Uniformly semidiscretized approximation for exact observability and controllability of one-dimensional Euler–Bernoulli beam

Jiankang Liu a, *, Bao-Zhu Guo b

a School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China
b School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

A R T I C L E   I N F O

Article history:
Received 6 May 2021
Received in revised form 23 July 2021
Accepted 1 August 2021
Available online xxxx

Keywords:
Euler–Bernoulli beam
Finite difference
Observability
Uniform approximation

A B S T R A C T

In this paper, a spatial semidiscretized finite difference scheme developed in our previous work (Liu and Guo, 2019) is used to approximate exact observability and controllability for Euler–Bernoulli beam control system. The uniform observability inequality is proved by discrete energy multiplier technique. The uniform controllability and uniform boundedness of the discrete controls are also developed. Compared with the existing literature, the proposed approach has achieved potentially the following advantages: (a) It removes the introduction of the numerical viscosity term to achieve uniformity; (b) It can deal with any type of boundary conditions without help of the spectral analysis which is limited only for some special boundary conditions; (c) The proofs of the uniform observability and controllability are simplified significantly with the similar techniques in dealing with the continuous counterpart.

1. Introduction

In this paper, we consider approximation of the following 1-d Euler–Bernoulli beam control system

\[
\begin{aligned}
\frac{d^2 y}{dt^2}(x,t) + y_{xxx}(x,t) &= 0, \quad 0 < x < 1, \quad t \in (0,T), \quad T > 0, \\
y(0,t) &= y_0(t) = y_{xx}(1,t) = 0, \\
y(x,0) &= y^0(x), \quad y(x,0) = y^1(x),
\end{aligned}
\]

(1.1)

where \((y^0, y^1)\) is the initial state and \(U(t)\) is the control (input). System (1.1) arises in the vibrating control of a flexible beam where one end is clamped and the other end is controlled, which is widely used in the fields of robot arms and aerospace engineering [1,2]. We consider system (1.1) in the energy space \(\mathcal{H} = H^2_0(0,1) \times L^2(0,1)\), with \(H^2_0(0,1) = \{ u \in H^2(0,1) | u(0) = u'(0) = 0 \}\). The exact controllability of system (1.1) can be stated as follows: For any initial state \((y^0, y^1) \in \mathcal{H}\), there exists a control \(U(t) \in L^2(0,T)\), such that the state of system (1.1) satisfies

\[
y(x,T) = y_1(x,T) = 0.
\]

(1.2)

The energy of system (1.1), denoted as \(E^y(t)\), is given by

\[
E^y(t) = \frac{1}{2} \int_0^1 \left[ y^2(t,x) + y_{xx}^2(t,x) \right] dx.
\]

By a standard duality argument, the exact controllability of system (1.1) is equivalent to the boundary observability of the following adjoint conservative system

\[
\begin{aligned}
w_{tt}(x,t) + w_{xxxx}(x,t) &= \frac{1}{2} \int_0^T \left[ w_1^2(t,x) + w_{xx}^2(t,x) \right] dx = E^w(0), \\
\end{aligned}
\]

(1.3)

where \(y_1(t)\) is the output and \((w_0(x), w_1(x))\) is the initial state. The energy of system (1.3), denoted by \(E^w(t)\), is conserved in time, i.e.,

\[
E^w(t) = \frac{1}{2} \int_0^1 \left[ w_1^2(t,x) + w_{xx}^2(t,x) \right] dx = E^w(0).
\]

(1.4)

The problem of the exact observability of system (1.3) can be characterize by the “observability inequality”: Find a time \(T > 0\) and \(C(T) > 0\) that depend only on \(T\) such that the solutions of (1.3) satisfy

\[
E^w(0) \leq C(T) \int_0^T y_1^2(t) dt.
\]

(1.5)

The main objective of this paper consists of two parts. Firstly, we study the discrete version of (1.5), i.e., for a given semidiscretized scheme of system (1.3) meaning that spacial variable
is discretized by finite difference scheme and the time is kept to be continuous, find a uniform constant $C_T$ such that $(1.5)$ holds with respect to all discrete step sizes. This uniformity of the observability inequality is essential for proving the convergence of the discrete controls to the continuous counterpart of system $(1.1)$. Secondly, we investigate the uniform controllability approximation of system $(1.1)$, i.e., for the semidiscretized scheme of system $(1.1)$, find uniformly bounded control $U_h(t) \in L^2(0, T)$ with respect to discrete step size $h$ such that the state of the semidiscretized system is controlled to zero at time $T$.

The answers of these two questions are usually negative for discretization of the standard finite difference scheme and finite element scheme. While for mixed finite element method [3,4] and polynomial-based Galerkin method [5], it was positive. This is highly attributed to the spurious oscillatory solutions with high frequencies in the spacial discretization. In order to eliminate the effect of these high frequencies, some remedies have been made, which include Tychonoff regularizations [6], two-grid algorithms [7,8] and filtering techniques [9–11]. The numerical viscosity method is another effective technique to damp out the high frequencies. The method incorporates a vanishing numerical viscosity term effectively at the whole domain. By virtue of this viscosity term, one can obtain the uniform observability of semidiscrete systems [6,10,12], the uniform boundary controllability, convergence of controls [13,14], and the uniformly exponential stability approximations for damped systems [15]. The uniformity of observability inequality was considered for the beam equation with hinged boundary conditions in [16] where it was shown that the corresponding observability inequality cannot be satisfied uniformly with respect to the discretization parameter $h$. Nevertheless, in the same paper, the authors proved that the system obtained by filtering the high-frequencies is uniformly observable. In [17], authors showed that by adding a vanishing viscosity term in the numerical scheme, the uniform controllability for beam systems can be achieved. A more general result, providing the uniform observability for semidiscrete hyperbolic type systems, was proved in [18] by filtering the highest frequencies. Both results in [16] and [17] rely on the explicit knowledge of the eigenvalues and eigenvectors of the discrete operators of the semi-discrete models. If the clamped boundary condition is considered like what we have in this paper, one could be difficult to get explicitly the eigenvalues and eigenvectors of the discrete operators, for example [19] where numerical estimates and asymptotic expansions are adopted approximately to localize all the eigenvalues of the corresponding discrete operators. In [12], the uniform observability was studied for a finite difference scheme of a clamped beam equation, which employs the discrete multiplier method and shows that the uniform observability inequality holds if the high eigenfrequencies are filtered. However, either filtration technique or numerical viscosity method depends on the extent of filtration for high frequencies, which is not very easily and friendly for engineers to deal with distributed parameter control systems. Recently, the paper [20] gave an optimal range of filtering for the approximation of boundary controls for one-dimensional wave equation. It is worth mentioning that the uniform observability can also be achieved by finite difference space semidiscretization without numerical viscosity, if a nonuniform numerical mesh is adopted [21]. In addition, for some fully discretized finite difference schemes, one can also obtain uniform observability [8,22] and uniformly exponential decay [23,24] without numerical viscosity term, provided that the space step coincides with the time step.

Unfortunately, the semi-discretized finite difference scheme with numerical viscosity has some problems in applications. First, the artificially introduced viscosity term is different from system to system, in particular, the coefficient of viscosity term specially selected for one system may not be effective to the other systems. Second, the proof of uniformly exponential decay relies on the analytic forms of the eigen-pairs, which are available only for some special boundary conditions. In this situation, seeking some natural schemes without using numerical viscosity is particularly important for engineering applications. In our previous paper [25], we found that the finite difference scheme derived from order reduction method can preserve the exponential stability of continuous beam system with potentially any type of boundary conditions. Meanwhile, in [26] we applied order reduction method to wave systems, constructing a novel finite difference scheme for wave equation with Neumann boundary controls, which preserves the exact controllability of continuous wave equation uniformly. In [27], it was proved that the finite difference scheme derived from order reduction method can preserve the exponential stability of wave equation with local viscosity uniformly.

In this paper we investigate the numerical approximation of the exact controllability and exact observability of systems $(1.1)$ and $(1.3)$, respectively. It is known that the observability/controllability time of beam system can be arbitrary small [28,29]. Since the proofs in this paper use mainly the energy multiply method in the continuous and discrete frameworks, the observability/controllability time should be $T > 2$. The semidiscretized finite difference scheme we use here is also derived from order reduction method, like that in [25]. The advantages of our numerical schemes include: (a) It is a finite difference scheme that is constructed on equidistant grids and removes the introduction of numerical viscosity to achieve uniform observability and controllability. (b) It can deal with any type of boundary conditions without help of the spectral analysis which is limited only for some special boundary conditions. (c) The proof of observability is simplified significantly with the similar techniques in dealing with the continuous counterpart.

The rest of this paper is organized as follows. Section 2 presents some known results of continuous system and gives the finite difference scheme of systems $(1.1)$ and $(1.3)$. The discrete multiplier method is adopted to prove the uniform observability in Section 3. The uniform approximation of controllability is discussed in Section 4, followed up the conclusions presented in Section 5.

2. Preliminaries

2.1. Continuous model

In this subsection, we present some known results about the exact controllability and exact observability of systems $(1.1)$ and $(1.3)$. In order to make an explicit comparison between continuous and semi-discrete counterparts, we give the proof of the continuous counterpart in Lemmas 2.1–2.3 (see, e.g., [30, p. 203]).

**Lemma 2.1** (Exact Observability). Let $T > 2$ and $E^w(t)$ be the energy of adjoint system $(1.3)$. Then, the following observability inequality holds true:

$$E^w(t) \leq \frac{1}{2(T-2)} \int_0^T w^2(1, t) \, dt.$$  

**Proof.** Multiply the first equation of system $(1.3)$ by $w_{xx}(x, t)$ and integrate over $[0, 1]$ with respect to $x$, to give

$$\int_0^1 w_{xx}(x, t)w_{xx}(x, t) \, dx + \int_0^1 w_{xx}(x, t)w_{xxx}(x, t) \, dx = 0. \quad (2.1)$$
The first term on the left-hand side of (2.1) implies that
\[ \int_0^1 xw_0(x,t)w_{it}(x,t)\,dx = \frac{d}{dt} \int_0^1 xw_0(x,t)w_i(x,t)\,dx - \int_0^1 xw_i(x,t)w_{it}(x,t)\,dx \]
\[ = \frac{d}{dt} \int_0^1 xw_0(x,t)w_i(x,t)\,dx \]
\[ - \frac{1}{2} w_i^2(1,t) + \frac{1}{2} \int_0^1 w_i^2(x,t)\,dx. \] (2.2)

Combining the boundary conditions in system (1.3), the second term on the left-hand side of (2.1) gives
\[ \int_0^1 xw_0(x,t)w_{xxx}(x,t)\,dx = -\int_0^1 xw_0(x,t)w_{xxx}(x,t)\,dx \]
\[ - \frac{1}{2} w_0^2(x,t)\,dx \]
\[ = \frac{3}{2} \int_0^1 w_0^2(x,t)\,dx. \] (2.3)

Plugging (2.2) and (2.3) into (2.1), we obtain
\[ \frac{1}{2} w_i^2(1,t) = \frac{d}{dt} \int_0^1 xw_0(x,t)w_i(x,t)\,dx + \frac{1}{2} \int_0^1 w_i^2(x,t)\,dx \]
\[ + \frac{3}{2} \int_0^1 w_0^2(x,t)\,dx \]
\[ \geq \frac{d}{dt} \int_0^1 xw_0(x,t)w_i(x,t)\,dx + E^w(t). \] (2.4)

By Cauchy’s inequality and the fact that \( w_i(x,t) = 0 \), we have
\[ \left| \int_0^1 xw_0(x,t)w_i(x,t)\,dx \right| \leq \frac{1}{2} \int_0^1 w_0^2(x,t)\,dx + \frac{1}{2} \int_0^1 w_i^2(x,t)\,dx \]
\[ \leq \frac{1}{2} \int_0^1 w_0^2(x,t)\,dx \]
\[ + \frac{1}{2} \int_0^1 w_i^2(x,t)\,dx = E^w(t). \] (2.5)

Integrating both sides of (2.4) over \([0, T]\) with respect to \(t\), we obtain
\[ \int_0^T \frac{1}{2} w_i^2(1,t)\,dt \geq \int_0^1 xw_0(x,t)w_i(x,t)\,dx \bigg|_{t=0}^{t=T} + \int_0^T E^w(t)\,dt, \]
from which and (1.4), we finally obtain the required result
\[ \int_0^T w_i^2(1,t)\,dt \geq 2(T - 2)E^w(0). \]

This ends the proof of the lemma. \(\square\)

**Lemma 2.2 (Exact Controllability).** System (1.1) is exactly controllable over \([0, T]\) for \(T > 2\). Precisely, for any \(T > 2\) and initial state \((y_0, y_1) \in \mathcal{H}\), there exists a control \(U(\cdot) \in L^2(0,T)\) such that the solution \((y, y_1)\) of system (1.1) satisfies
\[ y(x, T) = y_1(x, T) = 0, \quad x \in (0, 1). \] (2.6)

The control \(U(t)\) with property (2.6) satisfies
\[ \int_0^T \frac{1}{2} w_i(1,t)\,dt - \int_0^1 w_0^0(y)w_i(y)\,dx + \int_0^1 w_i^2(x)\,dx = 0, \] (2.7)
for any \((y_0, y_1) \in \mathcal{H}\), the initial state of adjoint system (1.3).

**Proof.** (2.6) is the consequence of the duality property and Lemma 2.1. Multiplying the first equation of system (1.1) by \(w_i(x,t)\) and integrating over \([0, 1]\) with respect to \(x\), with \((w, w_i)\) the solution of adjoint system (1.3), we get
\[ \int_0^1 w_i(x,t)y_{it}(x,t)\,dx + \int_0^1 w_{i}(x,t)y_{xxx}(x,t)\,dx = 0. \] (2.8)

Combining the boundary conditions in system (1.3), the first term of the left-hand side of (2.8) gives
\[ \int_0^1 w_i(x,t)y_{it}(x,t)\,dx = \frac{d}{dt} \int_0^1 u_i(x,t)y_i(x,t)\,dx \]
\[ - \int_0^1 w_i(x,t)y_i(x,t)\,dx \]
\[ = \frac{d}{dt} \int_0^1 u_i(x,t)y_i(x,t)\,dx \]
\[ + \int_0^1 w_{xxx}(x,t)y_i(x,t)\,dx \]
\[ = \frac{d}{dt} \int_0^1 u_i(x,t)y_i(x,t)\,dx \]
\[ + \int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx. \] (2.9)

Similarly, the second term of the left-hand side of (2.8) implies that
\[ \int_0^1 w_i(x,t)y_{xxx}(x,t)\,dx = u_i(1,t)U(t) + \int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx. \] (2.10)

Plugging (2.9) and (2.10) into (2.8), we obtain
\[ \frac{d}{dt} \int_0^1 u_i(x,t)y_i(x,t)\,dx \]
\[ + \frac{d}{dt} \int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx + w_i(1,t)U(t) = 0. \] (2.11)

Integrate both sides of (2.11) over \([0, T]\) with respect to \(t\), to give
\[ \int_0^T w_i(x,t)y_i(x,t)\,dx \bigg|_0^T + \int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx \bigg|_0^T \]
\[ + \int_0^T w_i(1,t)U(t)\,dt = 0. \] (2.12)

By (2.6) holds and
\[ \int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx = -\int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx \]
\[ = \int_0^1 w_{xxx}(x,t)y_{xxx}(x,t)\,dx, \]
it follows from (2.12) that
\[ \int_0^T w_i(1,t)U(t)\,dt - \int_0^1 w_i(x,t)y_{xxx}(x,t)\,dx - \int_0^1 w_i^0(x)w_i^0(x)\,dx = 0, \]
which completes the proof of the lemma. \(\square\)

For convenience, we introduce the 0-inner product \((\cdot, \cdot)_0 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}\) as
\[ ((w_0, w_1), (y_0, y_1))_0 = \int_0^1 w_0(x)y_1(x)\,dx + \int_0^1 w_0(x)y_0(x)\,dx, \]
from which we can induce the corresponding 0-norm \(\| \cdot \|_0\) as
\[ \| (w_0, w_1) \|_0 = \sqrt{((w_0, w_1), (w_0, w_1))_0}. \]
As is well known, among all the admissible controls (i.e., all controls having property (2.7)), there is a unique control with minimal $L^2$-norm, which is referred to as optimal control from the HUM control [31] and can be characterized by the minimizer of the following functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$:

$$
\mathcal{J}(w^0, w^1) = \frac{1}{2} \int_0^T w^2_I(t, 1) \, dt - \int_0^1 w^1(x) y^I(x) \, dx - \int_0^1 w^0_x(x) y^I_x(x) \, dx.
$$

(2.13)

It is obvious that $\mathcal{J}(\cdot)$ is continuous and strictly convex. Furthermore, $\mathcal{J}(\cdot)$ is also coercive, i.e.,

$$
\lim_{\|w^0, w^1\|_0 \to +\infty} \mathcal{J}(w^0, w^1) = +\infty,
$$

(2.14)

which follows from the observability inequality (1.5). Indeed, according to the definition of the 0-inner product, observability inequality (1.5) and Cauchy–Schwartz’s inequality, we have

$$
\mathcal{J}(w^0, w^1) = \frac{1}{2} \int_0^T w^2_I(t, 1) \, dt - \|w^0, w^1\|_0 \|\gamma^0, y^1\|_0 \\
\geq (T - 2) \mathcal{E}^N(0) - \|\gamma^0, y^1\|_0 \|\gamma^0, y^1\|_0 \\
= \frac{T - 2}{2} \|\gamma^0, y^1\|_0^2 - \|\gamma^0, y^1\|_0 \|\gamma^0, y^1\|_0 + \frac{\|\gamma^0, y^1\|_0^2}{2},
$$

(2.15)

which implies the coerciveness of the functional $\mathcal{J}(\cdot)$. Thus the functional $\mathcal{J}(\cdot)$ has a unique minimizer. Suppose that the unique minimizer of $\mathcal{J}(\cdot)$ is $(\hat{w}^0, \hat{w}^1)$. With the initial value $\hat{w}^0, \hat{w}^1$, we have the solution $(\hat{w}, \hat{w}_t)$ to system (1.3) and from well-established control theory,

$$U(t) = \hat{w}_t(1, t)
$$

is just the optimal control to be sought. Since the functional $\mathcal{J}(\cdot)$ is of class $C^1$, the gradient of $\mathcal{J}(\cdot)$ at the minimizer vanishes, i.e., for any $(w^0, w^1) \in \mathcal{H}$

$$0 = \lim_{w \to \hat{w}^0} \frac{\mathcal{J}(\hat{w}^0, \hat{w}^1) + \alpha \langle w^0, w^1 \rangle - \mathcal{J}(w^0, w^1)}{\alpha} = \int_0^T \hat{w}_t(1, t) w_I(1, t) \, dt - \int_0^1 w^1(x) y^I(x) \, dx - \int_0^1 w^0_x y^I(x) \, dx,
$$

which is just the condition (2.7) with $U(t) = \hat{w}_t(1, t)$.

Lemma 2.3. Suppose that $(\hat{w}^0, \hat{w}^1) \in \mathcal{H}$ is the unique minimizer of functional $\mathcal{J}(\cdot)$. For any given initial state $(y^0, y^1) \in \mathcal{H}$, let $U(t) = \hat{w}_t(1, t)$, where $(\hat{w}, \hat{w}_t)$ is the solution of adjoint system (1.3) with the initial value $(\hat{w}^0, \hat{w}^1)$. Then

$$
\|U(t)\|_{\mathcal{L}(2, 0)} \leq \frac{2}{\sqrt{T - 2}} \|\gamma^0, y^1\|_0,
$$

(2.15)

$$
\|\hat{w}^0, \hat{w}^1\|_0 \leq \frac{2}{T - 2} \|\gamma^0, y^1\|_0.
$$

(2.16)

Proof. Since $(\hat{w}^0, \hat{w}^1)$ is the unique minimizer of functional $\mathcal{J}(\cdot)$, we have

$$
0 = \mathcal{J}(0, 0) \geq \mathcal{J}(\hat{w}^0, \hat{w}^1)
$$

$$
= \frac{1}{2} \int_0^T \hat{w}_t^2(1, t) \, dt - \|\hat{w}^0, \hat{w}^1\|_0 \|\gamma^0, y^1\|_0 \\
\geq \frac{1}{2} \|\hat{w}_t(1, t)\|_{\mathcal{L}(2, 0)}^2 - \|\hat{w}^0, \hat{w}^1\|_0 \|\gamma^0, y^1\|_0
$$

(2.17)

$$
\geq \frac{1}{2} \|\hat{w}_t(1, t)\|_{\mathcal{L}(2, 0)}^2 - \frac{1}{T - 2} \|\hat{w}_t(1, t)\|_{\mathcal{L}(2, 0)} \|\gamma^0, y^1\|_0.
$$

(2.18)

where we used the following observability inequality

$$
\frac{1}{2} \|\hat{w}^0, \hat{w}^1\|_0^2 = E^\hat{w}(0) \leq \frac{1}{2(T - 2)} \|\hat{w}_t(1, t)\|_{\mathcal{L}(2, 0)}^2
$$

(2.19)

The (2.15) then follows from (2.18) for $U(t) = \hat{w}_t(1, t)$. From (2.17) and (2.19), we have

$$
\|\hat{w}^0, \hat{w}^1\|_0 \|\gamma^0, y^1\|_0 \geq \frac{1}{2} \|\hat{w}_t(1, t)\|_{\mathcal{L}(2, 0)}^2 \geq \frac{T - 2}{2} \|\hat{w}^0, \hat{w}^1\|_0^2,
$$

which implies (2.16). This completes the proof of the lemma. □

2.2. Construction of numerical schemes

In this subsection, we give a semi-discrete scheme for (1.1) following our previous work [25]. Let $N \in \mathbb{N}^+$ and let the spatial mesh size $h = \frac{1}{N + 1}$. The interval $[0, 1]$ is discretized equidistantly as

$$
x_0 = x_1 < \cdots < x_j = jh < \cdots < x_{N + 1} = 1.
$$

In order to discretize the boundary conditions of the system, we introduce two external points $x_{-1} = x_0 - h$ and $x_{N + 2} = x_{N + 1} + h$ outside the spatial domain $[0, 1]$. For notational simplicity, we introduce the following average operator and difference operators

$$
\delta_j w = \frac{w_j + w_{j+1}}{2}, \quad \delta_j^u w = \frac{w_{j+1} - w_j}{h},
$$

(2.20)

$$
\delta_j^u w = \frac{\delta_j^u w_{j+1} - \delta_j^u w_j}{h} = \frac{w_{j+1} - 2w_j + w_{j-1}}{h^2},
$$

$$
\delta_j^u w = \frac{\delta_j^u w_{j+1} - \delta_j^u w_j}{h} = \frac{w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}}{h^4}
$$

Combining order reduction method and finite difference method, similarly to that in [25], we can get the following semi-discretized finite difference scheme for control system (1.1) and adjoint system (1.3)

$$
\frac{1}{2} \left( y_{j+1}^r - y_{j-1}^r \right) + \delta_j^u y_j = 0, \quad j = 1, \ldots, N,
$$

(2.21)

$$
\frac{1}{2} \left( u_{j+1}^r - u_{j-1}^r \right) + \delta_j^u w_j = 0, \quad j = 1, \ldots, N,
$$

$$
\delta_j^u w_j = 0, \quad j = 1, \ldots, N.
$$

(2.22)

where we used the following observability inequality

$$
\frac{1}{2} \left( u_{j+1}^r - u_{j-1}^r \right) + \delta_j^u w_j = 0, \quad j = 1, \ldots, N.
$$

(2.23)

$$
\delta_j^u w_j = 0, \quad j = 0, \ldots, N + 1.
$$

(2.24)

respectively.

Lemma 2.4. Let $\{x_i\}_{i=0}^{N+1}$ be the grid points with $x_i = ih$ and $h = \frac{1}{N + 1}$. Let $\{u_i\}_{i=0}^{N+1}$ be grid functions on grid $\{x_i\}_{i=0}^{N+1}$ such that

$$
\delta_j^u w_j = 0, \quad j = 0, \ldots, N + 1.
$$

(2.25)
Lemma 3.1. \( \lambda \) and \( \beta \) with \( 0 \leq i < N \).

Then,

\[
\max_{0 \leq i \leq N} \left( \lambda_i u_{j+1} \right)^2 \leq h \sum_{j=0}^{N} \left( \delta_i u_{j+1} \right)^2.
\]

(2.22)

\[
\max_{0 \leq i \leq N} \left( \beta_i u_{j+1} \right)^2 \leq h \sum_{j=0}^{N} (\lambda_i^2 u_j)^2.
\]

3. Uniform observability

The energy \( E_n(t) \) of semidiscretized system (2.21) is defined as

\[
E_n(t) = \frac{h}{2} \sum_{j=0}^{N} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right)^2 + \frac{h}{2} \sum_{j=0}^{N} (\lambda_i^2 u_j)^2.
\]

Lemma 3.1 shows that the energy \( E_n(t) \) is conservative in time.

Lemma 3.1. The derivative of energy \( E_n(t) \) of system (2.21) satisfies

\[
\frac{dE_n(t)}{dt} = 0.
\]

(3.1)

Proof. The proof of the lemma is similar to that in [25, Lemma 3.1] by letting \( k = 0. \)

Theorem 3.1. Suppose that \( E_n(t) \) is the energy of (2.21), then for every \( T > 2 \), the following observability inequality holds true:

\[
E_n(0) \leq \frac{1}{2(T-2)} \int_0^{T} (w_{j+1})^2 \, dt.
\]

(3.2)

Proof. Multiplying the first equation of (2.21) by \( 2h (w_{j+1} - w_{j-1}) \), integrating with respect to \( t \) over \([0, T]\) and taking the sum over \( j = 1, \ldots, N \), we deduce

\[
0 = \frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \frac{(w_{j+1} - w_{j-1})}{2} \, dt
\]

\[
+ \left( \frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \delta_i w_{j+1} \left( w_{j+1} - w_{j-1} \right) \, dt \right)
\]

\[
= \frac{h}{2} \sum_{j=0}^{N} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \frac{(w_{j+1} - w_{j-1})}{2} \bigg|_0^T
\]

\[
- \frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \frac{(w_{j+1} - w_{j-1})}{2} \, dt
\]

\[
+ \left( \frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \delta_i w_{j+1} \left( w_{j+1} - w_{j-1} \right) \, dt \right).
\]

(3.3)

Multiply the second equation of (2.21) by \( \delta_i w_{j+1} \) and integrate with respect to \( t \) in \([0, T]\) to give

\[
0 = \frac{h}{2} \int_0^{T} w_{j+\frac{1}{2}} \delta_i w_{j+1} \, dt - \int_0^{T} \delta_i w_{j+\frac{3}{2}} \delta_i w_{j+\frac{1}{2}} \, dt
\]

\[
= \frac{h}{2} \int_0^{T} \left( w_{j+\frac{1}{2}} \delta_i w_{j+\frac{3}{2}} - h \right)
\]

\[
- \int_0^{T} \delta_i w_{j+\frac{3}{2}} \delta_i w_{j+\frac{1}{2}} \, dt.
\]

(3.4)

Adding (3.3) and (3.4) together, we obtain

\[
L_0(t) \bigg|_0^T + l_1 + l_2 = 0.
\]

(3.5)

where

\[
L_0(t) = \frac{h}{2} \sum_{j=0}^{N} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \frac{(w_{j+1} - w_{j-1})}{2} + \frac{h}{2} \left( w_{j+\frac{1}{2}} \delta_i w_{j+1} \right).
\]

\[
l_1 = -\frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \frac{(w_{j+1} - w_{j-1})}{2} \, dt
\]

\[
- \frac{h}{2} \int_0^{T} \delta_i w_{j+1} \delta_i w_{j+\frac{1}{2}} \, dt.
\]

\[
l_2 = \frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \delta_i w_{j+1} \delta_i w_{j+1} \, dt - \frac{1}{2} \int_0^{T} (w_{j+\frac{1}{2}})^2 \, dt.
\]

(3.6)

For \( l_1 \) in (3.5), we have

\[
l_1 = -\frac{h}{2} \sum_{j=0}^{N} \int_0^{T} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \frac{(w_{j+1} - w_{j-1})}{2} \, dt
\]

\[
- \frac{h}{2} \int_0^{T} \left( (w_{j+\frac{1}{2}}) + (w_{j+\frac{1}{2}}) \right) \delta_i w_{j+\frac{1}{2}} \, dt
\]

\[
- \frac{h}{2} \int_0^{T} (w_{j+\frac{1}{2}})^2 \, dt - \frac{1}{2} \int_0^{T} (w_{j+\frac{1}{2}})^2 \, dt.
\]

(3.7)
in which the inequality $2ab \leq a^2 + b^2$ was used. For $I_2$ in (3.5), since $\delta_x w_{-\frac{1}{2}} = 0$ and $\delta_x^2 w_{N+1} = 0$, we then get

$$h \sum_{j=1}^{N} \frac{\delta_x^2 w_j}{2} w_{j+1} - w_{j-1}$$

$$= h \sum_{j=1}^{N} \frac{\delta_x^2 w_j}{2} w_{j+1} - w_{j-1}$$

$$= h \sum_{j=1}^{N} \frac{\delta_x^2 w_j}{2} w_{j+1} + \frac{1}{2} \delta_x w_{N+\frac{1}{2}}$$

$$= -\left( \frac{h}{2} \sum_{j=1}^{N} \delta_x^2 w_{j+\frac{1}{2}} w_{j+1} + \frac{1}{2} \delta_x w_{N+\frac{1}{2}} \right)$$

$$- \left( \frac{h}{2} \sum_{j=1}^{N} \delta_x^2 w_{j+\frac{1}{2}} w_{j+1} - \frac{1}{2} \delta_x w_{N+\frac{1}{2}} \right)$$

$$+ \left( \frac{h}{2} \sum_{j=1}^{N} \delta_x^2 w_{j+\frac{1}{2}} w_{j+1} - \frac{1}{2} \delta_x w_{N+\frac{1}{2}} \right)$$

$$= \frac{h}{2} \sum_{j=1}^{N} \delta_x^2 w_{j+\frac{1}{2}} w_{j+1} - \frac{1}{2} \delta_x w_{N+\frac{1}{2}}$$

and

$$I_2 = h \sum_{j=1}^{N} \int_0^T \left( \delta_x^2 w_j \right)^2 dt + h \sum_{j=1}^{N} \int_0^T \delta_x w_{j+\frac{1}{2}} \delta_x^2 w_{j+1} dt$$

Plugging (3.6), (3.7) and (3.8) into (3.5) and noticing conservation property of $E_0(t)$ in time, we obtain, by $2ab \leq a^2 + b^2$, that

$$0 \geq -2E_0(0) + \frac{1}{2} \sum_{j=0}^{N} \int_0^T \left( w_{j+\frac{1}{2}} \right)^2 dt - \frac{1}{2} \int_0^T (w_{N+1})^2 dt$$

$$+ \frac{h}{2} \sum_{j=0}^{N} \int_0^T \delta_x^2 w_j^2 dt + \frac{h}{2} \sum_{j=0}^{N} \int_0^T \delta_x w_{j+\frac{1}{2}}^2 \delta_x^2 w_{j+1} dt$$

$$\geq -2E_0(0) + \int_0^T E_0(t) dt - \frac{1}{2} \int_0^T (w_{N+1})^2 dt$$

which leads to the observability inequality (3.2) for $T > 2$. $\square$

4. Uniform controllability

In this section, we show that the uniform observability implies the uniform controllability. To simplify the notation, we reformulate systems (2.20) and (2.21) into the following vectorial forms

$$\left\{ \begin{array}{l} M_h \dot{Y}_h(t) + A_h Y_h(t) = F_h(t), \\ Y_h(0) = Y_h^0, \quad Y_h'(0) = Y_h^1, \end{array} \right. \quad (4.1)$$

and

$$\left\{ \begin{array}{l} M_h \dot{W}_h(t) + A_h W_h(t) = 0, \\ W_h(0) = W_h^0, \quad W_h'(0) = W_h^1, \end{array} \right. \quad (4.2)$$

respectively, where

$$M_h = \frac{h}{4} \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix},$$

$$A_h = \frac{1}{h^2} \begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & -4 & 6 & -4 & 1 \\ 0 & \cdots & 0 & 1 & -4 & 5 & -2 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

and

$$Y_h(t) = (y_1(t), y_2(t), \ldots, y_{N+1}(t))^T, \quad F_h(t) = (0, 0, \ldots, -U_h(t))^T,$$

$$Y_h^0 = (y_1^0, y_2^0, \ldots, y_{N+1}^0)^T, \quad Y_h^1 = (y_1^1, y_2^1, \ldots, y_{N+1}^1)^T,$$

and similarly for $W_h(t), W_h^0, W_h^1$. Let $\langle \cdot, \cdot \rangle$ be the canonical inner product in $\mathbb{R}^{N+1}$:

$$\langle f, g \rangle = \sum_{i=1}^{N+1} f_i g_i.$$

We have Lemma 4.1, which is the discrete counterpart of Lemma 2.2.

**Lemma 4.1.** System (4.1) is uniformly exactly controllable over $[0, T]$ for $T > 2$ (and hence for any $T > 0$). Precisely, for any initial state $(Y_h^0, Y_h^1) \in \mathbb{R}^{2(N+1)}$ and $T > 2$, there exists a control $U_h(t) \in L^2(0, T)$ such that

$$Y_h(T) = Y_h^0 = 0.$$

(4.3)
The control $u_h(t)$ with property (4.3) satisfies
\[
\int_0^T u_h(t)w_{N+1}^t(t) \, dt = (M_h Y_0^t, W_0^t) + (A_h Y_0^t, W_0^t),
\]
for any $(W_0^t, W_0^t) \in \mathbb{R}^{2N+2}$, the initial state of adjoint system (4.2).

**Proof.** Multiplying the first equation of (4.1) by $W_0(t)$, the solution of system (4.2), and integrating in time over $[0, T]$, we obtain
\[
\int_0^T \langle F_0(t), W_0(t) \rangle \, dt = \int_0^T \langle M_0 Y_0^t(t), W_0^t(t) \rangle \, dt + \int_0^T \langle A_0 Y_0^t(t), W_0^t(t) \rangle \, dt \tag{4.4}
\]
for any $(Y_0^t, Y_0^t) \in \mathbb{R}^{2N+2}$. Integrating in time over $[0, T]$, we obtain
\[
\int_0^T \langle F_0(t), W_0(t) \rangle \, dt = \int_0^T \langle M_0 Y_0^t(t), W_0^t(t) \rangle \, dt + \int_0^T \langle A_0 Y_0^t(t), W_0^t(t) \rangle \, dt \tag{4.5}
\]
Since (4.3), it follows from (4.5) that
\[
\int_0^T \langle F_0(t), W_0(t) \rangle \, dt = -\langle M_0 Y_0^t(0), W_0(0) \rangle - \langle A_0 Y_0^t(0), W_0(0) \rangle,
\]
which implies (4.4). This completes the proof of the lemma. \qed

Define discrete 0-inner product $(\cdot, \cdot)_0 : \mathbb{R}^{2N+2} \times \mathbb{R}^{2N+2} \to \mathbb{R}$ as
\[
\langle (f^0, f^1), (g^0, g^1) \rangle_0 = \langle M_0 f^1, g^1 \rangle + \langle A_0 g^0, f^0 \rangle,
\]
where $f^i, g^i \in \mathbb{R}^{N+1}, i = 0, 1$, which is the nature discretization of continuous inner product $(\cdot, \cdot)_0$. The corresponding inner product induced norm is $\| \cdot \|_0$, by which it is easy to check that
\[
E^0_0(t) = \frac{1}{2} \| (w(t), w'(t)) \|^2_0.
\]
Let us introduce the discrete functional $\mathcal{J}_h : \mathbb{R}^{2N+2} \to \mathbb{R}$ by
\[
\mathcal{J}_h((W_0^t, W_0^t)) = \frac{1}{2} \int_0^T \langle (w_{N+1}^t)'(t), (w_{N+1}^t)'(t) \rangle \, dt - \langle M_h Y_0^t, W_0^t \rangle - \langle A_h Y_0^t, W_0^t \rangle,
\]
which is the discrete counterpart of continuous functional (2.13). It is clear that $\mathcal{J}_h(\cdot)$ is strictly convex and hence continuous. In addition, it is easy to check that $\mathcal{J}_h(\cdot)$ is also coercive, i.e.,
\[
\lim_{\| (W_0^t, W_0^t) \|_0 \to \infty} \mathcal{J}_h((W_0^t, W_0^t)) = \infty.
\]
Indeed, according to the uniformly observable inequality (3.2), we have
\[
\mathcal{J}_h((W_0^t, W_0^t)) = \frac{1}{2} \int_0^T \langle (w_{N+1}^t)'(t), (w_{N+1}^t)'(t) \rangle \, dt - \langle (Y_0^t, Y_1^t), (W_0^t, W_0^t) \rangle_0 \geq \frac{T - 2}{2} \| (W_0^t, W_0^t) \|_0^2 - \| (Y_0^t, Y_1^t) \|_0 \| (W_0^t, W_0^t) \|_0,
\]
which implies (4.7) as $\| (W_0^t, W_0^t) \|_0 \to \infty$. Thus the functional $\mathcal{J}_h(\cdot)$ has a unique minimizer. Suppose that $(\hat{W}_h^t, \hat{W}_h^t)$ is the unique minimizer of functional $\mathcal{J}_h(\cdot)$. Then, as continuous case, it is known that $u_h(t) = \hat{w}_{N+1}^t(t)$ is minimal $L^2$ norm control for system (4.1) to achieve (4.3), where $(\hat{W}_h^t, \hat{W}_h^t(t))$ is the solution of adjoint system (4.2) with the initial value $(W_0^t, W_0^t)$. Since the gradient of $\mathcal{J}_h(\cdot)$ vanishes at $(\hat{W}_h^t, \hat{W}_h^t)$, we get, for any $(W_0^t, W_0^t) \in \mathbb{R}^{2N+2}$, that
\[
0 = \lim_{\alpha \to 0} \frac{\mathcal{J}_h((\hat{W}_h^t, \hat{W}_h^t)) + \alpha (W_0^t, W_0^t) - \mathcal{J}_h((\hat{W}_h^t, \hat{W}_h^t))}{\alpha} = \int_0^T \hat{w}_{N+1}^t w_{N+1}^t \, dt - \langle M_h Y_0^t, W_0^t \rangle - \langle A_h Y_0^t, W_0^t \rangle,
\]
which is just (4.4) with $u_h(t) = \hat{w}_{N+1}^t$.

Theorem 4.1 is the discrete counterpart of Lemma 2.3.

**Theorem 4.1.** Suppose that $(\hat{W}_h^t, \hat{W}_h^t) \in \mathbb{R}^{2N+2}$ is the unique minimizer of functional $\mathcal{J}_h(\cdot)$ and $(W_0^t, W_0^t(t))$ is the solution of adjoint system (4.2) with the initial state $(\hat{W}_h^t, \hat{W}_h^t)$. Let $u_h(t) = \hat{w}_{N+1}^t(t)$. Then, the following estimates hold
\[
\| U_h(t) \|_{L^2(0, T)} \leq \frac{2}{\sqrt{T - 2}} \| (Y_0^t, Y_1^t) \|_0, \tag{4.8}
\]
\[
\| (\hat{W}_h^t, \hat{W}_h^t) \|_0 \leq \frac{2}{T - 2} \| (Y_0^t, Y_1^t) \|_0. \tag{4.9}
\]

**Proof.** Since $(\hat{W}_h^t, \hat{W}_h^t)$ is the unique minimizer of functional $\mathcal{J}_h(\cdot)$, we have
\[
0 = \mathcal{J}_h((0, 0)) \geq \mathcal{J}_h((\hat{W}_h^t, \hat{W}_h^t)) = \frac{1}{2} \int_0^T \langle \hat{w}_{N+1}^t(t)^2 \rangle \, dt - \langle (\hat{W}_h^t, \hat{W}_h^t), (Y_0^t, Y_1^t) \rangle_0 \geq \frac{1}{2} \| \hat{w}_{N+1}^t \|_{L^2(0, T)}^2 - \| (\hat{W}_h^t, \hat{W}_h^t) \|_0 \| (Y_0^t, Y_1^t) \|_0 \tag{4.10}
\]
\[
\geq \frac{1}{2} \| \hat{w}_{N+1}^t \|_{L^2(0, T)}^2 - \frac{1}{2} \| \hat{w}_{N+1}^t \|_{L^2(0, T)} \| (Y_0^t, Y_1^t) \|_0. \tag{4.11}
\]
In (4.11) we used the following observability inequality
\[
\frac{1}{2} \| (\hat{W}_h^t, \hat{W}_h^t) \|_0 \leq E_0^0(0) \leq \frac{1}{2(T - 2)} \| \hat{w}_{N+1}^t \|_{L^2(0, T)}^2. \tag{4.12}
\]
From (4.11), we can get (4.8) easily by letting $U_h(t) = \hat{w}_{N+1}^t$. Combining (4.10) and (4.12), we obtain
\[
\| (\hat{W}_h^t, \hat{W}_h^t) \|_0 \| (Y_0^t, Y_1^t) \|_0 \geq \frac{1}{2} \| \hat{w}_{N+1}^t \|_{L^2(0, T)} \| (Y_0^t, Y_1^t) \|_0 \geq \frac{T - 2}{2} \| (\hat{W}_h^t, \hat{W}_h^t) \|_0,
\]
which implies (4.9). This completes the proof of the theorem. \qed

5. Conclusions

In this paper, we investigate the numerical approximation of an Euler–Bernoulli beam system with shear force boundary control. Based on our previous study [25], a numerical scheme without using commonly numerical viscosity is constructed on equidistant grids based on order reduction method and finite difference schemes, which can preserve the uniform observability and controllability of the adjoint and control systems, respectively. The uniform observability of the discrete system is proved by discrete multiplier method, very similarly to the continuous counterpart. This simplifies significantly the spectral approach for finite difference scheme with numerical viscosity which works only for some special boundary conditions and is difficult to be applied to our case in this paper. The remaining future work is the convergence of the minimal $L^2$ controls of the discrete systems.
to achieve zero controllability to the continuous counterpart. The boundary conditions of the control system (1.1) enable us to apply the energy multiplier method and thus the Lyapunov function method can be constructed. The order reduction method can deal with different boundary conditions. However, if the hinged boundary condition or clamped boundary condition is considered in our framework, since the Lyapunov function method may not be constructed, one should adopt spectral analysis method to prove the uniform observability and uniform controllability.

Finally, we indicate that the difficulty we encounter in this paper is the uniform convergence of the minimal $L^2$-control of the semidiscrete system to the minimal $L^2$-control of the continuous counterpart. In order to prove the convergence of discrete control to continuous control, the idea of $L^2$-convergence obscurely used [32] might be applicable although we have difficulties in dealing with the average operator (1/4, 1/2, 1/4) of the time derivatives in the semidiscrete finite difference scheme.

**CRediT authorship contribution statement**

Jiankang Liu: Methodology, Validation, Writing - original draft. Bao-Zhu Guo: Conceptualization, Supervision, Writing - review & editing.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgments**

The authors would like to thank the anonymous referees for their careful reading and valuable suggestions to improve the manuscript.

**References**


