H∞ Fuzzy Control for Nonlinear Fourth-Order Parabolic Equation Subject to Input Delay

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Abstract—This article discusses H∞ fuzzy control for the nonlinear fourth-order parabolic equation with input delay via collocated actuator/sensor pairs. We suggest that the interval [0, 1] is divided into M subdomains, where sensors provide spatially averaged/point discrete-time state measurements. The control design strategy is proposed based on output measurements. We derive constructive conditions ensuring that the resulting closed-loop system is internally exponentially stable and has $H_{\infty}$ performance by means of the Lyapunov approach.

Index Terms—Fuzzy model, $H_{\infty}$ control, linear matrix inequalities (LMIs), nonlinear fourth-order parabolic equation.

I. INTRODUCTION

SPURRED by practical applications, the systems described by fourth-order partial differential equations (PDEs) have become an active research object. Such infinite-dimensional systems can be used to model many physical phenomena in engineering, biological mathematics, and materials science. The case of the fourth-order Schrödinger equation has been investigated in [1]. Stabilization for a class of fourth-order parabolic equation described by the Kuramoto–Sivashinsky equation (KSE) under periodic boundary conditions [2]–[5] or Dirichlet boundary conditions [5], [6] has been initiated from several aspects. In [2] and [3], based on a finite-dimensional system that captures the dominant (slow) dynamics of the infinite-dimensional system, a finite-dimensional controller has been introduced. These results aforementioned are confined to the boundary/distributed control of the fourth-order PDEs, yet many challenging problems are still open.

II. MATHEMATICAL PRELIMINARIES

Notation: Throughout this article, the $L^2(0, 1)$ stands for the Hilbert space of square integrable functions on the interval [0, 1].
vector functions $\psi(x) : [0, 1] \rightarrow \mathbb{R}^n$ with the norm $\|\psi\|_{L^2_2(x)}^2 = \int_0^1 |\psi(x)|^2 dx$. $H^k_n(0, 1)$ is the Sobolev space defined as $\|\psi\|_{H^k_n(0, 1)}^2 = \left\{ \sum_{0 \leq |a| \leq k} \|D^a \psi\|_{L^2_2(x)}^2 \right\}^{1/2}$ with $k \in \mathbb{Z}$. The symbol $*$ is used as an ellipsis for terms in matrix expressions induced by the symmetry. $I$ stands for the identity matrix with appropriate dimension, conv$(\cdot)$ stands for the convex hull. Denote the support of a function $f$ by supp $f$.

Some preliminary results are introduced in the following lemmas.

**Lemma 1:** Let $\Omega = [a, b]$ and let $\psi : \Omega \rightarrow \mathbb{R}^n$ and $\psi \in H^1_n(a, b)$.

1. **Agmon’s Inequality [41]:** If $\psi(a) = \psi(b) = 0$, then $\|\psi\|_{L^2_2(a, b)}^2 \leq 2\|\psi\|_{L^2_2(a, b)}\|\psi_s\|_{L^2_2(a, b)} \leq \|\psi\|_{H^1_n(a, b)}^2$.

2. **Poincaré’s Inequality [42]:** If $\int_a^b \psi(x)dx = 0$, then

$$\int_a^b \psi^T(x)Q\psi(x)dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b \psi_s^T(x, t)Q_s(x, t)dx$$

for any $Q \geq 0$.

3. **Wirtinger’s Inequality [43]:** If $\psi(a)$ or $\psi(b) = 0$, then

$$\int_a^b \psi^T(x)Q\psi(x)dx \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \psi_s^T(x)Q_s(x)dx$$

for any $Q \geq 0$. Furthermore, if $\psi(a) = \psi(b) = 0$, then

$$\int_a^b \psi^T(x)Q\psi(x)dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b \psi_s^T(x)Q_s(x)dx$$

for any $Q \geq 0$.

**Lemma 2 (Jensen’s Inequality [44]):** Let $\Omega = [a, b]$. Assume $\psi : \Omega \rightarrow \mathbb{R}^n$ and $\psi \in L^2_2(a, b)$. Then, the following inequality holds:

$$\int_a^b \psi^T(x)Q\psi(x)dx \geq -\frac{1}{b-a} \left( \int_a^b \psi^T(x)dx \right) Q \left( \int_a^b \psi(x)dx \right)$$

for any matrix $Q \geq 0$.

### III. Problem Formulation

We consider the following fourth-order parabolic equation with input delay and disturbance:

$$w_t(x, t) + v w_{xxxx}(x, t) + w_{xx}(x, t) + f(w(x, t)) + \sum_{j=1}^M f_j(x) u_j(t-h) + p(x, t-h) = 0$$

$$(x, t) \in (0, 1) \times (0, \infty)$$

$$w(0, t) = w(1, t) = 0$$

$$w_x(0, t) = w_x(1, t) = 0$$

$$w(x, 0) = w_0(x)$$

where $v > 0$, $h > 0$ is the time delay, $w(x, t) \triangleq \begin{bmatrix} w_1(x, t) & w_2(x, t) & \ldots & w_n(x, t) \end{bmatrix}^T \in \mathbb{R}^n$, and the delayed actuators are denoted by $u(t) \triangleq \begin{bmatrix} u_1(t) & u_2(t) & \ldots & u_n(t) \end{bmatrix}^T \in \mathbb{R}^n$, $(j = 1, \ldots, M)$. The $p(x, t)$ is a distributed perturbation, and $f(\cdot)$ is locally Lipschitz continuous and satisfies $f(0) = 0$.

Assumption 1: It is supposed that $p \in L^2(0, \infty; L^2_2(0, 1))$ and $p(\cdot, t) = 0, \ t \leq 0$.

Motivated by [4] and [38]–[40], we divide $[0, 1]$ into $M$ sampling subintervals $\Omega_j = [x_{j-1}, x_j]$ covering the whole domain (see Fig. 1), where $0 = x_0 < x_1 < \cdots < x_M = 1$

and the width of the each interval is supposed to be upper bounded by some constant $\Delta$

$$0 < x_j - x_{j-1} = \Delta_j \leq \Delta.$$

The spatial characteristic functions are taken as

$$F_j(x) = \begin{cases} 1, & x \in \Omega_j \\
0, & x \notin \Omega_j \end{cases} \quad j = 1, \ldots, M.$$

We further assume that the sensors provide point measurements of the state

$$y_j(t) = w(\bar{x}_j, t), \bar{x}_j = \frac{x_{j-1} + x_j}{2}, \quad j = 1, \ldots, M$$

or averaged measurements of the state

$$\bar{y}_j(t) = \frac{\int_{x_{j-1}}^{x_j} w(x, t)dx}{\Delta_j}, \quad j = 1, \ldots, M.$$

The objective is to construct a stabilizing distributed $H_{\infty}$ fuzzy control for system (1).

### IV. Main Results

In order to accurately represent the nonlinear fourth-order PDEs (1), a T–S fuzzy PDE model is employed. After that, a stabilizing $H_{\infty}$ fuzzy controller is proposed. Then, the LMI-based conditions ensuring the internal exponential stability and $H_{\infty}$ performance of the closed-loop system will be finally investigated.

#### A. $H_{\infty}$ Fuzzy Control Law

In this section, by use of the sector nonlinearity method, the following T–S fuzzy PDE model is constructed via fuzzy IF-THEN rules.

**Plant Rule $\alpha$:** IF $\Gamma_1(x, t)$ is $F_{a1}$, $\Gamma_2(x, t)$ is $F_{a2}$, \ldots, and $\Gamma_\beta(x, t)$ is $F_{a\beta}$, THEN

$$\begin{aligned}
&\begin{cases}
\begin{aligned}
w_t(x, t) + v w_{xxxx}(x, t) + w_{xx}(x, t) + A_w w(x, t) + \sum_{j=1}^M f_j(x) u_j(t-h) + p(x, t-h) &= 0 \\
w(0, t) = w(1, t) = 0 \\
w_x(0, t) = w_x(1, t) = 0 \\
w(x, 0) = w_0(x)
\end{aligned}
\end{cases}
\end{aligned}$$

\text{(4)}
where $A_{\alpha} \in \mathbb{R}^{m \times n}$, $\alpha \in \mathcal{S} \triangleq \{1, 2, \ldots, r\}$, and $r$ represents the number of IF-THEN fuzzy rules, and $F_{a1}, F_{a2}, \ldots, F_{arb}$ are the fuzzy sets. The premise variables $\Gamma_{g}(x, t)$, $g = 1, 2, \ldots, \beta$ are assumed to be functions of $w(x, t)$. The overall fuzzy PDE (1) can therefore be rewritten as

\[
\begin{align*}
    w_{1}(x, t) &+ v w_{xxx}(x, t) + w_{xt}(x, t) \\
    + \sum_{\alpha=1}^{r} h_{\alpha}(\Gamma(x, t)) A_{\alpha} w(x, t) + \sum_{j=1}^{M} F_{j}(x) u_{j}(t-h) \\
    + p(x, t-h) &= 0 \\
    w(0, t) &= w(1, t) = 0 \\
    w_{x}(0, t) &= w_{x}(1, t) = 0 \\
    w(x, 0) &= w_{0}(x)
\end{align*}
\]

where $\Gamma(x, t) = [\Gamma_{1}(x, t) \Gamma_{2}(x, t) \cdots \Gamma_{\beta}(x, t)]^{T}$, $h_{\alpha}(\Gamma(x, t)) = \frac{\prod_{i=g}^{\beta} F_{ag}(\Gamma_{i}(x, t))}{\sum_{g=1}^{\beta} \prod_{i=g}^{\beta} F_{ag}(\Gamma_{i}(x, t)), \alpha \in \mathcal{S}}$ and $F_{ag}$ represents the grade of the membership of $\Gamma_{g}(x, t)$ in $F_{ag}$ for $\alpha \in \mathcal{S}$. Assume that

\[
\prod_{g=1}^{\beta} F_{ag}(\Gamma_{g}(x, t)) \geq 0, \quad \alpha \in \mathcal{S}
\]

and

\[
\sum_{\alpha=1}^{r} \prod_{g=1}^{\beta} F_{ag}(\Gamma_{g}(x, t)) > 0.
\]

Then

\[
h_{\alpha}(\Gamma(x, t)) \geq 0, \quad \alpha \in \mathcal{S}, \quad \sum_{\alpha=1}^{r} h_{\alpha}(\Gamma(x, t)) = 1.
\]

From the PDC scheme, the following control strategy is proposed for the fuzzy model (4):

Control Rule $\alpha$: IF $\Gamma_{1}(x, t)$ is $F_{a1}$, $\Gamma_{2}(x, t)$ is $F_{a2}$, \ldots and $\Gamma_{\beta}(x, t)$ is $F_{arb}$, THEN

\[
u(t) = \begin{cases} K_{\alpha} y(t), & t > 0 \\ 0, & t \leq 0 \end{cases}
\]

where $K_{\alpha}$ are fuzzy control gain matrices to be determined hereafter, and $y(t)$ is defined by (2) or (3). Denote the characteristic function of the time interval $[0, h]$ by $\chi_{[0,h]}(t)$. Then

\[
u(t) = \sum_{\alpha=1}^{r} h_{\alpha}(\Gamma(x, t)) K_{\alpha} y(t), \quad t > 0 \\
0, \quad t \leq 0
\]

which leads to the closed-loop system

\[
\begin{align*}
    w_{1}(x, t) &+ v w_{xxx}(x, t) + w_{xt}(x, t) \\
    + \sum_{\alpha=1}^{r} h_{\alpha}(\Gamma(x, t)) A_{\alpha} w(x, t) \\
    + \sum_{\alpha=1}^{r} \sum_{j=1}^{M} F_{j}(x)(1 - \chi_{[0,h]}(t)) h_{\alpha}(\Gamma(x, t)) K_{\alpha} \\
    &\times [w(x, t-h) - p_{j}(x, t-h)] + p(x, t-h) = 0 \\
    w(0, t) &= w(1, t) = 0 \\
    w_{x}(0, t) &= w_{x}(1, t) = 0 \\
    w(x, 0) &= w_{0}(x)
\end{align*}
\]

where for (2)

\[
\rho_{j}(x, t-h) = \int_{t-h}^{t} w_{\xi}(\xi, t-h) d\xi
\]

and for (3)

\[
\rho_{j}(x, t-h) = \frac{\int_{t-h}^{t} [w(x, t-h) - w(\xi, t-h)] d\xi}{\Delta \tau}.
\]

In the course of the internally stabilizing the process, the influence of the admissible external disturbance $p(x, t)$ on the controlled output

\[
\eta(x, t) = C_{w}(x, t)
\]

with a constant matrix $C \in \mathbb{R}^{m \times n}$ to be attenuated.

**Definition 1:**

1. The system (9) subject to (10) or (11) is internally exponentially stable if it is exponentially stable with $p \equiv 0$.
2. The system (9) subject to (10) or (11) has an $L_{2}$-gain ($H_{\infty}$ gain) less than $\gamma$ if for $w_{0}(x) \equiv 0$ and all admissible external disturbances $p(x, t)$ satisfying Assumption 1, the following relation holds on the trajectories of (9), (10), or (11):

\[
J(T) = \int_{0}^{T} \left[ \|\eta(x, t)\|_{L_{2}^{2}(0, 1)}^{2} - \gamma^{2} \|p(x, t-h)\|_{L_{2}^{2}(0, 1)}^{2} \right] dt < 0
\]

for all $T > h$.

Our aim is to find conditions so that:

1. the system (9) subject to (10) or (11) is internally exponentially stable;
2. the system (9) subject to (10) or (11) has an $L_{2}$-gain ($H_{\infty}$ gain) less than $\gamma$.

### B. Well-Posedness

Define the Hilbert space $H_{\infty}^{2}(0, 1) = \{ \psi \in H_{\infty}^{2}(0, 1) : \psi(0) = \psi(1) = \psi'(0) = \psi'(1) = 0 \}$ and define the spatial differential operator $A: D(A) \subset L_{2}^{2}(0, 1) \rightarrow L_{2}^{2}(0, 1)$ as follows:

\[
\begin{align*}
    &D(A) = H_{\infty}^{2}(0, 1) \cap H_{0}^{2}(0, 1). \\
    &\text{The operator } A \text{ is dissipative and generates an analytic } C_{0} \text{ -semigroup } e^{At}. \text{ Since } -A \text{ is positive, } (-A)^{1/2} \text{ is positive, and } \text{D}((-A)^{1/2}) = H_{\infty}^{2}(0, 1). \text{ We then represent (9) as}
\end{align*}
\]

\[
\begin{align*}
    &\frac{d}{dt} Y(t) = A Y(t) + F(Y(t)) \\
    &Y(0) = w_{0}
\end{align*}
\]

where $Y(t) = w(\cdot, t)$

\[
F(w(\cdot, t)) = -w_{xx}(\cdot, t) - \sum_{\alpha=1}^{r} h_{\alpha}(\Gamma(\cdot, t)) A_{\alpha} w_{x}(\cdot, t) \\
- \sum_{\alpha=1}^{r} \sum_{j=1}^{M} F_{j}(\cdot)(1 - \chi_{[0,h]}(t)) h_{\alpha}(\Gamma(\cdot, t)) K_{\alpha}[w(\cdot, t-h) - p(\cdot, t-h)] - p(\cdot, t-h).
\]
Therefore, for any initial value \( w_0 \in H^2_{n_0}(0, 1) \), (9) admits a unique local strong solution

\[
\begin{align*}
& w_t \in L^2((0, \infty); L^2_n(0, 1)) \quad w \in C([0, \infty); H^2_{n_0}(0, 1) \cap H^1_n(0, 1)) \cap L^2((0, \infty); D(A)).
\end{align*}
\]

C. Stability and Performance Analysis

This section devotes the proof of the main result.

Remark 1: To avoid using Lyapunov–Krasovskii functionals that depend on \( \omega_t \), we introduce a novel augmented Lyapunov–Krasovskii functional (13) that depends on the state only. Otherwise, by differentiating Lyapunov–Krasovskii functionals that depend on \( \omega_t \) and substituting \( \omega_t \) from (9), we will arrive at a positive term that cannot be compensated.

We construct the Lyapunov–Krasovskii functional of the following form:

\[
E(t) = \sum_{i=1}^{5} E_i(t)
\]

with

\[
E_i(t) = \int_{0}^{1} w^T(t, x) P_i w(x, t) dx
\]

\[
E_1(t) = \int_{0}^{1} \int_{-h}^{t} w(x, s) dx P_1 w(x, t) ds
\]

\[
E_2(t) = \int_{0}^{1} \int_{-h}^{t} w(x, s) dx P_2 w(x, t) ds
\]

\[
E_3(t) = \int_{0}^{1} \int_{-h}^{t} w(x, s) dx P_3 w(x, t) ds
\]

\[
E_4(t) = \int_{0}^{1} \int_{-h}^{t} w(x, s) dx P_4 w(x, t) ds
\]

\[
E_5(t) = \int_{0}^{1} \int_{-h}^{t} w(x, s) dx P_5 w(x, t) ds
\]

where \( P_i > 0 \) (\( i = 1, 2, 3, 4, 5 \)).

The next result provides sufficient conditions in the form of the LMIs for (9).

Theorem 1: Consider the PDE system (9) subject to (10) or (11). Given positive scalars \( \gamma, \Delta, h, \) and \( \delta \), and positive tuning parameter \( \delta_1 \), let there exist scalars \( \lambda \geq 0, \alpha_0 > 0 \) and matrices \( P_i > 0 \) (\( i = 1, 2, 3, 4, 5 \), \( W_a \in \mathbb{R}^{n \times n} \) (\( a \in S \)) such that the following inequalities hold:

\[
\begin{align*}
& \lambda \in \mathbb{R}^{n \times n}, \\
& \Theta \in \mathbb{R}^{n \times n}, \\
& \Xi \in \mathbb{R}^{n \times n}
\end{align*}
\]

where

\[
\begin{align*}
& \Lambda_{11} = -P_1 A - A P_1 + P_4 + h P_3 + 2 h P_1 + C^T C - \lambda \pi^4 I \\
& \Lambda_{13} = -P_3 - \frac{1}{2} P_5 \\
& \Lambda_{33} = -2 \nu P_1 + \lambda I \\
& \Lambda_{44} = -e^{-2 h} P_3 + 2 \delta P_2 \\
& \Lambda_{55} = -\frac{\gamma^2}{\Delta} \alpha_0 I \\
& \Theta_{11} = -P_1 A - A P_1 + 2 \delta P_1 - \lambda \pi^4 I \\
& \Xi_{11} = -P_1 A - A P_1 + P_4 + h P_3 + 2 (\delta - \delta_1) P_1 - \lambda \pi^4 I \\
& \Xi_{44} = -e^{-2 h} P_3 + 2 (\delta - \delta_1) P_2.
\end{align*}
\]

Then, the closed-loop system is internally exponentially stable with a decay rate \( \delta \), and its \( L_2 \)-gain is less than \( \gamma \). In addition, the controller gain matrices satisfy

\[
K_a = P_1^T W_a, \quad a \in S.
\]

Proof: We split the proof into three steps.

Step 1 [Find an Estimate of \( E(h) \) in Terms of \( w_0 \)]: Note that the solution to (9) does not depend on the values \( w(x, t) \) for \( t < 0 \). As in [45], we choose

\[
w|_{t\leq0} = w_0(x).
\]

Let

\[
E_0(t) = E_1(t) + E_2(t).
\]

Given \( \delta > 0 \), for \( t \in [0, h] \), if there exists \( \delta_1 > 0 \) such that along (9)

\[
\dot{E}_0(t) - 2 \delta_1 E_0(t) \leq 0
\]

\[
\dot{E}(t) + 2 \delta E(t) - 2 \delta_1 E_0(t) \leq 0
\]

then

\[
E_0(t) \leq e^{2 h_1} E_0(0)
\]

\[
E(t) \leq e^{-2 h} E(0) + (e^{2 h_1} - 1) E_0(0).
\]

Hence

\[
E(h) \leq E(0) + (e^{2 h_1} - 1) E_0(0) \leq e^{2 h_1} E(0).
\]

Step 2 [Prove the LMIs (16) and (17) to Yield (19) and (20)]: From Lemma 1, we have

\[
\lambda \left[ \| w(x, t) \|_{L^2_0(0, 1)}^2 - \pi^4 \| w(\cdot, t) \|_{L^2_0(0, 1)}^2 \right] \geq 0
\]

for all \( \lambda \geq 0 \). Differentiating \( E_0(t), E(t) \) along (9) with \( t \in [0, h] \), and using (24), we obtain

\[
\dot{E}_0(t) - 2 \delta_1 E_0(t)
\]

\[
\leq \dot{E}_0(t) - 2 \delta_1 E_0(t)
\]

\[
+ \lambda \left[ \| w(x, t) \|_{L^2_0(0, 1)}^2 - \pi^4 \| w(\cdot, t) \|_{L^2_0(0, 1)}^2 \right]
\]

\[
\leq \sum_{a=1}^{r} \int_{0}^{1} h_a (\Gamma(x, t), \xi) \Xi \xi dx
\]
and
\[ \dot{E}(t) + 2\delta E(t) = 2\delta E(t) - 2\delta_1 E_0(t) + \lambda \left( \| w_{xx}(t, \cdot) \|_{L_2^2(0,1)}^2 - \pi^2 \| w(\cdot, t) \|_{L_2^2(0,1)}^2 \right) \leq \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) \phi^T \Xi \phi dx \]

where
\[ \xi = \text{col}(w(x, t), w_{xx}(x, t)) \]
\[ \phi = \text{col}(w(x, t), w(x, t-h), w_{xx}(x, t), \int_{t-h}^{t} w(x, s) ds) \].

This leads to (19) and (20) under the LMIs (16) and (17).

Step 3: Show that
\[ \dot{E}(t) + 2\delta E(t) + \| \eta(\cdot, t) \|_{L_2^2(0,1)}^2 - \pi^2 \| p(\cdot, t-h) \|_{L_2^2(0,1)}^2 < 0 \]

for all \( t > h \). This is to guarantee that system (9) subject to (10) or (11) satisfies \( H_\infty \) performance (12).

Now, differentiate \( E(t) \) along (9) subject to (10) or (11) for \( t > h \) to obtain
\[ \dot{E}(t) + 2\delta E(t) = 2 \int_{0}^{1} w^T(x, t) P_1 w(x, t) dx + 2 \int_{0}^{1} \left[ \int_{t-h}^{t} w(x, s) ds \right]^T P_2 \left[ \int_{t-h}^{t} w(x, s) ds \right] dx + 2\delta \int_{0}^{1} w^T(x, t) P_1 w(x, t) dx + 2\delta \int_{0}^{1} \left[ \int_{t-h}^{t} w(x, s) ds \right]^T P_2 \left[ \int_{t-h}^{t} w(x, s) ds \right] dx + h \int_{0}^{1} w^T(x, t) P_3 w(x, t) dx - \int_{0}^{1} \int_{t-h}^{t} e^{-\delta(t-s)} w^T(x, s) P_3 w(x, s) ds dx + \int_{0}^{1} w^T(x, t) P_4 w(x, t) dx - e^{-2\delta h} \int_{0}^{1} w^T(x, t-h) P_4 w(x, t-h) dx + \int_{0}^{1} w_{xx}^T(x, t) P_5 w_{xx}(x, t) dx - e^{-2\delta h} \int_{0}^{1} w_{xx}^T(x, t-h) P_5 w_{xx}(x, t-h) dx. \]

Performing the integration by parts over \([0, 1]\) leads to
\[ 2 \int_{0}^{1} w^T(x, t) P_1 w(x, t) dx = -2 \int_{0}^{1} w^T(x, t) P_1 \left[ w w_{xxx} + w_{xx} \right] dx - 2 \int_{0}^{1} w^T(x, t) P_1 p(x, t-h) dx \]
\[ -2 \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) w^T(x, t) P_1 A_a w(x, t) dx - 2 \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) w^T(x, t) P_1 K_a w(x, t-h) dx + 2 \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) w^T(x, t) P_1 K_a \rho_j(x, t-h) dx = -2 \int_{0}^{1} w_{xx}^T(x, t) P_1 w_{xx}(x, t) dx - 2 \int_{0}^{1} w^T(x, t) P_1 w(x, t-h) dx - 2 \int_{0}^{1} w^T(x, t) P_1 p(x, t-h) dx - 2 \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) w^T(x, t) P_1 A_a w(x, t) dx - 2 \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) w^T(x, t) P_1 K_a w(x, t-h) dx + 2 \sum_{a=1}^{r} \int_{0}^{1} h_a(\Gamma(x, t)) w^T(x, t) P_1 K_a \rho_j(x, t-h) dx \]
\[ \int_{0}^{1} w^T(x, t) P_3 w(x, t) dx = - \int_{0}^{1} w^T(x, t) P_5 w_{xx}(x, t) dx. \]

Moreover, by Jensen’s inequality, it holds
\[ -\int_{0}^{1} \int_{t-h}^{t} e^{-\delta(t-s)} w^T(x, s) P_3 w(x, s) ds dx \leq -e^{-2\delta h} \frac{1}{h} \int_{0}^{1} \left[ \int_{t-h}^{t} w(x, s) ds \right]^T P_3 \left[ \int_{t-h}^{t} w(x, s) ds \right] dx. \]

We have two cases. The first case is that for the point measurements, \( \rho_j(x, t-h) \) is given by (10). In this case, by invoking Wirtinger’s inequality in Lemma 1, it follows that:
\[ \| \rho_j(\cdot, t-h) \|_{L_2^2(\bar{\Omega}_j)}^2 \leq \Delta^2 \pi \| w(\cdot, t-h) \|_{L_2^2(\bar{\Omega}_j)}^2 \forall t > h. \]

The second case is that for the averaged measurements, \( \rho_j(x, t-h) \) is given by (11). In this case, by invoking the
Poincaré inequality in Lemma 1, it follows that
\[ \| \varphi_j(t, t-h) \|^2_{L^2_a(\Omega_i)} \leq \frac{\Delta^2}{\pi^2} \| w_{x}(t, t-h) \|^2_{L^2_a(\Omega_i)} \quad \forall t > h. \]
Thus, for both cases, one has
\[ \| \varphi_j(t, t-h) \|^2_{L^2_a(\Omega_i)} \leq \frac{\Delta^2}{\pi^2} \| w_{x}(t, t-h) \|^2_{L^2_a(\Omega_i)} \quad \forall t > h. \]
Multiplying the above inequality by some constant \( a_0 > 0 \), and summing from \( j = 1 \) to \( M \), we obtain
\[ a_0 \sum_{j=1}^{M} \left[ \| w_{x}(t, t-h) \|^2_{L^2_a(\Omega_i)} - \frac{\pi^2}{2} \| \varphi_j(t, t-h) \|^2_{L^2_a(\Omega_i)} \right] \geq 0. \tag{30} \]
Set \( \varphi = \text{col} \{ w(x, t), w(x, t-h), w_{x}(x, t), \int_{-h}^{t} w(x, s) ds, \varphi_j(t, t-h), p(x, t-h) \} \), and \( W_\alpha = P_i K_\alpha (\alpha \in S) \). Then, applying (24) and (26)–(30), we obtain
\[ \dot{E}(t) + 2\Delta E(t) + \| \eta(t) \|^2_{L^2_a(0, 1)} - \gamma^2 \| p(t, t-h) \|^2_{L^2_a(0, 1)} \leq \dot{E}(t) + 2\Delta E(t) \]
\[ + a_0 \sum_{j=1}^{M} \left[ \| w_{x}(t, t-h) \|^2_{L^2_a(\Omega_i)} - \frac{\pi^2}{2} \| \varphi_j(t, t-h) \|^2_{L^2_a(\Omega_i)} \right] \]
\[ + \| \eta(t) \|^2_{L^2_a(0, 1)} - \gamma^2 \| p(t, t-h) \|^2_{L^2_a(0, 1)} \]
\[ + \int_{x_1}^{x_2} \left[ \| w_{x}(x, t) \|^2_{L^2_a(0, 1)} - \pi^2 \| w_{x}(x, t) \|^2_{L^2_a(0, 1)} \right] \]
\[ \leq \sum_{j=1}^{M} \int_{x_1}^{x_2} \left[ \alpha_0 \Gamma(x, t) \varphi(t) \right] \]
\[ - \int_{x_1}^{x_2} \left[ \alpha_0 \Gamma(x, t) \right] \]
\[ = \left( \int_{x_1}^{x_2} \alpha_0 \Gamma(x, t) \right) \int_{x_1}^{x_2} \left( x, t - h \right) \]
\[ = \left( \int_{x_1}^{x_2} \alpha_0 \Gamma(x, t) \right) \int_{x_1}^{x_2} \left( x, t - h \right) \]
\[ \leq 0 \leq d_j \in L^2(\Omega_i), \quad \int_{x_1}^{x_2} d_j(x) dx = 1. \]
Indeed, there exists \( \bar{x}_j \in \text{conv}(\text{supp} \ d_j), \) \( j = 1, \ldots, M \) such that
\[ \int_{\Omega_i} d_j(x) w(x, t) dx = w(\bar{x}_j, t). \tag{32} \]
Thus, a similar result can be obtained as Theorem 1.

Remark 4: The presented result for the case of mixed boundary conditions \( \left[ \left( \partial^m w \right) / \left( \partial x^n \right) \right](0, t) = \left[ \left( \partial^m w \right) / \left( \partial x^n \right) \right](1, t) = 0, \) \( m = 0, 1, \ldots, 3 \) can be extended to (1) with periodic boundary conditions \( \left[ \left( \partial^m w \right) / \left( \partial x^n \right) \right](0, t) = \left[ \left( \partial^m w \right) / \left( \partial x^n \right) \right](1, t), \) \( m = 0, 1, 2, 3 \). It must be stressed that all the proofs of Theorem 1 are valid except Wirtinger’s inequality in the new case. Thus, the LMI-based conditions in Theorem 1 with \( \lambda = 0 \) can be obtained.

Remark 5: The computational complexity can be estimated being proportional to \( N_2^2 N_1 \), where \( N_2 = \left( 5n(n+1) / 2 \right) + mn^2 + 2 \) is the total number of scalar decision variables, and \( N_1 = 13 \) is the total row size of LMIs.

Remark 6: The obtained LMI conditions in Theorem 1 are sufficient. If these conditions are feasible, then we can find a desirable controller. Simulations show that the LMIs in Theorem 1 are usually feasible for appropriate decision variables, large enough \( \delta_1 \), and small enough \( h \).

Remark 7: In [2] and [3], based on Galerkin’s method, a finite-dimensional controller was suggested to stabilize KSE. In [6], via the spectral method, the boundary stabilization for KSE was studied. Compared with the existing results in the literature, the merit of this article is that an effective LMI approach is proposed for analysis and \( H_\infty \) fuzzy control design for the nonlinear fourth-order parabolic equation subject to input delay. Due to the disturbance in system, the situation becomes much more complicated.

Next, consider system (9) subject to (10) or (11) without delay and disturbance (i.e., \( h = 0 \) and \( p = 0 \)). To investigate sufficient LMI conditions for the stability, the following Lyapunov–Krasovskii functional is introduced:
\[ V(t) = \| w(\cdot, t) \|^2_{L^2_a(0, 1)} \].
\[ \text{Specifically, the following result holds.} \]

Proposition 1: Consider system (9) subject to (10) or (11) without delay and disturbance. Given a positive scalar \( \Delta \), there exist \( \alpha_0 > 0 \), \( \delta > 0 \), and \( K_\alpha \in \mathbb{R}^{n \times n} (\alpha \in S) \) such that \( \Upsilon \leq 0 \), where
\[ \Upsilon \Delta \left[ \begin{array}{cc} \Upsilon_1 & -\alpha_0 I \\ * & K_\alpha \end{array} \right] \]
\[ * -2\nu \begin{array}{c} 0 \\ * -\pi^2 \Delta \end{array} I \[ \right] \tag{34} \]
with \( \Upsilon_1 = -A_\alpha - A_\alpha^T - K_\alpha - K_\alpha^T + 2\Delta I \). Then
\[ \| w(\cdot, t) \|^2_{L^2_a(0, 1)} \leq e^{-2\Delta t} \| w(\cdot) \|^2_{L^2_a(0, 1)}, \quad t \geq 0. \]

Proof: For the energy function (33), by applying the S-procedure with \( \alpha_0 > 0 \) and employing (30) with \( h = 0 \),
one has
\[
\dot{V}(t) + 2\delta V(t) \\
\leq \dot{V}(t) + 2\delta V(t) \\
+ \rho_j(t) \frac{\pi^2}{\Delta^2} (\|w_j(t)\|^2_{L^2(\Omega_2)} - \sum_{j=1}^{M} \|w_j(t)\|^2_{L^2(\Omega_2)}) \\
\leq -2\nu \|w_{xx}(t)\|^2_{L^2(0,1)} \\
- 2 \sum_{a=1}^r \int_0^t h_a(\Gamma(x, t)) w^\top(x, t) A_a w(x, t) dx \\
- 2 \sum_{a=1}^r \int_0^t h_a(\Gamma(x, t)) w^\top(x, t) K_a w(x, t) dx \\
+ \sum_{j=1}^M \sum_{a=1}^r \int_{s_j-1}^{s_j} h_a(\Gamma(x, t)) w^\top(x, t) K_a \rho_j(t, x) dx \\
- \alpha_0 \int_0^t w^\top(x, t) w_{xx}(x, t) dx \\
- \alpha_0 \frac{\pi^2}{\Delta^2} \sum_{j=1}^M \|\rho_j(t)\|^2_{L^2(\Omega_2)} + 2\delta \|w(t)\|^2_{L^2(0,1)} \\
\leq \sum_{a=1}^r \int_{s_j-1}^{s_j} \int_{s_j-1}^{s_j} h_a(\Gamma(x, t)) \Psi^\top \Psi dx \leq 0
\]
provided that \( \Upsilon \leq 0 \) is satisfied, where \( \Psi = \text{col}[w, w_{xx}, \rho_j] \).

The latter inequality implies the required result. \( \blacksquare \)

**Remark 8:** For any \( \delta > 0 \), we can find \( K_a \) such that the LMI in Proposition 1 is always feasible for sufficiently small enough \( \Delta \).

**V. NUMERICAL ILLUSTRATION**

Consider the fourth-order PDE system described by (1):

\[
\begin{align*}
\frac{d}{dt} w_i(x, t) + w_{xxxx}(x, t) + w_{xx}(x, t) + \sin(w(x, t)) + \sum_{j=1}^M F_j(x) u_j(t - h) + p(x, t - h) &= 0, \\
(x, t) \in (0, 1) \times (0, \infty) \\
w(0, t) = w(1, t) = 0 \\
w_x(0, t) = w_x(1, t) = 0 \\
w_0(x) = 0, \quad 0 \leq x \leq 1
\end{align*}
\]

(35)

where \( f(\cdot) = \sin(\cdot) \). For numerical simulation, we choose the disturbance as

\[
p(x, t) = \begin{cases} 
(1 + x)e^{-0.02x}, & t > 0 \\
0, & t \leq 0
\end{cases}
\]

(36)

By arguments of [46], from T-S fuzzy rules, it follows that the system (35) can be rewritten as follows:

**Plant Rule 1:** IF \( w(\cdot, t) \) is “about 0,” THEN

\[
\begin{align*}
\frac{d}{dt} w_i(x, t) + w_{xxxx}(x, t) + w_{xx}(x, t) + a_1 w(x, t) + \sum_{j=1}^M F_j(x) u_j(t - h) + p(x, t - h) &= 0, \\
w(0, t) = w(1, t) = 0 \\
w_x(0, t) = w_x(1, t) = 0 \\
w_0(x) = 0
\end{align*}
\]

(37)

where

\[
h_1(w(\cdot, t), w(\cdot, t)) = \begin{cases} 
\frac{\sin(w(\cdot, t)) - \epsilon w(\cdot, t)}{(1 - \epsilon)w(\cdot, t)}, & w(\cdot, t) \neq 0 \\
1, & w(\cdot, t) = 0
\end{cases}
\]

(38)

For the controller (8) under the point/averaged measurement, by choosing \( C = I \) and using the Yalmip, we verify the LMI conditions of Theorem 1 with \( \delta = 0.2, \delta_1 = 0.1, \Delta = 0.1, \) and \( h = 0.2 \). In both cases, the closed-loop system preserves internally exponential stability and achieves the \( L_2 \)-gain \( \gamma_{\text{min}} = 0.02 \). We find the following feasible solutions for the LMIs: \( P_1 = 0.0208 \) and \( W_1 = W_2 = -0.0113 \). Thus, the controller gains can be obtained as \( K_1 = K_2 = -0.54 \).

Now, we propose a finite difference scheme. The steps of space and time are chosen as 0.05 and 0.0001, respectively. The spatial interval [0, 1] is divided into ten subintervals.
Fig. 2 shows that the evolution of
\[
J(T) = \int_{T}^{T+\tau} \left[ \|w(t)\|_{L_2(0,1)}^2 - \gamma^2 \|p(t-h)\|_{L_2(0,1)}^2 \right] dt
\]
for the system with \( h = 0.2, \gamma = 0.02, \) and the disturbance (36). As expected, the simulation of solutions under the \( H_\infty \) fuzzy control law
\[
K_j = \begin{cases} 
2 \sum_{r=1}^{2} h_0 (w(x,t)) K_2 w(x,t), & t > 0, \\
0, & t \leq 0
\end{cases}
\]
ensures that \( J(T) < 0 \) for all \( T > 0.2 \). Enlarging the value of \( h \) until 1, we find that the solution starting from the same initial conditions is unbounded.

VI. CONCLUSION

This article has discussed \( H_\infty \) fuzzy control for a fourth-order PDE subject to uncertain and bounded constant delay. By constructing an appropriate Lyapunov functional, sufficient LMI conditions have been investigated ensuring that the corresponding system is internally exponentially stable and has an \( L_2 \)-gain less than \( \gamma \).

The results are applicable to delayed high-dimensional distributed parameter systems with uncertainties. Our next future work will focus on \( H_\infty \) fuzzy control design for coupled PDE-PDE/PDE-ODE system and 2-D parabolic PDEs system subject to time delay with noncollocated actuators and sensors. Another direction for future research is the extension of the obtained results to various output control problems for nonlinear delayed fourth-order PDEs systems.

REFERENCES

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