A NEW SEMIDISCRETIZED ORDER REDUCTION FINITE DIFFERENCE SCHEME FOR UNIFORM APPROXIMATION OF ONE-DIMENSIONAL WAVE EQUATION

JIANKANG LIU† AND BAO-ZHU GUO‡

Abstract. In this paper, we propose a novel space semidiscretized finite difference scheme for approximation of the one-dimensional wave equation under boundary feedback. This scheme, referred to as the order reduction finite difference scheme, does not use numerical viscosity and yet preserves the uniform exponential stability. The paper consists of four parts. In the first part, the original wave equation is first transformed by order reduction into an equivalent system. A standard semidiscretized finite difference scheme is then constructed for the equivalent system. It is shown that the semidiscretized scheme is second-order convergent and that the discretized energy converges to the continuous energy. Very unexpectedly, the discretized energy also preserves uniformly exponential decay. In the second part, an order reduction finite difference scheme for the original system is derived directly from the discrete scheme developed in the first part. The uniformly exponential decay, convergence of the solutions, as well as uniform convergence of the discretized energy are established for the original system. In the third part, we develop the uniform observability of the semidiscretized system and the uniform controllability of the Hilbert uniqueness method controls. Finally, in the last part, under a different implicit finite difference scheme for time, two numerical experiments are conducted to show that the proposed implicit difference schemes preserve the uniformly exponential decay.

Key words. wave equation, finite difference discretization, stability, uniform approximation

AMS subject classifications. 39A12, 35L05, 34D20, 65L20

DOI. 10.1137/19M1246535

1. Introduction. For control systems described by partial differential equations (PDEs), numerical experiments and thus discretization PDEs are necessary for applications in many situations where PDEs are to be approximated by finite-dimensional systems. Among many discretization methods, the finite difference method is the most often used approach due to its simplicity in theory and versatility in implementation. It is also the one that is very familiar to most researchers. However, it has been acknowledged for a long time that for PDEs, the finite difference method or other similar numerical discretization methods may suffer from certain approximation problems. One such problem is that the exponential decay rate of an exponentially stable PDE cannot be preserved uniformly in the process of the discretization. Let us study the following one-dimensional wave equation under boundary control:

\[
\begin{align*}
    w_{tt}(x, t) - w_{xx}(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
    w(0, t) &= 0, \quad w_x(1, t) = U(t), \\
    w(x, 0) &= w^0(x), \quad w_t(x, 0) = w^1(x), \\
    y(t) &= w_t(1, t),
\end{align*}
\]

(1.1)
where \((w^0, w^1)\) is the initial state, \(U(t)\) is the control, and \(y(t)\) is the output. System (1.1) arises in vibration control of a vibrating string where one end is fixed and the vertical force at other end is actuated [14]. Consider system (1.1) in the energy state space \(\mathcal{H} = H^1_0(0, 1) \times L^2(0, 1),\) where \(H^1_0(0, 1) = \{ f \in H^1(0, 1)| f(0) = 0 \}\). The energy \(E(t)\) of system (1.1) is given by

\[ E(t) = \frac{1}{2} \int_0^1 \left[ w^2_t(x, t) + w^2_x(x, t) \right] dx. \]

Finding the derivative of \(E(t)\) along the solution of (1.1) gives

\[ \dot{E}(t) = U(t)y(t), \]

which means that system (1.1) is a passive system. A stabilizing output feedback control is then naturally designed as

\[ U(t) = -\alpha y(t), \quad \alpha > 0, \]

and the closed-loop system is therefore governed by

\[
\begin{align*}
   w_{tt}(x, t) - w_{xx}(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
   w(0, t) &= 0, \quad w_x(1, t) = -\alpha w_t(1, t), \\
   w(x, 0) &= w^0(x), \quad w_t(x, 0) = w^1(x).
\end{align*}
\]

Since the energy of the closed-loop satisfies

\[
\frac{dE(t)}{dt} = -\alpha w^2_t(1, t),
\]

the energy of the system is decreasing as time evolves. Actually, it has been known for a long time that for any given initial state \((w^0, w^1)\), the solution of (1.2) decays exponentially in time and the decay rate is uniform for all initial states \((w^0, w^1)\) in the state space \(\mathcal{H}:\)

\[
E(t) \leq Ke^{-\omega t}E(0) \quad \forall \, t \in [0, \infty)
\]

for some \(K, \omega > 0\) independent of the initial values; see, e.g., [5, 8, 16].

In the past two decades, there has been extensive literature on approximation of system (1.2) concerning the uniform decay rate, i.e., whether or not the exponential decay rate of the discretized energy is preserving uniformly with respect to the mesh size. Banks, Ito, and Wang in [3] first pointed out that the exponential decay of the discretized energy might not be uniform, with respect to the mesh size, for the classical finite difference and finite element schemes. This is attributed highly to the spurious oscillations of the high frequencies in discretization. To remedy this problem, the authors suggested to use mixed finite element methods or polynomial based Galerkin methods to obtain the uniformly exponential decay [9, 12]. Some other remedies have also been proposed to damp out these high frequencies, which include Tychonoff regularizations [15], two-grid algorithms [18, 22], and filtering techniques [2, 25, 26]. The numerical viscosity method is another effective technique to damp out the high frequencies. The method incorporates a numerical vanishing viscosity term effectively at the whole domain. By virtue of this viscosity term, one can obtain uniform observability of semidiscrete systems [7, 15, 25], uniform boundary...
controllability, convergence of controls [20, 21], and uniformly exponentially stable approximations for damped systems [23]. Unfortunately, the semidiscretized finite difference scheme using numerical viscosity has some problems in applications. First, the artificially introduced viscosity term is different from system to system, in particular, the coefficient of the viscosity term specially selected for one system may not be effective for different vanishing coefficients. Second, the proof of uniformly exponential decay relies on the analytic forms of the eigenpairs, which are available only for some special boundary conditions. In this situation, some natural schemes without using numerical viscosity have been proposed. A nonuniform numerical mesh was developed in [10] to preserve uniform observability. For some fully discretized finite difference schemes, one can also obtain uniform observability [11, 22] and uniformly exponential decay [6] without using numerical viscosity, provided that the space step is identical to the time step. To the best of our knowledge, there is no semidiscretized finite difference scheme that is on a uniform numerical mesh and does not use numerical viscosity that can preserve the uniformly exponential decay of system (1.2).

This paper aims to propose for the first time a new semidiscretized finite difference scheme for system (1.2) that enjoys uniformly exponential decay without using numerical viscosity. This new finite difference scheme provides a basic principle that not only is for numerical computation of the wave equation but is also potentially applicable to other types of PDEs. In addition, it has the following advantages: (a) it naturally keeps the finite difference discretization nature; (b) it can deal with various boundary conditions; and (c) the exponential convergence can be proved similarly as in the case of continuous counterpart. To this end, we introduce the following two intermediate variables:

\begin{equation}
\begin{aligned}
u(x, t) &= w_x(x, t), \quad \nu(x, t) = w_t(x, t).
\end{aligned}
\end{equation}

Then, system (1.2) can be reformulated as

\begin{equation}
\begin{aligned}
u_t(x, t) - v_x(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
v_t(x, t) - u_x(x, t) &= 0, \\
v(0, t) &= 0, \quad u(1, t) + \alpha v(1, t) = 0, \\
u(x, 0) &= u^0(x) = w^0_x(x), \quad v(x, 0) = v^0(x) = w^1_t(x),
\end{aligned}
\end{equation}

where \((u^0, v^0) \in L^2(0, 1) \times L^2(0, 1)\). Energy \(F(t)\) of system (1.6) now equivalently reads

\begin{equation}
F(t) = \frac{1}{2} \int_0^1 [u^2(x, t) + v^2(x, t)] \, dx
\end{equation}

and satisfies the following energy dissipation law:

\begin{equation}
\frac{dF(t)}{dt} = -\alpha v^2(1, t).
\end{equation}

Moreover, the exponential decay as in (1.4) holds true. The reason why we consider system (1.6) instead of system (1.2) is that no partial derivative appears in the right boundary condition in (1.6). It is well-known that how to discretize the partial derivatives at the right boundary condition in (1.2) is crucial to both the stability and the convergence order of the finite difference schemes from the numerical analysis point of
view. In the context of control theory, as described in [13, 4], an inadequate approximation of boundary control often contributes to the poor numerical approximation of the normal derivative on the boundary for high eigenfunctions. The equivalent system (1.6) avoids this sort of problem.

The paper is divided into four parts. In the first part, we show that the ordinary semidiscretized finite difference scheme preserves uniformly exponential decay for system (1.6). To the best of our knowledge, this makes a first successful PDE example for such discretization. In the second part, we show that the ordinary semidiscretized finite difference scheme for system (1.6) leads to a semidiscretized order reduction finite difference scheme for original system (1.2). It also turns out that the right semidiscretized finite difference scheme can preserve the uniformly exponential decay. In the third part, we show that other control properties like uniform observability and controllability can also be preserved by the semidiscretized order reduction finite difference scheme. The numerical simulation verifies the result in the last part.

The rest of the paper is organized as follows. In section 2 we construct a space semidiscretized finite difference scheme for equivalent system (1.6). The uniformly exponential decay of the discretized energy is proved in section 3 by the Lyapunov function method. The convergence of the discretized scheme to the original system is analyzed in section 4, which includes the convergence of the discretized solutions and the convergence of the discretized energy. In section 5, we derive the semidiscretized order reduction finite difference scheme for original system (1.2) directly from the discretized scheme for equivalent system. The uniformly exponential decay and weak convergence of the solutions are also proved. In section 6, the uniform observability of the semidiscrete system is proved. This leads to the uniformly exact controllability of control systems with the Hilbert uniqueness method (HUM) control. Numerical experiments are conducted in section 7, followed by concluding remarks in section 8.

2. Derivation of finite difference scheme. Before we conduct the finite difference semidiscretization for system (1.6), some notation is introduced. First of all, we denote sequences \( \{u_j(t)\}_{0 \leq j \leq N+1} \) by \( \{u_j(t)\}_j \) for simplicity. For every \( N \in \mathbb{N}^+ \) we consider an equidistant partition of the interval [0,1]:

\[
0 = x_0 < x_1 < \cdots < x_j = jh < \cdots < x_{N+1} = 1,
\]

where the mesh size \( h = \frac{1}{N+1} \). Let \( \{U_j(t)\}_j \), \( \{V_j(t)\}_j \), and \( \{W_j(t)\}_j \) be grid functions at grids \( \{x_j\}_j \), satisfying

\[
U_j(t) = u(x_j, t), \quad V_j(t) = v(x_j, t), \quad W_j(t) = w(x_j, t), \quad j = 0, 1, \ldots, N + 1.
\]

Denote the first-order and second-order finite difference operators by

\[
\delta_x U_{j+\frac{1}{2}}(t) = \frac{U_{j+1}(t) - U_j(t)}{h}, \quad \delta_x^2 U_j(t) = \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2},
\]

with the average operators being

\[
U_j(\frac{1}{2}) = \frac{U_{j+1}(t) + U_j(t)}{2}.
\]

Now, we begin to discretize the space derivatives of system (1.6) by the finite difference method. The first equation of system (1.6) is valid at \( (x_j+\frac{1}{2}, t) \), i.e.,

\[
u' \left( x_j + \frac{1}{2}, t \right) - v_x \left( x_j + \frac{1}{2}, t \right) = 0,
\]

where \( x_j+\frac{1}{2} = (j + \frac{1}{2})h \). Hereafter the prime ' represents the derivative with respect to time. Replace the differential operator with the difference operator to get that
(2.1) \[ U'_{j+\frac{1}{2}}(t) - \delta_x V_{j+\frac{1}{2}}(t) = r_{j+\frac{1}{2}}(t), \]
where
\[ r_{j+\frac{1}{2}}(t) = \left( -\frac{1}{8}w'_xx(x_1,t) + \frac{1}{24}v_xxxx(x_2,t) \right) h^2, \quad \xi_1, \xi_2 \in (x_j, x_{j+1}). \]

Similarly, for the second equation of system (1.6), we have
(2.3) \[ V'_{j+\frac{1}{2}}(t) - \delta_x U_{j+\frac{1}{2}}(t) = s_{j+\frac{1}{2}}(t), \]
where
\[ s_{j+\frac{1}{2}}(t) = \left( -\frac{1}{8}w'_xx(x_3,t) + \frac{1}{24}u_xxxx(x_4,t) \right) h^2, \quad \xi_3, \xi_4 \in (x_j, x_{j+1}). \]

By dropping the infinitesimal terms in (2.1) and (2.3), and replacing \( U_j(t) \) and \( V_j(t) \) by \( u_j(t) \) and \( v_j(t) \), respectively, we arrive at a semidiscretized finite difference scheme of system (1.6) as follows:
(2.5) \[
\begin{align*}
& u'_{j+\frac{1}{2}}(t) - \delta_x u_{j+\frac{1}{2}}(t) = 0, \quad j = 0, \ldots, N, \\
& v'_{j+\frac{1}{2}}(t) - \delta_x v_{j+\frac{1}{2}}(t) = 0, \\
& v_0(t) = 0, \quad u_{N+1}(t) + \alpha v_{N+1}(t) = 0, \\
& u_j(0) = u^0_j, \quad v_j(0) = v^0_j,
\end{align*}
\]
where \( u_j(t) \) and \( v_j(t) \) are approximations of \( u(x_j,t) \) and \( v(x_j,t) \), respectively, and \( u^0_j \) and \( v^0_j \) are approximations of the initial values \( u^0(x_j) \) and \( v^0(x_j) \), respectively. If the initial conditions are sufficiently smooth, we can set \( u^0(x_j) = u^0_j \) and \( v^0(x_j) = v^0_j \).

Remark 2.1. From the construction of difference schemes we can see that the semidiscretized finite difference schemes (2.5) is of second-order convergence to the transformed system (1.6).

The following two lemmas are straightforward yet very useful in proving the convergence of the semidiscretized finite difference scheme and the exponential decay of the discretized energy.

**Lemma 2.1 (Grönwall’s inequality; see, e.g., [17]).** Let \( y(t) \) be a nonnegative, absolutely continuous function over \( t \in [0,\infty) \) that satisfies the differential inequality
\[ y'(t) + f(t) \leq h(t) + g(t)y(t) \forall t \in (0,\infty) \text{ a.e.,} \]
where \( f(t) \), \( g(t) \), and \( h(t) \) are nonnegative, summable functions on \([0,\infty)\). Then,
\[ y(t) + \int_0^t f(s)ds \leq e^{\int_0^t g(s)ds} \left( y(0) + \int_0^t h(s)ds \right) \forall t \in [0,\infty). \]

**Lemma 2.2.** For any grid functions \( \{U_j\}_j \), \( \{V_j\}_j \), and \( \{W_j\}_j \) at mesh grids \( \{x_j\}_j \), we have the following two formulas of summation by parts:
By virtue of (2.6) in Lemma 2.2, we obtain
\begin{align*}
& (2.6) \quad \frac{h}{2} \sum_{j=0}^{N} \delta_x U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + h \sum_{j=0}^{N} U_{j+\frac{1}{2}} \delta_x V_{j+\frac{1}{2}} = U_{N+1} V_{N+1} - U_0 V_0, \\
& (2.7) \quad h \sum_{j=0}^{N} \delta_x U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} W_{j+\frac{1}{2}} + h \sum_{j=0}^{N} U_{j+\frac{1}{2}} \delta_x V_{j+\frac{1}{2}} W_{j+\frac{1}{2}} + h \sum_{j=0}^{N} U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} \delta_x W_{j+\frac{1}{2}} \\
& \quad = U_{N+1} V_{N+1} W_{N+1} - U_0 V_0 W_0 - \frac{1}{4} \sum_{j=0}^{N} (U_{j+1} - U_j)(V_{j+1} - V_j)(W_{j+1} - W_j).
\end{align*}

Proof. The first formula (2.6) follows from
\begin{align*}
& h \sum_{j=0}^{N} \delta_x U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + h \sum_{j=0}^{N} U_{j+\frac{1}{2}} \delta_x V_{j+\frac{1}{2}} \\
& \quad = \frac{1}{2} \sum_{j=0}^{N} (U_{j+1} - U_j)(V_{j+1} + V_j) + \frac{1}{2} \sum_{j=0}^{N} (U_{j+1} + U_j)(V_{j+1} - V_j) \\
& \quad = \sum_{j=0}^{N} (U_{j+1} V_{j+1} - U_j V_j) = U_{N+1} V_{N+1} - U_0 V_0,
\end{align*}
and the second formula (2.7) can be similarly obtained with some elementary operations.

3. Uniformly exponential decay. In this section, we analyze the uniformly exponential decay of the semidiscretized finite difference scheme (2.5) by the Lyapunov function method. Define the $L^2$-norm $\| \cdot \|$ of $\tilde{u}(t) = \{u_j(t)\}_j$ as
\begin{equation}
(3.1) \quad \| \tilde{u}(t) \| = \sqrt{h \sum_{j=0}^{N} u_{j+\frac{1}{2}}^2(t)}.
\end{equation}
The discretized energy of the finite difference scheme (2.5) reads
\begin{equation}
(3.2) \quad F_h(t) = \frac{1}{2} (\| \tilde{u}(t) \|^2 + \| \tilde{\nu}(t) \|^2), \quad \tilde{\nu}(t) = \{\nu_j(t)\}_j.
\end{equation}
Here, $F_h(t)$ is a natural discretization of the continuous energy $F(t)$ defined by (1.7).

**Lemma 3.1.** The discretized energy $F_h(t)$ defined by (3.2) satisfies
\begin{equation}
(3.3) \quad \frac{d}{dt} F_h(t) = -\alpha u_{N+1}^2(t) \quad \forall t \in [0, \infty).
\end{equation}

Proof. Multiply both sides of the first equation of (2.5) by $h u_{j+\frac{1}{2}}(t)$ and sum up $j$ from 0 to $N$ to obtain
\begin{equation}
(3.4) \quad h \sum_{j=0}^{N} u_{j+\frac{1}{2}}'(t) u_{j+\frac{1}{2}}(t) - h \sum_{j=0}^{N} \delta_x u_{j+\frac{1}{2}}(t) u_{j+\frac{1}{2}}(t) = 0.
\end{equation}
By virtue of (2.6) in Lemma 2.2, we obtain
\begin{equation}
(3.5) \quad \frac{1}{2} \frac{d}{dt} h \sum_{j=0}^{N} u_{j+\frac{1}{2}}^2(t) + h \sum_{j=0}^{N} \delta_x u_{j+\frac{1}{2}}(t) u_{j+\frac{1}{2}}(t) - u_{N+1}(t) \nu_{N+1}(t) = 0.
\end{equation}
Similarly, multiplying both sides of the second equation of (2.5) by \( hv_{j+\frac{1}{2}}(t) \) and summing up \( j \) from 0 to \( N \), we obtain

\[
(3.4) \quad \frac{1}{2} \frac{d}{dt} \left( \sum_{j=0}^{N} u^2_{j+\frac{1}{2}}(t) \right) - h \sum_{j=0}^{N} \delta_x u_{j+\frac{1}{2}}(t)v_{j+\frac{1}{2}}(t) = 0.
\]

Equations (3.3) and (3.4) together with the right boundary condition in (2.5) complete the proof.

Lemma 3.1 shows that discretized energy \( F_h(t) \) of system (2.5) is decreasing in time as long as the damping coefficient \( \alpha > 0 \). In order to establish the uniformly exponential decay of \( F_h(t) \) we construct the following Lyapunov function:

\[
(3.5) \quad G_h(t) = F_h(t) + \varepsilon \Phi_h(t), \quad 0 < \varepsilon < 1,
\]

where the auxiliary function \( \Phi_h(t) \) is defined by

\[
(3.6) \quad \Phi_h(t) = h \sum_{j=0}^{N} x_{j+\frac{1}{2}} u_{j+\frac{1}{2}}(t)v_{j+\frac{1}{2}}(t).
\]

**Lemma 3.2.** The Lyapunov function \( G_h(t) \) defined by (3.5) is equivalent to the discretized energy \( F_h(t) \), that is, there exist two constants \( C_1 \) and \( C_2 \) such that

\[
C_1 F_h(t) \leq G_h(t) \leq C_2 F_h(t),
\]

where \( C_1 = 1 - \varepsilon \) and \( C_2 = 1 + \varepsilon \) for some scalar \( \varepsilon \in (0, 1) \).

**Proof.** A direct use of Cauchy’s inequality and the triangle inequality shows that the function \( \Phi_h(t) \) defined by (3.6) satisfies

\[
|\Phi_h(t)| \leq h \sum_{j=0}^{N} |x_{j+\frac{1}{2}}||u_{j+\frac{1}{2}}(t)||v_{j+\frac{1}{2}}(t)| \leq \frac{h}{2} \sum_{j=0}^{N} u^2_{j+\frac{1}{2}}(t) + \frac{h}{2} \sum_{j=0}^{N} v^2_{j+\frac{1}{2}}(t) = F_h(t).
\]

The proof is completed by taking \( C_1 = 1 - \varepsilon \) and \( C_2 = 1 + \varepsilon \), according to the definition of \( G_h(t) \).

**Lemma 3.3.** The auxiliary function \( \Phi_h(t) \) defined by (3.6) satisfies

\[
\frac{d}{dt}\Phi_h(t) \leq -F_h(t) + \frac{1}{2} (1 + \alpha^2) v^2_{N+1}(t).
\]

**Proof.** Finding the derivative of \( \Phi_h(t) \) along the solution of (2.5) yields

\[
\frac{d}{dt}\Phi_h(t) = h \sum_{j=0}^{N} x_{j+\frac{1}{2}} u'_{j+\frac{1}{2}}(t)u_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_{j+\frac{1}{2}} u_{j+\frac{1}{2}}(t)v'_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_{j+\frac{1}{2}} \delta_x v_{j+\frac{1}{2}}(t)v_{j+\frac{1}{2}}(t)
\]

\[
= h \sum_{j=0}^{N} x_{j+\frac{1}{2}} \delta_x v_{j+\frac{1}{2}}(t)v_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_{j+\frac{1}{2}} u_{j+\frac{1}{2}}(t)\delta_x u_{j+\frac{1}{2}}(t).
\]

By (2.7) in Lemma 2.2, we further have
\[\frac{d}{dt} \Phi_h(t) = -h \sum_{j=0}^{N} v_{j+\frac{1}{2}}^2(t) - h \sum_{j=0}^{N} x_{j+\frac{1}{2}} u_{j+\frac{1}{2}}(t) \delta_x v_{j+\frac{1}{2}}(t) + v_{N+1}^2(t) - \frac{h}{4} \sum_{j=0}^{N} (v_{i+1} - v_i)^2 - \frac{h}{4} \sum_{j=0}^{N} (u_{i+1} - u_i)^2 \]

\[= -2F_h(t) - \frac{d}{dt} \Phi_h(t) + v_{N+1}^2(t) + u_{N+1}^2(t) - \frac{h}{4} \sum_{j=0}^{N} (v_{i+1} - v_i)^2 - \frac{h}{4} \sum_{j=0}^{N} (u_{i+1} - u_i)^2 \]

\[\leq -2F_h(t) - \frac{d}{dt} \Phi_h(t) + (1 + \alpha^2)v_{N+1}^2(t).\]

This gives the required result. \(\square\)

**Theorem 3.4.** For any damped coefficient \(\alpha > 0\), the discretized energy \(F_h(t)\) defined by (3.2) decays uniformly exponentially:

\[(3.7) \quad F_h(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} e^{-\frac{\alpha}{1 + \varepsilon}t} F_h(0),\]

where \(\varepsilon\) is independent of \(h\) and satisfies \(0 < \varepsilon < 2\alpha/(1 + \alpha^2) < 1\).

**Proof.** Taking the derivative of \(G_h(t)\) defined by (3.5) and combining with Lemmas 3.1, 3.2, and 3.3, we obtain

\[\frac{d}{dt} G_h(t) = \frac{d}{dt} F_h(t) + \varepsilon \frac{d}{dt} \Phi_h(t)\]

\[\leq -\alpha v_{N+1}^2(t) + \varepsilon \frac{1}{2} (1 + \alpha^2) v_{N+1}^2(t) - \varepsilon F_h(t)\]

\[\leq -\left(\alpha - \frac{\varepsilon}{2} (1 + \alpha^2)\right) v_{N+1}^2(t) - \frac{\varepsilon}{1 + \varepsilon} G_h(t)\]

\[\leq -\frac{\varepsilon}{1 + \varepsilon} G_h(t),\]

where \(0 < \varepsilon < 2\alpha/(1 + \alpha^2)\) implied \(\alpha - \frac{\varepsilon}{2}(1 + \alpha^2) > 0\). A direct use of Grönwall’s inequality in Lemma 2.1 leads to

\[G_h(t) \leq e^{-\frac{\alpha}{1 + \varepsilon}t} G_h(0).\]

Again according to Lemma 3.2, we have

\[F_h(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} e^{-\frac{\alpha}{1 + \varepsilon}t} F_h(0).\]

This completes the proof of the theorem. \(\square\)

**4. Convergence of semidiscretized scheme.** In this section, we consider the convergence of the semidiscretized finite difference scheme (2.5). It presents both convergence of the solution and convergence of the energy. For convergence of the solution, we show that the solutions of difference scheme (2.5) are of second-order convergence to the solutions of equivalent system (1.6) as \(h \to 0\) for any \(t \in [0, \infty)\). For convergence of the energy, we show that discretized energy \(F_h(t)\) converges to the continuous energy \(F(t)\) as \(h \to 0\) uniformly over \(t \in (0, \infty)\) as long as the initial energy is convergent.
Let $\theta_j(t) = U_j(t) - u_j(t)$ and $\eta_j(t) = V_j(t) - v_j(t)$ be the error functions. From (2.1), (2.3), and (2.5), we have the following error equation:

\[
\begin{align*}
\theta_j'(t) - \delta_x \eta_j(t) &= r_j(t), \\
\eta_j'(t) - \delta_x \theta_j(t) &= s_j(t), \\
\theta_0(t) &= 0, \\
\theta_j(0) &= u^0(x_j) - u_j^0, \\
\eta_j(0) &= v^0(x_j) - v_j^0,
\end{align*}
\]

where $r_j(t)$ and $s_j(t)$ are defined by (2.2) and (2.4), respectively. If the initial values are sufficiently smooth, we can set $\theta_j(0) = \eta_j(0) = 0$ for $j = 0, 1, \ldots, N + 1$. Denote the energy $e_h(t)$ of error by

\[
e_h(t) = \frac{1}{2} \|	ilde{\theta}(t)\|^2 + \frac{1}{2} \|\tilde{\eta}(t)\|^2,
\]

where $\tilde{\theta}(t) = \{\theta_j(t)\}_{j}, \tilde{\eta}(t) = \{\eta_j(t)\}_{j}$ and the auxiliary function $\varphi_h(t)$ is given by

\[
\varphi_h(t) = h \sum_{j=0}^{N} x_{j+\frac{1}{2}} \theta_j(t) \eta_j(t).
\]

Define the Lyapunov function $l_h(t)$ similarly with (3.5) as

\[
l_h(t) = e_h(t) + \varepsilon \varphi_h(t), \quad 0 < \varepsilon < 1.
\]

Similarly to Lemma 3.2, we have

\[
(1 - \varepsilon) e_h(t) \leq l_h(t) \leq (1 + \varepsilon) e_h(t).
\]

**Lemma 4.1.** The energy of the error defined by (3.1) satisfies

\[
\frac{d}{dt} e_h(t) = -\alpha \eta_{N+1}(t) + h \sum_{j=0}^{N} r_{j+\frac{1}{2}}(t) \theta_j(t) + h \sum_{j=0}^{N} s_{j+\frac{1}{2}}(t) \eta_j(t) \\
\forall t \geq 0.
\]

**Proof.** Multiply both sides of the first equation of (4.1) by $h \theta_{j+\frac{1}{2}}(t)$ and sum up $j$ from 0 to $N$ to obtain

\[
\frac{1}{2} \frac{d}{dt} \sum_{j=0}^{N} \theta_{j+\frac{1}{2}}(t) + \frac{1}{2} h \sum_{j=0}^{N} \delta_x \theta_j(t) \eta_{j+\frac{1}{2}}(t) - \theta_{N+1}(t) \eta_{N+1}(t) = h \sum_{j=0}^{N} r_{j+\frac{1}{2}}(t) \theta_j(t).
\]

Similarly, multiplying both sides of the second equation of (4.1) by $h \eta_{j+\frac{1}{2}}(t)$ and summing up $j$ from 0 to $N$, we obtain

\[
\frac{1}{2} \frac{d}{dt} \sum_{j=0}^{N} \eta_{j+\frac{1}{2}}(t) - \frac{1}{2} h \sum_{j=0}^{N} \delta_x \eta_j(t) \theta_{j+\frac{1}{2}}(t) = h \sum_{j=0}^{N} s_{j+\frac{1}{2}}(t) \eta_j(t).
\]

Adding (4.6) to (4.7) gives the required result. 

\[
\square
\]
Lemma 4.2. The auxiliary function \( \varphi_h(t) \) defined by (2.2) satisfies
\[
\frac{d}{dt} \varphi_h(t) \leq -e_h(t) + \frac{1}{2} (1 + \alpha^2) \eta^2_{N+1}(t) + h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} r_{j+\frac{1}{2}}(t) \eta_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \theta_{j+\frac{1}{2}}(t) \theta_{j+\frac{1}{2}}(t).
\]

Proof. Finding the derivative of \( \varphi_h(t) \) along the solution of (4.1) gives
\[
\frac{d}{dt} \varphi_h(t) = h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \theta_{j+\frac{1}{2}}(t) \eta_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \theta_{j+\frac{1}{2}}(t) \eta_{j+\frac{1}{2}}(t)
= h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \eta_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \theta_{j+\frac{1}{2}}(t) \delta_x \eta_{j+\frac{1}{2}}(t)
+ h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \eta_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \theta_{j+\frac{1}{2}}(t) \eta_{j+\frac{1}{2}}(t)
+ h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \eta_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \theta_{j+\frac{1}{2}}(t) \theta_{j+\frac{1}{2}}(t).
\]

By (2.7) in Lemma 2.2, we further have
\[
\frac{d}{dt} \varphi_h(t) = -h \sum_{j=0}^{N} \eta_{j+\frac{1}{2}}^2(t) - h \sum_{j=0}^{N} x_j^{+\frac{1}{2}} \eta_{j+\frac{1}{2}}(t) \delta_x \eta_{j+\frac{1}{2}}(t) + h \sum_{j=0}^{N} \eta_{N+1}^2(t) - h \sum_{j=0}^{N} \eta_{j+1} - \eta_{j})^2.
\]

This gives the required result.
where $e_h(t)$ is defined by (3.1) and $\varepsilon$ and $C$ are scalars independent of $h$ satisfying $0 < \varepsilon < 2\alpha/(1 + \alpha^2) \leq 1$.

Proof. Since $u, v \in C^1([0, \infty); C^3(0, 1])$, it follows from (2.2) and (2.4) that there exists a constant $C$ independent of $h$ such that

$$
|r_{j+\frac{1}{2}}(t)| \leq C h^2, \quad |s_{j+\frac{1}{2}}(t)| \leq C h^2, \quad j = 0, 1, \ldots, N, \quad \forall t \in [0, \infty).
$$

Denote $\tilde{r}(t) = \{r_{j+\frac{1}{2}}(t)\}_{0 \leq j \leq N}$ and $\tilde{s}(t) = \{s_{j+\frac{1}{2}}(t)\}_{0 \leq j \leq N}$. Finding the derivative of $l_h(t)$ defined by (2.4) along the solution of (4.1) and combining with Lemmas 4.1 and 4.2, we obtain

$$
\frac{d}{dt} l_h(t) = \frac{d}{dt} e_h(t) + \varepsilon \frac{d}{dt} \theta_h(t)
$$

$$
\leq -\varepsilon e_h(t) - \left(\alpha - \frac{\varepsilon}{2} (1 + \alpha^2)\right) \eta_{N+1}^2(t) + h \sum_{j=0}^{N} r_{j+\frac{1}{2}}(t) \theta_{j+\frac{1}{2}}(t)
$$

$$
+ h \sum_{j=0}^{N} s_{j+\frac{1}{2}}(t) \eta_{j+\frac{1}{2}} + \varepsilon h \sum_{j=0}^{N} x_{j+\frac{1}{2}} r_{j+\frac{1}{2}}(t) \eta_{j+\frac{1}{2}} + \varepsilon \sum_{j=0}^{N} x_{j+\frac{1}{2}} s_{j+\frac{1}{2}}(t) \theta_{j+\frac{1}{2}}(t)
$$

$$
\leq -\varepsilon e_h(t) + \frac{\varepsilon \xi}{4} h \sum_{j=0}^{N} \eta_{j+\frac{1}{2}}^2(t) + \frac{\|r(t)\|^2}{\xi} + \frac{\|s(t)\|^2}{\xi} + \frac{\|\tilde{r}(t)\|^2}{\xi} + \frac{\|\tilde{s}(t)\|^2}{\xi}
$$

$$
= -\varepsilon e_h(t) + \frac{\varepsilon \xi}{4} h \sum_{j=0}^{N} \eta_{j+\frac{1}{2}}^2(t) + \frac{\|r(t)\|^2}{\varepsilon} + \frac{\|s(t)\|^2}{\varepsilon} + \frac{\|\tilde{r}(t)\|^2}{\varepsilon} + \frac{\|\tilde{s}(t)\|^2}{\varepsilon}
$$

$$
\leq -\varepsilon e_h(t) + \frac{\xi}{2} (1 + \varepsilon) e_h(t) + \frac{1}{\xi} (1 + \varepsilon) \left(\|\tilde{r}(t)\|^2 + \|\tilde{s}(t)\|^2\right)
$$

$$
= -\varepsilon e_h(t) + \frac{1 + \varepsilon}{2} \left(\|\tilde{r}(t)\|^2 + \|\tilde{s}(t)\|^2\right)
$$

$$
\leq - \varepsilon \|\tilde{r}(t)\|^2 + \frac{1 + \varepsilon}{2} \|\tilde{s}(t)\|^2,
$$

where we used the fact $\alpha - \varepsilon (1 + \alpha^2)/2 > 0$ whenever $0 < \varepsilon < 2\alpha/(1 + \alpha^2)$, Young's inequality $ab \leq a^2/\xi + b^2/\xi$ with $\xi = \varepsilon/(1 + \varepsilon)$, and the equivalence (4.5). By (4.9),

$$
l_h(t) \leq e^{-\frac{\xi}{1 + \varepsilon} t} l_h(0) + \frac{(1 + \varepsilon)^2}{\varepsilon} \int_{0}^{t} e^{-\frac{\xi}{1 + \varepsilon} (t-s)} \left(\|\tilde{r}(\tau)\|^2 + \|\tilde{s}(\tau)\|^2\right) d\tau.
$$

According to (4.5), we have

$$
e_h(t) \leq e^{-\frac{\xi}{1 + \varepsilon} t} \left(\|\tilde{r}(0)\|^2 + \|\tilde{s}(0)\|^2\right) + \frac{(1 + \varepsilon)^2}{\varepsilon (1 - \varepsilon)} \int_{0}^{t} e^{-\frac{\xi}{1 + \varepsilon} (t-s)} \left(\|\tilde{r}(\tau)\|^2 + \|\tilde{s}(\tau)\|^2\right) d\tau
$$

$$
\leq e^{-\frac{\xi}{1 + \varepsilon} t} \left(\|\tilde{r}(0)\|^2 + \|\tilde{s}(0)\|^2\right) + \frac{2(1 + \varepsilon)^3}{\varepsilon^2 (1 - \varepsilon)} C^2 h^4 \quad \forall t \in [0, \infty).
$$

This completes the proof of the theorem. 

\[ \Box \]
Remark 4.1. Theorem 4.3 shows that for any \( t \in [0, \infty) \), if the initial conditions are exact, the semidiscretized finite difference scheme (2.5) is of second-order convergence.

From Theorem 4.3 we can further obtain the convergence of the discretized energy \( F_h(t) \) to the continuous counterpart \( F(t) \) immediately.

**Theorem 4.4 (convergence of energy).** Suppose that \( u, v \in C^1([0, \infty); C^3[0, 1]) \) solve system (1.6) and \( \{u_j(t)\}_j, \{v_j(t)\}_j \) solve difference equation (2.5). If the initial conditions \( \{u_j(0)\}_j, \{v_j(0)\}_j \) in system (2.5) are exact and the initial discretized energy \( F_h(0) \) converges to the initial continuous energy \( F(0) \) as \( h \to 0 \), then

\[
|F_h(t) - F(t)| \leq |F_h(0) - F(0)| + \alpha \sqrt{Ch^2 \sqrt{t}} \quad \forall t \geq 0,
\]

where \( C \) is a scalar independent of \( h \). Furthermore,

\[
\lim_{h \to 0} \|F_h - F\|_{C[0, \infty)} = 0.
\]

**Proof.** In what follows, we use \( C \) to represent a positive constant independent of \( h \) although it may have different values in different contexts. From Theorem 4.3, if the initial values \( \{u_j(0)\}_j, \{v_j(0)\}_j \) in system (2.5) are exact, then

\[
\frac{h}{2} |v_{N+1}(t) - v(1, t)|^2 = \frac{h}{2} |\eta_{N+1}(t)|^2 \leq \|\eta(t)\|^2 \leq 2e_h(t) \leq Ch^4
\]

for some \( C > 0 \). By dissipation law (1.8) and Lemma 3.1, it follows that

\[
F(t) = F(0) - \int_0^t \alpha v^2(1, \tau) d\tau, \quad F_h(t) = F_h(0) - \int_0^t \alpha v^2_{N+1}(\tau) d\tau.
\]

Since both \( F(t) \) and \( F_h(t) \) decay exponentially to zero as \( t \to \infty \), it holds that

\[
F(0) = \int_0^\infty \alpha v^2(1, t) dt, \quad F_h(0) = \int_0^\infty \alpha v^2_{N+1}(t) dt.
\]

In addition, since \( F_h(0) \to F(0) \) as \( h \to 0 \), both \( \int_0^\infty v^2(1, t) dt \) and \( \int_0^\infty v^2_{N+1}(t) dt \) are bounded. Therefore for any \( t \in [0, \infty) \), by (4.11) and (4.12), it follows that

\[
|F_h(t) - F(t)| \leq |F_h(0) - F(0)| + \int_0^t \alpha |v^2_{N+1}(\tau) - v(1, \tau)|^2 d\tau
\]

\[
\leq |F_h(0) - F(0)| + \alpha \left( \int_0^t |v_{N+1}(\tau) - v(1, \tau)|^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^\infty |v_{N+1}(t) + v(1, t)|^2 dt \right)^{\frac{1}{2}}
\]

\[
\leq |F_h(0) - F(0)| + \alpha \sqrt{Ch^2 \sqrt{t}} \left( \int_0^t \sqrt{2} \left( \int_0^\infty (v^2_{N+1}(t) + v^2(1, t)) dt \right)^{\frac{1}{2}}
\]

\[
\leq |F_h(0) - F(0)| + \alpha \sqrt{Ch^2 \sqrt{t}} \left( \int_0^\infty v^2_{N+1}(t) dt + \int_0^\infty v^2(1, t) dt \right)^{\frac{1}{2}}
\]

\[
\leq |F_h(0) - F(0)| + \alpha \sqrt{Ch^2 \sqrt{t}},
\]

where H"older’s inequality, (4.11), and the boundedness of \( \int_0^\infty v^2(1, t) dt \) and \( \int_0^\infty v^2_{N+1}(t) dt \) were employed. From (1.4) and (3.7), when \( t' > t \), both \( F_h(t') \) and \( F(t') \) can be sufficiently small for sufficiently large \( t \). We thus arrive at uniform convergence (4.10).
5. Order reduction finite difference scheme and its convergence. In this section, we show that the semidiscretized finite difference scheme (2.5) for equivalent system (1.6) can lead to a semidiscretized finite difference scheme for original system (1.2). It is this order reduction finite difference scheme that has universal significance for using the finite difference method to discretize PDEs.

**Lemma 5.1.** Suppose that \( w(x,t) \) solves system (1.2). The semidiscretized finite difference scheme (2.5) leads to the following semidiscretized finite difference scheme to the original system (1.2):

\[
\begin{aligned}
\frac{1}{4} (w''_{j-1}(t) + 2w''_j(t) + w''_{j+1}(t)) - \delta_x^2 w_j(t) = 0, & \quad j = 1, 2, \ldots, N, \\
\frac{h}{4} (w'_N(t) + w'_{N+1}(t)) + \delta_x w_{N+\frac{1}{2}} + \alpha w'_{N+1} = 0, \\
w_0(t) = 0, & \quad w_j(0) = w^0(x_j), \quad w'_j(0) = w^1(x_j), \quad j = 0, 1, \ldots, N + 1,
\end{aligned}
\]

(5.1)

where \( w_j(t) \) is an approximation of \( w(x_j, t) \).

**Proof.** From transformation (1.5), it has

\[
\begin{aligned}
u_{j+\frac{1}{2}} = \delta_x w_{j+\frac{1}{2}}, & \quad v_{j+\frac{1}{2}} = w'_{j+\frac{1}{2}}, \quad j = 0, \ldots, N.
\end{aligned}
\]

(5.2)

Multiplying both sides of the second equation of (2.5) by \( h/2 \) gives

\[
\frac{u_{j+1} - u_j}{2} = \frac{h}{2} v'_{j+\frac{1}{2}}, \quad j = 0, \ldots, N,
\]

(5.3)

which together with (5.2) yields

\[
u_{j+1} = \delta_x w_{j+\frac{1}{2}} + \frac{h}{2} w''_{j+\frac{1}{2}} \quad u_j = \delta_x w_{j+\frac{1}{2}} - \frac{h}{2} w''_{j+\frac{1}{2}}, \quad j = 0, \ldots, N.
\]

(5.4)

After cancellation of variable \( u_j \), we obtain

\[
\frac{h}{2} w''_{j+\frac{1}{2}} + \frac{h}{2} w''_{j-\frac{1}{2}} - \delta_x w_{j+\frac{1}{2}} + \delta_x w_{j-\frac{1}{2}} = 0, \quad j = 1, \ldots, N,
\]

(5.5)

which gives the first equation of (5.1). Letting \( j = N \) in the first equation of (5.4), we have

\[
u_{N+1} = \delta_x w_{N+\frac{1}{2}} + \frac{h}{2} w''_{N+\frac{1}{2}},
\]

(5.6)

which, together with the right boundary condition of (2.5), leads to the second equation of (5.1). The left boundary condition and initial conditions are trivial. This ends the proof of the lemma.

The finite difference scheme (5.1) is indeed the right semidiscretization of original system (1.2) that we are looking for. Also note that scheme (5.1) can be derived without using (2.5); the widely used order reduction finite difference scheme (see, e.g., [19]) can also be used for this purpose as follows. Introduce an intermediate variable

\[\zeta(x, t) = w_x(x, t).\]

Then, the second-order differentiation in spatial variable in system (1.2) is reduced to the first order as
Construct a standard second-order finite difference scheme for (5.7) to obtain
\[
\begin{align*}
\frac{1}{2} [w''(x_i, t) + w''(x_{i+1}, t)] &= \frac{1}{h^2} \left[ \zeta(x_{i+1}, t) - \zeta(x_i, t) \right] + \mathcal{O}(h^2), \\
\frac{1}{2} [\zeta(x_i, t) + \zeta(x_{i+1}, t)] &= \frac{1}{h} [w(x_{i+1}, t) - w(x_i, t)] + \mathcal{O}(h), \\
w(0, t) = 0, \quad \zeta(0, t) = -\alpha w_t(1, t),
\end{align*}
\]
(5.8)
Ignoring the infinitesimal terms gives
\[
\begin{align*}
\frac{1}{2} (w'' + w''_{i+1}) &= \frac{1}{h} (\zeta_{i+1} - \zeta_i), \\
\frac{1}{2} (\zeta_i + \zeta_{i+1}) &= \frac{1}{h} (w_{i+1} - w_i), \quad i = 0, \ldots, N, \\
w(0, t) = 0, \quad \zeta(x_{N+1}, t) = -\alpha w'_{N+1}.
\end{align*}
\]
(5.9)
Now we eliminate all $\zeta_i$ related terms in (5.9). First, multiplying by $-h/2$ the first equation of (5.9) and adding to the second one gives
\[
\zeta_{i+1} = \frac{h}{4} (w'' + w''_{i+1}) + \frac{1}{h} (w_{i+1} - w_i), \quad i = 0, \ldots, N.
\]
(5.10)
Second, multiplying by $h/2$ the first equation of (5.9) and adding to the second one yields
\[
\zeta_i = -\frac{h}{4} (w'' + w''_{i+1}) + \frac{1}{h} (w_{i+1} - w_i), \quad i = 0, \ldots, N.
\]
(5.11)
Shifting the subscript of (5.10) and combining with (5.11), we derive the first equation of (5.1). Finally, setting $i = N$ in (5.10) and substituting into the last equation $\zeta_{N+1} = -\alpha w'_{N+1}$ of (5.9) we obtain the last equation of (5.1). We have thus derived (5.1) by the standard order reduction finite difference method which turns out to be the right difference scheme for original system (1.2).

In this section, we denote the discretized energy of system (5.1) by
\[
E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[ \left| \frac{w'_{j+1}(t) + w'_j(t)}{2} \right|^2 + \left| \frac{w_{j+1}(t) - w_j(t)}{h} \right|^2 \right].
\]
(5.12)
In subsection 5.1, we show that the discretized energy of system (5.1) preserves the uniformly exponential decay.

5.1. **Uniformly exponential decay.** In this subsection, we show that the energy $E_h(t)$ of semidiscretized system (5.1) defined by (5.12) decays exponentially uniformly via the Lyapunov function method. Lemma 5.2 states that the discrete energy $E_h(t)$ is nonincreasing with respect to time.

**Lemma 5.2.** The discretized energy $E_h(t)$ defined by (5.12) satisfies
\[
\frac{dE_h(t)}{dt} = -\alpha (w'_{N+1})^2.
\]
(5.13)
Proof. Multiplying both sides of the first equation of (5.1) by \( hu_j \) and summing for \( j \) from 1 to \( N \), we obtain

\[
(5.14) \quad \frac{h}{4} \sum_{j=1}^{N} (w''_{j+1} + 2w''_{j} + w''_{j-1}) w_j' - h \sum_{j=1}^{N} \left( \frac{w_{j+1} - 2w_{j} + w_{j-1}}{h^2} \right) w_j' = 0.
\]

The first term of (5.14) can be expanded as

\[
\frac{h}{4} \sum_{j=1}^{N} (w''_{j+1} + 2w''_{j} + w''_{j-1}) w_j' = \frac{h}{4} \sum_{j=1}^{N} \left[ (w''_{j+1} + w''_{j}) w_j' + (w'_j + w''_{j-1}) w_j' \right]
\]

(5.15)

\[
= \frac{h}{4} \sum_{j=0}^{N} (w''_{j+1} + w''_{j})(w'_j + w'_{j+1}) - \frac{h}{4} w'_{N+1} w'_{N+1} - \frac{h}{4} w''_{N} w'_{N+1},
\]

and the second term of (5.14) can be expanded as

\[
\frac{h}{4} \sum_{j=1}^{N} \left( \frac{w_{j+1} - 2w_{j} + w_{j-1}}{h^2} \right) w_j' = \frac{1}{h} \sum_{j=0}^{N} (w_{j+1} - w_{j})(w'_j - w'_{j+1})
\]

\[
+ \frac{1}{h} (w_{N+1} - w_{N} w'_{N+1}).
\]

Multiplying both sides of the second equation of (5.1) by \( w'_{N+1} \) gives

\[
(5.17) \quad \frac{h}{4} (w''_{N} + w''_{N+1}) w'_{N+1} + \frac{w'_{N+1} - w_{N}}{h} w'_{N+1} + \alpha (w'_{N+1})^2 = 0.
\]

Plugging (5.15) and (5.16) into (5.14) and combining with (5.17) produce

\[
(5.18) \quad \frac{h}{4} \sum_{j=0}^{N} (w''_{j+1} + w''_{j})(w'_j + w'_{j+1}) + \frac{1}{h} \sum_{j=0}^{N} (w_{j+1} - w_{j})(w'_j + w'_{j+1}) + \alpha (w'_{N+1})^2 = 0,
\]

which gives the required result that

\[
\frac{dE_h(t)}{dt} = -\alpha (w'_{N+1})^2.
\]

This ends the proof.

In order to acquire the uniformly exponential decay of \( E_h(t) \), we define a Lyapunov function \( L_h(t) \) similarly with (3.5) as

\[
(5.19) \quad L_h(t) = E_h(t) + \varepsilon \psi_h(t), \quad 0 < \varepsilon < 1,
\]

where the auxiliary function \( \psi_h(t) \) is defined by

\[
(5.20) \quad \psi_h(t) = \frac{1}{h} \sum_{j=1}^{N} \frac{w''_{j+1} + 2w''_{j} + w''_{j-1}}{4} \frac{w_{j+1} - w_{j-1}}{2} + \frac{1}{4} (w'_{N} + w'_{N+1})(w_{N+1} - w_{N}).
\]

Lemma 5.3 shows that the Lyapunov function \( L_h(t) \) is equivalent to the discretized energy \( E_h(t) \).
LEMMMA 5.3. Suppose that $\alpha > 0$ in (5.1). For $0 < \varepsilon < 1$, the Lyapunov function $L_h(t)$ defined by (5.19) is equivalent to the discretized energy $E_h(t)$:

$$(1 - \varepsilon) E_h(t) \leq L_h(t) \leq (1 + \varepsilon) E_h(t).$$

Proof. For the auxiliary function $\psi_h(t)$ defined by (5.20), we have, by Cauchy’s inequality, that

$$(5.21) \quad |\psi_h(t)| \leq \frac{h}{2} \sum_{j=0}^{N} \left( \frac{w_{j+1} - w_{j-1}}{2h} \right)^2 + \frac{h}{2} \sum_{j=1}^{N} \left( \frac{w'_{j+1} + 2w'_j + w'_{j-1}}{4} \right)^2 + \frac{h}{4} \left( \frac{w_{N+1} - w_N}{h} \right)^2 + \frac{h}{4} \left( w'_N + w'_{N+1} \right)^2$$

$$+ \frac{h}{4} \sum_{j=0}^{N} \left( \frac{w'_j + w'_{j+1}}{2} \right)^2 + \frac{h}{4} \left( \frac{w_{N+1} - w_N}{h} \right)^2 + \frac{h}{4} \left( \frac{w'_N + w'_{N+1}}{2} \right)^2 + \frac{h}{2} \sum_{j=0}^{N} \left( \frac{w_{j+1} - w_j}{h} \right)^2 = E_h(t).$$

Therefore,

$$(1 - \varepsilon) E_h(t) \leq L_h(t) \leq (1 + \varepsilon) E_h(t).$$

LEMMMA 5.4. The auxiliary function $\psi_h(t)$ defined by (5.20) satisfies

$$\frac{d\psi_h(t)}{dt} \leq -E_h(t) + \frac{1}{2}(1 + \alpha^2) \left( w'_{N+1} \right)^2.$$

Proof. Finding the derivative of $\psi_h(t)$ along the solution of (5.1) gives

$$(5.22) \quad \frac{d\psi_h(t)}{dt} = \frac{h}{2} \sum_{j=1}^{N} \left( \frac{w_{j+1} - w_{j-1}}{2h} \right)^2 + \frac{h}{4} \left( \frac{w'_{N+1} + w'_N}{2} \right)^2 + \frac{h}{2} \sum_{j=1}^{N} \left( \frac{w'_{j+1} + w'_{j-1}}{2} \right)^2 + \frac{h}{2} \sum_{j=0}^{N} \left( \frac{w_{j+1} - w_j}{h} \right)^2 = \frac{h}{2} \sum_{j=1}^{N} \left( \frac{w_{j+1} + w'_j}{2} \right)^2 + \frac{h}{8} \sum_{j=1}^{N} \left( (w'_{j+1} + w'_j)^2 - (w'_j + w'_{j-1})^2 \right)$$

The first term on the right-hand side of (5.22) implies

$$(5.23) \quad \frac{h}{2} \sum_{j=1}^{N} \left( \frac{w_{j+1} + w'_j}{2} \right)^2 = \frac{h}{8} \sum_{j=1}^{N} \left( (w'_{j+1} + w'_j)^2 - (w'_j + w'_{j-1})^2 \right)$$

$$= \frac{h}{8} \sum_{j=1}^{N} j(w'_{j+1} + w'_j)^2 - \frac{h}{8} \sum_{j=0}^{N-1} (j + 1)(w'_{j+1} + w'_j)^2$$

$$= \frac{h}{8} \sum_{j=0}^{N} \left( \frac{w'_{j+1} + w'_j}{2} \right)^2 + \frac{1}{8} (w'_{N+1} + w'_N)^2.$$
The third term on the right-hand side of (5.22) together with the first equation of (5.1) yields

\[(5.24)\]
\[
\frac{1}{2h} \sum_{j=1}^{N} j(w_{j+1} - w_{j-1})(w_{j+1} - 2w_{j} + w_{j-1}) = \frac{1}{2h} \sum_{j=1}^{N} j(w_{j+1} - w_{j-1})^2 - \frac{1}{2h} \sum_{j=1}^{N} j(w_{j} - w_{j-1})^2
\]
\[
= -\frac{h}{2} \sum_{j=0}^{N} \left( \frac{w_{j+1} - w_{j}}{h} \right)^2 + \frac{1}{2} \left( \frac{w_{N+1} - w_{N}}{h} \right)^2.
\]

The fourth term on the right-hand side of (5.22) together with the second equation in (5.1) implies

\[(5.25)\]
\[
\frac{1}{4}(w''_{N} + w''_{N+1})(w_{N+1} - w_{N}) = -\left( \frac{w_{N+1} - w_{N}}{h} \right)^2 - \alpha w'_{N+1} \frac{w_{N+1} - w_{N}}{h}.
\]

Plugging (5.23), (5.24), and (5.25) into (5.22), we obtain

\[
\frac{d\psi_{h}(t)}{dt} = -E_{h}(t) + \frac{1}{8} (3 (w'_{N+1})^2 + 2 w'_{N} w'_{N+1} - (w'_{N})^2)
\]
\[
- \frac{1}{2} \left( \frac{w_{N+1} - w_{N}}{h} \right)^2 - \alpha w'_{N+1} \frac{w_{N+1} - w_{N}}{h}
\]
\[
\leq -E_{h}(t) + \frac{1}{2} (1 + \alpha^2) (w'_{N+1})^2,
\]

where in the last inequality, the Cauchy’s inequality was applied twice.

**Theorem 5.5.** Suppose that \(\alpha > 0\) in (5.1). There exists a constant \(\varepsilon\), independent of \(h\), satisfying \(0 < \varepsilon < 2\alpha/(1 + \alpha^2)\), such that for all initial conditions \(\{w^0_{j}\}, \{w^1_{j}\} \in \mathbb{R}^{N+1}\), the energy \(E_{h}(t)\) defined by (5.12) for semidiscretized finite difference scheme (5.1) satisfies

\[E_{h}(t) \leq \frac{1 + \varepsilon}{1 - \varepsilon} e^{-\frac{\alpha}{1 - \varepsilon} t} E_{h}(0) \quad \forall t > 0.\]

**Proof.** Finding the derivative of the Lyapunov function \(L_{h}(t)\) defined by (5.19) gives

\[
\frac{dL_{h}(t)}{dt} = \frac{dE_{h}(t)}{dt} + \varepsilon \frac{d\psi_{h}(t)}{dt}.
\]

According to Lemmas 5.2 and 5.4, there is

\[
\frac{dL_{h}(t)}{dt} \leq - \left( \alpha - \frac{\varepsilon}{2}(1 + \alpha^2) \right) (w'_{N+1})^2 - \varepsilon E_{h}(t).
\]

Since \(\alpha - \varepsilon(1 + \alpha^2)/2 > 0\) for \(0 < \varepsilon < 2\alpha/(1 + \alpha^2)\) it follows from the equivalence in Lemma 3.2 that

\[
\frac{dL_{h}(t)}{dt} \leq -\varepsilon E_{h}(t) \leq -\frac{\varepsilon}{1 + \varepsilon} L_{h}(t).
\]

A direct application of Gronwall’s inequality in Lemma 2.1 leads to
Moreover, if
\[ L_h(t) \leq e^{-\frac{t}{\Delta t}} L_h(0), \]
which together with Lemma 5.3 shows that
\[ E_h(t) \leq \frac{1 + e^{-\frac{t}{\Delta t}}}{1 - e^{-\frac{1}{\Delta t}}} E_h(0). \]

5.2. Convergence of the solution. In this subsection, we show that the solutions of the semidiscretized finite difference scheme (5.1) converge to the solutions of the continuous counterpart (1.2). To this purpose, set vectors \( w_h = (w_j)_j \), \( w_h^0 = (w_j^0)_j \) and \( w^1_h = (w^1_j)_j \). Introduce the extension operators defined by
\[
\begin{align*}
q_h^i v_h(x) &= v_i, \quad x \in [x_i, x_{i+1}], \ i = 0, 1, \ldots, N, \\
q_h^{i+1} v_h(x) &= v_{i+1}, \quad x \in [x_i, x_{i+1}], \ i = 0, 1, \ldots, N.
\end{align*}
\]
Define \( q_h v_h = \frac{1}{2} (q_h^i + q_h^{i+1}) v_h \). It is a routine task to check that
\[
\int_0^1 (q_h u_h)(q_h v_h) dx = h \sum_{j=0}^N u_{j+\frac{1}{2}} v_{j+\frac{1}{2}}.
\]
Introduce an extension operator
\[
p_h v_h(x) = v_i \frac{x-x_{i+1}}{x_i-x_{i+1}} + v_{i+1} \frac{x-x_i}{x_{i+1}-x_i}, \quad x \in [x_i, x_{i+1}], \ i = 0, \ldots, N.
\]
It verifies that
\[
\int_0^1 (p_h u_h)_x (p_h v_h)_x dx = h \sum_{j=0}^N \delta_x u_{j+\frac{1}{2}} \delta_x v_{j+\frac{1}{2}}.
\]
For every function \( v \in C[0, 1] \), if we set \( v_h = (v_j)_j = (v(x_j))_j \), it follows easily that
\[
\begin{align*}
q_h^i v_h &\to v \text{ strongly in } L^\infty(0, 1), \\
q_h^{i+1} v_h &\to v \text{ strongly in } L^\infty(0, 1), \\
qu_h v_h &\to v \text{ strongly in } L^\infty(0, 1).
\end{align*}
\]
Moreover, if \( v \in H^1(0, 1) \), then
\[
\| p_h v_h - q_h v_h \|^2_{L^2(0, 1)} = \frac{h^3}{12} \sum_{j=0}^N \left( \delta_x v_{j+\frac{1}{2}} \right)^2 = \mathcal{O}(h^2)
\]
and
\[
\| (p_h v_h)_x \|^2_{L^2(0, 1)} = h \sum_{j=0}^N \left( \delta_x v_{j+\frac{1}{2}} \right)^2 = \mathcal{O}(1),
\]
which, together with (5.30), imply that
\[
p_h v_h \to v \text{ weakly in } H^1(0, 1).
\]
Now we are in a position to state the convergence of the discrete solution to the continuous counterpart.
Theorem 5.6. Let \( w_h(t) \) be the solution of (5.1). Assume that \( w_h^0 \) and \( w_h^1 \) satisfy for \( h \to 0 \) that
\[
\begin{cases}
p_h w_h^0 \to w^0 \text{ weakly in } H^1_2(0, 1), \\
q_h w_h^1 \to w^1 \text{ weakly in } L^2(0, 1).
\end{cases}
\]
Then, as \( h \to 0 \),
\[
\begin{cases}
p_h w_h \to w \text{ weakly in } L^\infty(0, \infty; H^1_2(0, 1)), \\
q_h w_h' \to w' \text{ weakly in } L^\infty(0, \infty; L^2(0, 1)),
\end{cases}
\]
where \( w(x, t) \) is the solution of system (1.2). In addition, if \( E_h(0) \to E(0) \) as \( h \to 0 \), then
\[
\lim_{h \to 0} \|E_h - E\|_{C[0, \infty)} = 0.
\]

Proof. By definitions of \( p_h \) and \( q_h \), for every \( t \geq 0 \) and the energy \( E_h(t) \) defined by (5.12), it has
\[
E_h(t) = \frac{1}{2} \left( \|q_h w_h'(t)\|^2_{L^2(0, 1)} + \|p_h w_h(t)\|^2_{H^1_2(0, 1)} \right).
\]
Since \( E_h(t) \) is nonincreasing with respect to time \( t \), by the convergence of the initial state (5.34), we know that \( p_h w_h \) is bounded in \( L^\infty(0, \infty; H^1_2(0, 1)) \) and \( q_h w_h' \) is bounded in \( L^\infty(0, \infty; L^2(0, 1)) \). Moreover, it follows from Lemma 5.2 that for all \( t > 0 \)
\[
E_h(0) = E_h(t) + \alpha \int_0^t (w_{N+1}'^2) \, dt,
\]
from which we can see that \( w_{N+1}'(t) \) is bounded in \( L^2(0, \infty) \). By the Sobolev embedding theorem and up to extraction of a subsequence if necessary, we have, by Lemma 4 of [1], that
\[
\begin{cases}
p_h w_h \to w \text{ weakly* in } L^\infty(0, \infty; H^1_2(0, 1)), \\
w_{N+1}' \to w'(1, \cdot) \text{ weakly in } L^2(0, \infty), \\
q_h w_h \to w \text{ weakly* in } L^\infty(0, \infty; L^2(0, 1)), \\
q_h w_h' \to w' \text{ weakly* in } L^\infty(0, \infty; L^2(0, 1)),
\end{cases}
\]
where we implicitly assumed that the limits of \( p_h w_h \) and \( q_h w_h \) are the same, because for every \( t \geq 0 \),
\[
\int_0^1 |(p_h w_h - q_h w_h)(x, t)|^2 \, dx = \frac{h^3}{12} \sum_{j=0}^N \left( \delta_x w_{j+1/2} \right)^2 \leq \frac{h^2}{6} E_h(t) = O(h^2).
\]
The second convergence in (5.38) can be proved as follows. Suppose that \( w_{N+1}' \to g \) weakly in \( L^2(0, \infty) \), i.e., for all \( \phi \in \mathcal{D}(0, \infty) \), it has
\[
\int_0^\infty w_{N+1}' \phi \, dt \to \int_0^\infty g \phi \, dt.
\]
Then,
\[
\int_0^\infty w_{N+1}' \phi \, dt = - \int_0^\infty w_{N+1}' \phi' \, dt \to - \int_0^\infty w(1, t) \phi' \, dt.
\]
Thus, \( g(t) = w'(1, t) \) in the sense of distribution.

Next we show that the limit \( w(x, t) \) is the (weak) solution of (1.2). Let \( \varphi \in \mathcal{D}([0, 1] \times (0, \infty)) \) with \( \varphi(0, \cdot) = 0 \) and let \( \varphi_h = (\varphi(x_j, \cdot))_j \). First, multiplying the first equation of (5.1) by \( h\varphi_j \), performing the integration by parts over \((0, \infty)\), and summing up from \( j = 1, \ldots, N \), we obtain

\[
(5.39) \quad \int_0^\infty \frac{h}{4} \sum_{j=1}^N (w_{j-1} + 2w_j + w_{j+1}) \varphi_j'' dt - \int_0^\infty h \sum_{j=1}^N \delta_x^2 w_j \varphi_j dt = 0.
\]

Second, multiplying the second equation of (5.1) by \( \varphi_{N+1} \) and performing integration by parts over \((0, \infty)\), we have

\[
(5.40) \quad \int_0^\infty \frac{h}{4} (w_N + w_{N+1}) \varphi_{N+1}'' dt + \int_0^\infty \delta_x w_{N+\frac{1}{2}} \varphi_{N+1} dt + \alpha \int_0^\infty w_{N+1} \varphi_{N+1} dt = 0.
\]

Adding (5.39) to (5.40) and performing summation by parts, we obtain

\[
(5.41) \quad \int_0^\infty h \sum_{j=0}^N w_{j+\frac{1}{2}} \varphi_j'' dt + \int_0^\infty h \sum_{j=0}^N \delta_x w_j \delta_x \varphi_j dt + \alpha \int_0^\infty w_{N+1} \varphi_{N+1} dt = 0.
\]

By definitions of \( q_h \) and \( p_h \), it is easy to check that (5.41) is equivalent to

\[
(5.42) \quad \int_0^1 \int_0^1 (q_h w_h)(q_h \varphi_h'') dx dt + \int_0^1 \int_0^1 (p_h w_h)(p_h \varphi_h)_x dx dt + \alpha \int_0^\infty w_{N+1} \varphi(1, t) dt = 0.
\]

If we can show that for every \( \varphi \in \mathcal{D}([0, 1] \times (0, \infty)) \),

\[
(5.43) \quad \begin{cases} 
  p_h \varphi_h \to \varphi & \text{strongly in } L^2 \left( 0, \infty; H^1_L(0, 1) \right), \\
  q_h \varphi_h \to \varphi & \text{strongly in } L^2 \left( 0, \infty; L^2(0, 1) \right),
\end{cases}
\]

then, with the convergence in (5.38), we can pass to the limit as \( h \to 0 \) for all terms in (5.42) to arrive at [1]

\[
(5.44) \quad \int_0^1 \int_0^1 w(x, t) \varphi''(x, t) dx dt + \int_0^\infty \int_0^1 w_x(x, t) \varphi_x(x, t) dx dt + \alpha \int_0^\infty w'(1, t) \varphi(1, t) dt = 0.
\]

This shows that \( w(x, t) \) is a weak solution to (1.2).

Now we show (5.43). The first convergence in (5.43) is similar to that in [24]. For the second one, observe that

\[
(5.45) \quad \int_0^1 \int_0^1 |q_h^+ \varphi_h - \varphi|^2 dx dt = \int_0^\infty h \sum_{j=0}^N \varphi_{j+1}^2 dt + \int_0^1 \varphi^2 dx dt - 2 \int_0^\infty h \sum_{j=0}^N \varphi_{j+1} \int_0^1 \varphi(x_j + sh, t) ds dt.
\]

By the Riemann sum nature, we have
(5.46) \[ \lim_{h \to 0} \int_0^1 \int_0^1 |q^-_h \varphi_h - \varphi|^2 \, dx \, dt = 0. \]

Similarly,

(5.47) \[ \lim_{h \to 0} \int_0^1 \int_0^1 |q^+_h \varphi_h - \varphi|^2 \, dx \, dt = 0. \]

Since

(5.48) \[ \int_0^1 \int_0^1 |q_h \varphi_h - \varphi|^2 \, dx \, dt \leq \frac{1}{2} \int_0^1 \int_0^1 |q^-_h \varphi_h - \varphi|^2 \, dx \, dt + \frac{1}{2} \int_0^1 \int_0^1 |q^+_h \varphi_h - \varphi|^2 \, dx \, dt, \]

the second convergence follows from (5.46) and (5.47).

It remains to show that \( w(x, t) \) satisfies the initial conditions \( w(\cdot, 0) = w^0 \) and \( w'(\cdot, 0) = w^1 \). For this purpose, we set \( v \in \mathcal{D}(0, 1) \) and \( l \in \mathcal{D}(0, \infty) \) and let \( \varphi_h = (v(x_j))_j \). First, multiplying the first equation of (5.1) by \( hv \), performing the integration by parts over \( [0, \infty) \), and summing up from \( j = 1, \ldots, N \), we obtain

(5.49) \[ -l(0) \frac{h}{4} \sum_{j=1}^N (w^1_{j-1} + 2w^1_j + w^1_{j+1}) v_j + l'(0) \frac{h}{4} \sum_{j=1}^N (w^0_{j-1} + 2w^0_j + w^0_{j+1}) v_j \]
\[ + \int_0^1 \int_0^1 |q_h \varphi_h - \varphi|^2 \, dx \, dt + \int_0^1 \int_0^1 |q^-_h \varphi_h - \varphi|^2 \, dx \, dt = 0. \]

Second, multiplying the second equation of (5.1) by \( v_{N+1}l \) and performing the integration by parts over \( [0, \infty) \), we have

(5.50) \[ -l(0) \frac{h}{4} (w^1_N + w^1_{N+1}) v_{N+1} + l'(0) \frac{h}{4} (w^0_N + w^0_{N+1}) v_{N+1} \]
\[ + \int_0^1 \int_0^1 |q_h \varphi_h - \varphi|^2 \, dx \, dt + \int_0^1 \int_0^1 |q^-_h \varphi_h - \varphi|^2 \, dx \, dt = 0. \]

Add (5.49) to (5.50) to obtain

(5.51) \[ -l(0)h \sum_{j=0}^N w^1_{j+\frac{1}{2}} v_{j+\frac{1}{2}} + l'(0)h \sum_{j=0}^N w^0_{j+\frac{1}{2}} v_{j+\frac{1}{2}} \]
\[ + \int_0^1 \int_0^1 |q_h \varphi_h - \varphi|^2 \, dx \, dt + \int_0^1 \int_0^1 |q^-_h \varphi_h - \varphi|^2 \, dx \, dt = 0. \]

Using the definition of \( p_h \) and \( q_h \), it is easy to check that (5.51) is equivalent to

(5.52) \[ -l(0) \int_0^1 q_h w^1_h q_h \varphi_h \, dx + l'(0) \int_0^1 q_h w^0_h q_h \varphi_h \, dx \]
\[ + \int_0^1 \int_0^1 |q_h \varphi_h - \varphi|^2 \, dx \, dt + \int_0^1 \int_0^1 |q^-_h \varphi_h - \varphi|^2 \, dx \, dt = 0. \]
Passing to the limit as $h \to 0$ in (5.52) gives

\begin{equation}
(5.53) \quad -l(0) \int_0^1 w^1 v dx + l'(0) \int_0^1 w^0 v dx + \int_0^\infty \int_0^1 w v \phi'' dx dt + \int_0^\infty \int_0^1 w_x v_x \phi dx dt = 0,
\end{equation}

from which we can easily derive $w(\cdot, 0) = w^0$, $w'(\cdot, 0) = w^1$. Since system (1.2) admits a unique solution, we can conclude that the convergence in (5.38) holds for the whole sequence \{h\}, not only for an extracted subsequence.

Finally, by (1.3) and (5.13), it follows that

\begin{equation}
(5.54) \quad E(0) = E(t) + \alpha \int_0^t (w'(1, t))^2 dt, \quad E_h(0) = E_h(t) + \alpha \int_0^t (w'_{N+1})^2 dt.
\end{equation}

Since both $E(t)$ and $E_h(t)$ are exponentially stable, we have

\begin{equation}
(5.55) \quad E(0) = \alpha \int_0^\infty (w'(1, t))^2 dt, \quad E_h(0) = \alpha \int_0^\infty (w'_{N+1})^2 dt.
\end{equation}

The assumption $E_h(0) \to E(0)$ implies that

\begin{equation}
(5.56) \quad \int_0^\infty (w'_{N+1})^2 dt \to \int_0^\infty (w'(1, t))^2 dt \text{ as } h \to 0,
\end{equation}

which together with the weak convergence in (5.38) yields

\begin{equation}
(5.57) \quad w'_{N+1} \to w'(1, \cdot) \text{ strongly in } L^2(0, \infty).
\end{equation}

By (5.54) and (5.57), we have for all $t > 0$ that

\begin{equation}
(5.58) \quad |E_h(t) - E(t)| \leq |E_h(0) - E(0)| + \alpha \int_0^t (w'_{N+1})^2 dt - \int_0^t (w'(1, t))^2 dt
\end{equation}

\begin{equation*}
\leq |E_h(0) - E(0)| + \alpha \left( \int_0^t |w'_{N+1} + w'(1, t)|^2 dt \right)^{1/2} \left( \int_0^t |w'_{N+1} - w'(1, t)|^2 dt \right)^{1/2}
\end{equation*}

\begin{equation*}
\leq |E_h(0) - E(0)| + \sqrt{2\alpha} \left( \|w'_{N+1}\|_{L^2(0, \infty)} + \|w'(1, \cdot)\|_{L^2(0, \infty)} \right) \|w'_{N+1} - w'(1, \cdot)\|_{L^2(0, \infty)}.
\end{equation*}

This, together with $E_h(0) \to E(0)$ and (5.56), shows that

\begin{equation*}
\lim_{h \to 0} \|E_h - E\|_{C([0, \infty))} = 0.
\end{equation*}

This completes the proof of the theorem. \hfill \Box

6. Uniform observability and controllability. In this section, we discuss two additional preservation properties of the uniform observability and uniform controllability of the finite difference semidiscrete scheme (5.1) to the following control system in $\mathcal{H} = H^1_I(0, 1) \times L^2(0, 1)$:

\begin{equation}
(6.1) \quad \begin{cases}
    w_t(x, t) - w_{xx}(x, t) = 0, & 0 < x < 1, \ t > 0, \\
    w(0, t) = 0, \ w_x(1, t) = U(t), \\
    w(x, 0) = w^0(x), \ w_t(x, 0) = w^1(x).
\end{cases}
\end{equation}

Before we turn to the uniform approximation of exact controllability of system (6.1), we introduce some results and notation for the continuous problem.
6.1. Continuous problem: Known results and notation. Lemmas 6.1 and 6.2 are available in [26].

**Lemma 6.1.** System (6.1) is exactly controllable over $[0, T], T > 2$. Precisely, for any $T > 2$ and any initial value $(w^0, w^1) \in \mathcal{H}$, there exists a control $U(\cdot) \in L^2(0, T)$ such that the solution $w(x, t)$ of system (6.1) satisfies

$$w(x, T) = w_t(x, T) = 0, \quad x \in (0, 1).$$

Control $U(t)$ having property (6.2) satisfies

$$\int_0^T U(t)\overline{w}_t(1, t) dt = - \int_0^1 w^1 \overline{w}_1 dx - \int_0^1 w^0_0 \overline{w}_2 dx,$$

for any $(\overline{w}^0, \overline{w}^1) \in \mathcal{H}$, where $(\overline{w}, \overline{w}_t)$ is the solution of the following adjoint system:

$$\begin{cases}
\overline{w}_{tt}(x, t) - \overline{w}_{xx}(x, t) = 0, \quad x \in (0, 1), \quad t > 0, \\
\overline{w}(0, t) = \overline{w}_x(1, t) = 0, \quad t > 0, \\
\overline{w}(x, 0) = \overline{w}^0(x), \quad \overline{w}_t(x, 0) = \overline{w}^1(x), \quad x \in [0, 1].
\end{cases}$$

It is known that among the admissible controls (i.e., all controls having property (6.2)), there is a unique control with minimal $L^2$-norm. The optimal control, also referred to as HUM control, can be characterized as the minimizer of a suitable functional. Let us introduce the functional $\mathcal{J} : \mathcal{H} \rightarrow \mathbb{R}$ by

$$\mathcal{J}((\overline{w}^0, \overline{w}^1)) = \frac{1}{2} \int_0^T |\overline{w}_t(1, t)|^2 dt + \int_0^1 w^1(x)\overline{w}^1(x) dx + \int_0^1 w^0_0(x)\overline{w}_2^0(x) dx \forall (\overline{w}^0, \overline{w}^1) \in \mathcal{H}.$$ 

It is well known that the observability inequality of system (6.4) holds true:

$$\overline{E}(0) \leq \frac{1}{2(T-2)} \int_0^T |\overline{w}_t(1, t)|^2 dt, \quad \overline{E}(t) = \frac{1}{2} \int_0^1 |\overline{w}_x^2(x, t) + \overline{w}_t^2(x, t)| dx,$$

which means that system (1.1) is exactly observable over $[0, T], T > 2$, and ensures that $\mathcal{J}(\cdot)$ is coercive and thus has a unique minimizer.

**Lemma 6.2.** For given initial state $(w^0, w^1) \in \mathcal{H}$, suppose that $(\widehat{w}^0, \widehat{w}^1) \in \mathcal{H}$ is the minimizer of $\mathcal{J}(\cdot)$. Then, $U(t) = \widehat{w}_t(1, t)$ is the HUM control of system (6.1) with property (6.2), where $(\widehat{w}(x, t), \widehat{w}_t(x, t))$ is the solution of adjoint system (6.4) with the initial value $(\widehat{w}^0(x), \widehat{w}^1(x))$.

6.2. Uniform observability inequality. Parallel to finite difference scheme (5.1), the semidiscretized finite difference scheme of control system (6.1) and adjoint system (6.4) reads

$$\begin{align*}
&\frac{1}{4} (w''_{j-1}(t) + 2w''_j(t) + w''_{j+1}(t)) - \delta^2_x w_j(t) = 0, \quad j = 1, 2, \ldots, N, \\
&\frac{h}{4} (w''_j(t) + w''_{N+1}(t)) + \delta_x w_{N+\frac{1}{2}} = U_h(t), \\
w_0(t) = 0, \quad w_j(0) = w^0(x_j), \quad w'_j(0) = w^1(x_j), \quad j = 0, 1, \ldots, N + 1,
\end{align*}$$
and
\[
\begin{aligned}
\frac{1}{4} \left( \overline{\omega}_{j-1}''(t) + 2\overline{\omega}_j''(t) + \overline{\omega}_{j+1}''(t) \right) - \delta_2^2 \overline{\omega}_j(t) = 0, & \quad j = 1, 2, \ldots, N, \\
\frac{h}{4} \left( \overline{\omega}_j''(t) + \overline{\omega}_{N+1}''(t) \right) + \delta_x \overline{\omega}_{N+\frac{1}{2}} = 0, \\
\overline{\omega}_0(t) = 0, & \quad \overline{\omega}_j(0) = \overline{\omega}^0(x_j), \quad \overline{\omega}_j'(0) = \overline{\omega}^1(x_j), \quad j = 0, 1, \ldots, N + 1,
\end{aligned}
\]

(6.8)

respectively. The discrete energy \( \tilde{E}_h(t) \) is also defined for (6.8) paralleling as (5.12), which is conservative,
\[
\frac{d\tilde{E}_h(t)}{dt} = 0,
\]

similar to that of Lemma 5.2 by setting \( \alpha = 0 \) in (5.13).

Theorem 6.3 means that the observability inequality (6.6) is uniformly preserved by the semidiscrete scheme (5.1), which is the discrete counterpart of the observation problem of original system (1.1).

**Theorem 6.3.** Let \( T > 2 \) and \( \tilde{E}_h(t) \) be the discrete energy corresponding to (6.8). Then,
\[
\tilde{E}_h(0) \leq \frac{1}{2(T - 2)} \int_0^T |\overline{\omega}_{N+1}'(t)|^2 dt.
\]

**Proof.** Multiplying the first equation of (6.8) by \( jh(\overline{\omega}_{j+1} - \overline{\omega}_{j-1})/2 \) and summing for \( j \) from 1 to \( N \) give
\[
0 = \frac{h}{4} \sum_{j=1}^N (\overline{\omega}_{j-1}'' + 2\overline{\omega}_j'' + \overline{\omega}_{j+1}'') \frac{\overline{\omega}_{j+1} - \overline{\omega}_{j-1}}{2} - \frac{1}{h} \sum_{j=1}^N (\overline{\omega}_{j-1} - 2\overline{\omega}_j + \overline{\omega}_{j+1}) j \frac{\overline{\omega}_{j+1} - \overline{\omega}_{j-1}}{2}
\]
\[
= \frac{d}{dt} \frac{h}{4} \sum_{j=1}^N (\overline{\omega}_{j-1}'' + 2\overline{\omega}_j'' + \overline{\omega}_{j+1}'') \frac{\overline{\omega}_{j+1} - \overline{\omega}_{j-1}}{2} + \frac{h}{8} \sum_{j=0}^N (\overline{\omega}_j'' + \overline{\omega}_{j+1}'')^2 - \frac{1}{8} (\overline{\omega}_N'' + \overline{\omega}_{N+1}'')^2
\]
\[
+ \frac{h}{2} \sum_{j=0}^N \left( \frac{\overline{\omega}_{j+1} - \overline{\omega}_j}{h} \right)^2 - \frac{1}{2} \left( \frac{\overline{\omega}_{N+1} - \overline{\omega}_N}{h} \right)^2.
\]

Multiply the second equation of system (6.8) by \((N + 1)(\overline{\omega}_{N+1} - \overline{\omega}_N)\) to give
\[
0 = \frac{d}{dt} \frac{1}{4} (\overline{\omega}_N' + \overline{\omega}_{N+1}'')(\overline{\omega}_{N+1} - \overline{\omega}_N) - \frac{1}{4} (\overline{\omega}_N'' + \overline{\omega}_{N+1}'')(\overline{\omega}_{N+1} - \overline{\omega}_N')
\]
\[
+ \left( \frac{\overline{\omega}_{N+1} - \overline{\omega}_N}{h^2} \right)^2.
\]

Adding (6.11) with (6.12), we obtain
\[
\frac{dX_h(t)}{dt} + \tilde{E}_h(t) - \frac{1}{8} (3|\overline{\omega}_{N+1}'|^2 + 2\overline{\omega}_N'\overline{\omega}_{N+1}' - |\overline{\omega}_N'|^2) + \frac{1}{2} \left( \frac{\overline{\omega}_{N+1} - \overline{\omega}_N}{h} \right)^2 = 0,
\]

(6.13)

where
\[
X_h(t) = \frac{h}{4} \sum_{j=1}^N (\overline{\omega}_{j-1}'' + 2\overline{\omega}_j'' + \overline{\omega}_{j+1}'') \frac{\overline{\omega}_{j+1} - \overline{\omega}_{j-1}}{2} + \frac{1}{4} (\overline{\omega}_N'' + \overline{\omega}_{N+1}'')(\overline{\omega}_{N+1} - \overline{\omega}_N).
\]
Since
\[
|X_h(t)| \leq h \sum_{j=1}^{N} \left( \frac{w'_{j-1} + 2w'_j + w'_{j+1}}{4} \right) \left| \frac{w_{j+1} - w_j}{2h} \right| + \frac{h}{2} \left| \frac{w_N' + w'_{N+1}}{2} \right| \left| \frac{w_{N+1} - w_N}{h} \right|
\]
and
\[
\leq \frac{h}{2} \sum_{j=1}^{N} \left( \frac{w'_{j-1} + 2w'_j + w'_{j+1}}{4} \right)^2 + \frac{h}{2} \sum_{j=1}^{N} \left| \frac{w_{j+1} - w_j}{2h} \right|^2 + \frac{h}{4} \left| \frac{w_N' + w'_{N+1}}{2} \right|^2
\]
integrating both sides of (6.13) from 0 to T with respect to time produces
\[
0 \geq X_h(t) \bigg|_0^T + \int_0^T E_h(t) dt - \frac{1}{2} \int_0^T |w'_{N+1}|^2 dt \geq (T - 2) E_h(0) - \frac{1}{2} \int_0^T |w'_{N+1}|^2 dt,
\]
where we used the fact (6.9). This gives the uniform observability inequality (6.10).

### 6.3. Uniform controllability

For notational simplicity, we introduce matrices
\[ P, B \in \mathcal{M}^{(N+1) \times (N+1)} \] as
\[
P = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \cdots & \cdots \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ -1 & 2 & -1 \\ -1 & 1 & 1 & \cdots & 1 \end{pmatrix}.
\]
Let \( P_h = hP \) and \( B_h = \frac{1}{h}B \). Then, the control systems (6.7) and adjoint system (6.8) can be rewritten respectively into the following vectorial forms:
\[
\begin{cases}
P_h W_h''(t) + B_h W_h(t) = F_h(t), \\
W_h(0) = W_h^0, \quad W_h'(0) = W_h^1,
\end{cases}
\]
and
\[
\begin{cases}
P_h \bar{W}_h''(t) + B_h \bar{W}_h(t) = 0, \\
\bar{W}_h(0) = \bar{W}_h^0, \quad \bar{W}_h'(0) = \bar{W}_h^1,
\end{cases}
\]
where \( W_h(t) = (w_1(t), w_2(t), \ldots, w_{N+1}(t))^T, \) \( W_h^0 = (w_1^0, w_2^0, \ldots, w_{N+1}^0)^T, \) \( W_h^1 = (w_1^1, w_2^1, \ldots, w_{N+1}^1)^T, \) \( F_h(t) = (0, 0, \ldots, U_h(t))^T \) and similarly for \( \bar{W}_h(t), \) \( \bar{W}_h^0, \) and \( \bar{W}_h^1. \)
The uniform observability (6.10) shows that for any \( h > 0 \), system (6.15) is exactly controllable over any \([0, T], T > 2\). Precisely, for any initial state \((W_0^0, W_1^1) \in \mathbb{R}^{2N+2}\) and \( T > 2 \), there exists a control \( F_h(t) \in L^2(0, T) \) such that

\[
W_h(T) = W_0^0(T) = 0.
\]

Let \( \langle \cdot, \cdot \rangle \) be the canonical inner product in \( \mathbb{R}^{N+1} \). Multiplying (6.15) by the solution \( W'_h \) of system (6.16) and integrating in time over \([0, T]\), we obtain

\[
\int_0^T \langle F_h, W'_h \rangle dt = \int_0^T ([P_h W''_h, W'_h] + \langle B_h W_h, W'_h \rangle) dt
\]

\[
= \langle P_h W'_h, W'_h \rangle|_0^T + \langle B_h W_h, W'_h \rangle|_0^T - \int_0^T ([\langle P_h W'_h, W''_h \rangle + \langle B_h W'_h, W''_h \rangle) dt
\]

\[
= \langle P_h W'_h, W'_h \rangle|_0^T + \langle B_h W_h, W'_h \rangle|_0^T - \int_0^T ([\langle W'_h, P_h \rangle W''_h + \langle W'_h, B_h \rangle W'_h)] dt
\]

\[
= \langle P_h W'_h(T), W'_h(T) \rangle - \langle P_h W'_h, W'_h \rangle + \langle B_h W_h(T), W'_h(T) \rangle - \langle B_h W'_h, W'_h \rangle.
\]

Since (6.17), it follows from (6.18) that

\[
\int_0^T U_h(t) W_{N+1}^0(t) dt + \langle P_h W'_h, W'_h \rangle + \langle B_h W'_h, W'_h \rangle = 0,
\]

which is just the condition that the discrete control \( U_h(t) \) should satisfy to achieve (6.17).

Let us introduce the inner product \( \langle \cdot, \cdot \rangle_0 \) in \( \mathbb{R}^{2N+2} \)

\[
\langle (f^0, f^1), (g^0, g^1) \rangle_0 = \langle P_h f^1, g^1 \rangle + \langle B_h f^0, g^0 \rangle, \quad f^i, g^i \in \mathbb{R}^{N+1}, i = 0, 1,
\]

and the corresponding inner product induced norm is \( \| \cdot \|_0 \). From the definition of \( \langle \cdot, \cdot \rangle_0 \), we can check that the energy of adjoint system (6.16) satisfies

\[
E_h(0) = \frac{1}{2} \left\| \left( \frac{W_0^0}{W'_h}, \frac{W_1^1}{W'_h} \right) \right\|_0^2.
\]

Introduce functional \( J_h : \mathbb{R}^{2N+2} \to \mathbb{R} \) by

\[
J_h \left( \left( \frac{W_0^0}{W'_h}, \frac{W_1^1}{W'_h} \right) \right) = \frac{1}{2} \int_0^T |W'_{N+1}|^2 dt + \langle P_h W'_h, W'_h \rangle + \langle B_h W'_h, W'_h \rangle,
\]

which is the discrete counterpart of continuous functional (6.5).

**Lemma 6.4.** The functional \( J_h(\cdot) \) defined by (6.21) has a unique minimizer \( (W_0^0, W_1^1) \).

**Proof.** First, we observe that \( J_h(\cdot) \) is strictly convex in \( \mathbb{R}^{2N+2} \) and hence is continuous. Next, we check that \( J_h \) is coercive, i.e.,

\[
\lim_{\| (W_0^0, W_1^1) \|_0 \to \infty} J_h \left( \left( \frac{W_0^0}{W'_h}, \frac{W_1^1}{W'_h} \right) \right) = +\infty.
\]
Indeed, from observability inequality (6.10), we have

\[
\mathcal{J}_h\left((W_h^0, W_h^1)\right) = \frac{1}{2} \int_0^T |\widehat{w}_{N+1}(t)|^2 dt + \langle (W_h^0, W_h^1), (\tilde{W}_h^0, \tilde{W}_h^1) \rangle_0 \\
\geq \frac{T}{2} \| (W_h^0, W_h^1) \|^2_0 - \| (W_h^0, W_h^1) \|_0 \| (\tilde{W}_h^0, \tilde{W}_h^1) \|_0,
\]

which implies that \( \mathcal{J}_h((W_h^0, W_h^1)) \to \infty \) as \( \| (W_h^0, W_h^1) \|_0 \to +\infty \). Therefore, \( \mathcal{J}_h(\cdot) \) has a unique minimizer \((\tilde{W}_h^0, \tilde{W}_h^1)\).

Theorem 6.5 means that the semidiscrete scheme (5.1) preserves uniformly the exact controllability of original system (1.1).

**Theorem 6.5.** Suppose that \((\tilde{W}_h^0, \tilde{W}_h^1)\) is the unique minimizer of \( \mathcal{J}_h(\cdot) \). Then, 

\[ U_h(t) = \widehat{w}_{N+1}(t) \]

is the control of system (6.15) satisfying property (6.17), where 

\[(\tilde{W}_h^0(t), \tilde{W}_h^1(t))\]

is the solution of adjoint system (6.16) with the initial state \((\tilde{W}_h^0, \tilde{W}_h^1)\).

Furthermore,

\[
\| U_h \|_{L^2(0,T)} \leq \sqrt{T - 2} \| (W_h^0, W_h^1) \|_0
\]

and

\[
\| (\tilde{W}_h^0, \tilde{W}_h^1) \|_0 \leq \frac{2}{T - 2} \| (W_h^0, W_h^1) \|_0.
\]

**Proof.** The fact that \( \widehat{w}_{N+1}(t) \) is just the control that drives the state of (6.15) from \((W_h^0, W_h^1)\) to rest at time \( T \) is the counterpart of Lemma 6.2 for discrete system (6.15), which comes from general theory of finite-dimensional linear systems; see, e.g., [26, pp. 200–201]. We only need to show that the control \( U_h(t) \) is bounded in \( L^2(0, T) \). Since \((\tilde{W}_h^0, \tilde{W}_h^1)\) is a minimizer of \( \mathcal{J}_h(\cdot) \), it has

\[
\mathcal{J}_h((\tilde{W}_h^0, \tilde{W}_h^1)) \leq \mathcal{J}_h((0,0)) = 0.
\]

By observability inequality (6.10), we have

\[
\frac{1}{2} \int_0^T |\widehat{w}_{N+1}^\prime|^2 dt \leq -\langle (W_h^0, W_h^1), (\tilde{W}_h^0, \tilde{W}_h^1) \rangle_0 \leq \| (W_h^0, W_h^1) \|_0 \| (\tilde{W}_h^0, \tilde{W}_h^1) \|_0
\]

\[
= \sqrt{2 E_h^0(0)} \| (W_h^0, W_h^1) \|_0 \leq \sqrt{T - 2} \left( \int_0^T |\widehat{w}_{N+1}^\prime|^2 dt \right)^{\frac{1}{2}} \| (W_h^0, W_h^1) \|_0,
\]

which implies that

\[
\| U_h \|_{L^2(0,T)} = \left( \int_0^T |\widehat{w}_{N+1}^\prime|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{4(T - 2)} \| (W_h^0, W_h^1) \|_0.
\]

Furthermore, from (6.25) and observability inequality (6.10), we obtain

\[
\frac{T - 2}{2} \| (\tilde{W}_h^0, \tilde{W}_h^1) \|_0^2 = (T - 2) E_h^0(0) \leq \frac{1}{2} \int_0^T |\widehat{w}_{N+1}^\prime|^2 dt \leq \| (W_h^0, W_h^1) \|_0 \| (\tilde{W}_h^0, \tilde{W}_h^1) \|_0,
\]

which implies that

\[
\| (\tilde{W}_h^0, \tilde{W}_h^1) \|_0 \leq 4 C_T \| (W_h^0, W_h^1) \|_0.
\]

This ends the proof of the theorem. \( \square \)
7. Numerical experiments. For a numerical experiment, we also need to discretize the time derivative in our semidiscretized finite difference schemes. We carry out a numerical experiment for fully discretized finite difference schemes of both (2.5) and (5.1), separately.

7.1. Full-discretization scheme for (2.5). First, we rewrite the semidiscretized system (2.5) into a vector form

\[
\begin{align*}
W'(t) &= \frac{2}{h} M^{-1} NW(t), \\
W(0) &= W^0,
\end{align*}
\]

where \( W(t) = (U(t)^T, V(t)^T)^T, U(t) = (u_0(t), u_1(t), \ldots, u_N(t))^T, V(t) = (v_1(t), v_2(t), \ldots, v_{N+1}(t))^T, \)
\[
M = \begin{pmatrix} A & C \\ O & A^T \end{pmatrix}, \quad N = \begin{pmatrix} O & B \\ -B^T & C \end{pmatrix}.
\]

The matrices \( A, B, C, O \in \mathcal{M}^{(N+1)\times(N+1)} \), where \( O \) is a zero matrix and \( A, B, C \) read

\[
A = \begin{pmatrix} 1 & 1 \\ \vdots & \ddots & \ddots \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 & 1 \\ \vdots & \ddots & \ddots \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.
\]

Introducing a parameter \( \theta \in [0, 1] \), we can get the following fully discretized finite difference scheme:

\[
\frac{W^{n+1} - W^n}{\tau} = \frac{2}{h} (1 - \theta) M^{-1} NW^n + \frac{2}{h} \theta M^{-1} NW^{n+1},
\]

where \( \tau \) is the time step. If we let the mesh grid ratio \( \lambda = \tau/h \) (the CFL condition number), we can get the computational form of \( W^{n+1} \)

\[
(I - 2\lambda \theta M^{-1} N)W^{n+1} = (I + 2\lambda (1 - \theta) M^{-1} N)W^n,
\]

where \( I \) is the \( 2N + 2 \) order identity matrix. From (7.2) we see that if \( \theta = 0 \), it is an explicit difference scheme. If \( 0 < \theta \leq 1 \), it is an implicit difference scheme. In the simulation process, we find that the proposed finite difference (7.2) is unconditionally stable when \( \theta \in [0.5, 1] \). When \( \theta \in [0, 0.5] \), the finite difference (7.2) is conditionally stable, although we cannot obtain the sharp CFL condition number which depends on \( \theta \) and \( \alpha \).

Now we set the initial conditions \((u^0(x), v^0(x))\) of system (2.5) to check the exponential decay of difference scheme (7.2). Here we take

\[
u^0(x) = 1 - |2x - 1|, \quad v^0(x) = 0, \quad x \in (0, 1).
\]

The logarithmic energy decay curves with \( \theta = 0.4, \theta = 0.5 \) and \( \theta = 1 \) are depicted in Figures 1, 2, and 3, respectively.

It is seen in Figure 1 that the difference scheme (7.2) is conditionally stable when \( \theta = 0.4 \). A smaller CFL condition number gives better stability of the scheme. If the
CFL condition number $\lambda = 1$, it diverges to infinity as $N$ goes to infinity. Figures 2 and 3 show that if $\theta \in [0.5, 1]$, the difference scheme is unconditionally stable. In each subfigure of Figures 2 and 3, we can see that the discretized energy decays nearly at the same exponential decay rate for $N = 20, 40, 60,$ and $80$.

7.2. Full-discretization scheme for (5.1). The semidiscretized finite difference scheme (5.1) can be rewritten into a vector form
Also let mesh grid ratio \( \lambda = \tau / h \). Then, (7.5) implies that

\[
\begin{align*}
PW''(t) + \frac{1}{h^2} BW(t) + \frac{1}{h} AW'(t) &= 0, \\
W(0) &= W^0, \quad W'(0) = W^1,
\end{align*}
\]

where \( W(t) = (w_1(t), w_2(t), \ldots, w_{N+1}(t))^T, \; W^0 = (w^0_1, w^0_2, \ldots, w^0_{N+1})^T, \; W^1 = (w^1_1, w^1_2, \ldots, w^1_{N+1})^T, \; P, B \) are as defined in (6.14), and the \( N + 1 \) order diagonal matrix \( A = \text{diag}\{0, \ldots, 0, \alpha\} \). Let the time step be \( \tau \), and discretize the time derivative by the central difference operator and take an average operator for the space direction, to produce the following full-discretization finite difference scheme:

\[
P\delta^2 W^k + \frac{1}{4h^2} B (W^{k-1} + 2W^k + W^{k+1}) + \frac{1}{2\tau h} A (W^{k+1} - W^{k-1}) = 0, \quad k = 1, 2, \ldots
\]

Also let mesh grid ratio \( \lambda = \tau / h \). Then, (7.5) implies that

\[
\left(P + \frac{\lambda^2}{4} B + \frac{\lambda}{2} A\right) W^{k+1} = \left(2P - \frac{\lambda^2}{2} B\right) W^k - \left(P + \frac{\lambda^2}{4} B - \frac{\lambda}{2} A\right) W^{k-1}, \quad k = 1, 2, \ldots
\]

which is the actual difference scheme for our numerical experiment. We set the initial conditions \((w^0(x), w^1(x))\) of system (5.1) as

\[
w^0(x) = 1 - |2x - 1|, \quad w^1(x) = 0, \quad x \in (0, 1),
\]

to check the exponential decay of difference scheme (7.6). The logarithmic energy decay curves with \( \alpha = 2 \) and \( \lambda = 1, 2 \) are displayed in Figure 4, which shows both the unconditional stability and the uniformly exponential decay of difference scheme (7.6).

8. Concluding remarks. In this paper, a novel space semidiscretized finite difference scheme is proposed for approximating the exponentially stable one-dimensional boundary damped wave equation uniformly. The original system is first transformed into an equivalent system by a universally used equivalent transformation for wave equations. An ordinary semidiscretized difference scheme is then constructed for the equivalent system. The convergence of the difference scheme to the equivalent system is shown to be of second-order convergence. The uniformly exponential decay is established for the semidiscretized finite difference scheme. Of equal significance, the
A semidiscretized difference scheme for the equivalent system is found to be a widely used order reduction finite difference scheme to the original system. Both uniformly exponential decay and the convergence of the solutions are established rigorously for this new discrete scheme to the original system. A numerical experiment is conducted to verify the theoretical analysis under two different implicit time finite difference schemes. In contrast to other fully discretized finite difference schemes of preserving the uniformly exponential decay, the proposed implicit difference scheme possesses the unconditional stability of the finite difference scheme and the uniformly exponential decay of discretized energy, without using any numerical viscosity. In addition, this finite difference semidiscrete scheme also preserves two additional important properties of uniformly exact observability and exact controllability.

This new finite difference scheme provides a basic principle for numerical computation of PDE control systems such as wave equations and any other types of PDEs. There are at least four good features for this new scheme: (a) It preserves the uniform exponential stability and other control properties such as uniformly observability and controllability; (b) it can be used to deal with any other boundary conditions; (c) the proof for the semidiscretized finite difference system is completely analogous to the continuous counterpart; and (d) it naturally keeps the finite difference nature. We believe that this scheme can be used to study the convergence of the minimal energy control for driving a given state to the origin in finite time, which is our future work. It particularly tells what a proper finite difference scheme should be when one needs to approximate PDEs in numerical simulations or practical implementation.

REFERENCES