Fuzzy Observer for 2-D Parabolic Equation With Output Time Delay

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Abstract—This article addresses fuzzy observer design for a nonlinear parabolic equation over an unit square domain $\Omega$ in terms of the time delayed spatially averaged measurement, where the observer is composed of $m$-chain of subobservers. Due to 2-D domain, special emphases are made to the computational complexity. A Lyapunov argument is utilized to give constructive conditions ensuring the exponential stability of the resulting error system. The method used for the continuous-time fuzzy observer is applicable to the sampled-data implementation. Consistent simulation results that support the proposed theoretical statements are presented.

Index Terms—Lyapunov design, nonlinear 2-D parabolic equation, observer.

I. INTRODUCTION

Two-Dimensional (2-D) parabolic equations which have strong background in engineering, physics, fluid mechanics, biological mathematics, and materials science have attracted increasing attention in decades. These infinite dimensional systems can be used to model many physical, biological, and chemical systems. Stabilization for 1-D parabolic equations has been investigated by many researchers like those in [1]–[6] among many others. In [2], a backstepping method has been used to stabilize an unstable heat equation with large input time delay. The same approach has also been generalized to reaction-diffusion equations with state time delay (see, e.g., [3]). Very recently, distributed sampled control design for heat equation [4], [5] and Kuramoto–Sivashinsky equation [1] with point/averaged measurements has been investigated. The work in [6] has considered distributed control design of a Korteweg-de Vries-Burgers equation subject to input time delay. However, many challenging problems for 2-D case are still open to date. Concurrently, for practical applications, substantial efforts have been made to develop estimation/control for high-dimensional partial differential equations (PDEs) (see, e.g., [7]–[9]). A recent work [7] has dealt with network-based $H_\infty$ filtering problem for diffusion equation. In [8] and [9], boundary feedback for 2-D channel flow has been studied. One of the interesting problems is about design of fuzzy observer for 2-D parabolic equations in terms of delayed spatially averaged measurement.

The fuzzy control has many applications in various of fields such as robotics, motor industry, signal processing, control engineering, and so on. Actually, the fuzzy control systems have been extensively investigated in many articles [10]–[23], in particular, on stability and stabilization for delayed fuzzy systems (see, e.g., [21]–[23]). In addition, for practical applications, the study of adaptive fuzzy fault tolerant control for Markov jump systems has become an active research topic (see, e.g., [20]). However, most of works focus on finite dimensional systems. The problems of infinite dimensional systems are much more complicated. From a practical point of view, advanced fuzzy observer design for 2-D parabolic equations subject to delayed spatially averaged measurement is challenging, which motivates the present article. Compared with the previous work [1], [6], [24], in this article, a sector nonlinearity approach is adopted to deal with the nonlinearity of a PDE system through construction of a Takagi–Sugeno (T–S) fuzzy PDE model. To enlarge the delay size, the concept of chain of subobservers has been extended to a 2-D parabolic equation. With the help of an appropriate Lyapunov-based argument, we develop a stability analysis for error system.

In comparison to the existing known results, new special challenges and main differences of this work are as follows.

1) A sampled-data fuzzy observer design for high-dimensional distributed parameter system is addressed, which becomes much more complicated for 2-D case. We design a new fuzzy observer which is composed of m-chain of subobservers.

2) In the presence of output time delay, we consider two cases: sampled-data fuzzy observer and continuous-time fuzzy observer design for 2-D parabolic equation. For both cases, we give exponential stability conditions for the corresponding error system in terms of linear matrix inequalities (LMIs). Due to 2-D domain, special emphases are made to the computational complexity. The stability has been established based on 2-D Wirtinger’s, Poincaré’s inequalities (Lemma 1 later), together with time-delay approach and descriptor method.

The rest of this article is organized as follows. In Section II, some preliminary lemmas are introduced. In Section III, an observer setting is presented. Section IV is devoted to the fuzzy observer design in terms of the delayed spatially averaged measurements, and the well-posedness and stability analysis are established. Moreover, sufficient LMI-based conditions are presented for the error system via the Poincaré’s and Wirtinger’s inequalities. In Section V, consistent simulation results which support the proposed theoretical statements are presented. Finally, Section VI concludes this article.

II. MATHEMATICAL PRELIMINARIES

Notation. The following notations are used throughout this article.

1) $\mathbb{R}_+ := [0, \infty)$.

2) The superscript $\mathrm{T}$ stands for the transposition of matrix. For a partitioned matrix, the symbol $*$ stands for symmetric blocks.

3) $I$ stands for the identity matrix with appropriate dimension.
4) When \( \mathcal{O} \subset \mathbb{R}^2 \) is used for a computational domain, the \( L^2_{\mathcal{O}}(\mathcal{O}) \equiv L^2(\mathcal{O}; \mathbb{R}^n) \) stands for the space of measurable squared-integrable functions over \( \mathcal{O} \) with the norm \( \| g \|^2_{L^2_{\mathcal{O}}(\mathcal{O})} = \int_{\mathcal{O}} |g(x)|^2 \, dx \). Let \( H^1_{\mathcal{O}}(\mathcal{O}) = \{ g : \partial^\alpha g \in L^2_{\mathcal{O}}(\mathcal{O}), \forall \alpha \leq k \} \) denote the Sobolev space with the norm \( \| \partial^\alpha g \|^2_{L^2_{\mathcal{O}}(\mathcal{O})} = \sum_{\alpha \leq k} \| \partial^\alpha g \|^2_{L^2_{\mathcal{O}}(\mathcal{O})} \).

**Lemma 1:** Let \( \mathcal{O} = (0, L_j) \times (0, L_j) \). Suppose that \( \mu \in H^1_{\mathcal{O}}(\mathcal{O}) \).

(i) (Wirtinger’s inequality) \([7]\) If \( \mu|_{\partial\mathcal{O}} = 0 \), then, the following inequality holds:

\[
\| \mu \|^2_{L^2_{\mathcal{O}}(\mathcal{O})} \leq \frac{L^2_j + L^2_j}{\pi^2} \| \nabla \mu \|^2_{L^2_{\mathcal{O}}(\mathcal{O})}.
\]  

(ii) (Poincaré’s inequality) \([25]\) If \( \int_{\partial \mathcal{O}} \mu(x) \, dx = 0 \), then,

\[
\| \mu \|^2_{L^2_{\mathcal{O}}(\mathcal{O})} \leq \frac{L^2_j + L^2_j}{\pi^2} \| \nabla \mu \|^2_{L^2_{\mathcal{O}}(\mathcal{O})}.
\]

**Lemma 2:** (Halánay’s inequality) \([26]\) Let \( 0 < \delta_1 < 2\delta \) and let \( E : [-\delta, \infty) \to \mathbb{R}_+ \) be an absolutely continuous function satisfying

\[
\dot{E}(t) + 2\delta E(t) - \Delta \sup_{\theta \leq \delta \theta \leq \theta} E(t + \theta) \leq 0, \quad t \geq 0.
\]

Then,

\[
E(t) \leq e^{-\delta t} \sup_{\theta \leq \delta \theta \leq \theta} E(0), \quad t \geq 0
\]

where \( \sigma \) is a unique solution of

\[
\sigma = \delta - \frac{\delta_1 e^{2\delta h}}{2}.
\]

**Lemma 3:** (Jensen’s inequality \([32]\)) Let \( \mathcal{O} = [a, b] \). Suppose \( \psi : \mathcal{O} \to \mathbb{R}^n \) and \( \psi \in L^1_{\mathcal{O}}(\mathcal{O}) \). Then, the following inequality holds:

\[
\int_a^b \psi^T(x)Q\psi(x) \, dx \geq \frac{1}{b-a} \left( \int_a^b \psi^T(x) \, dx \right) Q \left( \int_a^b \psi(x) \, dx \right)
\]

for any matrix \( Q \geq 0 \).

### III. Problem Formulation

In this article, we consider the following parabolic equation in a 2-D square domain:

\[
\begin{align*}
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + f(y(x,t)), \quad (x,t) \in \Omega \times (0, \infty) \\
y(x,0) &= y_0(x)
\end{align*}
\]  

where \( x = (x_1, x_2) \in \Omega, \Omega = [0,1] \times [0,1] \subset \mathbb{R}^2 \). \( \partial \Omega \) stands for the boundary of \( \Omega \), and \( \Delta \) is the Laplacian operator:

\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.
\]

We denote the state by \( y(x,t) \equiv (y_1(x,t), y_2(x,t), \ldots, y_n(x,t)) \in \mathbb{R}^n \). The nonlinear function \( f(\cdot) \) is of \( C^1 \) and is globally Lipschitz continuous with \( f(0) = 0 \).

Motivated by \([4, 5, 28]\), we divide the whole region \( \Omega \) into \( M \) square subdomains \( \Omega_j \), as shown in Fig. 1, where

\[
\bigcup_{j=1}^M \Omega_j = \Omega.
\]

The sampling subdomains in space are supposed to be bounded in the sense that

\[
\text{Omega}_j = \{ x = (x_1, x_2) \in \Omega | x_i \in [x_{i}^{\min}(j), x_{i}^{\max}(j)], i = 1, 2 \}
\]

\[
j = 1, 2, \ldots, M
\]

where \( \Delta \) is the corresponding upper bound, and the Lebesgue measure of their intersections is zero.

**Remark 1:** For the sake of simplicity, we consider each subdomain \( \Omega_j \) to be a square. Indeed, \( \Omega_j \) can be a rectangular and the results of this work are applicable to the case of nonsquare subdomains.

It is further supposed that the sensors provide time delayed spatially averaged measurements

\[
y_{\text{sd}}(t) = \left\{ \begin{array}{ll}
\frac{\int_{\Omega_j} y(x,t-h) \, dx}{|\Omega_j|}, & t \geq h, \  j = 1, \ldots, M \\
0, & t < h
\end{array}\right.
\]

\((5)\)

where \( |\Omega_j| \) is the Lebesgue measure of the domain \( \Omega_j \), and \( h > 0 \) is the time delay which may be large.

We aim at constructing a fuzzy observer which is composed of a chain of several subobservers for (4) to achieve exponentially convergence of the error system for an arbitrary time delay \( h \).

### IV. Proposed Observer Strategy

Due to the nonlinearity in system, we employ T–S fuzzy PDE model and design a fuzzy observer. LMI-based conditions will be then developed for the exponential stability of the error system.

**A. Fuzzy Observer Design**

The fuzzy IF-THEN rules play an instrumental role in constructing the following T–S fuzzy PDE model:

**Plant Rule \( \xi \):**

\[
\begin{align*}
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + A_{\xi} y(x,t), \quad (x,t) \in \Omega \times (0, \infty) \\
y|_{\partial \Omega} &= 0,
\end{align*}
\]

**Observer Rule \( \xi \):**

\[
\begin{align*}
\frac{\partial y_i(x,t)}{\partial t} &= \Delta y_i(x,t) + \sum_{\xi=1}^p \theta_{i,\xi} (\Gamma(x,t)) A_{\xi} y(x,t) + I_i(t) \\
y|_{\partial \Omega} &= 0,
\end{align*}
\]

where \( A_{\xi} \) is the corresponding upper bound, and the Lebesgue measure of their intersections is zero.

**Remark 1:** For the sake of simplicity, we consider each subdomain \( \Omega_j \) to be a square. Indeed, \( \Omega_j \) can be a rectangular and the results of this work are applicable to the case of nonsquare subdomains.

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We aim at constructing a fuzzy observer which is composed of a chain of several subobservers for (4) to achieve exponentially convergence of the error system for an arbitrary time delay \( h \).
where \( \Gamma(x, t) = [\Gamma_1(x, t), \Gamma_2(x, t), \ldots, \Gamma_q(x, t)]^T \)

\[
h_{\xi}(\Gamma(x, t)) = \frac{\prod_{s=1}^{q} F_{\xi s}(\Gamma_s(x, t))}{\sum_{s=1}^{q} \prod_{s=1}^{q} F_{\xi s}(\Gamma_s(x, t))}, \quad \xi \in S
\]

and \( F_{\xi s} \) represents the grade of the membership of \( \Gamma_s(x, t) \) in \( F_{\xi s} \) for \( \xi \in S \). Here, we suppose that

\[
\prod_{s=1}^{q} F_{\xi s}(\Gamma_s(x, t)) \geq 0, \quad \xi \in S
\]

\[
\sum_{\xi=1}^{p} \prod_{s=1}^{q} F_{\xi s}(\Gamma_s(x, t)) > 0.
\]

Then,

\[
h_{\xi}(\Gamma(x, t)) \geq 0, \quad \xi \in S, \quad \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) = 1. \tag{8}
\]

As in [35], we first denote

\[
y^0(x, t) = y(x, t - h)
\]

\[
y^k(x, t) = y \left( x, t + \frac{k}{m}h - h \right), \quad k = 1, \ldots, m
\]

that is,

\[
y^m(x, t) = y(x, t)
\]

\[
y^{k-1}(x, t) = y^{k} \left( x, t - \frac{h}{m} \right), \quad k = 1, \ldots, m. \tag{9}
\]

From the parallel distributed compensation scheme, we introduce the following observer for the fuzzy model (6) in terms of the delayed spatially averaged measurements (5):

\[
\frac{\partial \hat{y}^1(x, t)}{\partial t} = \Delta \hat{y}^1(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A_{\xi} \hat{y}^1(x, t)
\]

\[
- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \chi_{j}(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} \frac{\hat{y}^{j}(x, t - \frac{h}{m})}{|\Omega_j|} - \hat{y}^j_{\text{obs}}(t) dx
\]

\[k = 1, \quad t > 0\]

\[
\frac{\partial \hat{y}^k(x, t)}{\partial t} = \Delta \hat{y}^k(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A_{\xi} \hat{y}^k(x, t)
\]

\[
- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \chi_{j}(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} \frac{\hat{y}^{j}(x, t - \frac{h}{m}) - \hat{y}^{j-1}(x, t)}{|\Omega_j|} dx
\]

\[k = 2, \ldots, m, \quad t > 0. \tag{10}
\]

subject to

\[
\begin{cases}
\hat{y}^1_{|\Omega_0} = \hat{y}^2_{|\Omega_0} = \cdots = \hat{y}^m_{|\Omega_0} = 0 \\
\hat{y}^1(x, t) = \hat{y}^2(x, t) = \cdots = \hat{y}^m(x, t) = 0, \quad t \leq 0
\end{cases} \tag{11}
\]

where the spatial characteristic functions are taken as

\[
\begin{cases}
\chi_{j}(x) = 1, \ x \in \Omega_j \\
\chi_{j}(x) = 0, \ x \notin \Omega_j
\end{cases} \quad j = 1, \ldots, N
\]

\( m \) is the number of the subobservers, and \( L_{\xi} \in \mathbb{R}^{n \times n} \) are observer gain parameters that will be determined later.

Let \( e^k(x, t) = \hat{y}^k(x, t) - y^k(x, t) \). Then, the estimation error is governed by

\[
\frac{\partial e^1(x, t)}{\partial t} = \Delta e^1(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A_{\xi} e^1(x, t)
\]

\[
- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \chi_{j}(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} e^j(x, t - \frac{h}{m}) dx
\]

\[k = 1, \quad \forall t > 0 \tag{12}
\]

\[
\frac{\partial e^k(x, t)}{\partial t} = \Delta e^k(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A_{\xi} e^k(x, t)
\]

\[
- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \chi_{j}(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} [e^j(x, t - \frac{h}{m}) - e^{j-1}(x, t)] dx
\]

\[k = 2, \ldots, m, \quad \forall t > 0 \tag{13}
\]

subject to

\[
e^1_{|\Omega_0} = e^2_{|\Omega_0} = \cdots = e^m_{|\Omega_0} = 0. \tag{14}
\]

From (9), it follows that

\[
\hat{y}^k \left( x, t - \frac{h}{m} \right) - \hat{y}^{k-1}(x, t)
\]

\[
= \hat{y}^k \left( x, t - \frac{h}{m} \right) - y^k \left( x, t - \frac{h}{m} \right) + y^k \left( x, t - \frac{h}{m} \right) - \hat{y}^{k-1}(x, t)
\]

\[
= y^k \left( x, t - \frac{h}{m} \right) - \hat{y}^k(x, t) + \hat{y}^k(x, t) - \hat{y}^{k-1}(x, t)
\]

\[
= e^k \left( x, t - \frac{h}{m} \right) - e^{k-1}(x, t).
\]

Hence, for \( k = 2, \ldots, m, \) (13) becomes

\[
\frac{\partial e^k(x, t)}{\partial t} = \Delta e^k(x, t) + \sum_{\xi=1}^{p} h_{\xi}(\Gamma(x, t)) A_{\xi} e^k(x, t)
\]

\[
- \sum_{\xi=1}^{p} \sum_{j=1}^{M} \chi_{j}(x) h_{\xi}(\Gamma(x, t)) L_{\xi} \int_{\Omega_j} [e^j(x, t - \frac{h}{m}) - e^{j-1}(x, t)] dx
\]

\[\forall t > 0. \tag{15}
\]

We redefine the initial condition to be a function:

\[
y(x, t) = y_0(x), \quad t \leq 0 \tag{16}
\]

because the solution of (7) is independent of the values of \( y(x, t) \) for \( t \leq 0 \). Moreover, (16) implies that

\[
e^k(x, t) = -y_0(x), \quad t \leq 0. \tag{17}
\]

Therefore, the error system becomes (12), (14), and (15) initialized by (17).

### B. Well-Posedness

Now we use step method to establish the well-posedness of the error system (12) and (15) subject to (14) and (17). Define the Hilbert space

\[
H = \{ \psi \in H^2_{\delta}(\Omega) : \psi_{|\Omega_0} = 0 \}.
\]
and the spatially differential operator $\mathcal{A}: D(\mathcal{A}) \subset L^2_a(\Omega) \to L^2_a(\Omega)$ as follows:

\[
\begin{align*}
\mathcal{A}\psi &= \Delta \psi, \forall \psi \in D(\mathcal{A}), \\
D(\mathcal{A}) &= H^2_a(\Omega) \cap H.
\end{align*}
\]

A direct computation yields

\[
\mathcal{A}^* = \mathcal{A}, \quad \text{Re}(\mathcal{A}\psi, \psi) = -\int_\Omega |\nabla \psi|^2 dx \leq 0. \quad (18)
\]

Hence, $\mathcal{A}$ is self-adjoint, dissipative and generates an analytic $C_0$-semigroup $e^{t\mathcal{A}}$. Since $-\mathcal{A}$ is positive, $(-\mathcal{A}^* \mathcal{A})^{1/2}$ is positive and $D((-\mathcal{A}^* \mathcal{A})^{1/2}) = H$. Set $X(t) := e^t \cdot t$. Then, for $t \in [0, \frac{1}{m}]$, we can rewrite (12) as the evolution equation

\[
\begin{align*}
\frac{d}{dt} X(t) &= \mathcal{A} X(t) + F(X(t)) \quad (19)
\end{align*}
\]

where

\[
F(X(t)) = \sum_{k=1}^p h_k(\Gamma(\cdot, t)) A_k X(t) + \sum_{k=1}^p \sum_{j=1}^M \mathcal{L}_j(\cdot, \Gamma(\cdot, t)) L_k - \int_{\Omega} y_0(x) dx. \quad (20)
\]

Since $F$ is Lipschitz continuous, for any initial value $y_0 \in H$, there exists a unique strong solution $e^t \in C([0, \frac{1}{m}]; H) \cap L^2([0, \frac{1}{m}; D(\mathcal{A}))$.

The same line of reasoning is applied step-by-step to the time segments $[\frac{1}{m}, \frac{2}{m}]; [\frac{2}{m}, \frac{3}{m}], \ldots$ Following the procedure, we arrive at that there exists a unique strong solution $e^t \cdot t$ for all $t \geq 0$.

By the same arguments, for $k = 2, \ldots, m$, and any initial value $y_0 \in H$ there exists a unique strong solution $e^t \cdot t$ for all $t \geq 0$.

C. Stability Analysis

In this section, we establish the stability of the error system (12) and (15) subject to (14) and (17). To this purpose, we build the following Lyapunov–Krasovskii functionals:

\[
V_k(t) = \sum_{i=1}^4 V_{k1}(t), \quad k = 1, 2, \ldots, m \quad (21)
\]

where

\[
V_{k1}(t) = \int_\Omega [e^k(x, t)]^T P_k [e^k(x, t)] dx \quad (22)
\]

\[
V_{k2}(t) = \int_\Omega [\nabla e^k(x, t)]^T P_k [\nabla e^k(x, t)] dx \quad (23)
\]

\[
V_{k3}(t) = \int_\Omega \int_{t}^{t^+} e^{-2\theta(s)} [e^k(x, s)]^T P_k [e^k(x, s)] ds dx \quad (24)
\]

\[
V_{k4}(t) = \int_\Omega \int_{t}^{t^+} e^{-2\theta(s)} [e^k(x, s)]^T P_k [e^k(x, s)] ds dx \quad (25)
\]

with $0 < P_k \in \mathbb{R}^{n \times n} (i = 1, 2, 3, 4)$ to be determined.

The succeeding Theorem 1 provides sufficient conditions in form of LMIs for the error system (12) and (15) subject to (14) and (17).

\textbf{Theorem 1:} For error system (12) and (15) subject to (14) and (17) and given positive scalars $\Delta$, $\delta$, $\delta_1 < 2\delta$, suppose that there are nonnegative scalars $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and matrices $0 < P_i \in \mathbb{R}^{n \times n}, (i = 1, 2, 3, 4)$, $W_k \in \mathbb{R}^{n \times n} (k \in S)$ satisfying

\[
\begin{align*}
\Lambda &= \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\
\lambda_{22} & \lambda_{23} & \lambda_{24} & * \\
\lambda_{33} & \lambda_{34} & * & * \\
* & * & * & * \\
\end{bmatrix} < 0 \quad (26)
\end{align*}
\]

\[
\begin{align*}
\Theta_1 &= -2P_2 + 2\delta P_2 + \lambda_1 I \leq 0 \quad (27)
\end{align*}
\]

and the observer gain matrices satisfy

\[
\begin{align*}
L_k &= P_k^{-1} W_k, \quad k \in S. \quad (28)
\end{align*}
\]

\textbf{Proof:} The proof will be split into two steps.

\textbf{Step 1: Derive sufficient conditions such that $V_1(t) + 2\delta V_1(t) + \delta_1 V_1(t) (t - \frac{1}{m}) \leq 0$ holds.}

Differentiating $V_1(t)$ along the solution of (12) subject to (17), we get

\[
V_1(t) + 2\delta V_1(t) = 2 \int_\Omega [e^1(x, t)]^T P_1 [e^1(x, t)] dx \quad (29)
\]

with $0 < P_1 \in \mathbb{R}^{n \times n} (i = 1, 2, 3, 4)$ to be determined.

The succeeding Theorem 1 provides sufficient conditions in form of LMIs for the error system (12) and (15) subject to (14) and (17).

\textbf{Theorem 1:} For error system (12) and (15) subject to (14) and (17) and given positive scalars $\Delta$, $\delta$, $\delta_1 < 2\delta$, suppose that there are nonnegative scalars $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and matrices $0 < P_i \in \mathbb{R}^{n \times n}, (i = 1, 2, 3, 4), W_k \in \mathbb{R}^{n \times n} (k \in S)$ satisfying
Now, we give several estimates for the aforementioned terms. First, applying Jensen’s inequality, we have
\[
-\int_0^\tau \int_\Omega e^{-2\delta(t-s)} \left[ e_x^* (x,s) \right]^T P_4 \left[ e_x (x,s) \right] dsdx \\
\leq -\frac{m}{h^2} \frac{2\pi^2}{2} \int_\Omega \left[ e^1 (x,t) - e^1 \left( x, t - \frac{h}{m} \right) \right]^T \\
\times P_4 \left[ e^1 (x,t) - e^1 (x, t - \frac{h}{m}) \right] dx.
\]
Next, the Wirtinger’s inequality leads to
\[
\lambda_1 \left\| \nabla e^1 (\cdot, t) \right\|^2_{L_2 (\Omega)} - \frac{\pi^2}{2} \left\| e^1 (\cdot, t) \right\|^2_{L_2 (\Omega)} \geq 0
\]
for any \( \lambda_1 \geq 0. \)

Applying Jensen’s inequality, we have
\[
2 \int_\Omega \left[ e^1 (x,t) \right]^T \left[ e^1_t (x,t) \right]^T P_2 \\
\times \left[ \Delta e^1 (x,t) - e^1_t (x,t) + \sum_{\xi = 1}^p h_\xi (\Gamma (x,t)) \Lambda_\xi e^1 (x,t) \right. \\
- \sum_{\xi = 1}^p \sum_{j = 1}^m \mathcal{X}_j (x) h_\xi (\Gamma (x,t)) L_\xi \int_{\Omega_j} e^1 (x,t - \frac{h}{m}) dx \right] = 0.
\]
Performing the integration by parts in \( x, \) we obtain
\[
\int_\Omega \left[ e^1 (x,t) \right]^T P_2 [\Delta e^1 (x,t)] dx \\
= - \int_\Omega \left[ \nabla e^1_t (x,t) \right]^T P_2 [\nabla e^1 (x,t)] dx
\]
\[
\int_\Omega \left[ e^1_t (x,t) \right]^T P_2 [\Delta e^1 (x,t)] dx \\
= - \int_\Omega \left[ \nabla e^1_t (x,t) \right]^T P_2 [\nabla e^1 (x,t)] dx.
\]
Denote
\[
f_j (x,t) = e^1 (x,t) - \frac{\int_{\Omega_j} e^1 (x,t) dx}{[\Omega_j]}.\]
Since \( \int_{\Omega_j} f_j (x,t) dx = 0, \) the Poincaré’s inequality yields
\[
\left\| f_j (\cdot, t) \right\|^2_{L_2 (\Omega_j)} \leq \frac{2 \pi^2}{\sigma^2} \left\| \nabla e^1 (\cdot, t) \right\|^2_{L_2 (\Omega_j)}.
\]
Hence,
\[
\lambda_2 \sum_{j = 1}^M \left\| \nabla e^1 (\cdot, t - \frac{h}{m}) \right\|^2_{L_2 (\Omega_j)} - \frac{\pi^2}{2 \pi^2} \left\| f_j \left( \cdot, t - \frac{h}{m} \right) \right\|^2_{L_2 (\Omega_j)} \geq 0
\]
for any \( \lambda_2 \geq 0. \)

Finally, set
\[
\varphi = \col \left\{ e^1 (x,t), e^1_t (x,t), e^1 (x,t - \frac{h}{m}), f_j \left( x, t - \frac{h}{m} \right) \right\}
\]
and \( W \varphi = P_2 L_\xi (\xi \in S). \)

Adding (32), (33), and (37) into \( \dot{V}_1 + 2 \delta V_1, \) and applying (30), (31), (34), and (35), we arrive at
\[
\dot{V}_1 + 2 \delta V_1 - \delta_1 V_1 \left( t - \frac{h}{m} \right) \\
\leq \dot{V}_1 + 2 \delta V_1 - \delta_1 V_1 \left( t - \frac{h}{m} \right) \\
+ \lambda_1 \left[ \left\| \nabla e^1 (\cdot, t) \right\|_{L_2^2 (\Omega)} - \frac{\pi^2}{2} \left\| e^1 (\cdot, t) \right\|_{L_2^2 (\Omega)} \right] \\
+ \lambda_2 \sum_{j = 1}^M \left[ \left\| \nabla e^1 (\cdot, t - \frac{h}{m}) \right\|_{L_2^2 (\Omega_j)} - \frac{\pi^2}{2 \pi^2} \left\| f_j \left( \cdot, t - \frac{h}{m} \right) \right\|_{L_2^2 (\Omega_j)} \right] \\
\leq \sum_{k = 1}^m \sum_{j = 1}^M \int_{\Omega_j} h_\xi (\Gamma (x,t)) \varphi^T L \varphi dx \\
- \int_\Omega \left[ \nabla e^1 (x,t) \right]^T \Theta_1 [\nabla e^1 (x,t)] dx \\
- \int_\Omega \left[ \nabla e^1 (x,t - \frac{h}{m}) \right]^T \Theta_2 [\nabla e^1 (x,t - \frac{h}{m})] dx \leq 0
\]
provided that the LMIs (25)–(27) hold.

Step 2: Show that the LMIs (25)–(27) hold:
\[
\dot{V}_k (t) + \left( 2 \delta - \nu \sigma^2 \right) V_k (t) - \nu V_k \left( t - \frac{h}{m} \right) \leq 0
\]
for \( k = 2, 3, \ldots, m, \) where \( \nu \sigma^2 \) is large enough and \( \nu \) is small enough such that
\[
\delta_1 < 2 \delta - \nu \sigma^2.
\]
To this end, we construct the following Lyapunov–Krasovskii functional:
\[
V (t) = \sum_{k = 1}^m e^{2(k-1)} V_k (t).
\]
Differentiating \( V (t) \) along the solution of (12) and (15) subject to (14) and (17), we have
\[
\dot{V} (t) + \left( 2 \delta - \nu \sigma^2 \right) V (t) - \delta_1 \sup_{-h \leq s \leq 0} V (t + s) \\
\leq \dot{V} (t) + \left( 2 \delta - \nu \sigma^2 \right) V (t) - \delta_1 V \left( t - \frac{h}{m} \right) \leq 0.
\]
Furthermore, from (40) and Halanay’s inequality, it follows that
\[
V (t) \leq e^{2 \sigma t} \sup_{-h \leq \theta \leq 0} V (\theta), \quad t \geq 0
\]
where \( \sigma \) is a unique solution of
\[
\sigma = \delta - \delta_1 e^{2 \sigma h}. \]

**Remark 2:** Here, the advantages of the descriptor method are summarized as follows:
1. Less conservative LMI-based conditions are obtained for the error system.
2. Simple delay-dependent conditions can be derived for the error system.
Remark 3: Please note that our objective is to derive constructive stability conditions in terms of LMIs. Halanay’s inequality allows us to do this. The general Gronwall’s lemma usually leads to qualitative results in terms of bounds on the norms of solutions, which seem conservative in terms of upper bounds on sampling/delays that preserve the stability.

Remark 4: Given the time delay \( h > 0 \). The \( \Lambda \) depending on \( \frac{h}{m} \) means that if the LMI-based conditions of Theorem 1 hold for some \( \frac{h}{m} > 0 \), they also hold with the same decision variables for all \( m \). Hence, the feasibility of the LMIs in Theorem 1 guarantees the exponential stability of the system for large enough \( m \).

D. Extension to Sampled-Data Measurements

Let

\[
0 = t_0 < t_1 < \cdots < t_i \cdots \lim_{i \to \infty} t_i = \infty
\]

be sampling time instants. Suppose that the sampling subintervals in time are bounded meaning that

\[
0 \leq t_{i+1} - t_i \leq t_M
\]

where \( t_M \) is the corresponding upper bound. Further suppose that the sensors provide delayed sampled measurements of the state

\[
y_j(t) = \begin{cases} 
\int_{t_j}^{t_{j+1}} \frac{E(y_j(t,h) - h) \, dt}{m_j}, & t \in [t_j + h, t_{j+1} + h) \\
0, & t < h 
\end{cases}
\]

(44)

Similar to section A, an observer under the sampled-data measurements is constructed as follows:

\[
\frac{\partial \bar{y}^1(x, t)}{\partial t} = \Delta \bar{y}^1(x, t) + \delta P \sum_{i=1}^{M} J_i \bar{x}(\Gamma(x, t)) A \bar{x}(x, t) \\
- \frac{\sum_{j=1}^{M} M_j \bar{x}(x, t) L \left[ \int_{t_j}^{t_{j+1}} \frac{\bar{y}(x, t_j + h - m) \, dt}{\partial [\Omega_j]} - \bar{y}(t) \right]}{1_k = 1, \forall t \in [t_j, t_{j+1}]}
\]

(44)

Then, the error system is governed by

\[
\frac{\partial e^1(x, t)}{\partial t} = \Delta e^1(x, t) + \delta \sum_{i=1}^{M} \bar{x}(\Gamma(x, t)) A \bar{x}(x, t) + \frac{\sum_{j=1}^{M} M_j \bar{x}(x, t) L \left[ \int_{t_j}^{t_{j+1}} \frac{e^1(x, t_j + h - m) \, dt}{\partial [\Omega_j]} - e^1(t) \right]}{1_k = 1, \forall t \in [t_j, t_{j+1}]}
\]

(44)

Theorem 2: Consider error system (46) subject to (14) and (17). Given positive scalars \( h, m, \Delta, \delta \), and a positive tuning parameter \( \delta_1 < 2 \delta \), suppose that there exist nonnegative scalars \( \lambda_1 \geq 0, \lambda_2 \geq 0 \) and matrices \( 0 < P_i \in \mathbb{R}^{n_x \times n_x} \), \( (i = 1, 2, 3, 4) \), \( W \in \mathbb{R}^{n_x \times n_x} \), \( (i \in S) \), \( W > 0 \) such that

\[
\bar{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 + \frac{\delta}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & P_i 
\end{bmatrix} < 0
\]

(47)

\[
\Theta_1 = -2P_2 + 2 \delta P_2 + \lambda_1 I \leq 0
\]

(48)

\[
\Theta_2 = -4 \delta \lambda_2 I \leq 0
\]

(49)

where \( \{ A_{ij} \} \) are given by (28). Then, for any initial state \( y_0 \in H \), the error system is exponentially stable with a decay rate \( \sigma \), where \( \sigma \) is a unique solution of \( \sigma = -\frac{\delta}{2} \frac{1}{2} e^{2 \delta h} \), and the observer gain matrices satisfy

\[
L \xi = P_2^{-1} W \xi, \xi \in S.
\]

(50)

Proof: For \( k = 1 \), we build a Lyapunov functional (using simplified notations)

\[
\tilde{V}_1(t) = \tilde{V}_1(t) + V_W(t), t \in [t_k, t_{k+1})
\]

(51)

where

\[
V_W(t) = e^{2 \delta t} \int_{t_k}^{t} \frac{e^{2 \delta(s-t)}[e^1(x, s)]^\top W[e^1(x, s)] \, ds \, dx}{4}
\]

\[
- \frac{\pi^2}{4} \int_{t_k}^{t} \frac{e^{2 \delta(s-t)}[e^1(x, s) - e^1(x, t - h/m)]^\top}{4} \times W[e^1(x, s) - e^1(x, t - h/m)] \, ds \, dx
\]

(52)

By Wirtinger’s inequality, \( V_W \) is nonnegative definite. Moreover,

\[
\tilde{V}_1(t) + 2 \delta V_1(t)
\]

\[
\leq \frac{\pi^2}{4} \int_{t_k}^{t} \frac{e^{2 \delta(s-t)}[e^1(x, s)]^\top W[e^1(x, s)] \, ds \, dx}{4}
\]

\[
- \frac{\pi^2}{4} \int_{t_k}^{t} \frac{e^{2 \delta(s-t)}[e^1(x, s) - e^1(x, t - h/m)]^\top}{4} \times W[e^1(x, s) - e^1(x, t - h/m)] \, ds \, dx
\]

(53)

By Theorem 1, the LMIs (25)–(27) guarantee (38). Substituting \( e^1(x, t - h/m) \to e^1(x, t - h/m) \) in Theorem 1, we have

\[
\tilde{V}_1(t) + 2 \delta \tilde{V}_1(t) - \delta_1 \tilde{V}_1(t - h/m)
\]

\[
\leq \tilde{V}_1(t) + 2 \delta \tilde{V}_1(t - h/m)
\]

\[
+ \frac{\pi^2}{4} \int_{t_k}^{t} \frac{e^{2 \delta(s-t)}[e^1(x, s)]^\top W[e^1(x, s)] \, ds \, dx}{4}
\]

\[
- \frac{\pi^2}{4} \int_{t_k}^{t} \frac{e^{2 \delta(s-t)}[e^1(x, s) - e^1(x, t - h/m)]^\top}{4} \times W[e^1(x, s) - e^1(x, t - h/m)] \, ds \, dx
\]

\[
\leq \delta \sum_{j=1}^{M} \int_{t_j}^{t_{j+1}} \frac{h_\xi(\Gamma(x, t)) \tilde{\Lambda} \tilde{\nu} dx}{4}
\]
Thus, the observer gains are found to be $\Omega = \{ 0 \}$ and $\Omega = \{ 0 \}$. For $k = 2, \ldots, m$, the error system is the same as (15). By arguments of Step 2 in Theorem 1 with replacement of $V_i$ by $\hat{V}_i$, we conclude that the LMI conditions (47)–(49) lead to the exponential stability of the error system.

V. NUMERICAL ILLUSTRATION

In this section, we present some simulations for the following nonlinear 2-D parabolic PDE:

\[
\begin{aligned}
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + \sin(y(x,t)), \quad (x,t) \in \Omega \times (0, \infty) \\
y|_{\partial \Omega} &= 0 \\
y(x,0) &= y_0(x)
\end{aligned}
\]  
(55)

where $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$.

By arguments of [33], it follows from the following T-S fuzzy rules:

**Plant Rule 1:**

\[
\begin{aligned}
\text{IF } y(x,t) \text{ is "about 0," THEN} \\
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + A_1 y(x,t), \quad (x,t) \in \Omega \times (0, \infty) \\
y|_{\partial \Omega} &= 0 \\
y(x,0) &= y_0(x)
\end{aligned}
\]

**Plant Rule 2:**

\[
\begin{aligned}
\text{IF } y(x,t) \text{ is "about } -\pi \text{ or } \pi," \text{ THEN} \\
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + A_2 y(x,t), \quad (x,t) \in \Omega \times (0, \infty) \\
y|_{\partial \Omega} &= 0 \\
y(x,0) &= y_0(x)
\end{aligned}
\]

where $A_1 = 1$, $A_2 = \epsilon \triangleq 0.01/\pi$. Then, (55) can be represented as follows:

\[
\begin{aligned}
\frac{\partial y(x,t)}{\partial t} &= \Delta y(x,t) + \sum_{j=1}^{2} h_j(x,t) A_j y(x,t) \\
y|_{\partial \Omega} &= 0 \\
y(x,0) &= y_0(x)
\end{aligned}
\]  
(56)

where the fuzzy membership are defined by

\[
h_1(y(x,t)) = \begin{cases} 
\frac{\sin(\pi y(x,t)/\delta)}{\pi(y(x,t)/\delta)}, & y(x,t) \neq 0 \\
1, & \text{otherwise}
\end{cases}
\]

\[
h_2(y(x,t)) = 1 - h_1(y(x,t)).
\]

With the averaged measurements (5) or (44), for system (56), we use observer (10) or (45) subject to (11) composed of a chain of subobservers. We verify LMI conditions of Theorems 1 and 2 via Yalmip with $\delta = 0.2, \delta_1 = 0.1, \Delta = 0.1$. Tables I and 2 show feasible solutions for LMI of Theorems 1 and 2 that guarantee the exponential stability of the error system for different values of $m$ and $h$, respectively.

For instance, in Table 2 for $h = 1$ and $m = 1$, we find the following feasible solutions for the LMI of Theorem 2: $P_2 = 0.7039$, $W_1 = 0.0972$, and $W_2 = 0.0505$. Thus, the observer gains are found to be $L_1 = 0.1380$, $L_2 = 0.0717$.

Now a finite difference method is adopted in discretization. The steps of space and time are set to be 0.05 and 0.0001, respectively.

**TABLE I**

<table>
<thead>
<tr>
<th>$h$</th>
<th>$m$</th>
<th>$P_2$</th>
<th>$W_1$</th>
<th>$W_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2.072</td>
<td>0.488</td>
<td>0.172</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2.659</td>
<td>0.787</td>
<td>0.596</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>1.916</td>
<td>0.659</td>
<td>0.414</td>
</tr>
<tr>
<td>0.25</td>
<td>3</td>
<td>1.912</td>
<td>0.677</td>
<td>0.420</td>
</tr>
</tbody>
</table>

**TABLE II**

For $\tau_M = 0.1$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$m$</th>
<th>$P_2$</th>
<th>$W_1$</th>
<th>$W_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.703</td>
<td>0.097</td>
<td>0.050</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.646</td>
<td>0.145</td>
<td>0.100</td>
</tr>
<tr>
<td>0.5</td>
<td>4</td>
<td>2.029</td>
<td>0.504</td>
<td>0.308</td>
</tr>
<tr>
<td>0.25</td>
<td>3</td>
<td>2.025</td>
<td>0.513</td>
<td>0.306</td>
</tr>
</tbody>
</table>

![Fig. 2. Error system (46) subject to (14), (17) with $m = 1$ and $h = 1$. (a) Evolution of the $L^2(\Omega)$-norm $\|e^1(\cdot, t)\|_{L^2(\Omega)}$. (b) Snapshots of the state $e^1(x,t)$ at different time $t \in \{ 0, 2.5, 7.5, 10\}$.](image-url)
VI. CONCLUSION

This article discussed fuzzy observer design for 2-D parabolic PDEs with delayed spatially averaged measurements. Differently from 1-D parabolic equation; here, we managed with a new fuzzy observer design for high-dimensional PDEs. In addition, this observer was composed of $m$-chain of subobservers. By constructing an appropriate Lyapunov functional, sufficient LMI-based conditions were found to achieve the exponential stability of the error system. A numerical example demonstrates the efficiency of the proposed scheme.

The obtained theoretical results were applicable to cascaded PDE–PDE/PDE–ODE system and other high-dimensional distributed parameter system. Our next step places its main focus on coupled PDE–PDE/ PDE–ODE system under the averaged measurement subject to time delay. It should be noticed that the time delay considered in the present article is a constant, which can be arbitrarily large. If the time delay becomes time-varying one, the situation becomes more complicated. It seems unclear whether the proposed method is applicable to this case, which will be our next future investigation.

REFERENCES


Let the domain $\Omega = [0, 1]^2$ be divided into $N = 4$ squares of side length $1/\sqrt{N} = 0.5$. In the case of $m = 1$ and $h = 1$, Fig. 2 shows that the evolution of the $L^2(\Omega)$-norm $\|e^1(., t)\|_{L^2(\Omega)}$ and snapshots of the state $e^1(x, t)$ at different time for the error system (46) subject to (14) and (17) initialized by $y_0(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$, $(x_1, x_2) \in \Omega$. Fig. 3 shows that the evolution of the $L^2(\Omega)$-norm $\|e^k(., t)\|_{L^2(\Omega)}$ $(k = 1, 2, \ldots, m)$ for the error system (46) subject to (14) and (17) with the same initial condition for different values of $m$ and $h$. As expected, simulation of solutions demonstrates that the error system is exponentially stable.


