Arbitrary decay for boundary stabilization of Schrödinger equation subject to unknown disturbance by Lyapunov approach

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Abstract

This paper deals with the design of boundary control to stabilize one-dimensional Schrödinger equation with general external disturbance. The backstepping method is first applied to transform the anti-stability from the free end to the control end. A variable structure feedback stabilizing controller is then designed to achieve arbitrary assigned decay rate. The Galerkin approximation scheme is used to show the existence of the solution to the closed-loop system. The exponential stability of the closed-loop system is obtained by the Lyapunov functional method. A numerical example demonstrates the efficiency of the proposed control scheme.

1. Introduction

In quantum mechanics, Schrödinger equation is a mathematical equation that describes the changes over time of a physical system in which quantum effects, such as wave-particle duality, are significant. Boundary control of linearized Schrödinger equation has been studied in Guo and Liu (2014), Krstic, Guo, and Smyshlyaev (2011) and Machtyngier and Zuazua (1994). Recent results for the Schrödinger equation include the multiplier technique (see e.g., Machtyngier & Zuazua, 1994), the backstepping method (Krstic et al., 2011), sliding mode control (Guo & Liu, 2014) and extension of backstepping to heat-Schrödinger equation cascade (Wang, Ren, & Krstic, 2012) and ODE-Schrödinger equation cascade (Ren, Wang, & Krstic, 2013). Stabilization of linear Schrödinger equation with time delay by boundary or internal feedbacks has been dealt with in Nicaise and Rebiai (2011). Regional boundary stabilization of nonlinear Schrödinger equation with state delay and bounded internal disturbance can be found in Kang and Fridman (2018). Very recently, a stabilizing output feedback control for Schrödinger equation subject to internal unknown dynamic and external disturbance has been developed in Peng, Feng, and Chai (2018). Differently from Zhang et al. (2018), we focus on the design of state feedback to reject the unknown disturbance by using backstepping method and Lyapunov method directly, with arbitrarily assigned decay rate whereas in Zhang et al. (2018), only asymptotic stability can be achieved.

Stabilization of systems described by partial differential equations (PDEs) subject to unknown disturbance has attracted considerable attention (see e.g., Guo & Kang, 2014; Jin & Guo, 2015; Kang & Fridman, 2016, 2017; Pisano & Orlov, 2012 and the references therein) in recent years. Many different approaches have been developed to deal with disturbance such as the internal model principle for output regulation, the robust control for systems with uncertainties from both internal and external disturbance, the adaptive control for systems with unknown parameters, to name just a few. In this paper, unknown bounded disturbances are modeled into the boundary control terms. The objective of this work is to design a stabilizing controller for perturbed Schrödinger equation by using backstepping transformation and appropriate Lyapunov–Krasovskii functional.

The paper is organized as follows. In the following section, the problem statement is presented and the backstepping transformation is introduced. We transform original system into an equivalent target system where the internal instability terms are transformed...
into the control boundary. A state feedback is thus constructed to stabilize the system. In Section 3, a Filippov type solution of the closed-loop system is determined by a Galerkin approximation scheme. In Section 4, by using Lyapunov–Krasovskii method for the target system we show that the resulting closed-loop system is exponentially stable. A numerical example is presented in Section 5 for illustration of the effectiveness of the method. Some concluding remarks are presented in Section 6.

**Notation.** Throughout the paper, Re{·}, Im{·} stand for real and imaginary parts of associated complex variable respectively, i.e., $\sqrt{-1}$ denotes the imaginary unit. The subscripts $x$ and $t$ of $u(x, t)$ are partial derivatives with respect to $x$ and $t$ respectively. $w_1(x, t) = \partial u(x, t)/\partial t$, $w_2(x, t) = \partial u(x, t)/\partial x$, and $w_3(x, t) = \partial^2 u(x, t)/\partial x^2$, respectively. $L^2(0, 1)$ is the Hilbert space of square integrable scalar functions $f(x), x \in (0, 1)$ with the corresponding norm $\|f\|^2_{L^2(0, 1)} = \int_0^1 |f(x)|^2 dx$. $H^1(0, 1)$ denotes the Sobolev space of continuously differentiable functions on $[0, 1]$ with measurable, square integrable first derivative.

2. State feedback controller design

2.1. Problem formulation

Consider the following anti-stable one dimensional Schrödinger equation with an external disturbance:

\[
\begin{align*}
\dot{w}_1(x, t) &= -iu_{x}(x, t), \quad x \in (0, 1), \quad t > 0, \\
\dot{w}_2(x, t) &= -i\dot{w}_3(x, t), \quad t > 0, \\
\dot{w}_3(x, t) &= U(t) + d(t), \quad t > 0, \\
w(x, 0) &= w_0(x), \quad x \in [0, 1],
\end{align*}
\]  
(2.1)

where $q > 0$, $w(x, t)$ is the complex-valued state, and $U(t)$ is the control input. The unknown disturbance $d(t) \in \mathbb{C}$ is supposed to be uniformly bounded measurable and satisfies $d \in H_{loc}^2(0, \infty)$, $|d| \leq M$ for some constant $M > 0$. The open-loop system (2.1) (subject to $U(t) = d(t) \equiv 0$) is anti-stable, which has all of its eigenvalues in the right half-plane. In the rest of the paper, we drop the obvious domains for both $x$ and $t$.

The objective of this work is to find a control law such that the resulting closed-loop system with disturbance is exponentially stable with an arbitrary decay rate by using backstepping transformation and Lyapunov approach.

Since the feedback control is discontinuous with multi-value, we actually consider (2.1) as

\[
\begin{align*}
\dot{w}_1(x, t) &= -iu_{x}(x, t), \quad x \in (0, 1), \quad t > 0, \\
\dot{w}_2(x, t) &= -i\dot{w}_3(x, t), \quad t > 0, \\
\dot{w}_3(x, t) &= U(t) + d(t), \quad t > 0, \\
w(x, 0) &= w_0(x), \quad x \in [0, 1],
\end{align*}
\]  
(2.1*)

2.2. Backstepping transformation

First we consider a transformation

\[
u(x, t) = w(x, t) - \int_0^x l(x, y)w(y, t)dy,
\]
(2.2)

where the kernel $l(x, y)$ satisfies the following PDE:

\[
\begin{align*}
l_{yx}(x, y) - ly_y(x, y) = icl(x, y), \\
l_y(0, x) = -iql(0, x), \\
l(x, x) = -ic - iq.
\end{align*}
\]  
(2.3)

**Lemma 2.1.** (Guo & Liu, 2014). The problem (2.3) admits a unique solution that is twice continuously differentiable in $0 \leq y \leq x \leq 1$. The transformation (2.2) transforms the system (2.1*) into the following target system:

\[
\begin{align*}
\dot{u}_1(x, t) &= -iu_{x}(x, t) - cu(x, t), \\
\dot{u}_2(x, t) &= cu_{x}(x, t), \\
\dot{u}_3(x, t) &= \int_0^t l(x, y)\dot{u}_1(y, t)dy, \\
u(x, 0) &= u_0(x), \\
\end{align*}
\]  
(2.4)

where $c$ is an arbitrary positive constant. By selecting the feedback controller

\[
U(t) = -ic_1u_1(1, t) + M\text{sign}\left(\frac{u_1(1, t) - \tilde{u}(1, t)}{2i}\right)
\]
(2.5)

one arrives at the target system

\[
\begin{align*}
\dot{u}_1(x, t) &= -iu_{x}(x, t) - cu(x, t), \quad x \in (0, 1), \quad t > 0, \\
\dot{u}_2(x, t) &= cu_{x}(x, t), \\
\dot{u}_3(x, t) &= \int_0^1 l(x, y)\dot{u}_1(y, t)dy, \\
u(x, 0) &= u_0(x),
\end{align*}
\]  
(2.6)

where $c_1$ is a positive constant, and the symbol function is a set-valued function defined by

\[
\text{sign}(z) = \begin{cases}
\frac{z}{|z|} & z \neq 0, \\
\{z \in \mathbb{C} : |z| \leq 1\} & z = 0.
\end{cases}
\]  
(2.7)

In order for the function on the right-hand side of (2.5) to be measurable, for any $T > 0$ and $f \in L^2(0, T)$, we restrict the set-valued composition of the symbolic function as follows:

\[
\begin{align*}
\text{sign}(f(t)) = \begin{cases}
\text{sign}(f(t)), & f(t) \neq 0, \\
\{f \in L^2([0, T] ; \mathbb{R}) : |f| \leq 1, \quad f(t) = 0\}, & f(t) = 0.
\end{cases}
\end{align*}
\]  
(2.8)

**Remark 2.1.** Our control law (2.5) is different from the one introduced by Guo and Liu (2014), where SMC (slide mode control) and ADRC (active disturbance rejection control) method were applied to deal with the disturbance. Here we focus on designing a feedback via backstepping approach and Lyapunov approach directly.

3. Well-posedness of closed-loop system

In this section, we show that the existence and uniqueness of solution to closed-loop system (2.6) which will be accomplished in Theorem 3.1. To this purpose, we define

\[
\mathcal{H} = \{f \in H^2(0, 1) | f'(0) = 0\}
\]
It is obvious that $\mathcal{H}$ is a closed subspace of Sobolev space $H^2\{0,1\}$.

**Theorem 3.1.** Given $T > 0$. For any initial value $u_0 \in \mathcal{H}$ satisfying the compatible conditions:

\[
\begin{aligned}
    & u_0(0) = 0, \\
    & u_0'(1) = -ic_1u_0(1) + M\text{sign}\left(\frac{u_0(1) - u_0(1)}{2i}\right) \\
    & -iM\text{sign}\left(\frac{u_0(1) + u_0(1)}{2}\right) + d(0). \\
\end{aligned}
\]

Eq. (2.6) admits a (Filippov type) solution.

**Proof.** We start by showing that the classical solution must be locally unique if $u(1, t) \neq 0$ locally. Otherwise, there are two different solutions $u$ and $\tilde{u}$ to (2.6) with the same initial value. Set $p = u - \tilde{u}$. Then $p$ satisfies

\[
\begin{aligned}
    & p_t(x, t) = -ip_{xx}(x, t) - cp(x, t), \\
    & p_x(0, t) = 0, \\
    & p_x(1, t) = -ic_1p(1, t) + M\text{sign}\left(\frac{u(1, t) - \tilde{u}(1, t)}{2i}\right) \\
    & -iM\text{sign}\left(\frac{u(1, t) + \tilde{u}(1, t)}{2}\right) + iM\text{sign}\left(\frac{\tilde{u}(1, t) + \tilde{u}(1, t)}{2}\right), \\
    & p(\cdot, 0) = 0.
\end{aligned}
\]

Define a Lyapunov functional as follows

\[
V(t) = \frac{1}{2}\|p(\cdot, t)\|_{L^2(0,1)}^2 + \frac{1}{2} \int_0^1 [p(x, t)]^2 dx.
\]

Taking the time derivative of $V(t)$ along (3.2), we obtain

\[
\dot{V}(t) = \Re\langle p(\cdot, t), p(\cdot, t) \rangle = -c_1|p(\cdot, t)|^2 - c\|p(\cdot, t)\|_{L^2(0,1)}^2
\]

\[\times M\left[\text{sign}\left(\frac{u(1, t) - \tilde{u}(1, t)}{2i}\right)
    -\text{sign}\left(\frac{u(1, t) + \tilde{u}(1, t)}{2i}\right)
    -i \cdot \text{sign}\left(\frac{u(1, t) + \tilde{u}(1, t)}{2}\right)
    +i \cdot \text{sign}\left(\frac{\tilde{u}(1, t) + \tilde{u}(1, t)}{2}\right)\right].
\]

Since $u(1, t) = \Re[u(1, t)] + \im[u(1, t)]$, and $\tilde{u}(1, t) = \Re[\tilde{u}(1, t)] + \im[\tilde{u}(1, t)]$, we obtain

\[
\dot{V}_1(t) = -c_1|p(\cdot, t)|^2 - c\|p(\cdot, t)\|_{L^2(0,1)}^2
\]

\[\times M\left[\text{sign}\left(\frac{u(1, t) - \tilde{u}(1, t)}{2i}\right)
    -\text{sign}\left(\frac{u(1, t) + \tilde{u}(1, t)}{2i}\right)
    -i \cdot \text{sign}\left(\frac{u(1, t) + \tilde{u}(1, t)}{2}\right)
    +i \cdot \text{sign}\left(\frac{\tilde{u}(1, t) + \tilde{u}(1, t)}{2}\right)\right].
\]

where we used the following facts:

\[
\Re\{u(1, t)\} = \frac{u(1, t) + \tilde{u}(1, t)}{2},
\]

\[
\Im\{u(1, t)\} = \frac{u(1, t) - \tilde{u}(1, t)}{2i},
\]

and

\[
\Re\{\text{sign}(a) - \text{sign}(b)||a - b||\} > 0, \quad \forall \text{ } a, b \in \mathbb{C}.
\]

Hence, for $t \geq 0$, $V_1(t) \equiv V_1(0) = 0$ or $u = \tilde{u}$.

Next we prove the existence of a Filippov type solution to (2.6) by a Galerkin approximation scheme. Multiplying the first equation of (2.6) by $\phi \in H^1(0,1)$ and integrating from 0 to 1 with respect to $x$, we obtain

\[
\langle u_t(\cdot, t), \phi \rangle - i\langle u(\cdot, t), \phi \rangle + c\langle u(\cdot, t), \phi \rangle
\]

\[\in \im\phi(1)\left[-ic_1u(1, t) + M\text{sign}\left(\frac{u(1, t) - \tilde{u}(1, t)}{2i}\right)
    -i\text{sign}\left(\frac{\tilde{u}(1, t) + \tilde{u}(1, t)}{2}\right)
    +i \cdot \text{sign}\left(\frac{\tilde{u}(1, t) + \tilde{u}(1, t)}{2}\right)\right].
\]

Suppose that $\{\phi_n(x)\}_n^{\infty}$ is an orthonormal basis for $H^1(0,1)$ and for any $N \in \mathbb{Z}^+$ define a finite-dimensional subspace of $\mathcal{H}$ by $V_N = \text{span}[\phi_1, \phi_2, \ldots, \phi_N]$. Since $u_0 \in \mathcal{H}$, we may assume without loss of generality that $u_0 \in \text{span}[\phi_1]$. A Galerkin approximation solution to (2.6) is constructed as follows:

\[
u_N(x, t) = \sum_{n=1}^N a_{n}(t)\phi_n(x).
\]
which satisfies
\[
\begin{align*}
\langle u^N_i(\cdot, t), \phi\rangle - i\langle u^N_i(\cdot, t), \phi_i\rangle + c\langle u^N(\cdot, t), \phi_m\rangle \\
\in -i\phi(t)[H_i - iC_iU^N_i(1, t)] \\
+ M\text{sign}\left(\frac{u^N(1, t) - u^N(1, t)}{2i}\right) \\
- iM\text{sign}\left(\frac{u^N(1, t) + u^N(1, t)}{2}\right) + d(t)], \forall \phi \in V_N,
\end{align*}
\]
(3.9)

where
\[
U_1(X) = \text{sign}(\langle C_1, X(t)\rangle), \quad U_2(X) = \text{sign}(\langle C_2, X(t)\rangle).
\]

When \(k_3 = 0\), (3.13) admits only classical solution:
\[
\dot{X}(t) = A^{-1}t(A_2 + A_3)X(t).
\]

When \(k_3 \neq 0\), set
\[
S_1(X) = \langle C_1, X(t)\rangle, \quad S_2(X) = \langle C_2, X(t)\rangle.
\]

Then \(U_1\) is discontinuous only on \(S_1 = 0\) \((i = 1, 2)\). Now we use the equivalent control method to find the sliding mode solution (locally) for (3.13). That is, we need to find a continuous function \(U^q_1(i = 1, 2)\) that is called the equivalent control, so that
\[
\dot{X}(t) = A^{-1}t(A_2 + A_3 - k_3C_1k_1^2)X(t) + U^q_1(t)A^{-1}_1k_1 + U^q_2(t)A^{-1}_2k_2 + d(t)A^{-1}_1k_3.
\]
(3.14)

Since the equivalent control is needed only when \(S_1(t) = 0\) \((i = 1, 2)\) and hence \(S_1(t) = \langle C_1, \dot{X}(t)\rangle = 0, S_2(t) = \langle C_2, \dot{X}(t)\rangle = 0\), we can find \(U^q_1\) and \(U^q_2\) directly. Returning back to (3.9), the sliding mode is \(S_1(t) = S_2(t) = 0\) on which we have
\[
\begin{align*}
\langle u^N_i(\cdot, t), \phi\rangle - i\langle u^N_i(\cdot, t), \phi_i\rangle + c\langle u^N(\cdot, t), \phi_m\rangle \\
\in -i\phi(t)[H_i - iC_iU^N_i(1, t) + MU^q_1(t)A^{-1}_1k_1 - iMU^q_2(t)A^{-1}_2k_2 + d(t)], \forall \phi \in V_N,
\end{align*}
\]
(3.15)

The remaining proof will be split into several lemmas.

**Lemma 3.1.** For either the classical solution or the Filippov solution of Eq. (3.9),
\[
\max_{i,\phi} \sup_{0 \leq t \leq T} \|u^N(\cdot, t)\| < \infty.
\]
(3.16)

**Proof.** Substituting \(\phi = u^N(\cdot, t)\) into (3.9), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|u^N(\cdot, t)\|^2 &\leq -c_1\|u^N(\cdot, t)\|^2 - c_2\|u^N(\cdot, t)\|^2 \\
- \text{Re}\left\{\frac{\langle u^N(1, t) - u^N(1, t)\rangle}{2i} + \text{sign}\left(\frac{u^N(1, t) + u^N(1, t)}{2}\right) + d(t)\right\} \\
&\leq -c_1\|u^N(\cdot, t)\|^2 - c\|u^N(\cdot, t)\|^2 \leq 0.
\end{align*}
\]
(3.17)

Therefore, for either classical solution or the Filippov solution, we can always draw the conclusion (3.16).

**Lemma 3.2.** For either the classical solution or the Filippov solution of Eq. (3.9),
\[
\sup_{0 \leq t \leq T} \|u^N(\cdot, 0)\| < \infty.
\]
(3.18)
\textbf{Proof.} Setting $t = 0$ in (3.9), and taking the compatible conditions (3.1) into account, we obtain
\begin{align}
&\langle ˇu^N(\cdot, 0), \phi \rangle - i(u_0, \phi_x) + c(u_0, \phi) \\
&\in -i\phi(0)\langle -i\alpha_1u_0(1) + M \text{sign}(u_0(1) - u_0(T)) + d(0) \rangle.
\end{align}
(3.19)
Substituting $\phi(x) = ˇu^N(x, 0)$ into the latter equation we obtain
\begin{align}
||\dot{u}^N(x, 0)||^2 &\in i(u_0^N, \dot{u}^N(\cdot, 0)) - c(u_0, \dot{u}^N(x, 0)) \\
&\quad - \dot{w}^N(1, 0) \langle -i\alpha_1u_0(1) + M \text{sign}(u_0(1) - u_0(T)) + d(0) \rangle \\
&\quad + M \text{sign}(u_0(1) - u_0(T)) \\
&\quad - iM \text{sign}(u_0(1) + u_0(T)) + d(0) \\
&\quad = -c(u_0, \dot{u}^N(x, 0)).
\end{align}
(3.20)
By compatible condition (3.1), we have
\begin{align}
||\dot{u}^N(\cdot, 0)||^2 &\leq c||u_0||. \hspace{1cm} \square
\end{align}

\textbf{Lemma 3.3.} For either the classical solution or the Filippov solution of (3.9),
\begin{align}
sup_{N} \|u^N(\cdot, t)\| &< \infty, \forall \, t \in [0, T] \text{ a.e.} \hspace{1cm} (3.21)
\end{align}
\textbf{Proof.} For fixed $t, \xi > 0$ with $\xi < T - t$, replace $t$ by $t + \xi$ in (3.9) and subtract the first equation of (3.9) to have
\begin{align}
\langle \dot{u}^N(\cdot, t + \xi) - \dot{u}^N(\cdot, t), \phi \rangle - i(u_0^N(\cdot, t + \xi) - u_0^N(\cdot, t), \phi_x) \\
&+ c(u_0^N(\cdot, t + \xi) - u_0^N(\cdot, t), \phi) \\
&\in -i\phi(0)\langle -i\alpha_1u_0(1, t + \xi) + M \text{sign}(u_0(1, t + \xi) - u_0(T)) + d(t + \xi) \rangle \\
&\quad + M \text{sign}(u_0(1, t + \xi) - u_0(T)) \\
&\quad - iM \text{sign}(u_0(1, t + \xi) + u_0(T)) + d(t + \xi) \\
&\quad + i\phi(0)\langle -i\alpha_1u_0(1, t) + M \text{sign}(u_0(1, t) - u_0(T)) + d(t) \rangle \\
&\quad + M \text{sign}(u_0(1, t) - u_0(T)) \\
&\quad - iM \text{sign}(u_0(1, t) + u_0(T)) + d(t), \hspace{1cm} (3.22)
\end{align}
Substituting $\phi(x) = u^N(\cdot, t + \xi) - u^N(\cdot, t)$ into (3.22) we have
\begin{align}
\frac{1}{2} \frac{d}{dt} ||u^N(\cdot, t + \xi) - u^N(\cdot, t)||^2 &\leq I_1 + I_2, \hspace{1cm} (3.23)
\end{align}
where
\begin{align}
I_1 &= -c||u^N(\cdot, t + \xi) - u^N(\cdot, t)||^2 \\
&\quad - c_1||u^N(1, t + \xi) - u^N(1, t)||^2 \leq 0, \\
I_2 &= \text{Re}\left\{ -i||u^N(1, t + \xi) - u^N(1, t)|| \times \right. \\
&\quad M \text{sign}(u^N(1, t + \xi) - u^N(1, t + \xi)) \\
&\quad \left. - iM \text{sign}(u^N(1, t + \xi) + u^N(1, t + \xi)) + d(t + \xi) \right\} \\
&\leq \text{Re}\{ -i||u^N(1, t + \xi) - u^N(1, t)||[d(t + \xi) - d(t)] \}
\end{align}
(3.24)
Integrating (3.23) over $[0, t]$ with respect to $t$, we obtain
\begin{align}
||u^N(\cdot, t + \xi) - u^N(\cdot, t)||^2 &\leq ||u^N(\cdot, 0)||^2 + \frac{2}{c_1} \int_0^t |d(s + \xi) - d(s)|^2 ds \\
&\leq \frac{1}{c_1} ||u^N(\cdot, 0)||^2 + \frac{2}{c_1} \int_0^t |d(s)|^2 ds, \hspace{1cm} (3.25)
\end{align}
in which we choose $\xi$ sufficiently small. Dividing $\xi$ on both sides (3.25) and passing to the limit as $\xi \to 0$, we have
\begin{align}
||\dot{u}^N(\cdot, t)||^2 &\leq ||u^N(\cdot, 0)||^2 \hspace{1cm} (3.26)
\end{align}
By Lemma 3.2, we obtain
\begin{align}
||\dot{u}^N(\cdot, t)|| &< \infty, \forall \, t \in [0, T]. \hspace{1cm} \square
\end{align}
We also need the following Lemma 3.4 from Jin and Guo (2015). For the sake of complete, the simple proof is given.

\textbf{Lemma 3.4.} For any $T > 0$, suppose that $f_i \to f$ in $L^2(0, T)$ strongly as $i \to \infty$ and $g \in L^2(0, T)$. Then under definition (2.8) for $\text{sign}(f(t))$, there exists a subsequence $\{f_{ij}\}$ of $\{f_i\}$ such that
\begin{align}
\lim_{j \to \infty} \int_0^T \text{sign}(f_{ij}(t))g(t)dt &\subset \int_0^T \text{sign}(f(t))g(t)dt, \hspace{1cm} (3.27)
\end{align}
which is understood in the following sense:
\begin{align}
\lim_{j \to \infty} \int_0^T \text{sign}(f_{ij}(t))g(t)dt &= \int_0^T \text{sign}(f(t))g(t)dt, \\
\int_Q \text{sign}(f_{ij}(t))g(t)dt &= \int_Q \text{sign}(f(t))g(t)dt \\
\int_Q \text{sign}(f(t))g(t)dt &= \left\{ \int_Q \text{sign}(f(t))g(t)dt \right\}
\end{align}
(3.28)
where
\begin{align}
P &= \{ t \in [0, T] \mid f(t) \neq 0 \}, \\
Q &= \{ t \in [0, T] \mid f(t) = 0 \}.
\end{align}
\textbf{Proof.} From (2.8) and $\bigcup_{t \in Q} \text{sign}(f_{ij}(t)) = \text{sign}(f(t))$, the second part is obtained. We need only show the first part of (3.28). Let $P^+ = \{ t \in R
Continuation of proof of Theorem 3.1. From Lemmas 3.1 to 3.3, we can extract a subsequence \(N_k\), still denoted by \(N\) without confusion, such that

\[
\begin{align*}
\{u^N \rightharpoonup u \text{ in } L^\infty(0, T; H^1(0, 1)) \} \text{ weak*}, \\
\{\hat{u}^N \rightharpoonup \hat{u} \text{ in } L^\infty(0, T; L^2(0, 1)) \} \text{ weak*}.
\end{align*}
\]

We first show that there exists a subsequence of \(u^N(1, t)\), still denote by itself without confusion, such that \(u^N(1, t) \rightharpoonup u(1, t)\) in \(L^2(0, T)\). To this end, by Adams and Fournier (2003, Theorem 2.32), we only need to show: (i) \(u^N(1, t)\) in \(L^2(0, T)\); (ii) \(\{u^N(1, t)\}^N_{N=1}\) is equicontinuous, that is, \(\|u^N(\cdot, t) - u^N(\cdot, 0)\|^2 \to 0\) as \(\xi \to 0\) uniformly with respect to \(N\). The uniform boundedness of \(\{u^N(1, t)\}^N_{N=1}\) is trivial fact from Lemma 3.1. Now we show that \(\|u^N(1, t)\|^2 \to 0\) uniformly in \(L^2(0, T)\). From the proof of Lemma 3.3 in (3.23) and (3.24), we have

\[
\frac{c_1}{2} \int_0^T |u^N(1, t + \xi) - u^N(1, t)|^2 dt \\
\leq \frac{1}{2} \|u^N(0, \cdot) - u^N(0, 0)\|^2 + \frac{1}{2} \int_0^T |d(t + \xi) - d(t)|^2 dt.
\]

Since \(d(t)\) is a continuous function, it is sufficient to show that, for \(\tau = 0, T\),

\[
\|u^N(\cdot, \tau + \xi) - u^N(\cdot, \tau)\|^2 \to 0, \quad \text{as } \xi \to 0,
\]

uniformly with respect to \(N\). It follows from Lemma 3.3 that for \(\tau = 0, T\), and for some \(C_3 > 0\)

\[
\|u^N(\cdot, \tau + \xi) - u^N(\cdot, \tau)\|^2 \leq \|u^N(\cdot, \tau_0)\|^2 |\xi| \leq C_3 |\xi|.
\]

Hence, (3.34) holds. As a result, there exists a subsequence \(u^N(\cdot, t)\) such that \(u^N(\cdot, t) \to u(\cdot, t)\) in \(L^2(0, T)\) strongly. Now, for any \(\psi \in D(0, T)\) and \(\phi \in \mathcal{H}\),

\[
\int_0^T (u^N(\cdot, t), \phi) dt - \int_0^T (u(\cdot), \phi) dt + \int_0^T (\nabla^N(\cdot, t) - \nabla(\cdot, t)) \psi dt
\]

\[
\in -i\phi(t) \int_0^T \left\{ -ic_1 u^N(1, t) \right\} \psi(t) dt + iM \left( \frac{u^N(1, t) - u(1, t)}{2i} ight) + d(t) \psi(t) dt.
\]

By Lemma 3.4, there exists a sequence of integer \(\{N_k\}^\infty_{k=1}\) such that

\[
\lim_{k \to \infty} k \int_0^T \left( u^N(1, t), \phi \right) dt = \lim_{k \to \infty} \frac{u(1, t) + u^N(1, t)}{2} \right( \frac{u(1, t) - u^N(1, t)}{2i} 
\]

\[
- iM \left( \frac{u^N(1, t) - u(1, t)}{2i} \right) + d(t) \psi(t) dt.
\]

Due to (3.32), passing to the limit as \(k \to \infty\) in (3.35) for the subsequence \(\{u^N\}^N_{N=1}\), we obtain

\[
\langle u_0, \phi \rangle - ic_1 u_0 + c \langle u, \phi \rangle = 0, \quad \forall \phi \in D(0, 1).
\]

This shows that the generalized derivative \(u_{\text{max}}\) exists, and \(u_t = -u_{\text{max}}\) holds for almost all \(t \in (0, T)\). In particular, \(u_t(\cdot, t) \in H^1(0, 1)\). Integration by parts of (3.37) over \([0, 1]\) with respect to \(x\) leads to

\[
\langle u_0, \phi \rangle - ic_1 u_0(t) + c \langle u, \phi \rangle = 0, \quad \forall \phi \in D(0, 1).
\]
4. Stability analysis

In this section, we study the global stability of the closed-loop system (2.6) as in Krstic and Smyshlyaev (2008).

**Theorem 4.1.** The closed-loop system (2.6) is exponentially stable with the decay rate $c$:

$$
\|u(\cdot, t)\|^2_{L^2(0, 1)} \leq e^{-2ct} \|u_0\|^2_{L^2(0, 1)}, \quad \forall t \geq 0.
$$

**Proof.** Set

$$
u(x, t) = p(x, t) + ik(x, t),
$$

$$
p(x, t) = \text{Re}\{u(x, t)\} = \frac{u(x, t) + \overline{u(x, t)}}{2},
$$

$$
k(x, t) = \text{Im}\{u(x, t)\} = \frac{u(x, t) - \overline{u(x, t)}}{2i},
$$

where $p(x, t)$ is the real part of $u(x, t)$, and $k(x, t)$ is the imaginary part of $u(x, t)$. Then, $p$ and $k$ satisfy the following coupled PDEs:

$$
\begin{align*}
p_t(x, t) &= \kappa \kappa x, \quad x \in (0, 1), \ t > 0, \\
k_t(x, t) &= -p_{xx}(x, t) - c\kappa(x, t), \quad x \in (0, 1), \ t > 0, \\
p_0(0, t) &= 0, \\
k_0(0, t) &= 0,
\end{align*}
$$

(4.5)

In order to prove the stability of (4.5), we employ the following Lyapunov–Krasovskii function:

$$
V(t) = \frac{1}{2} \int_0^1 |u(x, t)|^2 dx = \frac{1}{2} \int_0^1 [p^2(x, t) + k^2(x, t)] dx.
$$

Differentiating (4.6) along (2.6), we have

$$
\begin{align*}
\dot{V}(t) &= \int_0^1 p(x, t)p_t(x, t) dx + \int_0^1 k(x, t)k_t(x, t) dx \\
&= \int_0^1 p(x, t)\kappa\kappa x(x, t) dx - \int_0^1 \kappa(x, t)p_{xx}(x, t) dx \\
&\quad -c \int_0^1 [p^2(x, t) + k^2(x, t)] dx \\
&= [p(x, t)\kappa(x, t)]_{x=0}^{x=1} - k(x, t)p_{xx}(1, t) \\
&\quad -c \int_0^1 [p^2(x, t) + k^2(x, t)] dx \\
&\leq -2cV(t) + c2V(t) - c[p(1, t) - k(1, t)p_{xx}(1, t)] \\
&\quad - [M - |d(t)|]|p(1, t)| - [M - |d(t)|]|k(1, t)| \\
&\leq -2cV(t).
\end{align*}
$$

(4.7)

From (4.7) it follows that the norm of this system decays exponentially at the arbitrarily defined rate $c$, which implies the target system (2.6) is exponentially stable in $L^2(0, 1)$. This completes the proof of the theorem. □

Returning to the original system by the transformation (2.2), we obtain the following result:

**Theorem 4.2.** Consider the system (2.1*) with the following control law

$$
\begin{align*}
U(t) &= -ic_{1}g(t) + M\text{sign}\left(\frac{g(t) - \overline{g(t)}}{2i}\right) \\
&\quad - i\text{sign}\left(\frac{g(t) + \overline{g(t)}}{2}\right) + \|1, 1\|g(t) \\
&\quad + \int_0^1 \nu_{1}(y, t)\left[\nu(y, t) - \int_0^y \nu_{2}(y, z)\nu(z, t) dz\right] dy,
\end{align*}
$$

(4.8)

where

$$
g(t) = w(1, t) - \int_0^1 \|1, 1\|w(y, t) dy.
$$
Then the closed-loop system (2.1*) under the feedback control (4.8) is exponentially stable with the decay rate $c$.

5. Example

To illustrate the satisfactory and better performance of the proposed design method, this section considers an anti-stable Schrödinger equation (2.1) with the parameters $q = 3$ and the initial condition $u_0(x) = \sin x, x \in [0, 1]$. Fig. 1 provides open-loop numerical simulation result: profile of evolution of $w(\cdot, t)$. It is seen that the open-loop system is unstable.

Consider next the closed-loop system (2.6) under the feedback controller (2.5) with the parameters $c = 10$, $c_1 = 5$, and $M = 1$. Here, we simply specify $\text{sign}(0) = 0$. The disturbance is chosen as $d(t) = e^{-t}$. A finite difference method is applied to compute the displacement of the closed-loop system (2.6). We choose initial condition $u(x, 0) = \sin x, 0 \leq x \leq 1$. The steps of space and time are taken as 0.1 and 0.0001, respectively. Fig. 2 shows that the closed-loop system is exponentially stable. The simulations of the solutions confirm the theoretical results.

6. Conclusion

This paper develops a boundary controller combined with a backstepping technique for Schrödinger equation subject to unknown bounded disturbances on the controlled boundary. The technique introduces a new variable transformation that makes it possible to apply the backstepping technique to provide arbitrary exponential decay of the system to its origin in a way that is robust to bounded disturbances. Observer-based boundary control of perturbed Schrödinger equation to achieve arbitrary decay rate, which is different to asymptotic stability obtained in Zhang et al. (2018), may be a topic for future research.

References


