}, \quad \hat{A}_{rjj} &= \begin{bmatrix} A_{rj} & 0 \\ B_{fj}C_{ri} & 0 \end{bmatrix}, \\
\hat{A}_{rjj}(t) &= \begin{bmatrix} 0 & 0 \\ \Delta B_{fj}(t)C_{ri} & 0 \end{bmatrix}, \quad \hat{B}_{ij} &= \begin{bmatrix} B_i \\ B_{fj}D_i \end{bmatrix}, \quad \hat{B}_{ij}(t) &= \begin{bmatrix} 0 \\ \Delta B_{fj}(t)D_i \end{bmatrix}, \\
\tilde{C}_{ij} &= [L_i - D_{fj}C_i, -C_{fj}], \quad \tilde{C}_{rjj} &= [L_{ri} - D_{fj}C_{ri}, 0], \quad \tilde{D}_{ij} &= F_i - D_{fj}D_i. \\
\hat{\alpha}(t) &= \alpha(t) - \bar{\alpha}, \quad \hat{\beta}(t) = \beta(t) - \bar{\beta}.
\end{align*}
\]

**Definition 1.** The filtering error system (2.6) with \(w(t) = 0\) is said to be asymptotically stable in the mean-square sense if for any initial condition,
\[ \lim_{t \to \infty} E \{ \| \eta(t) \|^2 \} = 0. \]

**Definition 2.** For a given positive scalar \( \gamma \), the filtering error system (2.6) is said to be asymptotically stable in mean square with guaranteed \( H_\infty \) performance \( \gamma \) if it is asymptotically stable and the filtering error \( e(t) \) satisfies

\[ \int_0^L E \{ \| e(t) \|^2 \} dt \leq \gamma^2 \int_0^L E \{ \| w(t) \|^2 \} dt \quad (2.7) \]

for all \( L > 0 \) and nonzero \( w(t) \in L_2[0, \infty) \) subject to the zero initial condition.

Now, we state the fuzzy \( H_\infty \) filtering problem as follows.

**Filtering problem:** Design a filter with formula (2.5) such that the filtering error system (2.6) is asymptotically stable in the mean-square sense and it achieves a prescribed \( H_\infty \) performance level.

In the following, some lemmas are introduced as preliminaries before the proof of our main results.

**Lemma 1.** \([33]\) For a positive definite matrix \( R \) and a differentiable function \( \{ x(u) | u \in [a, b] \} \),

\[ -(b - a) \int_a^b \dot{x}(s) R \dot{x}(s) ds \leq -\Omega_1^T R \Omega_1 - 3\Omega_2^T R \Omega_2 - 5\Omega_3^T R \Omega_3, \]

where

\[ \Omega_1 = x(b) - x(a), \quad \Omega_2 = x(b) + x(a) - \frac{2}{b - a} \int_a^b x(s) ds, \]
\[ \Omega_3 = x(b) - x(a) + \frac{6}{b - a} \int_a^b x(s) ds - \frac{12}{(b - a)^2} \int_a^b \int_a^b x(\alpha) d\alpha d\beta. \]

**Lemma 2.** \([34]\) Let \( f_1, f_2, \cdots, f_N : \mathbb{R}^m \to \mathbb{R} \) have positive values in an open subset \( D \) of \( \mathbb{R}^m \). Then, the reciprocally convex combination of \( f_i \) over \( D \) satisfies

\[ \min_{\{ \alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1 \}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{ij}(t)} \sum_{i \neq j} g_{ij}(t) \]

subject to

\[ \left\{ g_{ij} : \mathbb{R}^m \to \mathbb{R}, \ g_{j,i}(t) \triangleq g_{i,j}(t), \ \left( \begin{array}{cc} f_i(t) & g_i(t) \\ g_i(t) & f_j(t) \end{array} \right) \succeq 0 \right\}. \]

**Lemma 3.** \([35]\) Let \( X, Y, \) and \( F \) be real matrices with appropriate dimensions and \( F^T F \leq I \). Then, for any scalar \( \sigma > 0 \),

\[ X F Y + Y^T F^T X^T \leq \sigma^{-1} X X^T + \sigma Y Y^T. \]

**Lemma 4.** \([26]\) If

\[ M_{ii} < 0, \ i = 1, 2, \ldots, r, \]
\[ \frac{2}{r - 1} M_{ii} + M_{ij} + M_{ji} < 0, \ i \neq j, \ j = 1, 2, \ldots, r, \]

then
where \( h_i(\theta(t)), i = 1, 2, \ldots, r \) satisfy (2.3).

3. Main results

In this section, we first present a delay-dependent condition for a non-fragile \( \mathcal{H}_\infty \) filter design in a T–S fuzzy system with interval time-varying delay and randomly occurring gain variations. For the sake of simplicity, we define \( \hat{e}_i \in \mathbb{R}^{(22n+q) \times n} \) as a block entry matrix. For example, \( \hat{e}_2 = [0, I, 0, \ldots, 0, 0]^T \). We collect some scalars, vectors, and matrices as

\[
\bar{M}_{aij} = \begin{bmatrix} 0 \\ M_{aij} \end{bmatrix}, \quad \bar{M}_{bij} = \begin{bmatrix} 0 \\ M_{bij} \end{bmatrix},
\]

\[
\bar{N}_{aij} = [0, N_{aij}], \quad \bar{B}_{aij} = [N_{aij} C_{ri}, 0],
\]

\[
\bar{R}_2 = \begin{bmatrix} R_2 & 0 & 0 \\ * & 3R_2 & 0 \\ * & * & 5R_2 \end{bmatrix}, \quad G \bar{A}_{ij} = \begin{bmatrix} G_1 A_i + \hat{B}_{fj} C_i & \hat{A}_{fj} \\ G_2^T A_i + \hat{B}_{fj} C_i & \hat{A}_{fj} \end{bmatrix},
\]

\[
G \bar{A}_{rij} = \begin{bmatrix} G_1 A_{rij} + \hat{B}_{fj} C_{rij} & 0 \\ G_2^T A_{rij} + \hat{B}_{fj} C_{rij} & 0 \end{bmatrix}, \quad G \bar{B}_{ij} = \begin{bmatrix} G_1 B_i + \hat{B}_{fj} D_i \\ G_2^T B_i + \hat{B}_{fj} D_i \end{bmatrix},
\]

\[
\Phi_{ij} = [G \bar{A}_{ij}, 0, G \bar{A}_{rij}, 0, \ldots, 0, \bar{B}_{ij}],
\]

\[
\Psi_{ij} = [\bar{C}_{ij}, 0, \bar{C}_{rij}, 0, \ldots, 0, \bar{D}_{ij}],
\]

\[
\Gamma_{ij} = \left[ \bar{a}_e e_1 G \bar{M}_{aij}, \bar{b}(e_2 + e_3) e_1 G \bar{M}_{bij}, \bar{a}_e e_4 e_{11} G \bar{M}_{aij}, \bar{b}_e (e_5 + e_6) e_{11} G \bar{M}_{bij} \right],
\]

\[
\Gamma_2 = \text{diag}\{-\bar{a}_e I, -\bar{b}(e_2 + e_3) I, -\bar{a}_e e_4 I, -\bar{b}_e (e_5 + e_6) I\}
\]

\[
\Gamma_{3j} = e_1 \bar{a}_e (e_1^{-1} + e_4^{-1}) \bar{N}_{aj} e_1 + e_3 \bar{b}_e (e_2^{-1} + e_5^{-1}) \bar{N}_{bij} e_3 + e_1 \bar{b}_e (e_3^{-1} + e_6^{-1}) G \bar{D}_j \bar{N}_{bij} G \bar{D}_j e_2,
\]

\[
\Pi_{1} = [e_1, \tau(t) - \tau_{1} e_6 + (\tau_2 - \tau(t)) e_7, \tau_2^2 e_8],
\]
\[ \Pi_2 = [e_{11}, e_{1} - e_{2}, e_2 - e_4, \tau_1 e_1 - \tau_1 e_3], \]
\[ \Pi_3 = [e_2 - e_3, e_2 + e_3 - 2e_6, e_2 - e_3 + 6e_6 - 12e_9], \]
\[ \Pi_4 = [e_3 - e_4, e_3 + e_4 - 2e_7, e_3 - e_4 + 6e_7 - 12e_{10}], \]
\[ \Xi_{ij} = \Pi_1 \mathcal{P} \Pi_2^\top + \Pi_2 \mathcal{P} \Pi_1^\top + e_2 \mathcal{Q}_1 e_3 - (1 - \mu) e_3 \mathcal{Q}_1 e_3 + e_1 \mathcal{Q}_2 e_1^\top - e_4 \mathcal{Q}_2 e_4^\top + \tau_1^\top e_1^\top R_1 e_1^\top + (\tau_2 - \tau_1)^2 e_1^\top R_1 e_1^\top - [e_1 - e_2] R_1 [e_1 - e_2]^\top - 3[e_1 + e_2 - 2e_5] R_1 [e_1 + e_2 - 2e_5]^\top \]
\[ - 5[e_1 - e_2 + 6e_5 - 12e_8] R_1 [e_1 - e_2 + 6e_5 - 12e_8]^\top - [\Pi_3, \Pi_4] \begin{bmatrix} \bar{R}_2 & \chi^\top \Pi_3, \Pi_4 \end{bmatrix}^\top + [e_1 + \varepsilon e_{11}] \Phi_{ij} + \Phi_{ij}^\top [e_1 + \varepsilon e_{11}]^\top - \gamma_2^2 e_{12}^\top e_{12}. \]

Now, we are in a position to state the main result of this study.

**Theorem 1.** For given constants \( \tau_2 > \tau_1 > 0, \mu, \gamma > 0, \varepsilon > 0 \) and \( \varepsilon_1 > 0 (i = 1, 2, \ldots, 6) \), an admissible non-fragile \( H_{\infty} \) filter (2.5) exists such that the filtering error system (2.6) is asymptotically stable in mean square for \( w(t) = 0 \) and the filtering error \( e(t) \) satisfies (2.7) under the zero initial condition for any nonzero \( w(t) \in \mathcal{L}_2[0, \infty) \), provided that the matrices \( \mathcal{P} > 0, \mathcal{Q}_1 > 0, \mathcal{Q}_2 > 0, R_1 > 0, R_2 > 0, \mathcal{X}, \hat{A}_{fj}, \hat{B}_{fj}, \hat{C}_{fj}, \hat{D}_{fj} \), \( G = \begin{bmatrix} G_1^\top & G_2 \end{bmatrix} \) exist that satisfy the following LMIs:
\[ \Upsilon_{ii} < 0, \Upsilon_{ij} + \Upsilon_{ji} < 0, i < j, \begin{bmatrix} \bar{R}_2 & \chi^\top \bar{R}_2 \end{bmatrix} \geq 0, \quad (3.1) \]
where
\[ \Upsilon_{ij} = \begin{bmatrix} \Xi_{ij} + \Gamma_{3j} & \Psi_{ij}^\top & \Gamma_{1j} \\ * & -I & 0 \\ * & * & \Gamma_2 \end{bmatrix}. \quad (3.2) \]

In addition, if the LMIs admit feasible solutions, then matrices of the admissible \( H_{\infty} \) filter in formula (2.5) can be obtained by
\[ A_{fj} = G_2^{-1} \hat{A}_{fj}, B_{fj} = G_2^{-1} \hat{B}_{fj}, C_f = \hat{C}_{fj}, D_f = \hat{D}_{fj}. \]

**Proof.** Construct the following Lyapunov functional
\[ V(t) = \sum_{i=1}^{5} V_i(t), \quad (3.3) \]
where
\[ V_1(t) = \begin{bmatrix} \int_{t-\tau_2}^{t} \eta(t) \eta(s) ds \\
\int_{t-\tau_1}^{t} \eta(t) \eta(s) ds \\
\int_{t-\tau_1}^{t} \int_{0}^{\theta} \eta(s) ds d\theta \end{bmatrix}^\top \mathcal{P} \begin{bmatrix} \int_{t-\tau_2}^{t} \eta(t) \eta(s) ds \\
\int_{t-\tau_1}^{t} \eta(t) \eta(s) ds \\
\int_{t-\tau_1}^{t} \int_{0}^{\theta} \eta(s) ds d\theta \end{bmatrix}^\top, \]
\[ V_2(t) = \int_{t-\tau_1}^{t} \eta(s) Q_1 \eta(s) ds, \]
\[ V_3(t) = \int_{t-\tau_2}^{t} \eta(s) Q_2 \eta(s) ds, \]
\[ V_4(t) = \tau_1 \int_{t-\tau_1}^{t} \int_{\theta}^{\hat{\eta}} \eta(s) R_1 \eta(s) ds d\theta, \]
\[ V_5(t) = (\tau_2 - \tau_1) \int_{t-\tau_2}^{t} \int_{\theta}^{\hat{\eta}} \eta(s) R_2 \eta(s) ds d\theta. \]
The time derivative of $V(t)$ along the trajectory of (2.6) is computed as

$$
\mathbb{E}\{\dot{V}_1(t)\} = 2\mathbb{E}\left\{\begin{bmatrix}
\eta(t) \\
\int_{t-\tau_1}^{t} \eta(s)ds \\
\int_{t-\tau_1}^{t} \int_{t-\tau_2}^{t} \eta(s)dsd\theta
\end{bmatrix}^T \mathcal{P} \begin{bmatrix}
\dot{\eta}(t) \\
\eta(t) - \eta(t - \tau_1) \\
\eta(t-\tau_1) - \tau(t-\tau_2)
\end{bmatrix}\right\}, \tag{3.4}
$$

$$
\mathbb{E}\{\dot{V}_2(t)\} \leq \mathbb{E}\left\{\eta^T(t-\tau_1)Q_1\eta(t-\tau_1) - (1-\mu)\eta^T(t-\tau(t))Q_1\eta(t-\tau(t))\right\}, \tag{3.5}
$$

$$
\mathbb{E}\{\dot{V}_3(t)\} = \mathbb{E}\left\{\eta^T(t)Q_2\eta(t) - \eta^T(t-\tau_2)Q_2\eta(t-\tau_2)\right\}, \tag{3.6}
$$

$$
\mathbb{E}\{\dot{V}_4(t)\} = \mathbb{E}\left\{\tau_2^2\dot{\eta}^T(t)R_1\dot{\eta}(t) - \tau_2 - \tau_1 \int_{t-\tau_1}^{t} \dot{\eta}^T(s)R_1\dot{\eta}(s)ds\right\}, \tag{3.7}
$$

$$
\mathbb{E}\{\dot{V}_5(t)\} = \mathbb{E}\left\{(\tau_2 - \tau_1)^2\dot{\eta}^T(t)R_2\dot{\eta}(t) - (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau_1} \dot{\eta}^T(s)R_2\dot{\eta}(s)ds\right\}
$$

$$
- (\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{\eta}^T(s)R_2\dot{\eta}(s)ds\right\}. \tag{3.8}
$$

By Lemma 1, we can obtain

$$
-\tau_1 \int_{t-\tau_1}^{t} \dot{\eta}^T(s)R_1\dot{\eta}(s)ds \leq -\gamma_1^T R_1\gamma_1 - 3\gamma_2^T R_1\gamma_2 - 5\gamma_3^T R_1\gamma_3,
$$

where

$$
\gamma_1 = \eta(t) - \eta(t - \tau_1), \quad \gamma_2 = \eta(t) + \eta(t - \tau_1) - \frac{2}{\tau_1} \int_{t-\tau_1}^{t} \eta(s)ds,
$$

$$
\gamma_3 = \eta(t) - \eta(t - \tau_1) + \frac{6}{\tau_1} \int_{t-\tau_1}^{t} \eta(s)ds - \frac{12}{\tau_1^2} \int_{t-\tau_1}^{t} \int_{\theta} \eta(s)d\theta.
$$

By applying Lemmas 1 and 2, we obtain

$$
-\tau_2 \int_{t-\tau_2}^{t-\tau(t)} \dot{\eta}^T(s)R_2\dot{\eta}(s)ds \leq -\tau_2 \int_{t-\tau_2}^{t-\tau(t)} \dot{\eta}^T(s)R_2\dot{\eta}(s)ds
$$

$$
\leq -\frac{\tau_2 - \tau_1}{\tau(t) - \tau_1} \dot{\gamma}_1^T R_2\dot{\gamma}_1 - \frac{\tau_2 - \tau_1}{\tau_2 - \tau(t)} \dot{\gamma}_2^T R_2\dot{\gamma}_2
$$

$$
\leq -\begin{bmatrix}
\dot{\gamma}_1^T \\
\dot{\gamma}_2^T
\end{bmatrix} \begin{bmatrix}
R_2 & \mathcal{X} \\
\ast & R_2
\end{bmatrix} \begin{bmatrix}
\dot{\gamma}_1 \\
\dot{\gamma}_2
\end{bmatrix}, \tag{3.9}
$$

where
If 

\[ y(t) = \gamma(t) + \eta(t) \]

Furthermore, for any appropriate dimensional matrix \( G \), from (2.6), it follows that

\[ \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t))[y(t)G + \varepsilon y(t)G] = 0. \]

For positive scalars \( \varepsilon_i (i = 1, 2, \ldots, 6) \), by Lemma 3, it follows that

\[ \sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t))[y(t)G + \varepsilon y(t)G] = 0. \]

Now, the \( H_\infty \) performance is established for the filtering error system (2.6). If the derivative of \( V(t) \) is negative, then \( z(t) \to 0 \) as \( t \to \infty \). Next, by assuming zero initial conditions for the filtering error system, the performance index is

\[ J = \int_0^L \mathbb{E}\{e(t)^T e(t) - \gamma^2 w(t)^T w(t)\} dt \leq \int_0^L \mathbb{E}\{e(t)^T e(t) - \gamma^2 w(t)^T w(t) + \dot{V}(t)\} dt. \]

If \( \mathbb{E}\{e(t)^T e(t) - \gamma^2 w(t)^T w(t) + \dot{V}(t)\} < 0 \), then (2.7) holds. This verifies \( H_\infty \).

Taking \( G \) as

\[ G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \]

and by a simple matrix calculation, it is straightforward to verify that

\[ G\bar{A}_{ij} = \begin{bmatrix} G_1A_{ij} + G_2B_{ij}C_i & G_2A_{ij} \\ G_2^TA_{ij} + G_2B_{ij}C_j & G_2A_{ij} \end{bmatrix}, \quad G\bar{A}_{rij} = \begin{bmatrix} G_1A_{ri} + G_2B_{ij}C_i & 0 \\ G_2^TA_{ri} + G_2B_{ij}C_j & 0 \end{bmatrix}, \]

(3.12)
Define a set of variables as
\[
\hat{A}_{fj} = G_2 A_{fj}, \quad \hat{B}_{fj} = G_2 B_{fj}, \quad \hat{C}_{fj} = C_{fj}, \quad \hat{D}_{fj} = D_{fj}.
\]
(3.13)

Combining with (3.4)–(3.13) gives
\[
\begin{align*}
E \left\{ \dot{V}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) \right\} & \leq \sum_{i,j} \left( \hat{A}_{ij} \hat{\eta}(t) + \hat{A}_{ij} \hat{\eta}(t - \tau(t)) + \hat{B}_j(\hat{w}(t)) \right), \\
\end{align*}
\]
(3.14)

By the Schur complement,
\[
\begin{align*}
\sum_{i,j} \left( \hat{A}_{ij} \hat{\eta}(t) + \hat{A}_{ij} \hat{\eta}(t - \tau(t)) + \hat{B}_j(\hat{w}(t)) \right) & = \sum_{i,j} \left( \hat{A}_{ij} \hat{\eta}(t) + \hat{A}_{ij} \hat{\eta}(t - \tau(t)) + \hat{B}_j(\hat{w}(t)) \right),
\end{align*}
\]
(3.15)

which is equivalent to (3.2). By Lemma 4, if (3.1) holds, then \( J < 0 \) for any nonzero \( \hat{w}(t) \in L_2[0, \infty) \), i.e., the filtering error system has a guaranteed \( \gamma \) level of disturbance attenuation. This completes the proof of the theorem. \( \Box \)

Next, to compare the \( \mathcal{H}_\infty \) filter design with previous results, we consider the following filter free of randomly occurring uncertainties
\[
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^r h_i(\theta(t))[A_{fi}x(t) + B_{fi}y(t)], \\
\dot{z}(t) & = \sum_{i=1}^r h_i(\theta(t))[C_{fi}x(t) + D_{fi}y(t)], \\
x(t) & = \psi(t), \quad t \in [-\tau_2, 0],
\end{align*}
\]
(3.16)

which was studied by [17–24]. The filtering error system is then reduced to
\[
\begin{align*}
\dot{\eta}(t) & = \sum_{i=1}^r h_i(\theta(t)) \left[ \sum_{j=1}^r h_j(\theta(t)) [\hat{A}_{ij} \eta(t) + \hat{A}_{ij} \eta(t - \tau(t)) + \hat{B}_j(\hat{w}(t))] \right], \\
e(t) & = \sum_{i=1}^r h_i(\theta(t)) \left[ \sum_{j=1}^r h_j(\theta(t)) [\hat{C}_{ij} \eta(t) + \hat{C}_{ij} \eta(t - \tau(t)) + \hat{D}_j(\hat{w}(t))] \right].
\end{align*}
\]
(3.17)

By Theorem 1, the \( \mathcal{H}_\infty \) filter design for a delayed T–S fuzzy system without randomly occurring gain variations is derived easily, as stated as Corollary 1.

\textbf{Corollary 1.} For given constants \( \tau_2 > \tau_1 > 0 \), \( \mu > 0 \), and \( \varepsilon > 0 \), an admissible \( \mathcal{H}_\infty \) filter (2.5) exists such that the filtering error system (2.6) is asymptotically stable in mean square for \( w(t) = 0 \) and the filtering error \( e(t) \) satisfies (2.7) under the zero initial condition for any nonzero \( w(t) \in L_2[0, \infty) \), provided that the matrices \( P > 0 \), \( Q_1 > 0 \), \( Q_2 > 0 \), \( R_1 > 0 \), \( R_2 > 0 \), \( \chi \), \( \hat{A}_{fj} \), \( \hat{B}_{fj} \), \( \hat{C}_{fj} \), \( \hat{D}_{fj} \), \( G = \begin{bmatrix} G_1 & G_2 \\ G_2 & G_2 \end{bmatrix} \) exist that satisfy the following LMIs
\[
\begin{align*}
\bar{\Upsilon}_{ij} & < 0, \\
\bar{\Upsilon}_{ij} + \bar{\Upsilon}_{ji} & < 0, i < j, \\
\begin{bmatrix} \bar{R}_2 & \chi \\ * & \bar{R}_2 \end{bmatrix} & \geq 0,
\end{align*}
\]
(3.18)

where
\[
\begin{align*}
\bar{\Upsilon}_{ij} & = \begin{bmatrix} \Xi_{ij} & \Psi_{ij}^T \\ * & I \end{bmatrix},
\end{align*}
\]
(3.19)

and the other notations are the same as those used in Theorem 1.

In addition, if the LMIs admit feasible solutions, then the matrices of the admissible \( \mathcal{H}_\infty \) filter in formula (2.5) can be obtained by
\[
A_{fj} = G_2^{-1} \hat{A}_{fj}, \quad B_{fj} = G_2^{-1} \hat{B}_{fj}, \quad C_{fj} = \hat{C}_{fj}, \quad D_{fj} = \hat{D}_{fj}.
\]
Remark 2. When the upper bound of the derivative of the time delay \( \mu \) is unknown or \( h(t) \) is not differentiable, then we can take \( Q_1 = 0 \) to easily obtain the corresponding results. Furthermore, when the lower bound of the time delay \( h_1 \) is 0, by Theorem 1 and Corollary 1, it is also easy to derive the corresponding results.

Remark 3. In [16–20], the lower bound of the time delay was assumed to be 0. Information about the derivative of time delay was not given by [23]. Therefore, we consider various cases for the time delay information.

Remark 4. Regarding the conservativeness of the \( H_\infty \) design, according to previous studies and our discussion, the possible ways for reducing conservativeness can be summarized as follows.

1) In terms of the Lyapunov functional, the Lyapunov functional constructed by (3.3) is more general compared with those in [20,22,24] because the augmented vector of \( V_1(t) \) was applied in (3.3).

2) In terms of estimating the Lyapunov function’s derivative, only Jensen’s inequality was employed to deal with the upper bound of the cross term in [16–24], whereas in the present study, Lemma 1 is used to estimate the upper bound of the cross term \( \int_{t-t_1}^{t} \dot{\eta}(s) R_1 \eta(s) ds \). Lemmas 1 and 2 are employed to estimate the upper bound of the cross term \( \int_{t-t_1}^{t} \dot{\eta}(s) R_2 \eta(s) ds \), which divides into two integral terms.

3) In terms of the filter design, the zero inequality (3.10) is first applied to the \( H_\infty \) design problem, which is different from the approach used by [16–24]. The advantage of our method proposed is that the designed filter does not need to rely on the parameter matrix \( P \) of the constructed Lyapunov function, which is more flexible compared with those described by [16–24].

4. Numerical examples

Example 1. Consider the same fuzzy system as [23,24] with the following parameters

\[
A_1 = \begin{bmatrix} -2.1 & 0.1 \\ 1 & -2 \end{bmatrix}, \quad A_{r1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}, \\
A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \quad A_{r2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}, \\
B_1 = \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}^T, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
C_{r1} = \begin{bmatrix} -0.8 & 0.6 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.5 & -0.6 \end{bmatrix}, \quad C_{r2} = \begin{bmatrix} -0.2 & 1 \end{bmatrix}, \\
D_1 = 0.3, \quad D_2 = -0.6, \quad L_1 = \begin{bmatrix} 1 & -0.5 \end{bmatrix}, \quad L_{r1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\
L_2 = \begin{bmatrix} -0.2 & 0.3 \end{bmatrix}, \quad L_{r2} = \begin{bmatrix} 0 & 0.2 \end{bmatrix}, \\
h_1(\theta(t)) = \sin^2(t), \quad h_2(\theta(t)) = \cos^2(t).
\]

Case I: When \( \Delta_1(t) = \Delta_2(t) = 0 \), the designed filter reduces to those studied by [17–20,22–24]. In this case, the minimum \( H_\infty \) performance \( \gamma \) is given in [17–20], which relies on the constant parameter \( \delta \). We note that the scalar \( \delta \) is not needed in the present study. Furthermore, when the lower bound \( \tau_1 \neq 0 \), the results proposed in [17–20] are infeasible. In addition, information regarding the derivative of the time delay \( \mu \) was not considered by [23]. The minimum \( H_\infty \) performance indices \( \gamma \) obtained by employing Corollary 1 and the methods in [22,23], and [24] for different cases are shown in Table 1, which shows that our method yields less conservative results than the previously reported results.

We provide simulation results to demonstrate the effectiveness of our proposed method. By taking the time-varying delay as \( \tau(t) = 1.025 + 0.225 \sin(1.7778t) \), the lower bound, upper bound, and the derivative of time delay are \( \tau_1 = 0.8 \), \( \tau_2 = 1.25 \), \( \mu = 0.4 \), respectively. Let \( \varepsilon = 0.2 \) and let the minimum \( H_\infty \) performance index \( \gamma \) be 0.2503. By solving the LMIs in Corollary 1, the corresponding filter parameters are obtained as follows:
Table 1
Comparison of the minimum $\mathcal{H}_\infty$ performance for different cases with $\tau_2 = 1.25$.

<table>
<thead>
<tr>
<th>$\tau_1$</th>
<th>Method</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1/unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[22]</td>
<td>0.32</td>
<td>0.49</td>
<td>0.84</td>
<td>1.14</td>
</tr>
<tr>
<td></td>
<td>[23]</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>[24]</td>
<td>0.2942</td>
<td>0.3158</td>
<td>0.3597</td>
<td>0.4099</td>
</tr>
<tr>
<td></td>
<td>Corollary 1</td>
<td>0.2497</td>
<td>0.2742</td>
<td>0.2956</td>
<td>0.3005</td>
</tr>
<tr>
<td></td>
<td>[22]</td>
<td>0.32</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>[23]</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.37</td>
</tr>
<tr>
<td>0.8</td>
<td>[24]</td>
<td>0.2905</td>
<td>0.2949</td>
<td>0.2949</td>
<td>0.2949</td>
</tr>
<tr>
<td></td>
<td>Corollary 1</td>
<td>0.2503</td>
<td>0.2516</td>
<td>0.2516</td>
<td>0.2516</td>
</tr>
<tr>
<td></td>
<td>[22]</td>
<td>0.28</td>
<td>0.28</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>[23]</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.36</td>
</tr>
<tr>
<td>1</td>
<td>[24]</td>
<td>0.2792</td>
<td>0.2792</td>
<td>0.2792</td>
<td>0.2792</td>
</tr>
<tr>
<td></td>
<td>Corollary 1</td>
<td>0.2356</td>
<td>0.2356</td>
<td>0.2356</td>
<td>0.2356</td>
</tr>
</tbody>
</table>

Fig. 1. Filtering error $e(t)$ for case I in Example 1.

$$A_{f1} = \begin{bmatrix} -1.9272 & -1.2183 \\ 1.4660 & -3.6573 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -1.4442 \\ 0.0761 \end{bmatrix},$$
$$C_{f1} = \begin{bmatrix} -0.7428 & 0.5102 \end{bmatrix}, \quad D_{f1} = 0.0855,$$
$$A_{f2} = \begin{bmatrix} -2.9107 & 1.0077 \\ -1.2860 & -0.8735 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} -0.4389 \\ 0.7989 \end{bmatrix},$$
$$C_{f2} = \begin{bmatrix} 0.1909 & -0.3441 \end{bmatrix}, \quad D_{f2} = 0.1693.$$

For the simulation, we set the initial condition as $x(0) = [-0.5, 0.5]^\top$ and

$$w(t) = \begin{cases} \sin(t), & 0 < t < 15 \\ 0, & \text{otherwise.} \end{cases}$$

Using the filter parameters given above, the error $e(t)$ is shown in Fig. 1, which indicates that the designed $\mathcal{H}_\infty$ filter can stabilize the system considered. Furthermore, the responses of $z(t)$ and $z_f(t)$ are shown in Fig. 2.

Case II: When $\Delta_1(t) = \Delta_2(t) = \sin(t)$, the uncertainty parameters are given as follows:

$$M_{a1} = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}^\top, \quad M_{a2} = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix}^\top, \quad N_{b1} = 0.5,$$
$$M_{b1} = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix}^\top, \quad M_{b2} = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}^\top, \quad N_{b2} = 0.2,$$
Let $\alpha = 0.8, \beta = 0.5$. The stochastic variables $\alpha(t)$ and $\beta(t)$ are shown in Figs. 3 and 4, respectively. By taking $\varepsilon = 0.2, \varepsilon_i = 1(i = 1, 2, \ldots, 6)$ and $\tau(t) = 0.9 + 0.4 \sin(2.5t)$, it is easy to show that the lower bound and upper bound of the time delay $\tau(t)$ are $\tau_1 = 0.5$ and $\tau_2 = 1.3$, respectively. The upper bound of the derivative of the time delay is $\mu = 1$. By solving the LMIs (3.1) in Theorem 1, the minimum $\mathcal{H}_\infty$ performance $\gamma$ obtained is 0.7321 and the corresponding filter parameters are given as follows:

$$
N_{a1} = \begin{bmatrix} 0.3 & 0 \end{bmatrix}, \quad N_{a2} = \begin{bmatrix} 0.2 & 0 \end{bmatrix}.
$$

$$
A_{f1} = \begin{bmatrix} -1.8942 & -0.5040 \\ 2.1763 & -3.0611 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -0.6744 \\ 0.6135 \end{bmatrix},
$$

$$
C_{f1} = \begin{bmatrix} -0.8064 & 0.4061 \end{bmatrix}, \quad D_{f1} = 0.0090,
$$

$$
A_{f2} = \begin{bmatrix} -1.9488 & -0.3635 \\ 0.6972 & -2.5353 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} 0.1743 \\ 1.3991 \end{bmatrix},
$$

$$
C_{f2} = \begin{bmatrix} 0.4448 & -0.4528 \end{bmatrix}, \quad D_{f2} = 0.2103.
$$

For simulation purposes, we assumed that the initial condition and disturbance $w(t)$ were the same as Case I. Using the filtering parameters given above, and the stochastic variables $\alpha(t)$ and $\beta(t)$ in Figs. 3 and 4, the filtering error $e(t)$ is presented in Fig. 5, which shows that the designed $\mathcal{H}_\infty$ filter can stabilize the considered system. Furthermore, the responses of $z(t)$ and $z_f(t)$ are shown in Fig. 6.
Fig. 4. The stochastic variable $\beta(t)$ in Example 1.

Fig. 5. Filtering error $e(t)$ for case II in Example 1.

Fig. 6. Responses of $z(t)$ and $z_f(t)$ for case II in Example 1.
Example 2. To further illustrate the effectiveness of the proposed design method, we consider a computer-simulated truck–trailer [22,24]. In this system, it is assumed that the state \( x_1(t) \) is perturbed by a time-varying delay

\[
\dot{x}_1(t) = -\lambda \frac{\bar{v} t}{L_0} x_1(t) - (1 - \lambda) \frac{\bar{v} t}{L_0} x_1(t - \tau(t)) + \frac{\bar{v} t}{L_0} x_1(t) u(t) + \nu(t),
\]

\[
\dot{x}_2(t) = \lambda \frac{\bar{v} t}{L_0} x_1(t) + (1 - \lambda) \frac{\bar{v} t}{L_0} x_1(t - \tau(t)) + 0.5 \nu(t),
\]

\[
\dot{x}_3(t) = \frac{\bar{v} t}{L_0} \sin \left( x_2(t) + \lambda \frac{\bar{v} t}{2L_0} x_1(t) + (1 - \lambda) \frac{\bar{v} t}{2L_0} x_1(t - \tau(t)) \right),
\]

where \( x_1(t) \) is the angle difference between the truck and the trailer, \( x_2(t) \) is the angle of the trailer, \( x_3(t) \) is the vertical position of the rear end of the trailer, \( u(t) \) is the steering angle, and \( \nu(t) \) is the disturbance signal.

The model parameters are given as \( l = 2.8, L = 5.5, v = -1.0, \bar{t} = 2.0, t_0 = 0.5, \) and \( \lambda = 0.7 \). The time-varying delay is given by \( \tau(t) = 2 + 1.05 \sin(t) \). Let \( \theta(t) = x_2(t) + \lambda \frac{\bar{v} t}{2L_0} x_1(t) + (1 - \lambda) \frac{\bar{v} t}{2L_0} x_1(t - \tau(t)) \), \( |\theta(t)| \leq 179.4270^\circ \), \( \bar{g} = 0.01/\pi \), and the membership functions are selected as follows:

\[
h_1(\theta(t)) = \begin{cases} 
\sin(\theta(t)) - \bar{g} \theta(t) & \text{if } \theta(t) \neq 0, \\
1, \theta(t) = 0,
\end{cases}
\]

\[
h_2(\theta(t)) = \begin{cases} 
\theta(t) - \sin(\theta(t)) & \text{if } \theta(t) \neq 0, \\
0, \theta(t) = 0.
\end{cases}
\]

From the membership functions given above, it is easy to show that \( h_1(\theta(t)) = 1, h_2(\theta(t)) = 0 \) when \( \theta(t) \) is about 0 rad; \( h_1(\theta(t)) = 0, h_2(\theta(t)) = 1 \) when \( \theta(t) \) is about \( \pi \) or \( -\pi \) rad. Therefore, the nonlinear system described above can be represented exactly by the following T–S fuzzy model.

Plant Rule 1: IF \( \theta(t) \) is about 0 rad, THEN

\[
\dot{x}(t) = A_1 x(t) + A_{r1} x(t - \tau(t)) + B_1 w(t),
\]

Plant Rule 2: IF \( \theta(t) \) is about \( \pi \) or \( -\pi \) rad, THEN

\[
\dot{x}(t) = A_2 x(t) + A_{r2} x(t - \tau(t)) + B_2 w(t),
\]

where \( x(t) = [x_1^T(t), x_2^T(t), x_3^T(t)]^T \) and

\[
A_1 = \begin{bmatrix} -1.6371 & 0.5741 & -0.0157 \\ -0.5091 & 0 & 0 \\ 0.5091 & -4 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.6086 & 0.5284 & -0.0150 \\ -0.5091 & 0 & 0 \\ 0.0016 & -0.0127 & 0 \end{bmatrix},
\]

\[
A_{r1} = \begin{bmatrix} 0.2182 & 0 & 0 \\ -0.2182 & 0 & 0 \\ 0.2182 & 0 & 0 \end{bmatrix}, \quad A_{r2} = \begin{bmatrix} 0.2182 & 0 & 0 \\ -0.2182 & 0 & 0 \\ 0.0007 & 0 & 0 \end{bmatrix},
\]

\[
B_1 = B_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \quad D_1 = D_2 = -0.5.
\]

Now, we suppose that the parameters of the measurement and the signal that need to be estimated are given by

\[
C_1 = C_2 = [4.9, -2, 0.04], \quad C_{r1} = C_{r2} = [2.1, 0, 0],
\]

\[
L_1 = L_2 = [0, 1, 0], \quad L_{r1} = L_{r2} = [0, 0, 0],
\]

\[
G_1 = C_2 = 0, \quad M_{a1} = M_{a2} = [0.2, 0.4, 0.1]^T,
\]

\[
M_{b1} = M_{b2} = [0.3, 0.5, 0.2]^T, \quad N_{a1} = N_{a2} = [1, 0, 0],
\]

\[
N_{b1} = N_{b2} = 0.2, \quad \Delta_1(t) = \Delta_2(t) = \sin(t).
\]
In this example, the main aim is to design a non-fragile $\mathcal{H}_\infty$ filter for the truck–trailer system defined above. Thus, let $\xi = 2, \xi_i = 1$ ($i = 1, 2, \ldots, 6$) and $\tau(t) = 2 + \sin(t)$. It is easy to show that the lower bound and upper bound of the time delay $\tau(t)$ are $\tau_1 = 1$ and $\tau_2 = 3$, respectively. The upper bound of the derivative of the time delay is $\mu = 1$. By solving the LMIs (3.1) in Theorem 1, the minimum $\mathcal{H}_\infty$ performance $\gamma$ obtained is $7.0948$ using the same stochastic variables employed in Example 1, and the corresponding filter parameters can also be obtained as follows:

$$A_{f1} = \begin{bmatrix} -7.5955 & 5.3071 & -0.1235 \\ -3.8738 & 1.8946 & -0.0330 \\ -4.1943 & 1.9533 & -0.2195 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} -1.1278 \\ -0.4048 \\ -0.4572 \end{bmatrix}, \quad D_{f1} = 0.0706,$$

$$C_{f1} = \begin{bmatrix} 0.5356 & -0.6497 & 0.0054 \end{bmatrix}, \quad A_{f2} = \begin{bmatrix} -7.4660 & 5.2317 & -0.1209 \\ -3.6947 & 1.6817 & -0.0390 \\ -8.2420 & 6.1317 & -0.1496 \end{bmatrix},$$

$$B_{f2} = \begin{bmatrix} -1.1448 \\ -0.4057 \\ -0.9391 \end{bmatrix}, \quad D_{f2} = 0.0561, \quad C_{f2} = \begin{bmatrix} 0.2117 & -0.1367 & -0.0250 \end{bmatrix}. $$
For simulation purpose, it is assumed that the initial condition is \( x(0) = \begin{bmatrix} -0.25\pi & 0.75\pi & -10 \end{bmatrix}^\top \), and the disturbance signal \( w(t) \) is given as \( w(t) = \frac{1}{15+3.14t}, t \geq 0 \). The filtering error response \( e(t) \) is depicted in Fig. 7, which shows that the designed \( \mathcal{H}_\infty \) filter is feasible and effective. Finally, the responses of \( z(t) \) and \( z_f(t) \) are shown in Fig. 8.

5. Conclusions

In this study, we addressed the non-fragile \( \mathcal{H}_\infty \) filtering problem for T–S fuzzy systems with an interval time-varying delay and randomly occurring gain variations. By constructing an augmented Lyapunov functional and employing an improved integral inequality combined with a reciprocally convex approach, we obtained a sufficient condition for designing the non-fragile \( \mathcal{H}_\infty \) filter such that the filtering error system is asymptotically mean-square stable with prescribed \( \mathcal{H}_\infty \) performance. Furthermore, we derived an improved result for \( \mathcal{H}_\infty \) filtering in a delayed T–S fuzzy system without randomly occurring gain variations. We provided two numerical examples and an application of a filter for a truck–trailer system to demonstrate the effectiveness of the proposed design method. Further improvement may be possible using a new method such as relaxed integral inequalities [36].

Acknowledgements

This study was supported by a 2016 Yeungnam University Research Grant, Republic of Korea. Also, this work was supported in part by the China Postdoctoral Science Foundation (2016M601154).

References