Robust Tracking Error Feedback Control for a One-Dimensional Schrödinger Equation

Jun-Jun Liu and Bao-Zhu Guo

Abstract—In this paper, we consider robust output tracking for a Schrödinger equation with external disturbances in all possible channels. The challenge of the problem comes from the fact that the observation operator is unbounded and the regulated output and the control are non-collocated. An observer-based approach is adopt in investigation. We first select specially some coefficients of the disturbances to obtain a nominal system which is a coupled PDE+ODE system. For this nominal system, we design a feedforward control by solving related regulator equation. An observer is then designed for the nominal system in terms of the tracking error only. As a result, an error feedback control is thus designed by replacing the state and disturbances in the feedforward control with their estimates obtained from the observer. We show that this observer based error feedback control is robust to disturbances in all possible channels and system uncertainty. The stability of the closed-loop and convergence are established by the Riesz basis approach. Some numerical simulations are presented to validate the results.

Index Terms—Schrödinger equation, output tracking, internal model principle, robust control.

I. INTRODUCTION

The Schrödinger equation is the cornerstone of quantum mechanics. Boundary control for Schrödinger equations has been active over the decades particularly for stabilization [16], [9], [15], [21]. However, in control engineering, it is most often not necessary to regulate all states of the system. In other words, we are only concerned with some output signals to be tracking the references and at the same time, to keep the system internally stable and all states to be uniformly bounded in the presence of external disturbance. The main systematic approach to solve output regulation is the internal model principle, which has been developed since from 1970s, first for the finite-dimensional systems [1], [4] and later on for the nonlinear ones [11] and then infinite dimensional linear systems [18], [19]. The general abstract output regulation for infinite-dimensional systems has been discussed in [12], [18] where a robust controller consists of a p-copy of the exosystem dynamics or satisfies so called G-condition. The controller is conditionally robust against perturbations in the system parameters provided that the closed loop remains stable.

On the other hand, there is still a gap between the abstract theory of the internal model principle for infinite-dimensional systems and the applications to systems described by the partial differential equations (PDEs). For finite-dimensional systems, the error feedback control is constructive but for infinite-dimensional systems, the error feedback control design has not a feasible way like finite-dimensional counterparts. The paper [2] considered a robust output feedback control for a parabolic system, where the reference signal was supposed to be known. A robust output regulation problem for a cascaded heat PDE-ODE system was considered in [14]. In [13], an extended state observer was designed to estimate the state, but the control and the performance output are collocated. A non-collocated output tracking problem was considered in [10] by means of the adaptive control method without the robustness.

In this paper, under the guidance of the internal model principle, we attempt to design a robust error feedback control for a one-dimensional Schrödinger equation where the output observation operator is unbounded and the regulated output and control are completely non-collocated, which bring some mathematical challenges. By abstract theory ([18], [19]), one can only expect very weak convergence with unbounded control and output operators. In this paper, however, we apply the Riesz basis approach to get more profound result on stability for closed-loop system and strong convergence for tracking error. The control design idea was recently introduced in [6] for a heat equation with bounded observation operator, which is motivated from observer based error feedback control theory for the output regulation of finite-dimensional systems in [11, Theorem 1.14, p.14] and [3], particularly the idea of [20] from abstract operator point of view. The problem of [6] utilizes the high regularity of the heat equation, which is not true for Schrödinger equation. The idea was also used to solve output regulation problem for Euler-Bernoulli beam equation in [7] where the regulated output and control are collocated.

This paper is organized as follows. In the next section, Section II, we formulate the problem and give a basic assumption. In Section III, we design an observer based error feedback control for a nominal system which is a couple PDE+ODE system, through feedforward control design, observer, and the separation principle. The 1-copy property is briefly discussed. The robustness to disturbances is discussed in Section IV where the Riesz basis property of the closed-loop system is developed which leads to the spectrum-determined growth condition for internal stability of the closed-loop. Some numerical simulations are presented in Section V to show the effectiveness of the proposed controller, followed up concluding remarks in Section VI.

At the end of this section, we introduce briefly the concept of the generalized eigenvector and algebraic multiplicity of the eigenvalues, which is used in latter discussion and can...
be found for instance in [5, p.141]. For a linear operator $A: D(A) \rightarrow H$ in a Hilbert space $H$, we say that $\phi$ is a 
generated eigenfunction if $H$ is a function space) of $A$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$ if there exists a positive integer $m$ such that $(\lambda - A)^m \phi = 0$. The minimal integer $m_\phi$ such that $(\lambda - A)^{m_\phi} \phi = 0$ if and only if $(\lambda - A)^{m_\phi+1} \phi = 0$ is called the algebraic multiplicity of $\lambda$. In particular, when $m_\phi = 1$, we say that $\lambda$ is an eigenvalue simple and in this case, the generalized eigenvector is just the usual eigenvector.

II. PROBLEM DESCRIPTION

The system that we consider in this paper is governed by the following one-dimensional Schrödinger equation with disturbances in all possible channels:

$$
\begin{cases}
  u_t(x, t) = -ju_x(x, t) + G(x)v(t), & x \in (0, 1), t > 0, \\
  u_x(0, t) = M_1 v(t), & t \geq 0, \\
  u_x(1, t) = U(t) + M_2 v(t), & t \geq 0, \\
  u(x, 0) = u_0(x), & 0 \leq x \leq 1, \\
  y_\phi(t) = u(0, t), & t \geq 0,
\end{cases}
$$

(2.1)

where $u(x, t) \in \mathbb{C}$ is the complex-valued state, $j$ is the imaginary unit: $j = \sqrt{-1}, G \in L^\infty(0, 1; \mathbb{C}^{1 \times n})$ is an unknown coefficient in-domain function, $M_i \in \mathbb{C}^{1 \times n}, i = 1, 2$ are unknown coefficients of the boundary disturbances, $U(t)$ is the control (input), $u_0(x)$ is the initial state, and $y_\phi(t)$ is the output to be regulated. The main difficulty for output regulation of system (2.1) lies in the fact that the control $U(t)$ and the output are non-collected. We consider system (2.1) in the state space $H = L^2(0, 1)$ with the norm $\| \cdot \|$ induced by the inner product: $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ for all $f, g \in H$. It is seen that in $H$, the observation operator is unbounded.

We suppose that both the disturbance $v(t)$ and reference signal $y_{\text{ref}}(t)$ are generated from a finite-dimensional exosystem described by

$$
\begin{cases}
  \dot{v}(t) = S v(t), & t \geq 0, \\
  v(0) = v_0 \in \mathbb{C}^n, \\
  y_{\text{ref}}(t) = N v(t), & t \geq 0,
\end{cases}
$$

(2.2)

where the matrix $S \in \mathbb{C}^{n \times n}$ is supposed to be known yet the initial state $v_0$ and $N \in \mathbb{C}^{1 \times n}$ are unknown, which make $y_{\text{ref}}(t)$ also unknown. The only measurement available for the control design is the tracking error $e(t)$ between the output $y_\phi(t)$ and the reference signal $y_{\text{ref}}(t): e(t) = y_\phi(t) - y_{\text{ref}}(t)$. The target of the output regulation is to design a tracking error feedback control $U(t)$ so that

$$
\int_0^\infty e^{\alpha t}|e(t)|^2 dt < \infty,
$$

(2.3)

for some $\alpha > 0$, regardless of the unknown parameters $M_i, i = 1, 2, G(x)$ and $N$.

The following assumption is made throughout the paper.

**Assumption II.1.** All eigenvalues $\sigma(S) = \{\lambda = \pm j\omega_i\}$ of $S$ are located on the imaginary axis and are distinct.

From general theory of the output regulation theory for linear systems, the eigenvalues of $S$ in exosystem (2.2) cannot contain zeros of the transfer function of system (2.1) without disturbance. This is satisfied because a simple computation shows that there is no zero for system (2.1). Actually let $v(t) = 0$ and take the Laplace transform for (2.1) to obtain

$$
\begin{cases}
  s \hat{u}(x, s) = -j \hat{u}_x(x, s), \\
  \hat{u}_x(0, s) = M_1 v(t), \\
  \hat{u}_x(1, s) = \hat{U}(s), \\
  \hat{y}_\phi(s) = \hat{u}(0, s),
\end{cases}
$$

(2.4)

where $\hat{y}_\phi(s), \hat{u}(x, s)$ and $\hat{U}(s)$ are the Laplace transforms of $y_\phi(t), u(x, t)$ and $U(t)$ with respect to $t$ respectively. Then,

$$
\hat{y}_\phi(s) = G(s)\hat{U}(s), G(s) = \frac{2e^{\mu}}{\mu(e^{2\mu} - 1)}, \mu = \sqrt{j\sigma}.
$$

Therefore, there is no zero for system (2.1).

For notation simplicity, all obvious domains for both time and spatial variables are omitted in equations hereafter.

III. ERROR FEEDBACK CONTROL DESIGN

To begin with, we select specially some coefficients of the uncertainties as follows:

$$
G(x) = 0, \quad N = N_0, \quad M_1 = j c_2 N_0, \quad M_2 = 0,
$$

(3.1)

to obtain a nominal (frozen) system of (2.1)-(2.2):

$$
\begin{cases}
  u_t(x, t) = -ju_x(x, t), \\
  u_x(0, t) = j c_2 u(0, t) - j c_2 e(t), \\
  u_x(1, t) = U(t), \\
  \dot{v}(t) = S v(t), \\
  e(t) = u(0, t) - N_0 v(t),
\end{cases}
$$

(3.2)

which is a coupled PDE+ODE system. The subsections in what follows discuss only the nominal system (3.2).

A. Feedforward control for nominal system

Introduce standardly the error for nominal system (3.2):

$$
\varepsilon(x, t) = u(x, t) - g_0(x) e(t),
$$

(3.3)

where the term $g_0(x) e(t)$ describes the steady state for $u(x, t)$ to achieve output regulation. Then, $\varepsilon(x, t)$ satisfies

$$
\begin{cases}
  \dot{\varepsilon}(x, t) = -j \varepsilon_x(x, t), \\
  \varepsilon_x(0, t) = 0, \\
  \varepsilon_x(1, t) = U(t) - g_0(1) v(t), \\
  \varepsilon(1, 0) = e(0, 0),
\end{cases}
$$

(3.4)

provided that $g_0(x) \in \mathbb{C}^{1 \times n}$ is the solution of the following regulator equation:

$$
\begin{cases}
  g_0'(x) = j g_0(x) S, \\
  g_0(0) = N_0, g_0(0) = j c_2 N_0,
\end{cases}
$$

(3.5)

which has solution

$$
g_0(x) = [N_0 \ j c_2 N_0] \exp \left( \begin{bmatrix} 0 & j S \\ I & 0 \end{bmatrix} x \right).
$$

(3.6)

Naturally, we design the following feedforward feedback stabilizing control for $\varepsilon$-subsystem of (3.4), by cancelling the disturbance, as

$$
U(t) = -j c_1 \varepsilon(1, t) + g_0'(1) v(t), \quad c_1 > 0.
$$

(3.7)
The closed-loop $\varepsilon$-subsystem of (3.4) under control (3.7) becomes
\[
\begin{align*}
\varepsilon_t(x, t) &= -j\varepsilon \varepsilon_x(x, t), \\
\varepsilon_x(0, t) &= 0, \\
\varepsilon_x(1, t) &= -j c_1 \varepsilon(1, t).
\end{align*}
\] (3.8)

We consider system (3.8) in the state space $H$. Define the system operator $A_1 : D(A_1) \subseteq H \rightarrow H$ as follows
\[
\begin{align*}
A_1 f(x) &= -j f''(x), \\
D(A_1) &= \{ f \in H^2(0, 1) \mid f'(0) = 0, f'(1) = -j c_1 f(1) \}.
\end{align*}
\] (3.9)

Then, (3.8) is written as
\[
\frac{d}{dt} \varepsilon(t, t) = A_1 \varepsilon(t, t). 
\] (3.10)

From Lemma 3.1 of [9], $A_1$ generates an exponentially stable $C_0$-semigroup on $H$:
\[
\|e^{A_1 t}\| \leq \tilde{M}_0 e^{-\omega_0 t}, \forall t \geq 0,
\] (3.11)
for some $\tilde{M}_0, \omega_0 > 0$. However, the exponential stability (3.11) is not enough for convergence of the tracking error.

To study the output convergence of (3.8), we need the following Proposition 3.1 which is a direct consequence of [5, Theorem 2.34, p. 133]. It can also be deduced from the theorem on exponentials of [17, p. 261] by noticing that the separation condition (3.13) below implies that the Carleson constant is positive ([5, p. 92-93]):
\[
\delta(\Lambda) = \inf_{\lambda_n \in \Lambda} \left\{ \prod_{\lambda_n \neq \lambda_k} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \lambda_k} \right| \right\} > 0.
\]

**Proposition 3.1.** Suppose that a countable set $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is located in a strip in the upper half plane parallel to the real axis that
\[
\Lambda \subseteq \{ z \mid 0 < \beta_1 < \text{Im}(z) \leq \beta_2 \},
\] (3.12)
and $\Lambda$ is separated:
\[
\inf_{n \neq m} |\lambda_n - \lambda_m| > 0.
\] (3.13)

Then, the exponential family $\{e^{j\lambda_n t}\}_{n \in \mathbb{Z}}$ forms a Riesz basis for its span in $L^2(0, \infty)$.

**Proposition 3.2.** The solution of (3.8) satisfies
\[
\int_0^\infty e^{\alpha_1 t} \|\varepsilon(0, t)\|^2 dt \leq C \|\varepsilon(\cdot, 0)\|^2,
\] (3.14)
for some $\alpha_1, C > 0$ independent of the initial values.

**Proof.** By Lemma 3.1 of [9], there exists a sequence of generalized eigenfunctions of $A_1$, which forms a Riesz basis for $H$. All eigenvalues with large modulus are algebraically simple and the eigenpairs $(\lambda_k, \phi_k)$ have the following asymptotic expressions:
\[
\begin{align*}
\lambda_k &= -2c_1 + j(k\pi)^2 + O(k^{-1}), \text{Re}(\lambda_k) < 0, \\
\phi_k(x) &= \cos(k\pi x) + O(k^{-1}),
\end{align*}
\] (3.15)
as integers $k \rightarrow \infty$. Therefore, there exists an integer $K$ such that the solution of (3.10) can be written as
\[
\varepsilon(x, t) = \sum_{k=0}^K f_k(t)e^{\lambda_k t} \phi_k(x) + \sum_{k=k+1}^\infty a_k e^{\lambda_k t} \phi_k(x),
\] (3.16)
where $f_k(t), k = 0, 1, \ldots, K$ are polynomials of $t$ and $f_k(0), 0 \leq k \leq K, a_k, k \geq K$ are uniquely determined by $\varepsilon(\cdot, 0) \in H$, which satisfies $\sum_{k=k+1}^\infty |a_k|^2 < \infty$. Conversely, any sequence $(f_k(0), a_k)$ satisfying $\sum_{k=k+1}^\infty |a_k|^2 < \infty$ determines $\varepsilon(\cdot, 0) \in H$ that
\[
\varepsilon(x, t) = \sum_{k=0}^K f_k(0) \phi_k(x) + \sum_{k=k+1}^\infty a_k \phi_k(x).
\]

In particular,
\[
\sum_{k=k+1}^\infty |a_k|^2 \leq C_2 \|\varepsilon(\cdot, 0)\|^2,
\] (3.17)
for some $C_2 > 0$ independent of the initial values. Since $\text{Re}(\lambda_k) < 0$ and
\[
\varepsilon(0, t) = \sum_{k=0}^K f_k(t)e^{\lambda_k t} \phi_k(0) + \sum_{k=k+1}^\infty a_k e^{\lambda_k t} \phi_k(0) = \delta_1(t) + \delta_2(t),
\] (3.18)
we have immediately that
\[
\int_0^\infty e^{\alpha_1 t} |\delta_1(t)|^2 dt \leq C_3 \|\varepsilon(\cdot, 0)\|^2,
\] (3.19)
for some $\alpha_1 > 0$. For $\delta_2(t)$, we write $\lambda_k + \alpha_2 = j(-j\lambda_k - j\alpha_2)$ and it is found from (3.15) that
\[
-j\lambda_k - j\alpha_2 = (2c_1 - \alpha_2)j + (k\pi)^2 + O(k^{-1}).
\]
Choose $0 < \alpha_2 < \alpha_1$ so that $\inf_{k \geq K+1} \{\text{Re}(\lambda_k) + \alpha_2\} < 0$. Then, $2c_1 - \alpha_2 > 0$ which shows that $\{j(-j\lambda_k - j\alpha_2)\}$ satisfies condition (3.12). It then follows from Proposition 3.1 that $\{e^{j\lambda_n t}\}_{n \geq K+1}$ forms a Riesz basis for its space in $L^2(0, \infty)$. Therefore,
\[
\int_0^\infty e^{\alpha_2 t} |\delta_2(t)|^2 dt \leq C_4 \sum_{k=k+1}^\infty |a_k|^2 \leq C_4 \|\varepsilon(\cdot, 0)\|^2.
\]
This, together with (3.18) and (3.17), leads to (3.14).

**B. Observer for nominal system**

In this subsection, we design an observer in terms of the tracking error $\varepsilon(t)$ for coupled nominal system (3.2) as follows:
\[
\begin{align*}
\dot{x}(x, t) &= -j \dot{\varepsilon}(x, t), \\
\dot{x}(0, t) &= j c_2 \dot{x}(0, t) - j c_2 \varepsilon(t), \\
\dot{x}(1, t) &= U(t), \\
\dot{\varepsilon}(t) &= S \dot{x}(t) + Q[e(t) - \hat{\varepsilon}(0, t) + N_0 \hat{v}(t)],
\end{align*}
\] (3.19)
where \( Q \in \mathbb{C}^{n \times 1} \) is chosen to ensure \( S + QN_0 \) Hurwitz to be confirmed in Lemma 3.2 later. Set the observer errors to be
\[
\hat{z}(x,t) = u(x,t) - \hat{z}(x,t), \hat{v}(t) = v(t) - \hat{v}(t),
\]
which are governed by
\[
\begin{cases}
\hat{z}_i(x,t) = -j\hat{z}_{xx}(x,t), \\
\hat{z}_i(0,t) = jc_2\hat{z}(0,t), \\
\hat{z}_S(1,t) = 0, \\
\hat{v}(t) = [S + QN_0]\hat{v}(t) - Q\hat{z}(0,t).
\end{cases}
\]

**Lemma 3.1.** Suppose that \( S + QN_0 \) is Hurwitz. Then, system (3.21) is exponentially stable in \( H \times \mathbb{C}^n \).

**Proof.** By letting \( \varepsilon(x,t) = \hat{z}(1-x,t) \), the \( \hat{z} \)-subsystem of (3.21) becomes (3.8) with \( c_1 \) being replaced by \( c_2 \). Hence,
\[
\|\hat{z}(\cdot,t)\| \leq M_we^{-\gamma t}\|\hat{z}(\cdot,0)\|,
\]
where \( M_w, \gamma > 0 \). In addition, similar to (3.14), we have
\[
\int_0^\infty e^{\beta t}\|\hat{z}(0,t)\|dt < C_5\|\hat{z}(\cdot,0)\|^2,
\]
for some \( \beta, C_5 > 0 \). For \( \hat{v} \)-subsystem of (3.21), its solution is
\[
\hat{v}(t) = e^{(S+QN_0)t}\hat{v}(0) - \int_0^t e^{(S+QN_0)(t-s)}Q\hat{z}(0,s)ds.
\]
Since \( S + QN_0 \) is Hurwitz, we can take \( \beta < -\max\text{Re} (\lambda(S+QN_0)) \) in (3.23), where \( \lambda(S+QN_0) \) represents the eigenvalue of \( S + QN_0 \). Then, (3.24) can also be written as
\[
e^{\beta t}\hat{v}(t) = e^{(\beta+S+QN_0)t}\hat{v}(0) - \int_0^t e^{(\beta+S+QN_0)(t-s)}Qe^{\beta s}\hat{z}(0,s)ds.
\]
By (3.23) and Lemma 1.1 of [22], it follows that
\[
\lim_{t \to \infty} e^{\beta t}\hat{v}(t) = 0,
\]
which shows that \( \hat{v}(t) \to 0 \) exponentially as \( t \to \infty \).

To make \( S + QN_0 \) Hurwitz, we need the detectability of system \((S,N_0)\), which is ensured by the following Lemma 3.2.

**Lemma 3.2.** Suppose that
\[
S_0 = J^{-1}SJ = \text{diag}\{j\omega_i\}, \\
H_0 = N_0J = (H_{10}, H_{20}, \ldots, H_{n0}),
\]
where \( J = [\varphi_1, \varphi_2, \ldots, \varphi_n] \) with \( \varphi_i \) being an eigenvector of \( S \) corresponding to \( j\omega_i \). Then, \( \Sigma(S,N_0) \) is detectable if and only if \( H_{i0} \neq 0 \) which is equivalent to \( N_0\varphi_i \neq 0 \) for all \( i = 1, 2, \ldots, n \) and is always realizable for instance by taking \( N_0 = [1, 1, \ldots, 1]J^{-1} \).

**Proof.** Since \((S,N_0)\) is detectable if and only if \((S_0,H_0)\) is detectable, for any \( j\omega_i \in \sigma(S), i = 1, 2, \ldots, n \) we have
\[
\text{rank}\left[\begin{array}{c}
S_0 - j\omega_i \\
H_0
\end{array}\right] = n,
\]
if and only if \( H_{i0} \neq 0, i = 1, 2, \ldots, n \), where \( \lambda_{ni} = j\omega_n - j\omega_i \). By (3.27),
\[
H_0 = N_0J = (N_0\varphi_1, N_0\varphi_2, \ldots, N_0\varphi_n),
\]
and so
\[
H_{i0} = N_0\varphi_i, i = 1, 2, \ldots, n.
\]
Hence, \( \Sigma(S,N_0) \) is detectable if and only if \( N_0\varphi_i \neq 0 \) for all \( i = 1, 2, \ldots, n \).

**C. Error feedback control and 1-copy property**

From the feedforward control (3.7) and observer (3.19) for nominal system (3.2), we design naturally an error feedback control as
\[
U(t) = -jc_1\hat{z}(1,t) + [jc_1g_0(1) + g_0(1)]\hat{v}(t),
\]
which, incorporating with observer (3.19), produces a tracking error feedback control:
\[
\begin{cases}
\hat{z}_i(x,t) = -j\hat{z}_{xx}(x,t), \\
\hat{z}_i(0,t) = jc_2\hat{z}(0,t) - jc_2\hat{v}(t), \\
\hat{z}_S(1,t) = U(t), \\
\hat{v}(t) = (S + Qg_0(0))\hat{v}(t) + Q[e(t) - \hat{z}(0,t)], \\
U(t) = -jc_1\hat{z}(1,t) + [jc_1g_0(1) + g_0(1)]\hat{v}(t).
\end{cases}
\]
Introduce the transformation
\[
y(x,t) = \hat{z}(x,t) - g_0(x)\hat{v}(t),
\]
which is parallel to (3.3). Then, (3.31) becomes
\[
\begin{align*}
y_t(x,t) &= -jy_{xx}(x,t) + g_0(x)Qy(0,t) - g_0(x)Qe(t), \\
y_x(0,t) &= jcy_x(0,t) - jce_x(t), \\
y_x(1,t) &= -jc_1y(1,t), \\
\hat{v}(t) &= S\hat{v}(t) - Qy(0,t) + Qe(t), \\
U(t) &= -jc_1y(1,t) + g_0(1)\hat{v}(t).
\end{align*}
\]
It is seen from the fourth equation of (3.33) that the above controller contains 1-copy of the exosystem, which is just the internal model that we need for robustness.

**Remark 3.1.** The controller (3.31) can be written into the dynamic error feedback form:
\[
\begin{cases}
d\hat{z}(t) \quad \hat{v}(t)
\end{cases} = \mathcal{G}_1 \begin{cases}
\hat{z}(t) \\
\hat{v}(t)
\end{cases} + \mathcal{G}_2e(t)
\]
\[
U(t) = K \begin{cases}
\hat{z}(t) \\
\hat{v}(t)
\end{cases},
\]
where \( \mathcal{G}_2 \) is unbounded considered abstractly in [19]. However, it is difficult to check all assumptions of [19], and hence we need to give a robustness analysis in next section. In addition, the convergence of the output in [19] is in a much weak sense due to unboundedness of the observation operators.
IV. ROBUSTNESS ANALYSIS

Now, under the tracking error dynamic feedback control (3.31), the closed-loop of system (2.1) becomes

\[
\begin{align*}
    u_t(x, t) &= -juxx(x, t) + G(x)v(t), \\
    u_x(x, t) &= M_1v(t), \\
    u_x(1, t) &= -jc_1z(1, t) + [jç(1) + g'0(1)]\dot{v}(t) + M_2v(t), \\
    z_t(x, t) &= -jzxx(x, t), \\
    z(x, t) &= jc_2\dot{z}(0, t) - jc_2e(t), \\
    z(1, t) &= -jc_1z(1, t) + [jç(1) + g'0(1)]\dot{v}(t), \\
    \dot{v}(t) &= (S + Qg0(0))\dot{v}(t) + Q[e(t) - \dot{z}(0, t)], \\
    e(t) &= u(0, t) - Ne(t).
\end{align*}
\]

(4.1)

Since each channel of (4.1) has disturbance, we first introduce the transformation

\[
\begin{align*}
    z(x, t) &= u(x, t) + f(x)v(t),
\end{align*}
\]

(4.2)

where \( f(x) \) satisfies the regulator equation

\[
\begin{align*}
    f''(x) &= jf(x)S + jG(x) \\
    f'(0) &= jc_2(f(0) + N) - M_1, \\
    f'(1) &= 0.
\end{align*}
\]

(4.3)

to cluster all the non-collocated uncertainties of (4.1) into the boundary:

\[
\begin{align*}
    z_t(x, t) &= -jzxx(x, t), \\
    z(x, 0) &= h(t)\dot{v}(t), \\
    z_x(1, t) &= -jc_1z(1, t) + [jç(1) + g'0(1)]\dot{v}(t) + M_2v(t), \\
    z(1, t) &= -jzxx(x, t), \\
    z(x, t) &= jc_2\dot{z}(0, t) - jc_2e(t), \\
    z(1, t) &= -jc_1z(1, t) + [jç(1) + g'0(1)]\dot{v}(t), \\
    \dot{v}(t) &= (S + Qg0(0))\dot{v}(t) + Q[e(t) - \dot{z}(0, t)], \\
    e(t) &= z(0, t) - h(0)v(t).
\end{align*}
\]

(4.4)

Next, we need to show that equation (4.3) admits a unique solution. Actually, let \( h(x) = g(x) + f(x) \) where \( g(x) \) satisfies

\[
\begin{align*}
    g''(x) &= jg(x)S - jG(x), \\
    g(0) &= N, g'(0) = M_1
\end{align*}
\]

(4.5)

which has a solution

\[
\begin{align*}
    g(x) &= \begin{bmatrix} N & M_1 \end{bmatrix} \exp \left[ \begin{bmatrix} 0 & jS & I \\ 0 & 0 & 0 \end{bmatrix} \int_{0}^{x} \begin{bmatrix} 0 & jS \end{bmatrix} (x - \tau) \begin{bmatrix} I & 0 \end{bmatrix} G(\tau) d\tau \right].
\end{align*}
\]

Then, \( h(x) \) satisfies

\[
\begin{align*}
    h''(x) &= jh(x)S, \\
    h'(0) &= jc_2h(0), \\
    h'(1) &= g'(1).
\end{align*}
\]

(4.6)

Lemma 4.1. The regulator equation (4.6) admits a unique solution and so does for (4.3).

Proof. Let \( J^{-1}S = \begin{bmatrix} j\omega_1, j\omega_2, \ldots, j\omega_n \end{bmatrix}, \omega_i \in \mathbb{R}, i = 1, 2, \ldots, n \) as (3.27). Postmultiplying (4.6) by \( \varphi_i \) obtains the following decoupled set of ODEs

\[
\begin{align*}
    h''_i(x) &= -\omega_i h_i(x), \\
    h'_i(0) &= jc_2h_i(0), \\
    h'_i(1) &= g'_i(1), \quad i = 1, 2, \ldots, n
\end{align*}
\]

(4.7)

where \( h_i(x) = h(x)\varphi_i, N_i = N\varphi_i, \) and \( g'_i(1) = g'(1)\varphi_i. \) There are two cases:

Case 1: \( \omega_i = 0. \) In this case, the solution of (4.7) is found to be

\[
\begin{align*}
    h_i(x) &= g'_i(1)x + \frac{g'_i(1)}{jc_2}.
\end{align*}
\]

(4.8)

Case 2: \( \omega_i \neq 0. \) We may suppose without loss of generality that \( \omega_i > 0. \) The general solution of (4.7) is

\[
\begin{align*}
    h_i(x) &= D_1e^{j\sqrt{\omega_i}x} + D_2e^{-j\sqrt{\omega_i}x}.
\end{align*}
\]

(4.9)

By boundary conditions of (4.7), \( D_1 \) and \( D_2 \) satisfy

\[
\begin{align*}
    D_2 &= \frac{j(\omega_1 + c_2)e^{j\sqrt{\omega_i}}}{{j\sqrt{\omega_i}} - c_2}, \\
    D_1 &= \frac{j\sqrt{\omega_i + c_2}e^{j\sqrt{\omega_i}}}{j\sqrt{\omega_i} + c_2}.
\end{align*}
\]

(4.10)

which is not true for \( \omega_i \neq 0 \) and \( c_2 > 0. \) This shows that \( [j\sqrt{\omega_i} - c_2] \neq [j\sqrt{\omega_i} + c_2]e^{j\sqrt{\omega_i}}, \) and \( D_1, D_2 \) above make sense. Therefore, the solution of (4.6) always exists for any \( c_2 > 0 \) and is expressed as

\[
\begin{align*}
    h(x) &= (h_1(x), h_2(x), \ldots, h_n(x))J^{-1}.
\end{align*}
\]

(4.11)

Now we go back to (4.4) where the uncertainty term \( h'(0)v(t) \) still appears at the left end. We introduce another new transformation:

\[
\begin{align*}
    \begin{bmatrix} s(\cdot, t) \\ \dot{s}(\cdot, t) \\ \dot{v}(t) \end{bmatrix} &= \begin{bmatrix} z(\cdot, t) \\ \dot{z}(\cdot, t) \\ \dot{v}(t) \end{bmatrix} - \begin{bmatrix} h(x) \\ h(x) \\ h_v \end{bmatrix} v(t), \quad h_v \in \mathbb{R}^{n \times n},
\end{align*}
\]

(4.12)

where \( h_v \), together with \( h(x) \) defined by (4.6) satisfies the following Sylvester equation:

\[
\begin{align*}
    h'(1) &= -jç(1) + M_2 + [jç(1) + g'0(1) - M_2]\dot{v}(t), \\
    h_vS - [S + Qg0(0)]h_v &= -Qh(0).
\end{align*}
\]

Then, \( (s(x, t), \dot{s}(x, t), \dot{q}(t)) \) is governed by

\[
\begin{align*}
    s_t(x, t) &= -jssxx(x, t), \\
    s_x(0, t) &= 0, \\
    s_x(1, t) &= -jc_1\dot{s}(1, t) + [g'_0(1) + jc_1g0(1) - M_2]\dot{v}(t), \\
    \dot{s}(x, t) &= -jssxx(x, t), \\
    \dot{s}(0, t) &= jc_2\dot{s}(0, t) - jc_2s(0, t), \\
    \dot{s}(1, t) &= -jc_1\dot{s}(1, t) + [g'_0(1) + jc_1g0(1) - M_2]\dot{v}(t), \\n    \dot{v}(t) &= [S + Qg0(0)]\dot{v}(t) + Q[s(0, t) - \dot{s}(0, t)], \\
    e(t) &= s(0, t),
\end{align*}
\]

(4.13)

where the disturbances disappear in all channels.
Lemma 4.2. The Sylvester equation (4.12) admits a solution $h_v$.

Proof. By (3.27), we suppose
\[
\begin{align*}
J^{-1} SJ &= S_0 = \text{diag}\{j\omega_i\}, \quad X = J^{-1} h_v J, Q_0 = J^{-1} Q,
\end{align*}
\]
\[
\begin{align*}
H(x) &= h(x) J = (H_1(x), H_2(x), \ldots, H_n(x)), \\
H_0(x) &= g_0(x) J = (H_0(1), H_{02}(x), \ldots, H_{0n}(x)).
\end{align*}
\]
(4.14)

Then, we can write the equivalent form of (4.12) as follows:
\[
\begin{align*}
& \left\{ \begin{array}{l}
H'(1) + j c_1 H(1) = M_2 J + [j c_1 H_0(1) + H'_0(1) - M_2 J] X, \\
X S - [S_0 + Q_0 H_0(0)] X = -Q_0 H(0),
\end{array} \right.
\end{align*}
\]
(4.15)

where $H(x)$ and $H_0(x)$ satisfy
\[
\begin{align*}
H_i(t) &= \left\{ \begin{array}{l}
H_i(0)(1 + j c_2), \omega_i = 0, \\
H_i(0) \left( \cos \frac{j c_2}{\sqrt{\omega_i}} + j \frac{j c_2}{\sqrt{\omega_i}} \sinh \sqrt{\omega_i} \right), \omega_i \neq 0,
\end{array} \right.
\end{align*}
\]
(4.16)

The boundary condition of (4.16) at $x = 1$ is found to be
\[
H_i(1) = \left\{ \begin{array}{l}
H_i(0)(1 + j c_2), \omega_i = 0, \\
H_i(0) \left( \cos \frac{j c_2}{\sqrt{\omega_i}} + j \frac{j c_2}{\sqrt{\omega_i}} \sinh \sqrt{\omega_i} \right), \omega_i \neq 0.
\end{array} \right.
\]
(4.17)

and
\[
H_0(1) = \left\{ \begin{array}{l}
H_0(0)(1 + j c_2), \omega_i = 0, \\
H_0(0) \left( \cos \frac{j c_2}{\sqrt{\omega_i}} + j \frac{j c_2}{\sqrt{\omega_i}} \sinh \sqrt{\omega_i} \right), \omega_i \neq 0.
\end{array} \right.
\]
(4.18)

Since by Lemma 3.2, $H_0(0) \neq 0$ for all $i = 1, 2, \ldots, n$, we can find the solution of (4.15) to be
\[
X = \text{diag}\{x_i\}, \quad x_i = \frac{H_i(0)}{H_0(0)}, \quad i = 1, 2, \ldots, n.
\]
(4.19)

and hence
\[
h_v = J X J^{-1}.
\]

This completes the proof of the lemma.

Now we are in a position to consider system (4.13) in the state space $\mathcal{H} = (L^2(0, 1))^2 \times C^n = H^2 \times C^n$. Let
\[
\hat{s}(x, t) = s(x, t) - \hat{s}(x, t).
\]
(4.20)

Then, system (4.13) is further reduced to
\[
\begin{align*}
\hat{s}_t(x, t) &= -j \hat{s}_{xx}(x, t), \\
\hat{s}_x(0, t) &= j c_2 \hat{s}(0, t), \\
\hat{s}_x(1, t) &= 0, \\
\hat{s}_t(x, t) &= -j \hat{s}_{xx}(x, t), \\
\hat{s}_x(0, t) &= -j c_2 \hat{s}(0, t), \\
\hat{s}_x(1, t) &= -j c_1 \hat{s}(1, t) + [g_0'(1) + j c_1 g_0(1) - M_2] \hat{v}(t), \\
\hat{v}(t) &= \hat{s}(0, t) + Q \hat{s}(0, t), \\
e(t) &= \hat{s}(0, t) + \hat{s}(0, t).
\end{align*}
\]
(4.21)

We consider (4.21) in $\mathcal{H}$ with the inner product
\[
\langle (\phi_1, \phi_2, \phi_3)^T, (\varphi_1, \varphi_2, \varphi_3)^T \rangle
\]
\[
= \int_0^1 [\phi_1(x) \varphi_1(x) + \phi_2(x) \varphi_2(x) + 2\text{Re} \phi_1(x) \varphi_2(x)] dx + \phi_3^T \varphi_3.
\]
(4.22)

Then, (4.21) is written as
\[
\frac{d}{dt} \begin{pmatrix} \hat{s}(\cdot, t) \\ \hat{s}(\cdot, t) \\ \hat{v}(t) \end{pmatrix} = \mathcal{A}_2 \begin{pmatrix} \hat{s}(\cdot, t) \\ \hat{s}(\cdot, t) \\ \hat{v}(t) \end{pmatrix},
\]
(4.23)

where the operator $\mathcal{A}_2$ is defined as
\[
\mathcal{A}_2(\phi_1, \phi_2, \phi_3) = \left( \begin{array}{c}
- j \phi_1'' \\
- j \phi_2'' \\
S \phi_3 + Q \phi_1(0)
\end{array} \right) = \left( \begin{array}{c}
f \\
g \\
h
\end{array} \right),
\]
(4.24)

Lemma 4.3. Let $\mathcal{A}_2$ be defined by (4.24). Then, $\mathcal{A}_2^{-1}$ exists and is compact, and hence $\sigma(\mathcal{A}_2)$, the spectrum of $\mathcal{A}_2$, consists of isolated eigenvalues of finite algebraic multiplicity only.

Proof. For any $(f, g, h) \in \mathcal{H}$, solving
\[
\begin{align*}
\mathcal{A}_2(\phi_1, \phi_2, \phi_3) &= \left( \begin{array}{c}
- j \phi_1'' \\
- j \phi_2'' \\
S \phi_3 + Q \phi_1(0)
\end{array} \right) = \left( \begin{array}{c}
f \\
g \\
h
\end{array} \right),
\end{align*}
\]
(4.25)

with $S = S + Q g_0(0)$, gives
\[
\begin{align*}
\phi_1(x) &= - j \int_0^x \int_0^1 f(\tau) d\tau d\sigma - \frac{1}{c_2} \int_0^1 f(\tau) d\tau, \\
\phi_2(x) &= - j \int_x^1 \int_0^1 g(\tau) d\tau d\sigma + j (x - 1) \int_0^1 f(s) ds + \frac{1}{c_2} \int_0^1 (f(\tau) + g(\tau)) d\tau, \\
\phi_3(x) &= \hat{S}^{-1}[h - Q \phi_1(0)],
\end{align*}
\]
(4.26)

which advises that $\mathcal{A}_2^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev imbedding theorem. Therefore, $\sigma(\mathcal{A}_2)$ contains only isolated eigenvalues.

From Lemma 4.3, $\sigma(\mathcal{A}_2)$ consists of isolated eigenvalues only. Solve the eigenvalue problem $\mathcal{A}_2 \Phi = - j \lambda^2 \Phi$ with $\Phi = (\varphi, \psi, q)^T$, to obtain
\[
\begin{align*}
\varphi''(x) &= \lambda^2 \varphi(x), \\
\varphi'(0) &= j c_2 \varphi(0), \\
\varphi'(1) &= 0, \\
\psi''(x) &= \lambda^2 \psi(x), \\
\psi'(0) &= - j c_2 \varphi(0), \\
\psi'(1) &= - j c_1 \psi(1) + \tilde{h} q, \\
-j \lambda^2 q &= \tilde{S} q + Q \varphi(0),
\end{align*}
\]
(4.26)

where $\tilde{h} = g_0'(1) + j c_1 g_0(1) - M_2$. We consider only those $\lambda$ satisfying
\[
-j \lambda^2 \notin \sigma(\tilde{S}).
\]
(4.27)
From the last equality of (4.26) and (4.27),
\[ q = (-j\lambda^2 - \tilde{S})^{-1}Q\varphi(0), \] (4.28)

which renders \((\varphi, \psi)\) satisfying
\[
\begin{align*}
\varphi''(x) &= \lambda^2\varphi(x), \\
\varphi'(0) &= j c_2 \varphi(0), \\
\varphi'(1) &= 0, \\
\psi''(x) &= \lambda^2 \psi(x), \\
\psi'(0) &= -j c_2 \psi(0), \\
\psi'(1) &= -j c_1 \psi(1) + \hat{h}\varphi(0),
\end{align*}
\]
(4.29)
with
\[ \hat{h} = (g_0(1) + j c_1 g_0(1) - M_2)(-j\lambda^2 - \tilde{S})^{-1}Q. \] (4.30)

Let
\[ \varphi(x) = ae^{\lambda x} + be^{-\lambda x}, \quad \psi(x) = ce^{\lambda x} + de^{-\lambda x}, \] (4.31)
where \(a, b\) and \(c, d\) are constants to be determined. Substituting (4.30) and (4.31) into the boundary conditions of (4.29), we obtain
\[
\begin{align*}
a(\lambda - j c_2) - b(\lambda + j c_2) &= 0, \\
ae^\lambda - be^{-\lambda} &= 0, \\
-j c_2 a - j c_2 b - c \lambda + d \lambda &= 0, \\
\hat{h} a + \hat{h} b - (j c_1 e^\lambda + \lambda e^\lambda)c - (j c_1 e^{-\lambda} - \lambda e^{-\lambda})d &= 0,
\end{align*}
\]
(4.32)
which admits nontrivial solution if and only if the characteristic equation \(\det(\Delta(\lambda)) = 0\), where
\[
\Delta(\lambda) = \begin{vmatrix} \lambda - j c_2 & -\lambda - j c_2 & 0 & 0 \\ e^\lambda & -e^{-\lambda} & 0 & 0 \\ -j c_2 & -j c_2 & -\lambda & \lambda \\ \hat{h} & \hat{h} & -(j c_1 + \lambda)e^\lambda & -(j c_1 - \lambda)e^{-\lambda} \end{vmatrix}.
\]
A direct computation gives
\[
\det(\Delta(\lambda)) = \begin{vmatrix} \lambda - j c_2 & -\lambda - j c_2 \\ e^\lambda & -e^{-\lambda} \end{vmatrix} \times \begin{vmatrix} -\lambda & -(j c_1 e^\lambda + \lambda e^\lambda) \\ -(j c_1 e^{-\lambda} - \lambda e^{-\lambda}) & [jc_2 - \lambda e^{-\lambda} + (\lambda + j c_2) e^\lambda] \times [\lambda j c_1 - \lambda^2] e^{-\lambda} + (\lambda j c_1 + \lambda^2) e^\lambda \end{vmatrix} = \begin{vmatrix} \lambda - j c_2 & -\lambda - j c_2 \\ e^\lambda & -e^{-\lambda} \end{vmatrix} \times \begin{vmatrix} (jc_2 - \lambda)e^{-\lambda} + (\lambda + j c_2) e^\lambda \\ \lambda j c_1 - \lambda^2 e^{-\lambda} + (\lambda j c_1 + \lambda^2) e^\lambda \end{vmatrix}. \] (4.33)

**Lemma 4.4.** Let \(A_2\) be defined by (4.24). Then, \(\sigma(A_2)\) consists of two families \(\{\lambda_{1k}\} \cup \{\lambda_{2k}\}\), where
\[
\begin{align*}
\lambda_{1k} &= -2c_2 + j(k\pi)^2 + O(k^{-1}), \\
\lambda_{2k} &= -2c_1 + j(k\pi)^2 + O(k^{-1}).
\end{align*}
\]
(4.34)

Therefore,
\[ \text{Re}(\lambda_{1k}) \to -2c_2, \text{Re}(\lambda_{2k}) \to -2c_1 \text{ as } k \to \infty. \] (4.35)

**Proof.** By (4.33), it follows that \(\det(\Delta(\lambda)) = 0\) if and only if
\[
(jc_2 - \lambda)e^{-\lambda} + (\lambda + j c_2) e^\lambda = 0 \quad \text{or} \quad (\lambda j c_1 - \lambda^2) e^{-\lambda} + (\lambda j c_1 + \lambda^2) e^\lambda = 0. \] (4.36)
We write (4.36) as
\[
e^{2\lambda} = \frac{\lambda - j c_2}{\lambda + j c_2} = 1 - \frac{2jc_2}{\lambda} + O(|\lambda|^{-2}) \text{ as } |\lambda| \to \infty, \quad (4.38)
\]
which has solutions
\[ \lambda = k\pi j + O(k^{-1}), k \to \infty. \] (4.39)
Substitute (4.39) into (4.38) to obtain \(O(k^{-1}) = -\frac{c_2}{k\pi} \) and hence
\[ \lambda = k\pi j - \frac{c_2}{k\pi} + O(k^{-2}). \] (4.40)
Since \(\lambda_{1k} = -j\lambda^2\), we have
\[ \lambda_{1k} = -2c_2 + j(k\pi)^2 + O(k^{-1}). \]
This is the first branch eigenvalues claimed by (4.34). The second branch can be obtained similarly from (4.37). \[ \square \]

**Lemma 4.5.** Suppose that \(c_1, c_2 > 0, c_1 \neq c_2\). Let \(A_2\) be defined by (4.24) and \(\sigma(A_2) = \{\lambda_{1k}\} \cup \{\lambda_{2k}\}\) be the eigenvalues of \(A_2\) obtained from Lemma 4.4. Then, there are two families of approximate normalized eigenfunctions of \(A_2\):

(i). One family \(\Psi_{1k}(x)\) corresponding to \(\lambda_{1k}\) has the following asymptotic expression:
\[
\Psi_{1k}(x) = \begin{pmatrix} (c_1 - c_2) \cos k\pi x \\ c_2 \cos k\pi x \end{pmatrix} + O(k^{-1}); \] (4.41)

(ii). The second family \(\Psi_{2k}(x)\) corresponding to \(\lambda_{2k}\) has the following asymptotic expression:
\[
\Psi_{2k}(x) = \begin{pmatrix} 0 \\ \cos k\pi x \end{pmatrix} + O(k^{-1}). \] (4.42)

**Proof.** Let \(\varphi(x) \equiv 0\). Then, (4.29) becomes
\[
\begin{align*}
\psi''(x) &= \lambda^2 \psi(x), \\
\psi'(0) &= 0, \\
\psi'(1) &= -j c_1 \psi(1),
\end{align*}
\]
(4.43)
which corresponds to the eigenvalue problem of system (3.8). By (3.15), the solution of (4.43) corresponds to \(\lambda_{2k}\) and \(\psi(x) = \cos k\pi x + O(k^{-1}).\) Since by (4.28) and counterpart of (4.39) for \(\lambda_{2k}, q = O(k^{-1}).\) Thus, \(\Psi_{2k}(x)\) has the expression (4.42).

For the first branch eigenvalues \(\{\lambda_{1k}\}\), we suppose that the eigenfunctions are \((\varphi(x), \psi(x))\). It is found that
\[
\varphi(x) = \begin{pmatrix} e^{\lambda x} & e^{-\lambda x} & 0 & 0 \\ e^{\lambda} & -e^{-\lambda} & 0 & 0 \\ -j c_2 & -j c_2 & -\lambda & \lambda \\ \hat{h} & \hat{h} & -(j c_1 + \lambda)e^{\lambda} & -(j c_1 - \lambda)e^{-\lambda} \end{pmatrix} \]
Therefore,

\[ \psi(x) = \begin{bmatrix} 0 & 0 & e^{\lambda x} & e^{-\lambda x} \\ e^\lambda & -e^{-\lambda} & 0 & 0 \\ -je\lambda & -je\lambda & -1 & 1 \\ e^\lambda & -e^{-\lambda} & e^{-\lambda} & e^{-\lambda} \end{bmatrix} + O(\lambda^{-1}) \]

\[ \Phi_{1k}(x) \begin{bmatrix} (c_1 - c_2) \\ c_2 \\ \cos k\pi x \\ 0 \end{bmatrix}, \quad k = 0, 1, \ldots, \]

\[ \Phi_{2k}(x) = \begin{bmatrix} 0 \\ \cos (k\pi x) \\ 0 \\ 0 \end{bmatrix}, \quad k = 0, 1, \ldots, \]

\[ \Phi_{3m}(x) = \begin{bmatrix} 0 \\ 0 \\ e_m \\ 0 \end{bmatrix}, \quad m = 1, 2, \ldots, n, \]

where \( \{e_m\}_{m=1}^n \) is the standard orthonormal basis for \( \mathbb{C}^n \). Then, \( \{c_2/(c_1-c_2)[\Phi_{1k}(x)-\Phi_{2k}(x)], \Phi_{2k}(x), \Phi_{3m}(x)\} \) forms an orthonormal basis for \( H = \mathbb{H}^2 \times \mathbb{C}^n \) and hence \( \{\Phi_{1k}(x), \Phi_{2k}(x), \Phi_{3m}(x)\} \) is a Riesz basis. From Lemma 4.5, one can find an \( N > 0 \) such that

\[ \sum_{n>N} [\|\Phi_{1k} - \Phi_{1k}\|_H + \|\Phi_{2k} - \Phi_{2k}\|^2_\mathcal{H}] < \infty. \]

By Theorem 2.38 of [5], we conclude assertions a)-c). Finally, by (4.34) and c), d) is valid if \( \text{Re}(\mu) < 0 \) for any \( \mu = -j\lambda^2 \in \sigma(A_2) \). But this can be seen from (4.29) that if \( \varphi(x) = 0 \), then, (4.43) has no zero solution, which means that \( \text{Re}(\mu) < 0 \) by (3.15) for \( \mu = -j\lambda^2 \in \sigma(A_2) \). If \( \varphi(x) \neq 0 \), then, \( (\lambda, \varphi) \) satisfies

\[ \varphi''(x) = \lambda^2 \varphi(x), \]

\[ \varphi'(0) = jc_2 \varphi(0), \]

\[ \varphi'(1) = 0, \]

which implies also that \( \text{Re}(\mu) < 0 \) by (3.15) for \( \mu = -j\lambda^2 \in \sigma(A_2) \). Except these two cases, other eigenvalues if any, must be eigenvalues of \( S \) by (4.27), which have negative real parts.

**Remark 4.1.** If we write the closed-loop system (4.1) as

\[ \begin{cases} \frac{d}{dt}\begin{bmatrix} \dot{u}(\cdot, t) \\ \dot{\varphi}(\cdot, t) \end{bmatrix} = A_e \begin{bmatrix} u(\cdot, t) \\ \varphi(\cdot, t) \end{bmatrix} + B_e v(t) \\ e(t) = C_e \begin{bmatrix} u(\cdot, t) \\ \varphi(\cdot, t) \end{bmatrix} + D_e v(t) \end{cases} \]

where \( B_e \) is an unbounded operator. The transformation from system (4.1) to (4.13) is equivalent to the existence of the solution to the Sylvester equation

\[ \Sigma S = A_e \Sigma + B_e, \]

in the space \( \Sigma \in \mathcal{L}(\mathbb{C}^n, \mathcal{H}) \). This can also be concluded from [19] from abstract operator point of view by the exponential stability of system (4.13). Here we find an analytic form of \( \Sigma \) which is obtained from (4.2) and (4.11). The exponential stability of system (4.13) or equivalently (4.21) is equivalent to the internal exponential stability of the closed-loop system (4.1), that is, \( A_e \) generates an exponentially stable \( C_0 \)-semigroup on \( \mathcal{H} \).
Theorem 4.2. Suppose that $c_1, c_2 > 0$, $c_1 \neq c_2$ and $S + Qg_0(0)$ is Hurwitz. Then, under Assumption II.1, the output tracking is guaranteed for any initial value in $\mathcal{H} = H^2 \times \mathbb{C}^n$ in the sense that
\[
\int_0^\infty e^{\alpha t}|e(t)|^2 dt < \infty,
\] (4.47)
for some $\alpha > 0$. When the initial value $(\hat{s}(\cdot, 0), \hat{s}(\cdot, 0), \hat{v}(\cdot, 0))^\top \in D(A_2)$,
\[
\lim_{t \to \infty} |e(t)| = 0
\] (4.48)
exponentially.

Proof. By (4.21), we need to show that $e(t) = \hat{s}(0, t) + \hat{s}(0, t)$ satisfies (4.47), or
\[
\int_0^\infty e^{\alpha t}|\hat{s}(0, t)|^2 dt < \infty \quad \text{and} \quad \int_0^\infty e^{\alpha t}|\hat{s}(0, t)|^2 dt < \infty.
\] (4.49)
The $\hat{s}$-subsystem of (4.21) is almost the same as (3.8) and is exponentially stable:
\[
F(t) = \frac{1}{2} \int_0^t |\hat{s}(x, t)|^2 \leq L_0 e^{-\omega_0 t} F(0),
\]
for some $L_0, \omega_0 > 0$ and
\[
\dot{F}(t) = -c_2 |\hat{s}(0, t)|^2.
\]
For any $0 < \alpha < \omega_0$, since
\[
dt e^{\alpha t}F(t) = \alpha e^{\alpha t}F(t) - c_2 e^{\alpha t}|\hat{s}(0, t)|^2,
\]
we have
\[
c_2 \int_0^\infty e^{\alpha t}|\hat{s}(0, t)|^2 dt = \alpha \int_0^\infty e^{\alpha t}F(t)dt + F(0) < \infty.
\] (4.50)
This is the first inequality of (4.49). Now we show the second inequality of (4.49). By Theorem 4.1, we may suppose without loss of generality that $\{\Psi_{1k}(x), \Psi_{2k}(x), \Phi_{3m}(x)\}$, $k = 1, 2, \cdots, m = 1, 2, \cdots, n$, is a set of generalized eigenfunctions of $A_2$ corresponding to the eigenvalues $\{\lambda_{1k}, \lambda_{2k}, \lambda_{3m}\}$, which forms a Riesz basis for $\mathcal{H} = H^2 \times \mathbb{C}^n$. Then, the solution of (4.21) can be written as
\[
\begin{pmatrix}
\hat{s}(x, t) \\
\hat{s}(x, t) \\
\hat{v}(t)
\end{pmatrix}
= \sum_{k=1}^{N-1} \sum_{s=1}^{2} f_{sk}(t)e^{\lambda_{sk}t}\Psi_{sk}(x)
+ \sum_{k=N}^{\infty} \sum_{s=1}^{2} a_{sk}e^{\lambda_{sk}t}\Psi_{sk}(x)
+ \sum_{m=1}^{n} f_{m}(t)e^{\lambda_{3m}t}\Phi_{3m}(x),
\] (4.51)
where $N > 0$ is an integer such that when $k \geq N$, the eigenvalues of $A_2$ are algebraically simple in terms of Theorem 4.1, and $f_{sk}(t)$ and $f_{m}(t)$ are polynomials of $t$. As a result,
\[
\sum_{k=N}^{\infty} \sum_{s=1}^{2} |a_{sk}|^2 \leq C \|\hat{s}(\cdot, 0), \hat{s}(\cdot, 0)\|_{H^2}^2,
\]
for some $C > 0$ independent of the initial value $(\hat{s}(\cdot, 0), \hat{s}(\cdot, 0), \hat{v}(\cdot, 0))$. Suppose that
\[
\Psi_{sk}(x) = (\phi_{sk}(x), \psi_{sk}(x), q_{sk}), s = 1, 2,
\]
\[
\Phi_{3m}(x) = (\phi_{3m}(x), \psi_{3m}(x), q_{3m}).
\]
It follows from Lemma 4.5 that
\[
\hat{s}(0, t) = \sum_{k=1}^{N-1} \sum_{s=1}^{2} f_{sk}(t)e^{\lambda_{sk}t}[\mathcal{O}(k^{-1})]
+ \sum_{k=N}^{\infty} \sum_{s=1}^{2} a_{sk}e^{\lambda_{sk}t}[1 + \mathcal{O}(k^{-1})]
+ \sum_{m=1}^{n} f_{m}(t)e^{\lambda_{3m}t}\psi_{3m}(0)
= \Delta_1(t) + \Delta_2(t) + \Delta_3(t).
\] (4.52)
Since all $\text{Re}(\lambda_{sk}) < 0, \text{Re}(\lambda_{3m}) < 0$ owing to Theorem 4.1, the first term $\Delta_1(t)$ and the third term $\Delta_3(t)$ of (4.52) satisfy
\[
\int_0^\infty e^{\alpha t}|\Delta_j(t)|^2 dt < \infty, j = 1, 3,
\] (4.53)
for some $\alpha > 0$.
Now it is seen from (3.44) that all our set $\{\lambda_{sk} + \alpha s = 1, 2, k \geq N\}$ where $\alpha > 0$ is chosen so that $\text{sup}_{s,k} \text{Re}(\lambda_{sk}) + \alpha < 0$ and (4.50) keeps valid. Write $\lambda_{sk} + \alpha = j(-j(\lambda_{sk} + \alpha))$. Then,
\[
\begin{cases}
-j(\lambda_{1k} + \alpha) = (2c_2 - \alpha)j + (k\pi)^2 + \mathcal{O}(k^{-1}), \\
-j(\lambda_{2k} + \alpha) = (2c_1 - \alpha)j + (k\pi)^2 + \mathcal{O}(k^{-1}),
\end{cases}
\] (4.54)
are located in a strip in the upper half plane parallel to the real axis satisfying (3.12). By Proposition 3.1, \(\{e^{(\lambda_{sk} + \alpha)t}\}_{s=1,2,k \geq N}\) forms a Riesz basis for its span in $L^2(0, \infty)$. Therefore,
\[
\|e^{\alpha t}\Delta_2(\cdot)\|^2_{L^2(0, \infty)} = \int_0^\infty e^{2\alpha t}|\Delta_2(t)|^2 dt
\leq C_7 \sum_{k=N}^{\infty} \sum_{s=1}^{2} |a_{sk}|^2 + \mathcal{O}(k^{-1})
\leq C_8 \sum_{k=N}^{\infty} \sum_{s=1}^{2} |a_{sk}|^2,
\] (4.55)
for some $C_i > 0, i = 7, 8$. This, together with (4.53), gives the second inequality of (4.49).
Finally, when $(\hat{s}(\cdot, t), \hat{s}(\cdot, t), \hat{v}(\cdot, 0))^\top \in D(A_2)$, the solution of (4.21) is classical solution and
\[
A_2 \begin{pmatrix}
\hat{s}(\cdot, t) \\
\hat{s}(\cdot, t) \\
\hat{v}(t)
\end{pmatrix}
= e^{A_2 t}A_2 \begin{pmatrix}
\hat{s}(\cdot, 0) \\
\hat{s}(\cdot, 0) \\
\hat{v}(0)
\end{pmatrix}
\to 0.
\] (4.56)
This implies in particular that \(s(\cdot, t) \in H^2(0, 1)\), and by the Sobolev trace-embedding theorem (see, e.g., [5, Theorem 1.43, p.22]), we have
\[
\lim_{t \to \infty} |s(0, t)|^2 \leq \lim_{t \to \infty} \|s(\cdot, t)\|_{H^2(0, 1)} = 0
\]
exponentially.

To end this section, we mention simply the robustness to the systematic uncertainties. Although the closed-loop system (4.1) contains the internal model, it is difficult to formulate generally the system uncertainties from PDEs perspective. Roughly speaking, the internal model means that when \((A_e, B_e, C_e)\) has a variation \((A'_e, B'_e, C'_e)\), where \((A_e, B_e, C_e)\) is defined in (4.45) and \(A'_e\) generates an exponentially stable \(C_0\)-semigroup, the output tracking claimed by (4.48) is still true. Here we only consider lower order perturbation for the closed-loop system (4.1), which now reads
\[
\begin{align*}
\dot{u}(x, t) &= -j u(x, t) + \Delta(x) u(x, t) + G(x)v(t), \\
u(0, t) &= M_1 v(t), \\
u(x, t) &= U(t) + M_2 v(t), \\
U(t) &= -j c_1 \hat{s}(1, t) + j c_1 g(0(1) + g_0(1)) \hat{v}(t), \\
\hat{s}(1, t) &= -j c_2 \hat{s}(0, t) - j c_2 e(t), \\
\hat{s}(0, t) &= j c_1 \hat{s}(1, t) + j c_1 g(0(1) + g_0(1)) \hat{v}(t), \\
\hat{v}(t) &= (S + Q g(0(0)) \hat{v}(t) + Q[e(t) - \hat{z}(0, t)], \\
e(t) &= u(0, t) - N v(t),
\end{align*}
\]
(4.57)
where \(\Delta(\cdot) \in L^\infty(0, 1)\) is supposed to be unknown uncertainty of the closed-loop system. For this case, we update the \(g(x)\) defined by (4.5) and \(f(x)\) defined by (4.3) with \(\hat{g}(x)\) and \(\hat{f}(x)\) respectively, which read
\[
\begin{align*}
\dot{\hat{g}}''(x) &= j \hat{g}(x) S - j G(x) - j \Delta(x) \hat{g}(x), \\
\hat{g}(0) &= N, \hat{g}'(0) = M_1,
\end{align*}
\]
(4.58)
and
\[
\begin{align*}
\dot{\hat{f}}''(x) &= j \hat{f}(x) S + j G(x) + j \Delta(x) \hat{f}(x), \\
\hat{f}'(0) &= j c_2 \hat{f}(0) + N - M_1, \\
\hat{f}'(1) &= 0.
\end{align*}
\]
(4.59)
It is seen that (4.58) and (4.59) are of the same forms of (4.5) and (4.3) respectively, and then they admit unique solutions. In addition, it is found that \(\hat{g}(x) + \hat{f}(x) = h(x)\) defined by (4.6).

Through (4.2) and (4.11) and Lemma 4.2, the system of (4.13) is updated as
\[
\begin{align*}
s_t(x, t) &= -j s_x(x, t) + \Delta(x) s(x, t), \\
s_x(0, t) &= 0, \\
s_x(1, t) &= -j c_1 \hat{s}(1, t) + [g_0'(1) + j c_1 g(0) - M_2] \hat{v}(t), \\
\hat{s}_t(x, t) &= -j s_x(x, t), \\
\hat{s}_x(0, t) &= j c_2 \hat{s}(0, t) - j c_2 s(0, t), \\
\hat{s}_x(1, t) &= -j c_1 \hat{s}(1, t) + [g_0'(1) + j c_1 g(0) - M_2] \hat{v}(t), \\
\hat{v}(t) &= [S + Q g(0(0)] \hat{v}(t) + Q[s(0, t) - \hat{s}(0, t)], \\
e(t) &= s(0, t).
\end{align*}
\]
(4.60)
Similarly to (4.13), system (4.60) can be written into the following evolution equation
\[
\begin{align*}
\frac{d}{dt} \left( \begin{array}{c}
s(\cdot, t) \\
\hat{s}(\cdot, t) \\
v(t)
\end{array} \right) &= \left( \begin{array}{ccc}
A_e & \Delta_e & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \left( \begin{array}{c}
s(\cdot, t) \\
\hat{s}(\cdot, t) \\
v(t)
\end{array} \right),
\end{align*}
\]
(4.61)
where \(\Delta_e\) is an operator defined by
\[
\Delta_e \left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3
\end{array} \right) = \left( \begin{array}{c}
\Delta(\cdot) \hat{\phi}_1 \\
0 \\
0
\end{array} \right),
\]
(4.62)
for all \((\phi_1, \phi_2, \phi_3)^T \in \mathcal{H}\), which is a bounded operator on \(\mathcal{H}\), satisfying \(\|\Delta_e\| = \|\Delta\|_{L^\infty(0, 1)}\). The following result is then a direct consequence of Theorem 4.1.

**Corollary 4.1.** Let \(A_e + \Delta_e\) be defined by (4.61). If \(A_e + \Delta_e\) generates an exponentially stable \(C_0\)-semigroup on \(\mathcal{H}\), then, for any initial value \((s(\cdot, 0), \hat{s}(\cdot, 0), v(0)) \in \mathcal{D}(A_2)\), the output regulation of the system (4.60) is still guaranteed
\[
\lim_{t \to \infty} |e(t)| = 0
\]
(4.63)
exponentially for all \(\Delta(x)\) with sufficiently small \(\|\Delta\|_{L^\infty(0, 1)}\).

**Proof.** Suppose that the semigroup \(e^{A\cdot t}\) satisfies
\[
\|e^{A\cdot t}\| \leq C_1 e^{-\omega_1 t},
\]
(4.64)
for some \(C_\omega, \omega_1 > 0\). If
\[
\sup_{t \leq T} \|\Delta\|_{L^\infty(0, 1)} < \frac{\omega_1}{C_\omega},
\]
(4.65)
then, \(e^{A\cdot t} = Pe^{A\cdot t}P^{-1}\) is an exponentially stable \(C_0\)-semigroup on \(\mathcal{H}\), where
\[
P = \begin{bmatrix}
0 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_n
\end{bmatrix}.
\]

Actually
\[
\|e^{A\cdot t}\| = \|P\| \|P^{-1}\| \|e^{A\cdot t}\| \leq \|e^{A\cdot t}\| \leq C_1 e^{-\omega_1 t}.
\]
Thus, \(A_e + \Delta_e\) generates an exponentially stable \(C_0\)-semigroup on \(\mathcal{H}\) that
\[
\|e^{(A_e + \Delta_e) t}\| \leq C_1 e^{-(\omega_1 - C_1 \|\Delta\|_{L^\infty(0, 1)}) t}.
\]
The result then follows from Theorem 4.1.

For non-smooth initial values in Corollary 4.1, we can also develop a similar convergence as (4.47) for which further spectral analysis is needed.

**Theorem 4.3.** Under conditions of Theorem 4.2, if
\[
\sup_{t \leq T} \|\Delta\|_{L^\infty(0, 1)} < \min \left\{ \sqrt{\frac{\omega_1}{C_\omega}} \right\},
\]
(4.66)
where \(\omega_1, C_\omega\) are defined by (4.65), then, the output tracking is still guaranteed for any initial value in \(\mathcal{H} = H^2 \times \mathbb{C}^n\) in the sense that
\[
\int_0^\infty e^{\gamma_0 t} |e(t)|^2 dt < \infty
\]
(4.67)
for some \(\gamma_0 > 0\).
Proof. By the Corollary 4.1 and Theorem 4.1, we know that $A_e + \Delta_e$ generates an exponentially stable $C_0$-semigroup on $\mathcal{H}$. Now, suppose $(A_e + \Delta_e)\Phi = -j\lambda^2\Phi$ with $\Phi = (\varphi, \psi, q)$, to obtain

$$\begin{cases}
\varphi''(x) = \lambda^2 \varphi(x) - j\Delta(x)\varphi(x), \\
\varphi'(0) = 0, \\
\varphi'(1) = \psi'(0), \\
\psi''(x) = \lambda^2 \psi(x), \\
\psi'(0) = j c_2 \psi(0) - j c_2 \varphi(0), \\
\psi'(1) = -j c_2 \psi(1) + \hat{h} q,
\end{cases}$$

Substituting these into the boundary conditions of (4.71) gives

$$\begin{align*}
a \left( \lambda + \frac{j}{2} \int_0^1 e^{2\lambda x} \Delta(x) dx + O(\lambda^{-2}) \right) \\
+ b (-\lambda + O(\lambda^{-2})) = 0,
\end{align*}$$

where $a, b$ are written as follows

$$\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} =
\begin{pmatrix}
-j c_2 + O(\lambda^{-2}) & -j c_2 + O(\lambda^{-2}) \\
h (1 + O(\lambda^{-2})) & h (1 + O(\lambda^{-2}))
\end{pmatrix},$$

with

$$\hat{h} = (g_0(1) + j c_1 g_0(1) - M_2)(-j\lambda^2 - \tilde{S})^{-1} Q.$$

Let

$$\varphi_1(x) = a \varphi_1(x) + b \varphi_2(x), \psi_1(x) = c e^{\lambda x} + d e^{-\lambda x},$$

where $a, b$ and $c, d$ are constants to be determined, and $\varphi_1(x), \varphi_2(x)$ are written as follows

$$\varphi_1(x) = e^{\lambda x} \left( 1 - \frac{j}{2\lambda} \int_0^x \Delta(\xi) d\xi - \frac{j}{2\lambda} \int_x^1 e^{-2\lambda(x-\xi)} \Delta(\xi) d\xi + O(\lambda^{-2}) \right),$$

$$\psi_1(x) = e^{-\lambda x} \left( 1 + \frac{j}{2\lambda} \int_0^x \Delta(\xi) d\xi + O(\lambda^{-2}) \right),$$

The facts $\frac{1}{2} \int_0^1 e^{2\lambda x} \Delta(x) dx = O(\lambda^{-1})$ and $\frac{1}{2} \int_0^1 e^{-2\lambda(x-\xi)} \Delta(x) dx = O(\lambda^{-1})$, together with (4.30), enables us to write $\det(B) = 0$ as

$$\det(B) = \lambda^4 (\det(B_1^2, B_2^2) + O(\lambda^{-2})).$$
where
\[
B_1^1 = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22} \\
  jc_2 + O(\lambda^{-2}) & jc_2 + O(\lambda^{-2}) \\
  0 & 0 \\
\end{pmatrix},
\]
\[
B_2^2 = \begin{pmatrix}
  0 & 0 \\
  jc_1 e^{\lambda} & 0 \\
  \lambda - jc_2 + O(\lambda^{-2}) & -(\lambda + jc_2) \\
  -jc_1 e^{\lambda} - \lambda e^{\lambda} & -jc_1 e^{\lambda} + \lambda e^{\lambda} \\
\end{pmatrix},
\]
and
\[
B_1^2 = \begin{pmatrix}
  e^{\lambda} \left(1 - \frac{j \int_0^1 \Delta(x)dx}{2\lambda}\right) & e^{-\lambda} \left(1 - \frac{j \int_0^1 \Delta(x)dx}{2\lambda}\right) \\
  \frac{jc_2}{\lambda} & \frac{jc_2}{\lambda} \\
\end{pmatrix},
\]
\[
B_2^3 = \begin{pmatrix}
  \frac{jc_2}{\lambda} & \frac{jc_2}{\lambda} \\
  0 & 0 \\
\end{pmatrix},
\]
\[
\begin{pmatrix}
  \frac{jc_2}{\lambda} & \frac{jc_2}{\lambda} \\
  0 & 0 \\
\end{pmatrix},
\]
\[
\begin{pmatrix}
  \frac{jc_2}{\lambda} & \frac{jc_2}{\lambda} \\
  0 & 0 \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
  \frac{jc_2}{\lambda} & \frac{jc_2}{\lambda} \\
  0 & 0 \\
\end{pmatrix},
\]
By Rouché’s theorem, the roots of (4.73) have the following asymptotic expressions:
\[
e^{2\lambda} = 1 - \frac{2jc_2\Delta}{\lambda} + O(\lambda^{-2}),
\]
\[
e^{2\lambda} = 1 - \frac{2jc_2\Delta}{\lambda} + O(\lambda^{-2}),
\]
where
\[
c_{2\Delta} = \left[\frac{c_1}{2} + \frac{c_2}{2} - \frac{1}{4} \int_0^1 \Delta(x)dx\right] \\
\]
\[
-\frac{1}{4} \sqrt{\left(\int_0^1 \Delta(x)dx\right)^2 + 4(c_1 + c_2) \int_0^1 \Delta(x)dx + 4(c_1 - c_2)^2}.
\]
\[
c_{1\Delta} = \left[\frac{c_1}{2} + \frac{c_2}{2} - \frac{1}{4} \int_0^1 \Delta(x)dx\right] \\
\]
\[
+\frac{1}{4} \sqrt{\left(\int_0^1 \Delta(x)dx\right)^2 + 4(c_1 + c_2) \int_0^1 \Delta(x)dx + 4(c_1 - c_2)^2}.
\]
It is seen that \(c_{2\Delta} = c_2\) and \(c_{1\Delta} = c_1\) whenever \(\Delta(x) = 0, c_1 > c_2\).

The (4.74) and (4.75) have the solutions respectively,
\[
\lambda = k\pi j - \frac{c_{2\Delta}}{k\pi} + O(k^{-2}),
\]
\[
\lambda = k\pi j - \frac{c_{1\Delta}}{k\pi} + O(k^{-2}),
\]
and
\[
\lambda_{1k} = -2 c_{2\Delta} + j(k\pi)^2 + O(k^{-1}),
\]
\[
\lambda_{2k} = -2 c_{1\Delta} + j(k\pi)^2 + O(k^{-1}).
\]
Hence, the normalized eigenfunction $\Phi_{1k}(x)$ corresponding to $\lambda$ satisfying (4.76) has the form
\[
\Phi_{1k}(x) = \frac{1}{-\lambda^2 4jc_2(2)^{-1}} \begin{pmatrix} \phi(x) \\ \psi(x) \\ q \end{pmatrix}
\]
\[
= \begin{pmatrix} c_1 + c_2 - c_2\Delta \cos k\pi x \\ c_2 \cos k\pi x \\ 0 \end{pmatrix} + O(k^{-1}).
\]
Similarly, the eigenfunction $\Phi(x) = (\varphi(x), \psi(x), q)^T$ corresponding to $\lambda$ satisfying (4.77) is computed as
\[
\Phi_{2k}(x) = \frac{1}{-\lambda^2 4jc_2(2)^{-1}} \begin{pmatrix} \phi(x) \\ \psi(x) \\ q \end{pmatrix}
\]
\[
= \begin{pmatrix} c_1 + c_2 - c_1\Delta \cos k\pi x \\ c_2 \cos k\pi x \\ 0 \end{pmatrix} + O(k^{-1}).
\]
We rewrite the system (4.21) with $\Delta(x) \neq 0$ as
\[
\frac{d}{dt} \begin{pmatrix} \dot{s}(\cdot,t) \\ \dot{\bar{v}}(t) \end{pmatrix} = (A_2 + \Delta_1) \begin{pmatrix} \dot{s}(\cdot,t) \\ \dot{\bar{v}}(t) \end{pmatrix},
\]
where $\Delta_1$ is the operator defined by
\[
\Delta_1 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \Delta(\cdot)\phi_1 + \Delta(\cdot)\phi_2 \\ 0 \\ 0 \end{pmatrix}.
\]
Introduce the invertible matrix $T$ such that
\[
A_2 + \Delta_1 = T^{-1}(Ae + \Delta e)\frac{}{}T,
\]
where
\[
T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}.
\]
Let $\bar{\Phi}_{1k}, \bar{\Phi}_{2k}$ and $\bar{\Phi}_{3m}$ are eigenfunctions of $A_2 + \Delta_1$. Then,
\[
(\bar{\Phi}_{1k}, \bar{\Phi}_{2k}, \bar{\Phi}_{3m}) = T^{-1}(\Phi_{1k}, \Phi_{2k}, \Phi_{3m}).
\]
By (4.80) and (4.81),
\[
\bar{\Phi}_{1n}(x) = \begin{pmatrix} (c_1 - c_2\Delta) \cos k\pi x \\ c_2 \cos k\pi x \\ 0 \end{pmatrix} + O(k^{-1}),
\]
\[
\bar{\Phi}_{2n}(x) = \begin{pmatrix} (c_1 - c_1\Delta) \cos k\pi x \\ c_2 \cos k\pi x \\ 0 \end{pmatrix} + O(k^{-1}),
\]
\[
\bar{\Phi}_{3m} = \begin{pmatrix} 0 \\ 0 \\ \epsilon_m \end{pmatrix},
\]
which are similar to (4.41) and (4.42) with $\Delta(x) = 0$. By the proof of Theorem 4.2,
\[
\int_0^\infty e^{\gamma_0 t} |e(t)|^2 dt < \infty
\]
for some $\gamma_0 > 0$. This completes the proof of the theorem. □

V. NUMERICAL SIMULATIONS

In this section, we present some numerical simulations to illustrate the effect of the designed controller. The finite difference method is adopted in computation of the simulation results for the closed-loop system (4.1) by taking the steps of time and space as 0.0009 and 0.34, respectively. The initial states are taken as
\[
v(0) = (0.6, 0, 4, 0.5, 0.6)^T, \quad \dot{v}(0) = (2, 4, 1, -5)^T,
\]
\[
u_0(x) = 0.3 \sin(\pi x) - (0.2 \cos(\pi x))j, 
\]
\[
\bar{z}_0(x) = 0.3 \cos(\pi x) - (0.2 \sin(\pi x))j,
\]
and the matrix $S$ is
\[
S = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 \pi \\ 0 & 0 & -0.8 \pi & 0 \end{bmatrix}.
\]
In this case, the solution of $v(t)$ is $v(t) = (\sin(0.5t), \cos(0.5t), \sin(0.8t), \cos(0.8t))$. We choose the parameters $c_1 = 1.7$ and $c_2 = 3.7$. Set $N_0 = (1, 2, 3, 4)$ and $M_0 = jc_2N_0$, where
\[
N_0J = (0.7071 + 1.4142j, 0.7071 - 1.4142j, 2.1213 + 2.8284j, 2.1213 - 2.8284j),
\]
where each component is nonzero to make $(S, N_0)$ detectable. Let
\[
Q = \frac{\pi}{220} (-175, -240, 232, -309).
\]
Then,
\[
[S + Qg_0(0)] = J = \begin{bmatrix} -10.91 & 0 & 0 & 0 \\ 0 & -5.60 & 0 & 0 \\ 0 & 0 & -0.28 & 0 \\ 0 & 0 & 0 & -0.28 \end{bmatrix} J^{-1},
\]
where \( J \) is the matrix with column vectors being the eigenvectors corresponding to eigenvalues $\lambda = \{-10.91, -5.6, -0.28, -0.28\}$ of $S + Qg_0(0)$, respectively.

Figures 1-3 depict the performance of the closed-loop system (4.1) with $M_1 = (2, 4, 3, 1), N = (1, 2, 2, 1)$, and $G(x) = (x, x^2, \sin(x), \cos x), M_2 = (6, 1, 8, 2)$.

Figures 4-6 depict the performance of the closed-loop system (4.1) with $M_1 = (3, 4, 2, 1), N = (3, 2, 2, 1)$, and $G(x) = (\sin(x), \cos(x), x^2), M_2 = (1, 6, 2, 8)$.

Figures 7-9 depict the performance of the closed-loop system (4.57) with $M_1 = (6, 5, 4, 3), N = (8, 7, 6, 5), G(x) = (\sin(x), \cos(x), x^2), M_2 \geq (1, 8, 2, 6)$ and $\Delta(x) = 0.02$ for looking at the robustness of the control.

From Figures 1 and 4, it is seen that $u(x, t)$ is bounded. From Figures 2 and 5, the tracking error $e(t)$ is regulated towards zero under this different set of disturbance. Figures 3 and 6 plot the control trajectory which is seen being bounded.

From Figures 7-9, we notice that the same controller can also stabilize the closed-loop system (4.57) with uncertainty term $\Delta(\cdot)$. As shown in Figures 7 and 8, the $u(x, t)$ is bounded and $u(0, t)$ is regulated to track $y_{ref}(t)$ as the time $t$ evolves. Figure 9 shows the boundedness of the control.
VI. CONCLUDING REMARKS

In this paper, under the guideline of the internal model principle, we consider robust output tracking problem for a one-dimensional Schrödinger equation with external disturbance in all possible channels. The problem is a typical problem for PDEs, where the observation operator is unbounded and the control and output are non-collocated. We adopt an observer based approach to design an error feedback control by selecting specially coefficients of the disturbances to get a nominal system which is a coupled PDE+ODE system. An observer is designed for the nominal system, which together with simple feedforward control gives a dynamic error feedback control. It is shown that the proposed feedback control is robust to all disturbances and conditionally robust to some system uncertainties. A much profound stability for the closed-loop and stronger convergence for tracking error then abstract result [19] are established by the Riesz basis approach, where some profound result is used due to non-collocated nature of the problem. This together with recent study [6], [7], [8] constitutes a systematic observer based error feedback designs...
for various output regulations of SISO PDEs.

Finally, one reviewer indicated the physical motivation of the setup of the controlled system (2.1)-(2.2). From Born’s explanation in quantum mechanics, \( |u(x,t)|^2 \) and \( u_x(x,t) \) represent respectively the probability and momentum of the particles appear at position \( x \) and time \( t \). From this, system (2.1)-(2.2) can be explained as regulation of the probability of the particles through momentum control at the right boundary \( x = 1 \) in terms of the probability of the particles at the left boundary \( x = 0 \), with possible external disturbances at all channels.

**References**


Jun-Jun Liu received his B.S. degree in Mathematics at Changzhi University, China, in 2008, M.S. degree from Xinjiang University in 2011, and Ph.D degree in Applied Mathematics from Beijing Institute of Technology in 2015. Since from 2015, he has been with Taiyuan University of Technology first as a lecturer (2015-2019) and subsequently as an associate professor (2019-). His research interests focus on distributed parameter systems control.

Bao-Zhu Guo received the Ph.D degree from Chinese University of Hong Kong in Applied Mathematics in 1991. From 1985 to 1987, he was a Research Assistant at Beijing Institute of Information and Control, China. During the period 1993-2000, he was with Beijing Institute of Technology, first as an associate professor (1993-1998) and subsequently a Professor (1998-2000). Since 2000, he has been with Academy of Mathematics and Systems Science, the Chinese Academy of Sciences, where he is a Research Professor in mathematical system theory. Since from 2019, he is also with School of Mathematics and Physics, North China Electric Power University, Beijing. His research interests include control theory of infinite-dimensional systems and active disturbance rejection control.