Fuzzy Observer for 2D Parabolic Equation with Output Time Delay

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Abstract—This paper addresses fuzzy observer design for nonlinear parabolic equation over an unit square domain Ω in terms of the time delayed spatially averaged measurement, where the observer is composed of m-chain of sub-observers. Due to 2D domain, special emphases are made to the computational complexity. A Lyapunov argument is utilized to give constructive conditions ensuring the exponential stability of the resulting error system. The method used for the continuous-time fuzzy observer is applicable to the sampled-data implementation. Consistent simulation results that support the proposed theoretical statements are presented.

Index Terms—Nonlinear 2D parabolic equation, observer, Lyapunov design.

I. INTRODUCTION

The 2D parabolic equations which have strong background in engineering, physics, fluid mechanics, biological mathematics, and materials science have attracted increasing attention in decades. These infinite dimensional systems can be used to model many physical, biological and chemical systems. Stabilization for 1D parabolic equations has been investigated by many researchers like those in [1]–[6] among many others. In [2], a backstepping method was used to stabilize an unstable heat equation with large input time delay. The same approach has also been generalized to reaction-diffusion equations with state time delay (see, e.g., [3]). Very recently, distributed sampled control design for heat equation ([4], [5]) and Kuramoto-Sivashinsky equation (KSE) ([1]) with point/averaged measurements has been investigated. The paper [6] considered distributed control design of a Korteweg-de Vries-Burgers equation (KdVB) subject to input time delay. However, many challenging problems for 2D case are still open to date. Concurrently, for practical applications, substantial efforts have been made to develop estimation /control for high dimensional partial differential equations (PDEs) (see, e.g., [7]–[9]). A recent paper [7] dealt with network-based $H_\infty$ filtering problem for diffusion equation. In [8], [9], boundary feedback for 2D channel flow has been studied. One of the interesting problems is about design of fuzzy observer for 2D parabolic equations in terms of delayed spatially averaged measurement.

The fuzzy control has many applications in various of fields such as robotics, motor industry, signal processing, control engineering and so on. Actually, the fuzzy control systems have been extensively investigated in many papers [10]–[23], in particular, on stability and stabilization for delayed fuzzy systems (see, e.g., [21]–[23]). In addition, for practical applications, the study of adaptive fuzzy fault tolerant control for Markov jump systems has become an active research topic (see, e.g., [20]). However, most of works focus on finite dimensional systems. The problems of infinite dimensional systems are much more complicated. From a practical point of view, advanced fuzzy observer design for 2D parabolic equations subject to delayed spatially averaged measurement is challenging, which motivates the present paper. Compared with the previous work [1], [6], [24], in this paper, a sector nonlinearity approach is adopted to deal with the nonlinearity of a PDE system through construction of a Takagi-Sugeno (T-S) fuzzy PDE model. To enlarge the delay size, the concept of chain of sub-observers has been extended to a 2D parabolic equation. With the help of an appropriate Lyapunov-based argument, we develop a stability analysis for error system.

In comparison to the existing known results, new special challenges and main differences of this work are as follows:

- A sampled-data fuzzy observer design for high dimensional distributed parameter system is addressed, which becomes much more complicated for 2D case. We design a new fuzzy observer which is composed of m-chain of sub-observers.
- In the presence of output time delay, we consider two cases: sampled-data fuzzy observer and continuous-time fuzzy observer design for 2D parabolic equation. For both cases, we give exponential stability conditions for the corresponding error system in terms of LMIs. Due to 2D domain, special emphases are made to the computational complexity. The stability has been established based on 2D Wirtinger’s, Poincaré’s inequalities (Lemma 1 later), together with time-delay approach and descriptor method.

We proceed as follows. In section II, some preliminary lemmas are introduced. An observer setting is presented in section III. Section IV is devoted to the fuzzy observer design in terms of the delayed spatially averaged measurements, and the well-posedness and stability analysis are established. Moreover, sufficient LMI-based conditions are presented for the error system via the Poincaré’s and Wirtinger’s inequalities. Consistent simulation results which support the proposed
II. MATHEMATICAL PRELIMINARIES

Notation. The following notations are used throughout the paper.

- $\mathbb{R}_+ := [0, \infty)$.
- The superscript "T" stands for the transposition of matrix. For a partitioned matrix, the symbol * stands for symmetric blocks.
- $I$ stands for the identity matrix with appropriate dimension.
- When $\mathcal{O} \subset \mathbb{R}^2$ is used for a computational domain, the $L^2_\mu(\mathcal{O}) \triangleq L^2(\mathcal{O}; \mathbb{R}^\mu)$ stands for the space of measurable squared-integrable functions over $\mathcal{O}$ with the norm $\|g\|^2_{L^2_\mu(\mathcal{O})} = \int_{\mathcal{O}} |g(x)|^2 \, dx$.

Lemma 1. Let $\mathcal{O} = (0, L_1) \times (0, L_2)$. Suppose that $\mu \in H^1_\mu(\mathcal{O})$.

(i) Wirtinger’s inequality [7] If $\mu|_{\partial \mathcal{O}} = 0$, then, the following inequality holds:

$$\|\mu\|^2_{L^2_\mu(\mathcal{O})} \leq \frac{L_1^2 + L_2^2}{\pi^2} \|\nabla \mu\|^2_{L^2_\mu(\mathcal{O})}. \quad (1)$$

(ii) Poincaré’s inequality [25] If $\int_{\mathcal{O}} \mu(x) \, dx = 0$, then,

$$\|\mu\|^2_{L^2_\mu(\mathcal{O})} \leq \frac{L_1^2 + L_2^2}{\pi^2} \|\nabla \mu\|^2_{L^2_\mu(\mathcal{O})}. \quad (2)$$

Lemma 2. (Halanay’s inequality) [26] Let $0 < \delta_1 < 2\delta$ and let $E : [-h, \infty) \rightarrow \mathbb{R}_+$ be an absolutely continuous function satisfying

$$E(t) + 2\delta E(t) - \delta_1 \sup_{-h \leq \theta \leq 0} E(t + \theta) \leq 0, \quad t \geq 0.$$  

Then,

$$E(t) \leq e^{-2\sigma t} \sup_{-h \leq \theta \leq 0} E(\theta), \quad t \geq 0,$$

where $\sigma$ is a unique solution of

$$\sigma = \delta - \frac{\delta_1}{2} e^{2\sigma h}. \quad (3)$$

Remark 1. For the sake of simplicity, we consider each subdomain $\Omega_j$ to be a square. Indeed, $\Omega_j$ can be a rectangular and the results of this work are applicable to the case of nonsquare sub-domains.
It is further supposed that the sensors provide time delayed spatially averaged measurements:
\[
y_{out}(t) = \begin{cases} \int_{\Omega_j} y(x, t - h) \, dx / |\Omega_j|, & t \geq h, \\ 0, & t < h, \end{cases}
\]
where \( |\Omega_j| \) is the Lebesgue measure of the domain \( \Omega_j \), and \( h > 0 \) is the time delay which may be large.

We aim at constructing a fuzzy observer which is composed of a chain of several sub-observers for (4) to achieve exponential convergence of the error system for an arbitrary time delay \( h \).

IV. PROPOSED OBSERVER STRATEGY

Due to the nonlinearity in system, we employ T-S fuzzy PDE model and design a fuzzy observer. An LMI-based condition will be then developed for the exponential stability of the error system.

A. Fuzzy observer design

The fuzzy observer rules play an instrumental role in constructing the following T-S fuzzy PDE model.

**Plant Rule 3:**
\[ \text{IF } \Gamma_1(x, t) = F_{\xi_1}, \Gamma_2(x, t) = F_{\xi_2}, \ldots, \Gamma_{\eta}(x, t) = F_{\xi_{\eta}}, \text{ THEN} \]
\[
\begin{align*}
\frac{\partial y_h(x, t)}{\partial t} &= \Delta y_h(x, t) + A \xi y_h(x, t), \quad (x, t) \in \Omega \times (0, \infty), \\
y_h(x, 0) &= 0, \\
y_h(x, t) &= \hat{y}_0(x),
\end{align*}
\]
(6)

where \( A \xi \in \mathbb{R}^{n \times n}, \xi \in \mathcal{S} \triangleq \{1, 2, \ldots, p\}, p \) represents the number of IF-THEN fuzzy rules, and \( F_{\xi_1}, F_{\xi_2}, \ldots, F_{\xi_{\eta}} \) are the fuzzy sets. The premise variables \( \Gamma_{\xi}(x, t), g = 1, 2, \ldots, \eta \) are supposed to be functions of \( y(x, t) \). In this way, (6) leads to the following equation:

\[
\begin{align*}
\frac{\partial y_h(x, t)}{\partial t} &= \Delta y_h(x, t) + \sum_{\xi=1}^{p} h_{\xi} (\Gamma(x, t)) A \xi y_h(x, t), \\
y_h(x, 0) &= 0, \\
y_h(x, t) &= \hat{y}_0(x),
\end{align*}
\]
(7)

where \( \Gamma(x, t) = [\Gamma_1(x, t) \quad \Gamma_2(x, t) \ldots \Gamma_{\eta}(x, t)]^\top \),

\[
h_{\xi}(\Gamma(x, t)) = \prod_{g=1}^{n} \prod_{1 \leq g \leq n} F_{\xi_g}(\Gamma_g(x, t))
\]
and \( F_{\xi_g} \) represents the grade of the membership of \( \Gamma_g(x, t) \) in \( F_{\xi_g} \) for \( \xi \in \mathcal{S} \). Here, we suppose that

\[
\begin{align*}
\prod_{g=1}^{n} F_{\xi_g}(\Gamma_g(x, t)) &> 0, \quad \xi \in \mathcal{S}, \\
\sum_{\xi=1}^{p} \prod_{1 \leq g \leq n} F_{\xi_g}(\Gamma_g(x, t)) &> 0.
\end{align*}
\]

Then,
\[
h_{\xi}(\Gamma(x, t)) \geq 0, \quad \xi \in \mathcal{S}, \quad \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) = 1.
\]
(8)

As in [35], we first denote
\[
y^0(x, t) = y(x, t - h),
\]
\[
y^k(x, t) = y(x, t + k \frac{h - h}{m}), \quad k = 1, \ldots, m,
\]
that is,
\[
y^m(x, t) = y(x, t),
\]
\[
y^{k-1}(x, t) = y^k(x, t - \frac{h}{m}), \quad k = 1, \ldots, m.
\]
(9)

From the PDC (Parallel Distributed Compensation) scheme, we introduce the following observer for the fuzzy model (6) in terms of the delayed spatially averaged measurements (5):

\[
\begin{align*}
\frac{\partial \hat{y}^1(x, t)}{\partial t} &= \Delta \hat{y}^1(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A \xi \hat{y}^1(x, t) \\
&- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \mathcal{A}_j(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} \hat{y}^1(x, t - \frac{h}{m}) \, dx \\
&- y^1_{out}(t),
\end{align*}
\]
(10)

subject to
\[
\begin{align*}
\hat{y}^1|_{\partial \Omega} &= \hat{y}^2|_{\partial \Omega} = \cdots = \hat{y}^m|_{\partial \Omega} = 0, \\
\hat{y}^1(x, t) &= \hat{y}^2(x, t) = \cdots = \hat{y}^m(x, t) = 0, \quad t \leq 0,
\end{align*}
\]
(11)

where the spatial characteristic functions are taken as
\[
\begin{align*}
\mathcal{X}_j(x) &= 1, \quad x \in \Omega_j, \\
\mathcal{X}_j(x) &= 0, \quad x \notin \Omega_j, \quad j = 1, \ldots, N,
\end{align*}
\]
\( m \) is the number of the sub-observers, and \( L_{\xi} \in \mathbb{R}^{n \times n} \) are observer gain parameters that will be determined later.

Let \( e^k(x, t) = \hat{y}^k(x, t) - y^k(x, t) \). Then, the estimation error is governed by

\[
\begin{align*}
\frac{\partial e^1(x, t)}{\partial t} &= \Delta e^1(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A \xi e^1(x, t) \\
&- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \mathcal{A}_j(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} e^1(x, t - \frac{h}{m}) \, dx \\
&- e^1|_{\partial \Omega},
\end{align*}
\]
(12)

subject to
\[
e^1|_{\partial \Omega} = e^2|_{\partial \Omega} = \cdots = e^m|_{\partial \Omega} = 0.
\]
(14)
From (9), it follows that
\[ y^k(x, t - \frac{h}{m}) - y^{k-1}(x, t) \]
\[ = \hat{y}(x, t - \frac{h}{m}) - y^k(x, t - \frac{h}{m}) + y^k(x, t - \frac{h}{m}) - y^{k-1}(x, t) \]
\[ = [\hat{y}(x, t - \frac{h}{m}) - y^k(x, t - \frac{h}{m})] + [y^{k-1}(x, t) - y^{k-1}(x, t)] \]
\[ = e^k(x, t - \frac{h}{m}) - e^{k-1}(x, t). \]

Hence, for \( k = 2, \ldots, m \), (13) becomes
\[ \frac{\partial e^k(x, t)}{\partial t} = \Delta e^k(x, t) + \sum_{\xi=1}^{p} h_\xi (\Gamma(x, t)) A_\xi e^k(x, t) \]
\[ - \sum_{\xi=1}^{p} \sum_{j=1}^{M} \mathcal{X}_j(x) h_\xi (\Gamma(x, t)) \int_{\Omega} |e^k(x, t - \frac{h}{m}) - e^{k-1}(x, t)| dx, \]
\[ \forall t > 0. \]
We redefine the initial condition to be a function:
\[ y(x, t) = y_0(x), \quad t \leq 0, \]
because the solution of (7) is independent of the values of \( y(x, t) \) for \( t \leq 0 \). Moreover, (16) implies that
\[ e^k(x, t) = -y_0(x), \quad t \leq 0. \]
Therefore, the error system has become (12), (15), (14) initialized by (17).

B. Well-posedness

Now we use step method to establish the well-posedness of the error system (12), (15) subject to (14) and (17). To this purpose, we build a Lyapunov-Krasovskii functional, for \( k = 1, 2, \ldots, m \), that
\[ V_k(t) = \sum_{i=1}^{4} V_{ki}(t) \]
where
\[ V_{k1}(t) = \int_{\Omega} [e^k(x, t)]^T P_1 [e^k(x, t)] dx, \]
\[ V_{k2}(t) = \int_{\Omega} [\nabla e^k(x, t)]^T P_2 [\nabla e^k(x, t)] dx, \]
\[ V_{k3}(t) = \int_{\Omega} \int_{t - \frac{h}{m}}^{t} e^{-\delta(t-s)} [e^k(x, s)]^T P_3 [e^k(x, s)] ds dx, \]
\[ V_{k4}(t) = \int_{\Omega} \int_{t - \frac{h}{m}}^{t} e^{-\delta(t-s)} [e^k(x, s)]^T P_4 [e^k(x, s)] ds dx, \]
with \( 0 < P_i \in \mathbb{R}^{n \times n} (i = 1, 2, 3, 4) \) being matrices to be determined.

The preceding Theorem 1 provides sufficient conditions in form of LMIs for the error system (12), (15) subject to (14), (17).

**Theorem 1.** For error system (12) and (15) subject to (14) and (17) and given positive scalars \( h, m, \Delta, \delta \) and a positive tuning parameter \( \delta_1 < 2\delta \), suppose that there are nonnegative scalars \( \lambda_1 \geq 0, \lambda_2 \geq 0 \) and matrices \( 0 < P_i \in \mathbb{R}^{n \times n} (i = 1, 2, 3, 4) \), \( \bar{W}_\xi \in \mathbb{R}^{n \times n} (\xi \in S) \) satisfying
\[ \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ * & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ * & * & \Lambda_{33} & \Lambda_{34} \\ * & * & * & \Lambda_{44} \end{bmatrix} < 0, \]
\[ \Theta_1 = -2P_2 + 2\delta P_2 + \lambda_1 I \leq 0, \]
\[ \Theta_2 = -\delta_1 P_2 + \lambda_2 I \leq 0, \]
where
\[ \Lambda_{11} = -\frac{m}{h} e^{-\frac{\pi^2}{h} P_4} + 2\delta P_1 + (1 - e^{-\frac{\pi^2}{h} P_3} - \frac{\pi^2}{2} \lambda_1 I) + \langle P_2 A_\xi + A_\xi^T P_2 \rangle - \delta_1 P_1, \]
\[ \Lambda_{12} = P_1 - P_2 + P_2 A_\xi, \]
\[ \Lambda_{13} = \frac{m}{h} e^{-\frac{\pi^2}{h} P_4} - W_\xi, \]
\[ \Lambda_{14} = W_\xi, \]
\[ \Lambda_{22} = \frac{h}{m} P_4 - 2P_2, \]
\[ \Lambda_{23} = -W_\xi, \]
\[ \Lambda_{24} = W_\xi, \]
\[ \Lambda_{33} = -\frac{m}{h} e^{-\frac{\pi^2}{h} P_4}, \]
\[ \Lambda_{44} = -\frac{\pi^2}{2\Delta^2} \lambda_2 I. \]

Then, for any initial state \( y_0 \in H \), the error system is exponentially stable with a decay rate \( \sigma \), where \( \sigma \) is a unique solution of
\[ \sigma = \delta - \delta_1 e^{2\sigma h}, \]
and the observer gain matrices satisfy
\[ L_\xi = P^{-1}_2 W_\xi, \xi \in S. \]

Proof. The proof will be split into two steps.

Step 1: Derive sufficient condition for \( \dot{V}_1(t) + 2\delta V_1(t) - \delta_1 V_1(t) - \frac{h}{m} \leq 0 \). Differentiating \( V_1(t) \) along the solution of (12) subject to (17) gives
\[ \dot{V}_1(t) + 2\delta V_1(t) = \frac{2}{\Omega} \int_{\Omega} \left[ e^1(x,t) \right]^T P_1 [e^1(x,t)] dx + \frac{2}{\Omega} \int_{\Omega} \left[ \nabla e^1(x,t) \right]^T P_2 \left[ \nabla e^1(x,t) \right] dx + 2\delta \int_{\Omega} \left[ e^1(x,t) \right]^T P_1 [e^1(x,t)] dx + \frac{2}{\Omega} \int_{\Omega} \left[ \nabla e^1(x,t) \right]^T P_2 \left[ \nabla e^1(x,t) \right] dx + \frac{2}{\Omega} \int_{\Omega} \left[ e^1(x,t) \right]^T P_3 [e^1(x,t)] dx - \frac{m}{h} \int_{\Omega} \left[ e^1(x,t) \right]^T P_4 [e^1(x,s)] ds dx. \]

Now, we give several estimates for these terms to bring into a matrix estimate. First, applying Jensen’s inequality, we have
\[ \frac{2}{\Omega} \int_{\Omega} \left[ e^1(x,t) \right]^T P_1 [e^1(x,t)] dx \leq \frac{\pi^2}{2} \| e^1(\cdot,t) \|^2_{L^2_\Omega} \]
\[ \times P_4 [e^1(x,t) - e^1(x,t - h/m)] dx. \]

Next, the Wirtinger’s inequality leads to
\[ \lambda_1 \left\| \nabla e^1(\cdot,t) \right\|^2_{L^2_\Omega} - \frac{\pi^2}{2} \left\| e^1(\cdot,t) \right\|^2_{L^2_\Omega} dx \geq 0, \]
for any \( \lambda_1 \geq 0 \).

An application of the descriptor approach [34] is made by adding to \( \dot{V}_1 + 2\delta V_1 \) the left-hand sides of the following equation:
\[ 2 \int_{\Omega} \left[ e^1(x,t) \right]^T + \left[ e^1(x,t) \right]^T P_2 \]
\[ \times \left[ \Delta e^1(x,t) - e^1(x,t) + \sum_{\xi=1}^p h_\xi \langle \Gamma(x,t) A_\xi e^1(x,t) \rangle \right]
\[ - \sum_{\xi=1}^p \alpha_j(x) h_\xi \langle \Gamma(x,t) \rangle L_\xi \left( x - \frac{h}{m} \right) dx \right] = 0. \]

Performing the integration by parts, we obtain
\[ \int_{\Omega} \left[ e^1(x,t) \right]^T P_2 \left[ \Delta e^1(x,t) \right] dx = \int_{\Omega} \left[ \nabla e^1(x,t) \right]^T P_2 \left[ \nabla e^1(x,t) \right] dx, \]
\[ \int_{\Omega} \left[ e^1(x,t) \right]^T P_2 \left[ \Delta e^1(x,t) \right] dx = \int_{\Omega} \left[ \nabla e^1(x,t) \right]^T P_2 \left[ \nabla e^1(x,t) \right] dx. \]

Denote
\[ f_j(x,t) = e^1(x,t) - \int_{\Omega} e^1(x,t) dx. \]

Since \( \int_{\Omega} f_j(x,t) dx = 0 \), the Poincaré’s inequality yields
\[ \| f_j(\cdot,t) \|^2_{L^2_\Omega} \leq \frac{2\Delta^2}{\pi^2} \| e^1(\cdot,t) \|^2_{L^2_\Omega}. \]

Hence,
\[ \lambda_2 \sum_{j=1}^M \| \nabla e^1(\cdot,t) \|^2_{L^2_\Omega} - \frac{\pi^2}{2\Delta^2} \| f_j(\cdot,t) - \frac{h}{m} \|^2_{L^2_\Omega} \geq 0 \]
for any \( \lambda_2 \geq 0 \).

Finally, set
\[ \varphi = \text{col} \left\{ e^1(x,t), e^1(x,t), e^1(x,t - \frac{h}{m}), f_j(x,t - \frac{h}{m}) \right\} \]
and \( W_\xi = P_2 L_\xi \ (\xi \in S) \). Adding (32), (33) and (37) into \( \dot{V}_1 + 2\delta V_1 \), and applying (30), (31), (34), (35), and (37), we arrive at
\[ \dot{V}_1 + 2\delta V_1 - \delta_1 V_1(t) - \frac{h}{m} \leq \dot{V}_1 + 2\delta V_1 - \delta_1 V_1(t) - \frac{h}{m} \]
\[ + \lambda_1 \left\| \nabla e^1(\cdot,t) \right\|^2_{L^2_\Omega} - \frac{\pi^2}{2} \| e^1(\cdot,t) \|^2_{L^2_\Omega} \]
\[ + \lambda_2 \sum_{j=1}^M \| \nabla e^1(\cdot,t - \frac{h}{m}) \|^2_{L^2_\Omega} - \frac{\pi^2}{2\Delta^2} \| f_j(\cdot,t - \frac{h}{m}) \|^2_{L^2_\Omega} \]
\[ \leq \sum_{\xi=1}^p \sum_{j=1}^M \int_{\Omega_j} h_\xi \langle \Gamma(x,t) \rangle \varphi^T A \varphi dx \]
\[ - \int_{\Omega} \left[ \nabla e^1(x,t) \right]^T \Theta_1 [\nabla e^1(x,t)] dx \]
\[ - \int_{\Omega} \left[ \nabla e^1(x,t - \frac{h}{m}) \right]^T \Theta_2 [\nabla e^1(x,t - \frac{h}{m})] dx \leq 0, \]
provided that the LMIls (25), (26), (27) hold.
Step 2: Show that the LMI s (25), (26), (27) lead to
\[
\dot{V}_k(t) + (2\delta - \varepsilon \nu^2)V_k(t) - \delta_1 V_k(t - \frac{h}{m}) - \nu^2 V_{k-1}(t) \leq 0, \tag{39}
\]
for \( k = 2, 3, \cdots, m \), where \( \nu^2 \) is large enough and \( \varepsilon \) is small enough such that
\[
\delta_1 < 2\delta - \varepsilon \nu^2. \tag{40}
\]

To this end, construct the following Lyapunov-Krasovskii functional:
\[
V(t) = \sum_{k=1}^{m} e^{k-1} V_k(t). \tag{41}
\]

Differentiating \( V(t) \) along the solution of (12) and (15) subject to (14) and (17) yields
\[
\dot{V}(t) + (2\delta - \varepsilon \nu^2)V(t) - \delta_1 \sup_{-h \leq s \leq 0} V(t + s) \leq 0, \tag{42}
\]
Furthermore, from (40) and Halanay’s inequality, it follows that
\[
V(t) \leq e^{-2\sigma t} \sup_{-h \leq \theta \leq 0} V(\theta), \quad t \geq 0, \tag{43}
\]
where \( \sigma \) is a unique solution of
\[
\sigma = \delta - \frac{\delta_1}{2} e^{2\sigma h}. \tag{44}
\]
This completes the proof of the theorem. \( \square \)

**Remark 2.** Here the advantages of the descriptor method are summarized as follows:
1) Less conservative LMI-based conditions are obtained for the error system.
2) Simple delay-dependent conditions can be derived for the error system.

**Remark 3.** Given the time delay \( h > 0 \). The \( \Lambda \) depending on \( \frac{h}{m} \) means that if the LMI based conditions of Theorem 1 hold for some \( \frac{h}{m} > 0 \), they also hold with the same decision variables for all \( \frac{h}{m} \geq m \). Hence, the feasibility of the LMI s in Theorem 1 guarantees the exponential stability of the system for large enough \( m \).

**D. Extension to sampled-data measurements**

Let
\[
0 = t_0 < t_1 < \cdots < t_l \cdots, \lim_{l \to \infty} t_l = \infty
\]
be sampling time instants. Suppose that the sampling subintervals in time are bounded meaning that
\[
0 \leq t_{l+1} - t_l \leq t_M,
\]
where \( t_M \) is the corresponding upper bound. Further suppose that the sensors provide delayed sampled-data measurements of the state:
\[
y_j^s(t) = \begin{cases} \int_{[t, t+h]} y_j(x, t-h) \, dx & , \quad t \in [t_l+h, t_{l+1}+h), \\ 0, & \quad t < h. \end{cases}, \quad j = 1, \cdots, M. \tag{44}
\]

Similar to subsection A, an observer under the sampled-data measurements is constructed as follows:
\[
\frac{d\hat{y}_1^s(x, t)}{dt} = \Delta \hat{y}_1(x, t) + \sum_{\xi=1}^{p} \sum_{l=1}^{M} \mathcal{X}_j(x) \mathcal{H}_\xi \left( \int_{[\xi]} \dot{y}_1^s(x, t-h) \, dx \right), \tag{45}
\]
where the advantages of the descriptor method are obtained for the error system. Finally, the error system is exponentially stable with a decay
\[
\begin{align*}
\frac{d\hat{y}_1(x, t)}{dt} &= \Delta e^{x}(x, t) + \sum_{\xi=1}^{p} \mathcal{X}_j(x) \mathcal{H}_\xi \left( \int_{[\xi]} e^{x}(x, t-h) \, dx \right), \\
\frac{d\hat{y}_k(x, t)}{dt} &= \Delta e^{k}(x, t) + \sum_{\xi=1}^{p} \mathcal{X}_j(x) \mathcal{H}_\xi \left( \int_{[\xi]} e^{k}(x, t-h) \, dx \right), \quad k = 2, \cdots, m. \tag{46}
\end{align*}
\]

**Theorem 2.** For error system (46) subject to (14) and (17), and given positive scalars \( h, m, \Delta, \delta \) and a positive tuning parameter \( \delta_1 < 2\delta \), suppose that there exist nonnegative scalars \( \lambda_1 \geq 0, \lambda_2 \geq 0 \) and matrices \( 0 < P_l \in \mathbb{R}^{n \times n}, (i = 1, 2, 3, 4) \), \( W_\xi \in \mathbb{R}^{n \times n} (\xi \in \mathcal{S}) \), \( W > 0 \) such that
\[
\mathcal{L} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & W_\xi \\
e & A_{22} + t_M^2 e^{2\delta t_M} W & A_{23} & A_{24} & W_\xi \\
e & * & A_{33} & A_{34} & 0 \\
e & * & * & A_{44} & 0 \\
e & * & * & * & -\frac{\delta}{4} W \end{bmatrix} < 0, \tag{47}
\]
\[
\Theta_1 = -2P_2 + 2\delta P_2 + \lambda_1 I \leq 0, \quad \Theta_2 = -\delta P_2 + \lambda_2 I \leq 0. \tag{48}
\]
where \( \{A_{ij}\} \) are given by (28). Then, for any initial state \( y_0 \in H, \) the error system is exponentially stable with a decay
rate \( \sigma \), where \( \sigma \) is a unique solution of \( \sigma = \delta - \frac{\delta_1}{2}e^{2\sigma h} \), and the observer gain matrices satisfy
\[
L_\xi = P_2^{-1}W_\xi, \ \xi \in S. \quad (50)
\]

**Proof.** For \( k = 1 \), we build a Lyapunov functional (using simplified notations):
\[
\dot{V}_1(t) \triangleq V_1(t) + V_W(t), \ t \in [t_1, t_{t+1}),
\]
where
\[
V_W(t) = t_M^2 e^{28t_M} \int_{\Omega} \left[ e^{28(s-t)}[e^1(x,s)]^TW[e^1(x,s)]dsdx
- \frac{\pi^2}{4} \int_{\Omega} [e^1(x,s) - e^1(x,t_1 - \frac{h}{m})]^TW[e^1(x,s)]dsdx.
\]
By Wirtinger's inequality, \( V_W \) is nonnegative definite. Moreover,
\[
\dot{V}_W(t) + 28V_W(t)
\leq t_M^2 e^{28t_M} \int_{\Omega} [e^1(x,t_1 - \frac{h}{m}) - e^1(x,t_1 - \frac{h}{m})]^TW[e^1(x,t_1 - \frac{h}{m})]dsdx
- \frac{\pi^2}{4} \int_{\Omega} [e^1(x,t_1 - \frac{h}{m}) - e^1(x,t_1 - \frac{h}{m})]^TW[e^1(x,t_1 - \frac{h}{m})]dsdx.
\]
By Theorem 1, the LMIs (25), (26), (27) guarantee (38). Substituting \( e^1(x,t_1 - \frac{h}{m}) \to e^1(x,t_1 - \frac{h}{m}) \) in Theorem 1, we have
\[
\dot{V}_1(t) + 28\dot{V}_1 - \delta_1 \dot{V}_1(t_1 - \frac{h}{m})
\leq \dot{V}_1(t) + 28\dot{V}_1 - \delta_1 \dot{V}_1(t_1 - \frac{h}{m})
+ t_M^2 e^{28t_M} \int_{\Omega} \left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]^TW\left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]dsdx
- \frac{\pi^2}{4} \int_{\Omega} \left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]^TW\left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]dsdx.
\]
By Theorem 1, the LMIs (25), (26), (27) guarantee (38). Substituting \( e^1(x,t_1 - \frac{h}{m}) \to e^1(x,t_1 - \frac{h}{m}) \) in Theorem 1, we have
\[
\dot{V}_1(t) + 28\dot{V}_1 - \delta_1 \dot{V}_1(t_1 - \frac{h}{m})
\leq \dot{V}_1(t) + 28\dot{V}_1 - \delta_1 \dot{V}_1(t_1 - \frac{h}{m})
+ t_M^2 e^{28t_M} \int_{\Omega} \left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]^TW\left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]dsdx
- \frac{\pi^2}{4} \int_{\Omega} \left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]^TW\left[ e^1(x,t_1 - \frac{h}{m}) - e^1(x,\frac{h}{m}) \right]dsdx.
\]
V. **Numerical Illustration**

In this section, we present some simulations for the following nonlinear 2D parabolic PDE:
\[
\begin{aligned}
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + \sin(y(x,t)), \ (x,t) \in \Omega \times (0, \infty) \\
y\mid_{\partial \Omega} &= 0,
\end{aligned}
\]
\begin{equation}
\begin{aligned}
y(x,0) &= g_0(x),
\end{aligned}
\end{equation}
where \( \Omega = [0,1] \times [0,1] \subset \mathbb{R}^2 \).

By arguments of [33], it follows from the following T-S fuzzy rules:

**Plant Rule 1:**
If \( y(\cdot,t) \) is “about 0”, THEN
\[
\begin{aligned}
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + A_1y(x,t), \ (x,t) \in \Omega \times (0, \infty) \\
y\mid_{\partial \Omega} &= 0,
\end{aligned}
\]
\begin{equation}
\begin{aligned}
y(x,0) &= g_0(x),
\end{aligned}
\end{equation}
where \( A_1 = 1, A_2 = \epsilon = 0.01/\pi \). Then, (55) can be represented as
\[
\begin{aligned}
&\frac{\partial y(x,t)}{\partial t} = \Delta y(x,t) + \sum_{\xi = 1}^{\pi} h_\xi(y(x,t))A_\xi y(x,t),
\end{aligned}
\]
\begin{equation}
\begin{aligned}
y\mid_{\partial \Omega} &= 0,
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
y(x,0) &= g_0(x),
\end{aligned}
\end{equation}
where the fuzzy membership are defined by
\[
\begin{aligned}
h_1(y(\cdot,t)) &= \begin{cases} 
\sin(y(\cdot,t)) - e\sin(\cdot,t) & , \quad y(\cdot,t) \neq 0, \\
1, & \text{otherwise,}
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
h_2(y(\cdot,t)) &= 1 - h_1(y(\cdot,t)).
\end{aligned}
\]

With the averaged measurements (5) or (44), for system (56), we use observer (10) or (45) subject to (11) composed of a chain of sub-observers. We verify LMI conditions of Theorems 1 and 2 via Yalmip with \( \delta = 0.2, \delta_1 = 0.1, \Delta = 0.1 \). Tables 1 and 2 show feasible solutions for LMIs of Theorems 1 and 2 that guarantee the exponential stability of the error system for different values of \( m \) and \( h \), respectively.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( m )</th>
<th>( P_2 )</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2.2072</td>
<td>0.4488</td>
<td>0.1721</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2.6591</td>
<td>0.7870</td>
<td>0.5896</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>1.9165</td>
<td>0.6597</td>
<td>0.4145</td>
</tr>
<tr>
<td>0.25</td>
<td>3</td>
<td>1.9192</td>
<td>0.6772</td>
<td>0.4202</td>
</tr>
</tbody>
</table>

Table 1

<table>
<thead>
<tr>
<th>( h )</th>
<th>( m )</th>
<th>( P_2 )</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.7039</td>
<td>0.0972</td>
<td>0.0505</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.6464</td>
<td>0.1453</td>
<td>0.1002</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>2.0292</td>
<td>0.5040</td>
<td>0.3083</td>
</tr>
<tr>
<td>0.25</td>
<td>3</td>
<td>2.0251</td>
<td>0.5138</td>
<td>0.3062</td>
</tr>
</tbody>
</table>
Table 2 for \( t_M = 0.1 \).

For instance, in Table 2 for \( h = 1 \) and \( m = 1 \), we find the following feasible solutions for the LMIs of Theorem 2: \( P_2 = 0.7039, W_1 = 0.0972, W_2 = 0.0505 \). Thus, the observer gains are found to be \( L_1 = 0.1380, L_2 = 0.0717 \).

Now a finite difference method is adopted in discretization. The steps of space and time are set to be 0.05 and 0.0001, respectively. Let the domain \( \Omega = [0, 1]^2 \) be divided into \( N = 4 \) squares of side length \( 1/\sqrt{N} = 0.5 \). In the case of \( m = 1 \) and \( h = 1 \), Fig. 2 shows that the evolution of the \( L^2(\Omega) \)-norm \( \| e^1(\cdot, t) \|_{L^2(\Omega)} \) and snapshots of the state \( e^1(x, t) \) at different time for the error system (46) subject to (14), (17) initialized by \( y_0(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2) \), \( (x_1, x_2) \in \Omega \). Fig. 3 shows that the evolution of the \( L^2(\Omega) \)-norm \( \| e^k(\cdot, t) \|_{L^2(\Omega)}(k = 1, 2, \cdots, m) \) for the error system (46) subject to (14), (17) with the same initial condition for different values of \( m \) and \( h \). As expected, simulation of solutions demonstrates that the error system is exponentially stable.

VI. CONCLUDING REMARKS

The present paper discusses fuzzy observer design for 2D parabolic PDEs with delayed spatially averaged measurements. Differently from 1D parabolic equation, here we manage with a new fuzzy observer design for high dimension PDEs. In addition, this observer is composed of \( m \)-chain of sub-observers. By constructing an appropriate Lyapunov functional, sufficient LMI-based conditions are found to achieve the exponential stability of the error system. A numerical example demonstrates the efficiency of the proposed scheme.

The obtained theoretical results are applicable to cascaded PDE-PDE/ PDE-ODE system and other high-dimensional distributed parameter system. Our next step places its main focus on coupled PDE-PDE/ PDE-ODE system under the averaged measurement subject to time delay. It should be noticed that the time delay considered in the present paper is a constant, which can be arbitrarily large. If the time delay becomes time-varying one, the situation becomes more complicated. It seems unclear whether the proposed method is applicable to this case, which will be our next future investigation.

REFERENCES

Fig. 3. Error system (46), (14), (17) with different values of $m$ and $h$


M. Krestic, A. Smyshlyaev (2008) Boundary control of PDEs: A course on backstepping design, SIAM.


