Results on stability of linear systems with time varying delay

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Abstract: The integral inequality approach has been widely used to obtain delay-dependent stability criteria for dynamic systems with delays, and finding integral inequalities for quadratic functions hence plays a key role in reducing conservativeness of corresponding stability conditions. In this study, an improved integral inequality which covers several well-known integral inequalities is introduced, and improves thereby stability for linear systems with time-varying delay. Three numerical examples are given to demonstrate the effectiveness and superiority of the proposed method.

1 Introduction

Time delays are frequently encountered in many physical, industrial and engineering systems. The existence of time delay may result in poor performance, oscillation or even instability for dynamical systems. The stability analysis for time-delay systems has therefore become one of the important issues in system control, and numerous results have been derived over the past few decades [1–31].

The main objective of stability analysis is to find an admissible maximal delay bound which is more accurate to guarantee asymptotic stability for time delay systems. One of the popular methods to address the problem is in the framework of Lyapunov–Krasovskii stability theory and linear matrix inequality (LMI). In order to find the admissible maximal delay bound, one important issue is to reduce the conservativeness of stability criteria. Naturally, the conservativeness of the Lyapunov functional approach depends on Lyapunov–Krasovskii functional itself and the upper bound of derivative of the constructed Lyapunov–Krasovskii functional. On the one hand, choosing an appropriate Lyapunov–Krasovskii functional is curial to derive less conservative criteria. Lyapunov–Krasovskii functionals with simple forms have been extensively used to investigate the stability problems [16]. It is well known that simple Lyapunov–Krasovskii functional will lead to conservative in some degree. Compared with the simple functionals [1], augmented-based [5, 6, 9] and delay-partition-based functionals [13, 14] have been constructed to improve the stability criteria. On the other hand, obtaining tighter bounds of derivative of the constructed Lyapunov–Krasovskii functional, especially interval terms appeared in the derivative, also plays a key role in reducing the conservativeness. There are two main techniques to deal with such integral terms, free-weighting matrix approach [9] and integral inequality method [11].

In general, the integral inequality method uses Jensen's inequality [15, 16, 19]. However, an alternative inequality based on Wirtinger-based inequality, which contains Jensen's inequality as a special case, was presented in [22]. Based on this alternative inequality [22], various inequalities were proposed and have been successfully applied to the stability of time-delay systems [23–25]. Furthermore, a new free-matrix-based inequality was developed to reduce the conservatism [28]. In [29], it is proved that the free-weighting matrix approach [28] is equivalent to the one proposed in [22]. Very recently, based on the Legendre polynomials and Bessel inequality, a novel integral inequality, called Bessel-Legendre (B-L) inequality encompassing the Jensen's inequality and the Wirtinger-based integral, has been proposed [30]. Two general inequalities have also been proposed in [31]. However, these methods suffer some common shortcomings. On one hand, B-L inequality only used for constant delay. On the other hand, inequalities in [28] only deal with the derivative of the state for the quadratic terms while upper bounds of the state for the quadratic terms cannot be estimated. Thus, it is necessary and important to further study the stability of time delay system, which is the motivation of this paper.

Based on the above discussions, a new integral inequality is developed, which covers Wirtinger-based integral inequality [22] and free matrix-based integral inequality [28] as special cases. Based on the new inequality and modified augmented Lyapunov–Krasovskii functional, some new delay-dependent stability criteria are presented and can be applied to systems with various kinds of time varying delay problems. The proposed criteria are less conservative than the existing results. Three numerical examples are provided to illustrate the effectiveness of the method and its superiority over the others.

The remaining of the paper is organised as follows. Section 2 gives the problem formulation and necessary preliminary. In Section 3, some less conservative stability criteria for various cases of time delay are presented. Three numerical examples are used to illustrate the benefits of the proposed criteria in Section 4 and a conclusion is given in Section 5.

Notations: Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; the superscripts ‘$-1$’ and ‘$T$’ stand for the inverse and transpose of a matrix, respectively. $\mathbb{R}^{m \times n}$ is the set of all $n \times m$ real matrices; $\text{Sym}(X) = X + X^T$. For a real matrix $X$, $X > 0$ and $X < 0$ mean that $X$ is a positive/negative definite symmetric matrix, respectively. We always use $I$ to denote identity matrix with appropriate dimension. $\ast$ is the symmetric part of given matrix, and diag(...) denotes the block diagonal matrix.

2 Problem statement

Consider the following linear system with time varying delay:

\[\begin{align*}
x(t) &= Ax(t) + A_x x(t - h(t)), \\
x(t) &= \phi(t), \ t \in \left[-h_0, 0\right],
\end{align*}\]

\((1)\)
where \( x(t) \in \mathbb{R}^n \) is the state vector, \( A \) and \( A_i \) are known constant matrices with appropriate dimensions, \( \phi(t) \) is a continuously differentiable function. The time delay \( h(t) \) satisfies the following conditions:

\[
0 \leq h_1 \leq h(t) \leq h_2, \quad \mu_1 \leq h(t) \leq \mu_2
\]

(2)

where \( h_2 > h_1 \geq 0, \mu_1, \) and \( \mu_2 \) are constants.

The first objective of this work is to introduce a new integral inequality to investigate the stability of system (1).

**Lemma 1** ([22]): For a symmetric positive definite matrix \( R \in \mathbb{R}^{n \times n} \), and a continuous function \( x(u) \) in \([a, b] \rightarrow \mathbb{R}^n\), the following inequality holds:

\[
(b-a) \int_a^b x^T(s)Rx(s) \, ds \geq \begin{bmatrix} \Omega_1^T \\ \Omega_2 \end{bmatrix} R \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix} \]

where

\[
\begin{align*}
\Omega_1 &= \int_a^b x(s) \, ds, \\
\Omega_2 &= -\Omega_1 + \frac{2}{b-a} \int_a^b \int_s^b x(u) \, du \, ds, \\
R &= \begin{bmatrix} R & 0 \\ * & 3R \end{bmatrix}
\end{align*}
\]

Inspired by Lemma 4 in [2], the following lemma can be derived.

**Lemma 2**: For a symmetric positive definite matrix \( R \in \mathbb{R}^{n \times n} \) and a continuous function \( x(u) \) in \([a, b] \rightarrow \mathbb{R}^n\). Taking \( \Phi \in \mathbb{R}^{n \times k} \) \((i = 1, 2)\) and a vector \( \xi \in \mathbb{R}^k \) such that

\[
\begin{align*}
\int_a^b x(s) \, ds &= \Phi_1 \xi, \\
-\int_a^b x(s) \, ds + \frac{2}{b-a} \int_a^b x(u) \, du \, ds &= \Phi_2 \xi,
\end{align*}
\]

then, for any constant matrices \( M_i \in \mathbb{R}^{n \times k} \) \((i = 1, 2)\), the following inequality holds:

\[
-\int_a^b x^T(s)Rx(s) \, ds \leq \xi^T \begin{bmatrix} \Phi_1 M + M^T \Phi & (b-a)M^T \hat{R}^{-1} M \\ \hat{R} \end{bmatrix} \xi,
\]

where \( \hat{R} = \begin{bmatrix} \Phi_1^T \Phi_2 \end{bmatrix}^T, M = \begin{bmatrix} M_1^T \\ M_2 \end{bmatrix}^T \).

**Proof**: Rewrite Lemma 1 as

\[
(b-a) \int_a^b x^T(s)Rx(s) \, ds \geq \xi^T \begin{bmatrix} \Phi_1^T \\ \Phi_2 \end{bmatrix} \begin{bmatrix} \Phi_1 M \\ \hat{R} \end{bmatrix} \xi = \xi^T \hat{R} \Phi \Phi^T \xi.
\]

For any constant matrix \( M = \begin{bmatrix} M_1^T \\ M_2 \end{bmatrix}^T \), the following inequality holds:

\[
[\hat{R} \Phi + (b-a)M]^T \hat{R}^{-1} [\hat{R} \Phi + (b-a)M] \geq 0.
\]

Hence

\[
\Phi^T \hat{R} \Phi \geq -(b-a)(\Phi^T M + M^T \Phi) - (b-a)^2 M^T \hat{R}^{-1} M,
\]

which completes the proof of the lemma. □

**Remark 1**: Taking \( \Phi_1 = [I \ 0], \Phi_2 = [-I \ 2I], \xi = [(1/(b-a))^\int_a^b x^T(s) \, ds \ (1/(b-a))^\int_a^b x^T(u) \, du] \), i.e. need to add Transpose \( M_1 = -(1/(b-a))R\Phi_1 \), and \( M_2 = -(3/(b-a))R\Phi_2, \) then Lemma 2 concludes Corollary 1.

**Remark 2**: When the function \( x(u) \) is continuously differentiable, replacing \( x(t) \) by \( x(t) \) then Lemma 2 can also be used to estimate the upper bound of \(-(b-a)^\int_a^b x^T(s)Rx(s) \, ds\).

**Remark 3**: The integral inequality proposed in Lemma 2 can cover the one presented in [22] and free-matrix-based integral inequality presented in [28]. Actually, replacing \( x(t) \) by \( x(t) \) as pointed out in Remark 2, and taking \( \xi = x^T(b) x^T(a), \) \((1/(b-a))^\int_a^b x^T(s) \) \( dx \), \( \Phi_1 = [I \ I], \Phi_2 = [I \ 1-2I], \) \( M_1 = -(1/(b-a))R\Phi_1 \), and \( M_2 = -(3/(b-a))R\Phi_2, \) then Lemma 2 concludes Corollary 5 of [22] which is equivalent to that of free-matrix-based integral inequality in [28], this fact proved in [29]. On the other hand, Lemma 2 can be used not only to estimate the upper bound of \(-(b-a)^\int_a^b x^T(s)Rx(s) \, ds\), but also to estimate the upper bound of \(-(b-a)^\int_a^b x^T(s)Rx(s) \, ds\). It is noted that the inequality proposed in [28] can only be used to estimate the upper bound of \(-(b-a)^\int_a^b x^T(s)Rx(s) \, ds\).

### 3 Main results

In this section, some delay-dependent stability criteria for system (1) are presented. First, Theorem 1 gives a stability criterion when the lower bound of time delay is not zero and the information of derivative of time delay is not known.

**Theorem 1**: For given scalars \( h_1 > h_2 > 0 \), system (1) is asymptotically stable if there exist matrices \( P > Q_i > 0, Q_i > 0, R_i > 0 \) \((i = 1, 2, 3)\) and \( M_i = \begin{bmatrix} M_{i1}^T \\ M_{i2} \end{bmatrix}(j = 1, 2, \ldots, 5) \) with appropriate dimensions such that

\[
\begin{bmatrix} [\Sigma_1 + \Sigma_2]_{h(t)-h_1} \\
[\Sigma_1 + \Sigma_2]_{h(t)-h_2} \end{bmatrix} \begin{bmatrix} h_1 M_{11} & (h_2 - h_1) M_{12} \\
(h_2 - h_1) M_{12} & (h_2 - h_1) M_{22} \end{bmatrix} < 0,
\]

(3)

\[
\begin{bmatrix} -h_i R_i & 0 & 0 \\
0 & -(h_2 - h_i) R_i & 0 \\
0 & 0 & -(h_2 - h_i) R_i \end{bmatrix} < 0.
\]

(4)

where \( \Sigma_1 \) and \( \Sigma_2 \) (see equation below) and \( e_i \in \mathbb{R}^{m \times n} \) denotes the block entry matrices: \( e_i^T \xi(t) = x(t - h_i) \) with \( \xi(t) \) (see equation below)

**Proof**: Construct the following Lyapunov–Krasovskii functional candidate:

\[
V(t) = \sum_{i=1}^6 V_i(t),
\]

where
The time-derivative of $\dot{x}$ along the trajectories of (1) is calculated as

$$\dot{V}_i(t) = \begin{bmatrix} x(t) \\ \pi \int_{t-h}^{t-h} x(s) \, ds \\ \int_{t-h}^{t-h} x(s) \, ds \end{bmatrix} \begin{bmatrix} p \int_{t-h}^{t-h} x(s) \, ds \\ \pi \int_{t-h}^{t-h} x(s) \, ds \\ \int_{t-h}^{t-h} x(s) \, ds \end{bmatrix} x(t)$$

$$\dot{V}_i(t) \leq x^T(t)Q_i x(t) - x^T(t-h_i)Q_i x(t-h_i),$$

$$\dot{V}_i(t) = x^T(t-h_i)Q_i x(t-h_i) - x^T(t-h_i)Q_i x(t-h_i).$$

The proof of the theorem. □

From (6)–(18), it follows that:

$$V_i(t) \leq \bar{\beta}_i^T(t) \bar{P}_i \bar{P}_i^T(t) + \dot{V}_i(t),$$

$$\dot{V}_i(t) \leq h_i x_i^T(t) R_i \dot{x}(t) - \int_{t-h_i}^{t-h_i} x_i^T(s) R_i \dot{x}(s) \, ds,$$

$$V_i(t) \leq (h_i - h_i) x_i^T(t) R_i \dot{x}(t) - \int_{t-h_i}^{t-h_i} x_i^T(s) R_i \dot{x}(s) \, ds$$

(see (11))

$$V_i(t) \leq (h_i - h_i) x_i^T(t) R_i \dot{x}(t) - \int_{t-h_i}^{t-h_i} x_i^T(s) R_i \dot{x}(s) \, ds$$

(see (13)) For any matrices $M_j = [M_{ij}, M_{ji}] (j = 1, 2, \ldots, 5)$, the following inequality can be obtained from Lemma 2:

$$\int_{t-h_i}^{t} x_i^T(s) R_i \dot{x}(s) \, ds \leq \xi_i(t) [E_i M_i^T + M_i E_i + h_i \bar{R}_i \bar{R}_i^T M_i] \xi_i(t),$$

(14)

From (6)–(18), it follows that:

$$V_i(t) \leq \xi_i(t) (\Sigma_i + \Sigma_i + \Sigma_i) \xi_i(t),$$

(19)

$$\Sigma_i = h_i M_i^T \bar{R}_i \bar{R}_i^T M_i + (h_i - h_i) (M_i^T \bar{R}_i \bar{R}_i^T M_i + M_i \bar{R}_i \bar{R}_i^T M_i)$$

Since $\Sigma_i + \Sigma_i + \Sigma_i$ is a convex combination of $h_i - h_i$ and $h_i(1 - h_i)$, combining with Schur complement, $\Sigma_i + \Sigma_i + \Sigma_i < 0$ if and only if (3) and (4) hold. This completes the proof of the theorem.

When $h_i$ is zero, the Lyapunov–Krasovskii functional (5) reduces to

$$V_i(t) = \bar{P}_i^T(t) \bar{P}_i \bar{P}_i^T(t) + \dot{V}_i(t),$$

(20)
where \( V_i(t) = \int_{t-h_i}^{t} x^T(s)Q_i x(s) \, ds + \int_{t-h_i}^{t} \beta_i x^T(s)R_i x(s) \, ds \, da + \int_{t-h_i}^{t} \beta_i x^T(s)R_i x(s) \, ds \, da \), \( \beta_i^T(t) = [x^T(t) \quad \int_{t-h_i}^{t} x^T(s) \, ds] \).

By Theorem 1, we can have the following corollary when the lower bound is zero and the information of derivative of time delay is unknown.

**Corollary 1:** For a given scalar \( h > 0 \), system (1) is asymptotically stable if there exist matrices \( P > 0 \), \( Q_i > 0 \), \( R_i > 0 \), \( i = 2, 3 \) and \( M_j = [M_{ij} \quad M_{ji}](j = 2, ..., 5) \) with appropriate dimensions such that

\[
\begin{bmatrix}
\{\Xi_1 + \Xi_2\}_{h(t)} = -h\bar{R}_2 & 0 \\
-\bar{h}R_2 & -\bar{h}R_2 \\
0 & 0
\end{bmatrix} < 0, \quad (21)
\]

\[
\begin{bmatrix}
\{\Xi_1 + \Xi_2\}_{h(t) = h} = hM_3 & hM_4 \\
-\bar{h}R_2 & -\bar{h}R_2 \\
0 & 0
\end{bmatrix} < 0, \quad (22)
\]

where

\[
\bar{\Xi}_i = \text{Sym}(\Pi_i P\Pi_i^T) + \bar{e}_i Q_i \bar{e}_i^T - \bar{e}_i Q_i \bar{e}_i^T + h_i (E_i R_i \bar{e}_i^T + \bar{e}_i R_i \bar{e}_i^T),
\]

\[
\Xi_2 = \text{Sym}(\bar{E}_i M_i^T + \bar{E}_i M_i^T + \bar{E}_i M_i^T + \bar{E}_i M_i^T),
\]

\[
\Pi_i = [\bar{e}_i - h(t)\bar{e}_i], \quad \Pi_i = [\bar{e}_i - \bar{e}_i],
\]

and \( \bar{e} \in \mathbb{R}_n^m \) means the block entry matrices:

\[
\bar{\Xi}_i^T (t) = x(t) - h(t)\]

\[
\frac{1}{h(t)} \left( \int_{t-h(t)}^{t} x^T(s) \, ds \right)
\]

\[
\frac{1}{h(t)} \left( \int_{t-h(t)}^{t} x^T(s) \, ds \right)
\]

\[
\frac{1}{h(t-h(t))} \left( \int_{t-h(t)}^{t} x^T(s) \, ds \right)
\]

\[
\frac{1}{h(t)} \left( \int_{t-h(t)}^{t} x^T(s) \, ds \right)
\]

\[
\frac{1}{h(t-h(t))} \left( \int_{t-h(t)}^{t} x^T(s) \, ds \right)
\]

Lastly, when \( h_i \) is zero and the derivative of time delay satisfies \( \mu_i \leq h_i \leq \mu_i \), let us consider the following Lyapunov–Krasovskii functional:

\[
V(t) = \bar{\beta}_i^T(t)P \bar{\beta}_i(t) + V_i(t) + \int_{t-h(t)}^{t} \eta^T(s, t)S_i \eta(s, t) \, ds
\]

\[
+ \int_{t-h(t)}^{t} \eta^T(s, t)S_i \eta(s, t) \, ds.
\]

where

\[
\beta_i^T(t) = [x^T(t) \quad \int_{t-h(t)}^{t} x^T(s) \, ds \quad \int_{t-h(t)}^{t} x^T(s) \, ds],
\]

\[
\eta^T(t, s) = [x^T(s) \quad \int_{t-h(t)}^{t} x^T(r) \, dr].
\]

**Corollary 2:** For given scalars \( h > 0 \), \( \mu_i \) and \( \mu_i \), system (1) is asymptotically stable if there exist matrices \( P > 0, \bar{S}_i > 0, \bar{S}_i > 0, \bar{Q}_i > 0, \bar{R}_i > 0 \), \( i = 2, 3 \), and \( M_j = [M_{ij} \quad M_{ji}](j = 2, ..., 5) \) with appropriate dimensions such that

\[
\begin{bmatrix}
\{\Xi_1 + \Xi_2\}_{h(t) = h} + \eta^T(s, t)S_i \eta(s, t) \, ds + \int_{t-h(t)}^{t} x^T(s) \, ds
\end{bmatrix} < 0, \quad (24)
\]

where

\[
\bar{\Xi}_i = \Xi_i - \text{Sym}(\Pi_i P\Pi_i^T) + \text{Sym}(\Pi_i P\Pi_i^T) + [\bar{e}_i - \bar{e}_i]S_i [\bar{e}_i - \bar{e}_i]^T
\]

\[
- (1 - h(t))[\bar{e}_i - \bar{e}_i]S_i [\bar{e}_i - \bar{e}_i]^T
\]

\[
+ h(t) \text{Sym}([\bar{e}_i - \bar{e}_i]S_i [\bar{e}_i - \bar{e}_i]^T)
\]

\[
+ (1 - h(t))[\bar{e}_i - \bar{e}_i]S_i [\bar{e}_i - \bar{e}_i]^T - [\bar{e}_i - \bar{e}_i]S_i [\bar{e}_i - \bar{e}_i]^T
\]

\[
+ (h_i - h(t)) \text{Sym}([\bar{e}_i - \bar{e}_i]S_i [\bar{e}_i - \bar{e}_i]^T)
\]

\[
\Pi_i = [\bar{e}_i - h_i \bar{e}_i], \quad \Pi_i = [\bar{e}_i - \bar{e}_i],
\]

**Remark 4:** In [16, 17, 18, 23, 25], the problem of stability for linear systems with interval time varying delay is considered, where the lower bound is not restricted to 0. In [18, 20, 22, 27, 28], the authors devote their effort to investigate the stability of linear
Table 1 Comparison of maximum delay bound \( h_1 \) for various \( h_1 \) in Example 1

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>0</th>
<th>0.3</th>
<th>0.6</th>
<th>1</th>
<th>2</th>
<th>Number of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[16]</td>
<td>1.52</td>
<td>1.59</td>
<td>1.69</td>
<td>1.90</td>
<td>2.56</td>
<td>( 18n^2 + 8n )</td>
</tr>
<tr>
<td>[17]</td>
<td>1.86</td>
<td>1.87</td>
<td>1.92</td>
<td>2.06</td>
<td>2.61</td>
<td>( 3.5n^2 + 2.5n )</td>
</tr>
<tr>
<td>[19]</td>
<td>1.70</td>
<td>1.78</td>
<td>1.89</td>
<td>2.09</td>
<td>2.69</td>
<td>( 21.5n^2 + 8.5n )</td>
</tr>
<tr>
<td>[23]</td>
<td>1.86</td>
<td>1.88</td>
<td>1.99</td>
<td>2.17</td>
<td>2.72</td>
<td>( 9.5n^2 + 5.5n )</td>
</tr>
<tr>
<td>[25]</td>
<td>2.14</td>
<td>2.17</td>
<td>2.22</td>
<td>2.33</td>
<td>2.80</td>
<td>( 21n^2 + 6n )</td>
</tr>
<tr>
<td>[26]</td>
<td>2.22</td>
<td>2.25</td>
<td>2.27</td>
<td>2.34</td>
<td>2.79</td>
<td>( 132.5n^2 + 3.5n )</td>
</tr>
<tr>
<td>Theorem 1 (Corollary 1)</td>
<td>2.24</td>
<td>2.26</td>
<td>2.28</td>
<td>2.34</td>
<td>2.80</td>
<td>( 97n^2 + 4n )</td>
</tr>
</tbody>
</table>

Table 2 Comparison of maximum delay bound \( h_2 \) for various \( \mu \) in Example 1

<table>
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<tr>
<th>( \mu )</th>
<th>0</th>
<th>0.1</th>
<th>0.5</th>
<th>0.8</th>
<th>1</th>
<th>Number of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[7]</td>
<td>3.604</td>
<td>2.008</td>
<td>1.364</td>
<td>0.999</td>
<td>( 5.5n^2 + 1.5n )</td>
<td></td>
</tr>
<tr>
<td>[9]</td>
<td>3.606</td>
<td>2.043</td>
<td>1.492</td>
<td>1.345</td>
<td>( 3n^2 + 3n )</td>
<td></td>
</tr>
<tr>
<td>[10]</td>
<td>3.605</td>
<td>2.043</td>
<td>1.492</td>
<td>1.345</td>
<td>( 6n^2 + 3n )</td>
<td></td>
</tr>
<tr>
<td>[16]</td>
<td>3.611</td>
<td>2.072</td>
<td>1.590</td>
<td>1.529</td>
<td>( 17.5n^2 + 7.5n )</td>
<td></td>
</tr>
<tr>
<td>[18]</td>
<td>3.86</td>
<td>2.33</td>
<td>1.93</td>
<td>1.86</td>
<td>( 46n^2 + 3n )</td>
<td></td>
</tr>
<tr>
<td>[20]</td>
<td>4.704</td>
<td>2.240</td>
<td>2.113</td>
<td>2.113</td>
<td>( 9n^2 + 3n )</td>
<td></td>
</tr>
<tr>
<td>[22]</td>
<td>4.703</td>
<td>2.420</td>
<td>2.137</td>
<td>2.128</td>
<td>( 10n^2 + 3n )</td>
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<td>[27]</td>
<td>4.753</td>
<td>2.429</td>
<td>2.183</td>
<td>2.182</td>
<td>( 27n^2 + 4n )</td>
<td></td>
</tr>
<tr>
<td>[28] (Corollary 1)</td>
<td>4.710</td>
<td>2.459</td>
<td>2.212</td>
<td>2.186</td>
<td>( 54n^2 + 9n )</td>
<td></td>
</tr>
<tr>
<td>Corollary 2</td>
<td>4.723</td>
<td>2.481</td>
<td>2.242</td>
<td>2.242</td>
<td>( 66n^2 + 5n )</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 Comparison of maximum delay bound \( h_2 \) for various \( h_1 \) in Example 2

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>0</th>
<th>0.3</th>
<th>0.5</th>
<th>0.8</th>
<th>1</th>
<th>Number of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[16]</td>
<td>0.87</td>
<td>1.07</td>
<td>1.21</td>
<td>1.45</td>
<td>1.61</td>
<td></td>
</tr>
<tr>
<td>[17]</td>
<td>1.06</td>
<td>1.24</td>
<td>1.38</td>
<td>1.60</td>
<td>1.75</td>
<td></td>
</tr>
<tr>
<td>[19]</td>
<td>1.04</td>
<td>1.24</td>
<td>1.39</td>
<td>1.61</td>
<td>1.77</td>
<td></td>
</tr>
<tr>
<td>[23]</td>
<td>1.11</td>
<td>1.29</td>
<td>1.43</td>
<td>1.64</td>
<td>1.79</td>
<td></td>
</tr>
<tr>
<td>[25]</td>
<td>2.66</td>
<td>2.94</td>
<td>3.12</td>
<td>3.40</td>
<td>3.59</td>
<td></td>
</tr>
<tr>
<td>[26]</td>
<td>2.74</td>
<td>3.04</td>
<td>3.23</td>
<td>3.52</td>
<td>3.70</td>
<td></td>
</tr>
<tr>
<td>Theorem 1 (Corollary 1)</td>
<td>3.26</td>
<td>3.52</td>
<td>3.68</td>
<td>3.90</td>
<td>4.04</td>
<td></td>
</tr>
</tbody>
</table>

Table 4 Comparison of maximum delay bound \( h_2 \) for various \( \mu \) in Example 3

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>3</th>
<th>Number of variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>[11]</td>
<td>1.99</td>
<td>1.81</td>
<td>1.75</td>
<td>1.61</td>
<td>1.60</td>
<td>( 9.5n^2 + 2.5n )</td>
</tr>
<tr>
<td>[18]</td>
<td>2.52</td>
<td>2.17</td>
<td>2.02</td>
<td>1.62</td>
<td>1.60</td>
<td>( 49n^2 + 3n )</td>
</tr>
<tr>
<td>[20]</td>
<td>3.03</td>
<td>2.55</td>
<td>2.36</td>
<td>1.69</td>
<td>1.66</td>
<td>( 9n^2 + 3n )</td>
</tr>
<tr>
<td>[28] (Corollary 1)</td>
<td>3.03</td>
<td>2.55</td>
<td>2.37</td>
<td>1.71</td>
<td>1.66</td>
<td>( 54n^2 + 9n )</td>
</tr>
<tr>
<td>Corollary 2</td>
<td>3.03</td>
<td>2.55</td>
<td>2.38</td>
<td>1.72</td>
<td>1.66</td>
<td>( 66n^2 + 5n )</td>
</tr>
</tbody>
</table>

Remark 5: For the constructed Lyapunov functional in [17–20, 22, 23, 25–28], \( V(x) \) in (6) or \( \int_0^\tau \int_{\tau}^\infty R(x)dx \text{d}s \) in (20) are not considered, which may lead to conservatism to some extent.

By using the proposed new lemma to deal with the quadratic term that appeared in the derivative of these terms, some less conservative stability criteria have been proposed, which will be demonstrated by the following numerical examples.

4 Numerical examples

In this section, three numerical examples are given to show the effectiveness of the proposed method.

Example 1: Consider system (1) with the following coefficient matrices:

\[
A = \begin{bmatrix}
-2 & 0 \\
0 & -0.9 \\
\end{bmatrix},
A_1 = \begin{bmatrix}
-1 & 0 \\
-1 & -1 \\
\end{bmatrix}.
\]

It is noted that the example is widely used in the literature. The purpose is to compare the maximum allowable upper bound of \( h(t) \) that guarantees the asymptotic stability of the above system. For given \( h_1 \), Table 1 lists the maximum allowable upper bounds derived by Theorem 1 (Corollary 1) along with those obtained by other methods. Furthermore, let \( \mu = \mu_2 = -\mu_1 \) and \( h_1 = 0 \), for various \( \mu \), the maximum allowable upper bounds obtained from Corollary 2 and the ones derived by other methods are shown in Table 2. The number of variables is also given to compare the computation complexity in Tables 1 and 2. From these tables, it is clear that the proposed techniques can provide larger upper bounds than those in the existing literature.

Example 2: Consider system (1) with the following coefficient matrices:

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & -2 \\
\end{bmatrix},
A_1 = \begin{bmatrix}
0 & 0 \\
0 & -1 \\
\end{bmatrix}.
\]

To demonstrate the effectiveness of the proposed method in this work, the comparative results on the maximal allowable \( h_2 \) for various \( h_1 \) are given in Table 3. As expected, larger maximum upper bounds are obtained by Theorem 1 (Corollary 1), which further demonstrates that it is less conservative than others.

Example 3: Consider system (1) with the following coefficient matrices:

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & -1 \\
\end{bmatrix},
A_1 = \begin{bmatrix}
0 & 0 \\
0 & -1 \\
\end{bmatrix}.
\]

For this example, the maximum allowable upper bounds of \( h(t) \) are compared among Corollary 2 and other conditions in the literature. The maximum allowable upper bounds for different \( \mu \) and the number of decision variables for different conditions is listed in Table 4. From Table 4, it is shown that the results of Corollary 2 give larger delay bound than those of the existing works listed therein.

5 Conclusion

In this paper, a novel integral inequality for quadratic functions based in Wirtinger inequality has been given. With this inequality, some improved delay-dependent stability criteria for time-delayed systems have been proposed in the form of linear matrix inequalities. The effectiveness of the results derived in this work has been shown by well-known three examples.

6 Acknowledgments

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7 References
