Order reduction-based uniform approximation of exponential stability for one-dimensional Schrödinger equation

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A B S T R A C T

This paper considers the uniform exponential stability approximation of a one-dimensional Schrödinger system with boundary damping. The continuous system is known to be exponentially stable. Firstly, the order reduction method is adopted to transform the original system into an equivalent one. Two second-order semi-discretized finite difference schemes are derived for both the transformed system and the original system, which are shown to be equivalent to each other. Secondly, the Lyapunov function method is used to prove the uniform exponential stability of the semi-discretized transformed system, which is parallel to the proof of the continuous transformed system. Finally, the solution of the semi-discretized system converges weakly to the solution of the original system and the discrete energy converges to the continuous energy.

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1. Introduction

For control systems described by partial differential equations (PDEs), numerical discretization is indispensable from theoretical research to engineering applications. Among all existing numerical methods, the finite difference (FD) method is one of the most popular ones for engineers due to its simplicity and universality. However, it has been known for a long time that for PDEs, traditional finite difference and even finite element methods may have some problems in the approximation process [1]. One of such problems is that the exponential decay rate of an exponentially stable PDE might not be preserved uniformly in the process of spatial discretization.

Let us consider the following one-dimensional Schrödinger control system

\[
\begin{align*}
    u_t(x, t) + i w_u(x, t) &= 0, \quad x \in (0, 1), \quad t \in (0, T), \\
    w(0, t) = 0, \quad w(1, t) &= -kw_1(1, t), \quad t \in (0, T), \\
    w(0, 0) &= w^0(x), \quad x \in (0, 1),
\end{align*}
\]

where \( i = \sqrt{-1} \) is the imaginary unit, \( k > 0 \) is a constant and \( w^0 \in H^2_0(0, 1) = \{ f \in H^4(0, 1), f(0) = 0 \} \). The energy \( F(t) \) of system (1.1) in the state space \( H^4_0(0, 1) \) is given by

\[
F(t) = \frac{1}{2} \int_0^1 |w_u(x, t)|^2 \, dx.
\]

Formally,

\[
\frac{dF(t)}{dt} = -k |w_1(1, t)|^2,
\]

that is, the energy of the system is decreasing as time evolves. Actually, it has been known that for any given initial state \( w^0(\cdot) \), Eq. (1.1) is associated with a \( C_0 \)-semigroup solution in \( H^2_0(0, 1) \), which decays exponentially in time and the decay rate is uniform for any initial state \( w^0(\cdot) \) in \( H^2_0(1, 1) \), i.e.,

\[
F(t) \leq Ke^{-\omega t}F(0), \quad \forall t \geq 0,
\]

for some \( K, \omega > 0 \), independent of the initial state \( w^0(\cdot) \), see, e.g., [2, Theorem 1.1].

The uniform approximation for exponentially stable infinite dimensional systems has been studied extensively since 1990s, for which a big concern is whether or not the exponential decay rate of the discretized energy is uniformly preserved. The paper [3] first pointed out that the exponential decay of the discretized energy might not be uniform with respect to the mesh size for classical finite difference and finite element schemes. This is largely due to the high frequency spurious oscillations present in the numerical schemes. To remedy this problem, the authors of [3] suggested to use the mixed finite element methods [4,5] or
the polynomial based Galerkin methods to preserve the uniform exponential stability \[6\]. There are also some other remedies to
damp out these spurious high frequencies like Tychonoff regularizations \[7\], two-grid algorithms \[8\], and filtering techniques
\[9-11\]. Another popular technique to damp out the spurious high frequencies is the numerical viscosity method which incorporates
a vanishing numerical viscosity term in the whole domain. By virtue of this viscosity term, one can obtain uniform observability of semi-discretized systems \[7,\, 11,\, 12\], uniform boundary controllability, convergence of controls \[10,\, 13\], and uniform exponential stability \[14\]. For the spatial semi-discretized approximation of one-dimensional Schrödinger control systems, paper \[15\] studied
the uniform controllability approximation with vanishing numerical viscosity term. The uniform approximation of the controllability of a one-dimensional Schrödinger system was studied in paper \[16\] by filtering techniques.

In the uniform controllability approximation of wave or beam equations, both numerical viscosity method and filtering techniques need to
calculate the eigenvalues and eigenvectors of the semi-discretized systems, which is most often possible only for problems with special boundary conditions. For some complex or more general boundary conditions, it is difficult to calculate these eigen-pairs. Recently, we developed a semi-discretized finite difference scheme by order reduction method, which can preserve the uniform exponential stability of semi-discretized systems of exponential PDEs with various boundary conditions. In addition, the proof of uniform exponential stability is parallel to that of the continuous counterpart, which simplifies significantly the proof by other finite difference semi-discretized schemes. In \[17\], the order reduction method was used to construct a semi-discretized finite difference scheme for a one-dimensional wave equation with damped boundary condition, which preserves the exponential stability of the continuous system uniformly. In \[18\], we used the same idea to study the uniform approximation of exponential stability of a one-dimensional Euler–Bernoulli beam system. In \[19\], the uniform exponential stability of wave equation with local viscosity was studied by order reduction method-based semi-discretized finite difference scheme.

To the best of our knowledge, there is no semi-discretized finite difference scheme that can preserve the uniform exponential stability of one-dimensional Schrödinger system \(1.1\), for which we have encountered difficulty for quite some time by our approach. The main reason is that it is difficult to find a suitable Lyapunov function for this essentially coupled (real and imaginary) system. In this paper, we propose a completely new proof for system \(1.1\), which is quite different from our previous works \[17-19\]. The main technique we adopt here is the order reduction method. The order reduced system is a coupled system and the proof of the exponential stability is equivalent and parallel that of the original system. The uniform exponential stability is also carried out for the order reduced semi-discretized system, which is shown to be much simpler. Since the order reduced system in discrete form is equivalent to the final semi-discretized system of the original one, the proof provides a different new way for the construction of the Lyapunov function for this Schrödinger system, which has potential applicability to deal with other PDEs, like observer-based control coupled PDEs \[20\].

First, we introduce the following intermediate variable
\[
v(x, t) = u_x(x, t).
\]
Then, system \(1.1\) can be reformulated as
\[
\begin{align*}
v_x(x, t) + iv_y(x, t) &= 0, \quad x \in (0, 1), \quad t \in (0, T),
v(x, t) - u_x(x, t) &= 0, \\
v(0, t) &= 0, \quad v(1, t) = -kw\{1, t\}, \\
w(0, t) &= w_0(x).
\end{align*}
\]
(1.5)
The energy \(E(t)\) of system \(1.5\) now equivalently reads
\[
E(t) = \frac{1}{2} \int_0^1 |v(x, t)|^2 dx, \tag{1.6}
\]
and satisfies the following energy dissipation law
\[
\frac{dE(t)}{dt} = -k|w_1(1, t)|^2. \tag{1.7}
\]
Moreover, the exponential stability as that for system \(1.4\) still holds true, i.e., there are \(K, \omega > 0\) that are independent of the initial state \(w^0(\cdot)\) such that
\[
E(t) \leq Ke^{-\omega t}E(0), \quad \forall t \geq 0, \tag{1.8}
\]
for all initial state \(w^0 \in H_1^2(0, 1)\), which will be proved in Section 2.

The rest of this paper is organized as follows. In Section 2, we prove the exponential stability of the equivalent continuous system \(1.5\) by the Lyapunov function method. In Section 3, we construct a standard spatial semi-discretized finite difference scheme for the equivalent system \(1.5\). Equivalently, we get a semi-discretized finite difference scheme for the original system \(1.1\). Section 4 is devoted to the proof of the uniform exponential stability of the semi-discretized system of \(1.5\), which leads to the uniform exponential stability of the semi-discretized system of the original system \(1.1\). Section 5 shows that the solution of semi-discretized system of the original system \(1.1\) converges weakly to the solution of the original continuous system \(1.1\), followed by the conclusions given in Section 6.

2. Exponential stability of system \(1.5\)

In this section, we adopt the Lyapunov function method to prove the exponential stability of system \(1.5\). Although this is similar to the proof for system \(1.1\) as \[2, \text{Theorem 1.1}\], we present here as a preparation and motivation of construction of the related functions for the uniform exponential stability of its semi-discretized counterpart in the next section. For this purpose, we construct the following Lyapunov function:
\[
L(t) = E(t) + \varepsilon \varphi(t), \quad (0 < \varepsilon < 1), \tag{2.1}
\]
where the auxiliary function \(\varphi(t)\) is given by
\[
\varphi(t) = -\frac{1}{2} \text{Im} \int_0^1 xw(x, t)\overline{v(x, t)}dx. \tag{2.2}
\]
In order to prove the uniform exponential stability of system \(1.5\), we rely on the following Lemmas 2.1–2.3.

Lemma 2.1. The derivative of energy \(E(t)\) defined by (1.6) satisfies
\[
\frac{dE(t)}{dt} = -k|w_1(1, t)|^2.
\]
Proof. Taking the derivative of \(E(t)\) defined by (1.6) with respect to \(t\) on both sides, we obtain
\[
\frac{dE(t)}{dt} = \frac{1}{2} \int_0^1 v_x(x, t)|v(x, t)|dx + \frac{1}{2} \int_0^1 v(x, t)|v_x(x, t)|dx
\]
\[
= Re \int_0^1 v_x(x, t)|v(x, t)|dx
\]
\[
= Re \int_0^1 w_x(x, t)|v(x, t)|dx
\]
\[
= Re \left\{ w_x(x, t)v(x, t) \right\}_{0}^{1} - \int_0^1 w_x(x, t)v_x(x, t)dx
\]
\[
= Re \left\{ -k|w_1(1, t)|^2 + i \int_0^1 |v_x(x, t)|^2 dx \right\}
\]
\[
= -k|w_1(1, t)|^2,
\]
which leads to the required result. □
Lemma 2.2. The Lyapunov function $L(t)$ defined by (2.1) is equivalent to the energy $E(t)$ of system (1.5), i.e.

$$(1 - \varepsilon)E(t) \leq L(t) \leq (1 + \varepsilon)E(t),$$

for any $0 < \varepsilon < 1$.

Proof. A direct use of Cauchy–Schwarz’s inequality and Poincaré’s inequality gives

$$\left| \psi(t) \right| \leq \frac{1}{2} \int_0^1 \left| \int_0^1 xw(x, t)v(x, t)dx \right| \leq \frac{1}{2} \int_0^1 \left| w(x, t) \right|^2 dx + \frac{1}{4} \int_0^1 \left| v(x, t) \right|^2 dx$$

and hence

$$(1 - \varepsilon)E(t) \leq L(t) \leq (1 + \varepsilon)E(t),$$

for $0 < \varepsilon < 1$. This completes the proof of the lemma. □

Lemma 2.3. The derivative of the auxiliary function $\psi(t)$ defined by (2.2) satisfies

$$\frac{d\psi(t)}{dt} \leq -E(t) + \frac{1 + 5k^2}{8} |w_1(t, t)|^2.$$  \hspace{1cm} (2.3)

Proof. Finding the derivative of $\psi(t)$ with respect to $t$ gives

$$\frac{d\psi(t)}{dt} = -\int_0^1 \left| xw(x, t)v(x, t) \right| dx - \int_0^1 \left| xw(x, t)v(x, t) \right| dx.$$ \hspace{1cm} \hspace{1cm} (2.4)

The first term on the right-hand side of (2.4) implies that

$$\frac{1}{2} \int_0^1 \left| xw(x, t)v(x, t) \right| dx = \frac{1}{2} \int_0^1 \left| bxv(x, t)v(x, t) \right| dx$$

The second term on the right-hand side of (2.4) leads to

$$\frac{1}{2} \int_0^1 \left| xw(x, t)v(x, t) \right| dx = \frac{1}{2} \int_0^1 \left| xw(x, t)v(x, t) \right| dx$$

and

$$\frac{1}{2} \int_0^1 \left| xw(x, t)v(x, t) \right| dx = \frac{1}{2} \int_0^1 \left| xw(x, t)v(x, t) \right| dx$$

Finally, by Lemma 2.2 and $1 - \varepsilon > 0$, we arrive at

$$E(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} E(0).$$

This completes the proof of the theorem. □

Remark 2.1. For the original system (1.1), the Lyapunov function is

$$L_0(t) = F(t) + \varepsilon \psi_0(t), \quad 0 < \varepsilon < 1,$$

where $F(\cdot)$ is given by (1.2) and the auxiliary function $\psi_0(\cdot)$ is given by

$$\psi_0(t) = \frac{1}{2} \int_0^1 xw(x, t)\psi_0(x, t)dx.$$
3. Semi-discretized finite difference schemes for both (1.5) and (1.1).

In this section, first, we construct a spatial semi-discretized finite difference scheme for the equivalent system (1.5). For fixed \( N \in \mathbb{N}^* \), we consider an equidistant partition of interval \([0, 1]\):
\[
0 = x_0 < x_1 < \cdots < x_j = jh < \cdots < x_{N+1} = 1.
\]
where \( h = \frac{1}{N+1} \) is the mesh size. To simplify the notations, we denote the sequence \( \{u_{j0}\}_{j=0}^{N+1} \) by \( \{u_j\} \) and introduce respectively the average operator, the first-order and second-order finite difference operators as
\[
\begin{align*}
  u_{j+\frac{1}{2}} &= \frac{u_j + u_{j+1}}{2}, \\
  \delta_0 u_{j+\frac{1}{2}} &= \frac{u_{j+1} - u_j}{h}, \\
  \delta_0^2 u_j &= \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}.
\end{align*}
\]
Suppose that \( u(x, t) \) and \( w(x, t) \) are the solutions of system (1.5). Let \( \{V_j(t)\} \) and \( \{W_j(t)\} \) be grid functions at grids \( \{x_j\} \), satisfying
\[
V_j(t) = w(x_j, t), \quad 0 \leq j \leq N + 1.
\]
The first equation of system (1.5) holds at \( (x_{j+\frac{1}{2}}, t), \) i.e.,
\[
w'(x_{j+\frac{1}{2}}, t) + i\nu w(x_{j+\frac{1}{2}}, t) = 0,
\]
where \( x_{j+\frac{1}{2}} = (j + \frac{1}{2})h \). Hereafter the prime "\'" represents the derivative with respect to \( t \). Replace the differential operator \( \delta_0 \) with difference operator \( \delta_0 \) to get
\[
W'_{j+\frac{1}{2}}(t) + i\nu W_{j+\frac{1}{2}}(t) = O(h^2).
\]
Similarly, for the second equation of system (1.5), we have
\[
V_{j+\frac{1}{2}}(t) - \delta_0 W_{j+\frac{1}{2}}(t) = O(h^3).
\]
By dropping the infinitesimal terms in (3.1) and (3.2), and replacing \( W_j(t) \) and \( V_j(t) \) by \( w_j(t) \) and \( v_j(t) \), respectively, we arrive at a semi-discretized finite difference scheme of system (1.5) as follows:
\[
\begin{align*}
  w_j' + i\nu w_j &= 0, \quad 0 \leq j \leq N, \quad t > 0, \\
  v_j' - \delta_0 w_j &= 0, \quad 0 \leq j \leq N, \quad t > 0, \\
  w_0(t) &= 0, \quad v_0(t) = -kw_{N+1}(t), \quad t > 0, \\
  w_j(0) &= w_j^0, \quad 0 \leq j \leq N + 1,
\end{align*}
\]
which means that (3.3) implies (3.4). Conversely, by letting
\[
\begin{align*}
  v_{j+\frac{1}{2}}(t) &= \delta_0 w_{j+\frac{1}{2}}(t), \quad 0 \leq j \leq N, \\
  v_{N+1} &= -kw_{N+1},
\end{align*}
\]
we obtain (3.3) from (3.4). In other words, systems (3.3) and (3.4) are equivalent to each other. Very interestingly, the proof of the uniform exponential stability for system (3.3) is much simpler than that of system (3.4). This is a new discovery of this paper because in our previous works [17–19], the proofs were done for system (3.4) directly without the help of the reduced order system (3.3). We shall go back to this point at the end of Section 4. This might be attributed to the peculiarity of system (1.1) which is actually a coupled system by its real and imaginary parts. We believe that the idea, and particularly the methodology of the proof provided in this paper, are potentially useful to study other types of PDEs.

4. Uniform exponential stability

In this section, we shall prove the uniform exponential stability of the semi-discretized system (3.4) of the original system (1.1). Now that the systems (3.3) and (3.4) are equivalent, it suffices to prove the uniform exponential stability of system (3.3), which is much simpler and feasible. The discretized energy \( E_h(t) \) of semi-discretized system (3.3) reads naturally as
\[
E_h(t) = \frac{h}{2} \sum_{j=0}^{N} |v_{j+\frac{1}{2}}(t)|^2. \tag{4.1}
\]
The following Lemmas 4.1–4.4 are crucial to the proof of the uniform exponential stability of the semi-discretized system (3.3).

Lemma 4.1 [17]. For any grid functions \( \{U_j\} \), \( \{V_j\} \) and \( \{W_j\} \) at mesh grids \( \{x_j\} \), the following two formulas of summation by parts hold:
\[
\begin{align*}
  h \sum_{j=0}^{N} \delta_0 U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + h \sum_{j=0}^{N} U_{j+\frac{1}{2}} \delta_0 V_{j+\frac{1}{2}} &= U_{N+1} V_{N+1} - U_0 V_0, \tag{4.2} \\
  h \sum_{j=0}^{N} \delta_0 U_{j+\frac{1}{2}} W_{j+\frac{1}{2}} + h \sum_{j=0}^{N} U_{j+\frac{1}{2}} \delta_0 W_{j+\frac{1}{2}} &= U_{N+1} W_{N+1} - U_0 W_0 \tag{4.3} \\
  - \frac{1}{4} \sum_{j=0}^{N} (U_{j+1} - U_j)(V_{j+1} - V_j)(W_{j+1} - W_j).
\end{align*}
\]

Lemma 4.2. The derivative of discrete energy \( E_h(t) \) of system (3.3) satisfies
\[
\frac{dE_h(t)}{dt} = -k|w_{N+1}(t)|^2.
\]

Proof. Finding the derivative of \( E_h(t) \) defined by (4.1) with respect to \( t \) on both sides yields
\[
\frac{dE_h(t)}{dt} = \frac{h}{2} \sum_{j=0}^{N} v_{j+\frac{1}{2}}'(t) v_{j+\frac{1}{2}}(t) \tag{4.4} \\
= \text{Re} \left\{ \frac{h}{2} \sum_{j=0}^{N} v_{j+\frac{1}{2}}'(t) v_{j+\frac{1}{2}}(t) \right\}
\]
\[ \psi(t) = \text{Re} \left\{ \frac{1}{h} \sum_{j=0}^{N} \delta_j w_{j+\frac{1}{2}}(t) v_{j+\frac{1}{2}}(t) \right\} \]

\[ = \text{Re} \left\{ w_{N+1}(t) w_{N+1} - h \sum_{j=0}^{N} w_{j+\frac{1}{2}}'(t) \delta_j v_{j+\frac{1}{2}}(t) \right\} \]

\[ = \text{Re} \left\{ w_{N+1}'(t) (\text{sgn}(x_{N+1}^2)) - h \sum_{j=0}^{N} w_{j+\frac{1}{2}}'(t) \left( -w_{j+\frac{1}{2}}(t) \right) \right\} \]

\[ = -k |w_{N+1}(t)|^2. \]

This completes the proof of the lemma. \( \square \)

From Lemma 4.2, we know that the discretized energy \( E_h(t) \) of system (3.3) is non-increasing in time as long as the damping coefficient \( k \) is positive. To show the uniform exponential decay of the discretized energy \( E_h(t) \), we construct the following Lyapunov function

\[ L_h(t) = E_h(t) + \varepsilon \psi_h(t), \quad (0 < \varepsilon < 1), \]

where the auxiliary function \( \psi_h(t) \) is given by

\[ \psi_h(t) = -\text{Re} \left\{ \frac{1}{h} \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}(t) v_{j+\frac{1}{2}}(t) \right\}. \]

which really is a natural discretization of \( \psi(t) \) defined in (2.2) for the continuous counterpart.

**Lemma 4.3.** The Lyapunov function \( L_h(t) \) defined by (4.4) is equivalent to the energy \( E_h(t) \) of system (3.3), i.e., for any \( 0 < \varepsilon < 1 \),

\[ (1 - \varepsilon) E_h(t) \leq L_h(t) \leq (1 + \varepsilon) E_h(t). \]

**Proof.** A direct use of Cauchy's inequality gives

\[ |\psi_h(t)| \leq \frac{h}{2} \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}(t) v_{j+\frac{1}{2}}(t)^2 \leq \frac{h}{4} \sum_{j=0}^{N} |w_{j+\frac{1}{2}}(t)|^2 \]

\[ + \frac{h}{4} \sum_{j=0}^{N} \left| v_{j+\frac{1}{2}}(t) \right|^2 \]

\[ \leq \frac{h}{8} \sum_{j=0}^{N} \left( |w_{j+1}(t)|^2 + |w_j(t)|^2 \right) + \frac{h}{4} \sum_{j=0}^{N} \left| v_{j+\frac{1}{2}}(t) \right|^2. \]

Since \( u_0(t) = 0 \), we have

\[ |w_{j+1}(t)|^2 = \sum_{k=0}^{j} (w_{k+1}(t) - u_k(t))^2 \]

\[ \leq \sum_{j=0}^{N} \left| 1 \right|^2 \sum_{j=0}^{N} |w_{j+1}(t) - w_j(t)|^2 = h \sum_{j=0}^{N} \left| \delta_j w_{j+\frac{1}{2}}(t) \right|^2. \]

Hence,

\[ |\psi_h(t)| \leq \frac{h}{4} \sum_{j=0}^{N} \left| \delta_j w_{j+\frac{1}{2}}(t) \right|^2 + \frac{h}{4} \sum_{j=0}^{N} \left| v_{j+\frac{1}{2}}(t) \right|^2 \leq E_h(t). \] (4.6)

According to (4.4), we further obtain the required inequalities:

\[ (1 - \varepsilon) E_h(t) \leq L_h(t) \leq (1 + \varepsilon) E_h(t). \] \( \square \)

**Lemma 4.4.** The derivative of the auxiliary function \( \psi_h(t) \) defined by (4.5) satisfies

\[ \frac{d \psi_h(t)}{dt} \leq -E_h(t) + \frac{1 + 5k^2}{8} |w_{N+1}'(t)|^2. \]

**Proof.** Finding the derivative of \( \psi_h(t) \) with respect to \( t \) on both sides gives

\[ \frac{d \psi_h(t)}{dt} = -\frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}'(t) v_{j+\frac{1}{2}}(t) \right\} \]

\[ - \frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}(t) v_{j+\frac{1}{2}}'(t) \right\}. \] (4.7)

According to (4.3), the first term on the right-hand side of (4.7) is simplified as

\[ \frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}'(t) v_{j+\frac{1}{2}}(t) \right\} \]

\[ = -\frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} \left( -i \delta_j v_{j+\frac{1}{2}}(t) \right) \overline{v_{j+\frac{1}{2}}(t)} \right\} \]

\[ = \frac{h}{2} \text{Re} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} \delta_j v_{j+\frac{1}{2}}(t) \overline{v_{j+\frac{1}{2}}(t)} \right\} \]

\[ = \frac{1}{4} |v_{N+1}(t)|^2 \leq \frac{h^2}{16} \sum_{j=0}^{N} |\delta_j v_{j+\frac{1}{2}}(t)|^2 - \frac{1}{2} E_h(t). \] (4.8)

Similarly, the second term on the right-hand side of (4.7) can be reduced to

\[ -\frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}(t) v_{j+\frac{1}{2}}'(t) \right\} \]

\[ = -\frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} w_{j+\frac{1}{2}}(t) \delta_j w_{j+\frac{1}{2}}(t) \right\} \]

\[ = -\frac{1}{2} \text{Im} \left\{ u_{N+1}(t) u_{N+1}'(t) + \frac{h^3}{8} \text{Im} \left\{ \sum_{j=0}^{N} \delta_j w_{j+\frac{1}{2}}(t) \overline{\delta_j w_{j+\frac{1}{2}}(t)} \right\} \right\} \]

\[ + \frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} \delta_j w_{j+\frac{1}{2}}(t) w_{j+\frac{1}{2}}(t) \right\} \]

\[ + \frac{h}{2} \text{Im} \left\{ \sum_{j=0}^{N} w_{j+\frac{1}{2}}(t) \overline{w_{j+\frac{1}{2}}(t)} \right\} \]

\[ = -\frac{1}{2} \text{Im} \left\{ u_{N+1}(t) u_{N+1}'(t) + \frac{h^3}{8} \text{Im} \left\{ \sum_{j=0}^{N} v_{j+\frac{1}{2}}(t) \overline{\delta_j v_{j+\frac{1}{2}}(t)} \right\} \right\} \]

\[ + \frac{h}{2} \text{Re} \left\{ \sum_{j=0}^{N} x_{j+\frac{1}{2}} v_{j+\frac{1}{2}}(t) \overline{\delta_j v_{j+\frac{1}{2}}(t)} \right\} \]

\[ + \frac{h}{2} \text{Re} \left\{ \sum_{j=0}^{N} w_{j+\frac{1}{2}}(t) \overline{\delta_j v_{j+\frac{1}{2}}(t)} \right\}. \] (4.9)

According to the first Eq. (4.2) of Lemma 4.1, we have

\[ \frac{h^3}{8} \text{Im} \left\{ \sum_{j=0}^{N} v_{j+\frac{1}{2}}(t) \delta_j w_{j+\frac{1}{2}}(t) \right\} \]

\[ = \frac{h^3}{8} \text{Im} \left\{ v_{N+1}(t) w_{N+1}'(t) - h \sum_{j=0}^{N} \delta_j v_{j+\frac{1}{2}}(t) w_{j+\frac{1}{2}}'(t) \right\} \]
$$\begin{align*}
&= \frac{h^2}{8} \text{Im} \left\{ -k |w'_{N+1}(t)|^2 - h \sum_{j=0}^{N} \delta_j v_{j+\frac{1}{2}}(t) \left( \delta_j v_{j+\frac{1}{2}}(t) \right) \right\} \\
&= -\frac{h^2}{8} \sum_{j=0}^{N} |\delta_j v_{j+\frac{1}{2}}(t)|^2, \quad (4.10)
\end{align*}$$

and
$$\begin{align*}
&= \frac{h}{2} \text{Re} \left\{ \sum_{j=0}^{N} w_{j+\frac{1}{2}}(t) \delta_j v_{j+\frac{1}{2}}(t) \right\} \\
&= \frac{1}{2} \text{Re} \left\{ w_{N+1}(t)x_{N+1}(t) - h \sum_{j=0}^{N} \delta_j w_{j+\frac{1}{2}}(t)v_{j+\frac{1}{2}}(t) \right\} \\
&= \frac{1}{2} \text{Re} \left\{ w_{N+1}(t)v_{N+1}(t) - \frac{h}{2} \sum_{j=0}^{N} |v_{j+\frac{1}{2}}(t)|^2 \right\} \\
&= \frac{1}{2} \text{Re} \left\{ w_{N+1}(t)v_{N+1}(t) - E_{N}(t) \right\}. \quad (4.11)
\end{align*}$$

By virtue of Poincaré’s inequality, plug (4.8)–(4.11) into (4.7) to get
$$\begin{align*}
\frac{d\phi_{N}(t)}{dt} &= -2E_{N}(t) + \frac{k^2}{2} |w'_{N+1}(t)|^2 \\
&\leq -2E_{N}(t) + \frac{k^2}{2} |w'_{N+1}(t)|^2 + \frac{1}{8} (1 + k^2) |w'_{N+1}(t)|^2 \\
&\leq -2E_{N}(t) + \frac{1 + 5k^2}{8} |w'_{N+1}(t)|^2 + E_{N}(t) \\
&= -E_{N}(t) + \frac{1 + 5k^2}{8} |w'_{N+1}(t)|^2.
\end{align*}$$

This completes the proof of the lemma. □

**Theorem 4.1.** For any $k > 0$, the energy $E_{N}(t)$ of the semi-discretized finite difference scheme (3.3) decays exponentially with a uniform decay rate. More precisely,
$$E_{N}(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} e^{-\frac{\varepsilon}{1 - \varepsilon}t} E_{N}(0),$$
where $\varepsilon$ is independent on $h$ and satisfies $0 < \varepsilon < \min \left\{ \frac{8k}{1 + 5k^2} \right\}$.

**Proof.** Finding the derivative of $L_{0}(t)$, together with Lemmas 4.2 and 4.4, we obtain
$$\begin{align*}
\frac{dL_{0}(t)}{dt} &= \frac{d\phi_{N}(t)}{dt} + \varepsilon \frac{d\phi_{N}(t)}{dt} \\
&\leq -\varepsilon E_{N}(t) + \left( \frac{1 + 5k^2}{8} \varepsilon - k \right) |w'_{N+1}(t)|^2.
\end{align*}$$

Since $0 < \varepsilon < \min \left\{ \frac{8k}{1 + 5k^2} \right\}$ implies $(1 + 5k^2)\varepsilon - k < 0$, it follows that
$$\frac{dL_{0}(t)}{dt} \leq -\varepsilon E_{N}(t) \leq -\frac{\varepsilon}{1 + \varepsilon} L_{0}(t).$$
A direct use of Grönwall’s inequality leads to
$$L_{0}(t) \leq e^{-\frac{\varepsilon}{1 + \varepsilon}t} L_{0}(0).$$

According Lemma 4.3 and $1 - \varepsilon > 0$, we finally arrive at
$$E_{N}(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} e^{-\frac{\varepsilon}{1 - \varepsilon}t} E_{N}(0).$$
This completes the proof of theorem. □

According to the equivalence between (3.3) and (3.4), we have the following uniform exponential stability for the semi-discretized system (3.4) of the original system (1.1).

**Corollary 4.1.** The discrete energy $F_{n}(t)$ of the semi-discretized system (3.4) reads
$$F_{n}(t) = \frac{h}{2} \sum_{j=0}^{N} |\delta_j w_{j+\frac{1}{2}}(t)|^2,$$
which satisfies, as a natural discretization of the continuous energy defined by (1.2),
$$\frac{dF_{n}(t)}{dt} = -k |w'_{N+1}(t)|^2,$$
and decays uniformly exponentially:
$$F_{n}(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} e^{-\frac{\varepsilon}{1 - \varepsilon}t} F_{n}(0),$$
for any $0 < \varepsilon < \min \left\{ \frac{8k}{1 + 5k^2} \right\}$.

To end this section, let us talk a little bit more why we consider the uniform approximation of the transformed system (1.5) rather than the original system (1.1), though these two systems are equivalent mathematically. Actually, in the proof of the uniform exponential stability of the semi-discretized system (3.3), Lemma 4.4 plays an important role which is embodied essentially in the construction of the discrete auxiliary function $\phi_{N}(t)$ given by (4.5). The discrete auxiliary function $\phi_{N}(t)$ should be a natural discretization of the continuous auxiliary function $\phi(t)$. Notice that for the original system (1.1), the continuous auxiliary function $\phi(t)$ is given by (2.7):
$$\phi_{0}(t) = -\frac{1}{2} \text{Im} \int_{0}^{1} xu(x, t)w_{N}(x, t)dx.$$
other technical difficulties arise. Although these two discrete auxiliary functions have the same limit \( \varphi(t) \) as \( h \to 0 \), they possess different efficiency in the process of numerical approximations.

To illustrate why the semi-discretized finite difference scheme (3.4) can uniformly preserve the exponential stability of the original system (1.1), we compare it with the following standard semi-discretized finite difference scheme

\[
\begin{align*}
    w_j' + i2\pi w_j &= 0, \quad 1 \leq j \leq N + 1, \\
    w_0 &= 0, \quad w_{N+2} = w_{N+1}, \\
    w_j(0) &= w_j^0, \quad 0 \leq j \leq N + 1.
\end{align*}
\]

(4.17)

As well-known, the nonuniform exponential stability of the semi-discretized system (4.17) is mainly attributed to the high frequency components produced by the semi-discretization. These high frequency eigenvalues have negative real parts, but vanish as \( h \to 0 \). Thus these high frequency components cannot decay uniformly with respect to \( h \). Fig. 1 shows the eigenvalue distribution of the semi-discretized system (4.17) and (3.4), respectively. As shown in Fig. 1 that the eigenvalues with high frequency of semi-discretized system (3.4) tend to infinity, while the high frequency eigenvalues of the semi-discretized system (4.17) accumulate to imaginary axis. That is why the semi-discretized system (3.4) possesses uniform exponential stability. However, due to the technical difficulty, we are not able to compute the eigenvalues of the semi-discretized system (3.4) analytically.

5. Convergence to continuous solution

In this section, we prove that the solution of the semi-discretized system (3.4) converges weakly to the solution of the original system (1.1), and so does the discrete energy \( F_h(t) \).

To begin with, we introduce some additional notations. Let \( v_h = (v_j)_j \) be a vector in \( \mathbb{R}^{N+2} \). Introduce an extension operator \( q_h : \mathbb{R}^{N+2} \to L^2(0, 1) \) as

\[
q_h v_h(x) = \frac{1}{2} (v_i + v_{i+1}), \quad x \in (x_i, x_{i+1}), \quad i = 0, 1, \ldots, N.
\]

(5.1)

It is easy to check that

\[
\int_0^1 (q_h w_h)(q_h v_h) dx = h \sum_{j=0}^N \frac{1}{2} |v_{j+\frac{1}{2}}|^2.
\]

(5.2)

Introduce another extension operator \( p_h : \mathbb{R}^{N+2} \to H^1(0, 1) \) as follows

\[
p_h v_h(x) = v_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + v_{i+1} \frac{x - x_i}{x_{i+1} - x_i}, \quad x \in (x_i, x_{i+1}), \quad i = 0, 1, \ldots, N.
\]

(5.3)

It is verified that

\[
\int_0^1 (p_h u_h)(p_h v_h) dx = h \sum_{j=0}^N \delta_x u_{j+\frac{1}{2}} \delta_x v_{j+\frac{1}{2}}.
\]

(5.4)

For every function \( v \in C(0, 1) \), if we set \( v_h = (v_j)_j \), it follows easily that

\[
q_h v_h \to v \quad \text{strongly in } L^\infty(0, 1).
\]

(5.5)

Moreover, if \( v \in H^1(0, 1) \), then

\[
\|p_h u_h - q_h v_h\|_{L^2(0, 1)}^2 = h^2 \sum_{j=0}^N (\delta_x v_{j+\frac{1}{2}})^2 = O(h^2).
\]

(5.6)

and

\[
\|(p_h u_h)_x\|_{L^2(0, 1)}^2 = h \sum_{j=0}^N (\delta_x v_{j+\frac{1}{2}})^2 = O(1),
\]

(5.7)

which, together with (5.5), implies that

\[
p_h v_h \to v \quad \text{weakly in } H^1(0, 1) \text{ as } h \to 0.
\]

(5.8)

Now we are in a position to state the convergence of the discrete solution to the continuous counterpart.

Theorem 5.1. Let \( u_h(t) \) be solution of (3.4). Assume that \( u_h^0 \) satisfies,

\[
p_h u_h^0 \to w^0 \quad \text{weakly in } H^1(0, 1).
\]

(5.9)

Then,

\[
p_h \varphi_h \to w \quad \text{weakly* in } L^\infty(0, \infty; H^1(0, 1)) \text{ as } h \to 0.
\]

(5.10)

where \( w(x, t) \) is the solution of system (1.1). In addition, if \( F_0(0) \to F(0) \) as \( h \to 0 \), then

\[
\lim_{h \to 0} \|F_h - F\|_{C(0, \infty)} = 0.
\]

Proof. By definition of \( q_h, p_h \) and \( F_h(t) \), for every \( t > 0 \), it has

\[
F_h(t) = \frac{1}{2} \|p_h w_h(t)\|_{H^1(0, 1)}^2.
\]

(5.11)

Since \( F_h(t) \) is non-increasing with respect to time \( t \), by the convergence of the initial state (5.9), we know that \( p_h u_h \) is bounded in \( L^2(0, \infty; H^1(0, 1)) \). Moreover, it follows from (4.13) that for all \( t > 0 \)

\[
F_0(0) = F_0(t) + \int_0^t w_h'(t)^2 dt,
\]

(5.12)

from which we can see that \( w_h'(t) \) is bounded in \( L^2(0, \infty) \). By the Sobolev embedding theorem and extracting a subsequence if
necessary [14], we have
\[
p_h w_h \rightarrow w \text{ weakly}^* \text{ in } L^\infty(0, \infty; H^1_0(0, 1)), \\
w_{N+1}^0 \rightarrow w'(1, \cdot) \text{ weakly in } L^1(0, \infty),
\]
(5.13)
where we assumed implicitly that \(p_h w_h\) and \(q_h w_h\) have the same limit. The second convergence in (5.13) can be proved similarly as that in [18].

Next, we show that the limit \(w(x, t)\) is a weak solutions of (1.1).

Let \(\psi \in \mathcal{D}(0, 1) \times (0, \infty)\) with \(\psi(0, \cdot) = 0\) and let \(\psi_h = (\psi(x, \cdot))\). Firstly, multiplying the first equation of (3.4) by \(\psi_j\) and summing up for \(j = 1, \ldots, N\), we obtain, after some manipulations, that
\[
\begin{aligned}
&\sum_{j=0}^{N} w_{N+1}^j \psi_j - i \sum_{j=0}^{N} \delta w_{j+1/2}^j \psi_{j+1/2} - \frac{h}{2} w_{N+1}^j \frac{\psi_{N+1}}{w_{N+1}} \\
&+ i \delta w_{N+1}^j \psi_{N+1} = 0.
\end{aligned}
\]
(5.14)
Plugging the second equation of (3.4) into (5.14) and performing an integration by parts over \((0, \infty)\) with respect to \(t\), we obtain
\[
\begin{aligned}
&\int_{0}^{\infty} \sum_{j=0}^{N} \frac{w_{j+1/2}^j \psi_j}{w_{j+1/2}} dt - i \int_{0}^{\infty} \sum_{j=0}^{N} \delta w_{j+1/2}^j \psi_{j+1/2} dt \\
&- i \int_{0}^{\infty} \delta w_{N+1}^j \psi_{N+1} dt = 0.
\end{aligned}
\]
(5.15)
By the definitions of \(p_h\) and \(q_h\), it is easy to check that (5.15) is equivalent to
\[
\begin{aligned}
&\int_{0}^{\infty} \int_{0}^{\infty} \frac{(q_h w_h)(q_h \psi_h)}{w_h} x dx dt + i \int_{0}^{\infty} \int_{0}^{\infty} \frac{(p_h w_h)(p_h \psi_h)}{w_h} x dx dt \\
&+ ik \int_{0}^{\infty} w_{N+1}^j \psi_{N+1} dt = 0.
\end{aligned}
\]
(5.16)
Similarly to that in [18], we can prove that for every \(\psi \in \mathcal{D}(0, 1) \times (0, \infty)\)
\[
\begin{align*}
&\{q_h \psi_h \rightarrow \psi \text{ strongly in } L^\infty(0, \infty; L^2(0, 1)), \\
&p_h \psi_h \rightarrow \psi \text{ strongly in } L^\infty(0, \infty; H^1_0(0, 1)).
\end{align*}
\]
Next, with the convergence in (5.13), we can pass to the limit as \(h \rightarrow 0\) for all terms in (5.16) to arrive at
\[
\begin{aligned}
&\int_{0}^{\infty} \int_{0}^{\infty} w(x, t)\psi(x, t) dt dx + i \int_{0}^{\infty} \int_{0}^{\infty} w'(x, t)\psi'(x, t) dt dx \\
&+ ik \int_{0}^{\infty} w'(1, t)\psi'(1, t) dt = 0.
\end{aligned}
\]
This shows that \(w(x, t)\) is the weak solutions of (1.1).

It remains to show that \(w(x, t)\) satisfies the initial state \(w(\cdot, 0) = w^0\). For this purpose, we pick \(s \in \mathcal{D}(0, 1)\) and \(t \in \mathcal{D}(0, \infty)\) and let \(s_h = (s(x))^h\).

Firstly, multiplying the first equation of (3.4) by \(\psi_j\) and summing up for \(j = 1, \ldots, N\), we obtain
\[
\begin{aligned}
&\sum_{j=0}^{N} w_{j+1/2}^j \psi_j - i \sum_{j=0}^{N} \delta w_{j+1/2}^j \psi_{j+1/2} - \frac{h}{2} \frac{w_{j+1/2}^j \psi_j}{w_{j+1/2}^j} \\
&+ i \delta w_{N+1}^j \psi_{N+1} = 0.
\end{aligned}
\]
(5.17)
Since \(s_{N+1}^j = 0\), performing integration by parts on (5.17) over \([0, \infty)\) with respect to \(t\), one has
\[
\begin{aligned}
&\int_{0}^{\infty} \sum_{j=0}^{N} w_{j+1/2}^j \psi_j dt - i \sum_{j=0}^{N} \delta w_{j+1/2}^j \psi_{j+1/2} dt \\
&- i \int_{0}^{\infty} \delta w_{N+1}^j \psi_{N+1} dt = 0.
\end{aligned}
\]
(5.18)
By definitions of \(p_h\) and \(q_h\), it is easy to check that (5.18) is equivalent to
\[
\begin{aligned}
&\ll 0 \rr \int_{0}^{1} (q_h w_h)(q_h s_h) x dx + \int_{0}^{\infty} \int_{0}^{1} (q_h w_h)(q_h s_h) x dx dt \\
&+ i \int_{0}^{\infty} \int_{0}^{1} (p_h w_h)(p_h s_h) x dx dt = 0.
\end{aligned}
\]
(5.19)
Passing to the limit as \(h \rightarrow 0\) in (5.19) gives
\[
\begin{aligned}
&\ll 0 \rr \int_{0}^{1} w^0(x) dx + \int_{0}^{\infty} \int_{0}^{1} w^0(x) dx dt + i \int_{0}^{\infty} \int_{0}^{1} w^0(x) dx dt = 0,
\end{aligned}
\]
which implies \(w(x, 0) = w^0(x)\) from the variational principle. Since system (1.1) admits a unique solution, we can conclude that the convergence in (5.13) holds for the whole sequence \((h)_\infty\), not only for the extracted subsequence.

Finally, by (1.3) and (4.13), it follows that
\[
F_0(t) = F(t) + k \int_{0}^{t} \left| w'(1, t) \right|^2 dt, \quad F_h(0) = F_h(t) + k \int_{0}^{t} \left| w_{N+1}^j \right|^2 dt.
\]
(5.20)
Since both \(F(t)\) and \(F_h(t)\) are exponentially stable, it has
\[
F_0(t) = k \int_{0}^{\infty} \left| w'(1, t) \right|^2 dt, \quad F_h(0) = k \int_{0}^{\infty} \left| w_{N+1}^j \right|^2 dt.
\]
(5.21)
The assumption \(F_0(0) \rightarrow F(0)\) implies that
\[
\int_{0}^{\infty} \left| w_{N+1}^j \right|^2 dt + \int_{0}^{\infty} \left| w'(1, t) \right|^2 dt as h \rightarrow 0,
\]
whereas the weak convergence in (5.13) yields
\[
w_{N+1}^j \rightarrow w'(1, \cdot) \text{ strongly in } L^2(0, \infty),
\]
(5.22)
By (5.20) and (5.23), we have, for all \(t > 0\), that
\[
\begin{aligned}
&\left| F_h(t) - F(t) \right| \leq \left| F_h(0) - F(0) \right| + k \int_{0}^{t} \left| w_{N+1}^j \right|^2 dt \\
&- \int_{0}^{t} \left| w'(1, t) \right|^2 dt \\
&\leq \left| F_h(0) - F(0) \right| + k \left( \int_{0}^{t} \left| w_{N+1}^j + w'(1, t) \right|^2 dt \right)^{1/2} \\
&\times \left( \int_{0}^{t} \left| w_{N+1}^j \right|^2 + \left| w'(1, t) \right|^2 dt \right)^{1/2} \\
&\leq \left| F_h(0) - F(0) \right| + \sqrt{2k} \left( \left| w_{N+1}^j \right|_{L^2(0, \infty)} + \left| w'(1, \cdot) \right|_{L^2(0, \infty)} \right) \\
&\times \left| w_{N+1}^j - w'(1, \cdot) \right|_{L^2(0, \infty)}.
\end{aligned}
\]
(5.24)
which, together with \(F_h(0) \rightarrow F(0)\) and (5.22), show that
\[
\lim_{h \rightarrow 0} \left| F_h - F \right|_{C(0, \infty)} = 0.
\]
(5.25)
This completes the proof of the theorem. \qed

6. Conclusions

In this paper, the uniform approximation of exponential stability of one-dimensional Schrödinger equation is considered. The order reduction method is adopted to transform the original continuous system (1.1) into an equivalent system (1.5). The transformed system (1.5) has no spatial derivative at the
right boundary condition, which plays a key role in the uniform stability approximation, without discretizing the derivative at boundary as indicated in the introduction of [1] that “It turns out that the key difficulty lies with the approximation of the normal derivative”. For the transformed system (1.5), we construct a standard spatial semi-discretized finite difference scheme (3.3), from which we derive the semi-discretized finite difference scheme (3.4) for the original system (1.1). The uniform exponential stability of the semi-discretized system (3.3) is proved by the Lyapunov function method, which is parallel to the proof of the transformed continuous system (1.5). According to the equivalence between (3.3) and (3.4), we get the uniform exponential stability approximation for the original continuous system (1.1). This is the first time to see another power of the order reduction method, because a direct proof for the uniform exponential stability of (3.4) is not straightforward. Furthermore, the (weak) solution of the semi-discretized system (3.4) converges to the solution of the original system (1.1). The order reduction method has potential advantages in dealing with complex boundary derivatives. The technique of constructing the scheme is also feasible for the uniform exponential stability approximation of other control problems of PDEs.

Finally, we point out that the big issue for system (1.1) is the exponential stability in the state space \( L^2(0, 1) \) where one is not able to find Lyapunov function even for continuous system let alone the semi-discrete counterpart. For the continuous part, the Riesz basis can give a simple proof which was presented in [21] but the Riesz basis is hard to be applied to discrete systems. This challenge would be our future work for this system. One reviewer indicated the relationship between continuous system (1.1) and the discrete one (3.4), which should be an interesting problem of spectral approximation because system (1.1) can be shown to be a Riesz spectral system and the spectrum-determined growth condition holds. Both reviewers mentioned the applicability for high dimensional problems which is what we should strive for in the future.

CRediT authorship contribution statement


Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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