Proof of three lemmas for the paper "Boundary output tracking for an Euler-Bernoulli beam equation with unmatched perturbations from a known exosystem"

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1 Lemma 2.1.

Consider the boundary value problem (BVP):

$$\begin{cases} g^{(4)}(x) + (S^\top)^2 g(x) + f(x)p_{d_1} = 0, \\ g''(0) = p_{d_2},\ g'''(0) = p_{d_3}, \\ g(1) = 0,\ g(0) = p_r. \end{cases} \tag{15}$$

Lemma 2.1. *Assume that S has distinct eigenvalues on the imaginary axis only. Then BVP (15) admits a unique solution.*

Proof. Since S is diagonalizable, there exists an invertible matrix $V = (v_1, v_2, \cdots, v_n)$ such that

$$V^{-1}SV = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{1.1}$$

where v_i is the eigenvector of S corresponding to the eigenvalue λ_i of S for $i = 1, 2, \cdots, n$.

Premultiplying (15) by $v_i^\top, i = 1, 2, \cdots, n$ in three equations, we obtain a system of ODEs as follows

$$\begin{cases} \dfrac{d^4 g_i^*(x)}{dx^4} + \lambda_i^2 g_i^*(x) + f(x)p_{d_1 i}^* = 0, \\ \left.\dfrac{d^2 g_i^*}{dx^2}\right|_{x=0} = p_{d_2 i}^*,\ \ \left.\dfrac{d^3 g_i^*}{dx^3}\right|_{x=0} = p_{d_3 i}^*, \\ g_i^*(1) = 0, g_i^*(0) = p_{ri}^*, \end{cases} \tag{1.2}$$

for $i = 1, 2, \cdots, n$, where $g_i^*(x) = g^\top(x)v_i, p_{d_1 i}^* = p_{d_1}^\top v_i, p_{d_2 i}^* = p_{d_2}^\top v_i, p_{ri}^* = p_r^\top v_i$. There are two cases for $\lambda_i (i = 1, 2, \cdots, n)$:

Case 1: $\lambda_i \neq 0$. In this case, $-\lambda_i^2 > 0$. Set $\beta_i^4 = -\lambda_i^2, \beta_i \in \mathbb{R}$. The general solution of (1.2) can be written as

$$\begin{aligned} g_i^*(x) &= C_{i1}\sinh(\beta_i x) + C_{i2}\cosh(\beta_i x) + C_{i3}\sin(\beta_i x) \\ &+ C_{i4}\cos(\beta_i x) - \frac{1}{2\beta_i^3}\int_0^x [\sinh(\beta_i(x-y)) - \sin(\beta_i(x-y))] f(y)dy p_{d_1 i}^*. \end{aligned} \tag{1.3}$$

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By the first and the fourth boundary conditions of (1.2), it has $C_{i2} = \frac{\beta_i^2 p_{ri}^* + p_{d_2 i}^*}{2\beta_i^2}$ and $C_{i4} = \frac{\beta_i^2 p_{ri}^* - p_{d_2 i}^*}{2\beta_i^2}$. The second boundary condition of (1.2) implies that $C_{i3} = C_{i1} - \frac{p_{d_3 i}^*}{\beta_i^3}$. The third boundary condition together with (1.3) gives

$$C_{i1} = \frac{-C_{i2}\cosh(\beta_i) - C_{i4}\cos(\beta_i) + \dfrac{p_{d3i}^* \sin(\beta_i)}{\beta_i^3}}{\sinh(\beta_i) + \sin(\beta_i)} + \frac{\displaystyle\int_0^1 [\sinh(\beta_i(1-y)) - \sin(\beta_i(1-y))] f(y) dy p_{d_1 i}^*}{2\beta_i^3(\sinh(\beta_i) + \sin(\beta_i))}. \tag{1.4}$$

The solution (1.3) of (1.2) is thus determined.

Case 2: $\lambda_i = 0$. In this case, the BVP (1.2) becomes

$$\begin{cases} \dfrac{d^4 g_i^*(x)}{dx^4} = -f(x) p_{d_1 i}^*, \\ \dfrac{d^2 g_i^*}{dx^2}\Big|_{x=0} = p_{d_2 i}^*, \dfrac{d^3 g_i^*}{dx^3}\Big|_{x=0} = p_{d_3 i}^*, \\ g_i^*(1) = 0, g_i^*(0) = p_{ri}^*. \end{cases} \tag{1.5}$$

A straightforward computation gives the solution of (1.5):

$$\begin{aligned} g_i^*(x) &= p_{ri}^* + \left[\int_0^1 \frac{(1-y)^3}{6} f(y) dy p_{d_1 i}^* - p_{ri}^* - \frac{p_{d_2 i}^*}{2} - \frac{p_{d_3 i}^*}{6}\right] x \\ &+ \frac{p_{d_2 i}^*}{2} x^2 + \frac{p_{d_3 i}^*}{6} x^3 - \int_0^x \frac{(x-y)^3}{6} f(y) dy p_{d_1 i}^*. \end{aligned} \tag{1.6}$$

We therefore obtain the solution to BVP (15):

$$g^\top(x) = [g_1^*(x), g_2^*(x), \cdots, g_n^*(x)] V^{-1}. \tag{1.7}$$

□

2 Lemma 3.1

Consider the boundary value problem (BVP) of the following:

$$\begin{cases} h^{(4)}(x) = -(S_d^2)^\top h(x) + f(x) q_{d_1}, \\ h''(0) = -q_{d_2}, h'''(0) = -q_{d_3}, h(1) = 0, \\ h''(1) + c_1 S_d^\top h'(1) + c_2 h'(1) = 0. \end{cases} \tag{31}$$

Lemma 3.1. *Assume that all eigenvalues of S_d are distinct and are located on the imaginary axis. Then the BVP (31) admits a unique solution.*

Proof. Since S is diagonalizable, so is S_d. There exists an invertible matrix $W = (w_1, w_2, \cdots, w_{n_1})$ such that

$$W^{-1} S_d W = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_{n_1}), \tag{2.1}$$

where w_i is the eigenvector of S_d corresponding to the eigenvalue μ_i of S_d for $i = 1, 2, \cdots, n_1$.

Similar to the proof of Lemma 2.1, premultiplying by $w_i^\top, i = 1, 2, \cdots, n_1$ in three equations of (31), we obtain a system of n_1 ODEs as follows

$$\begin{cases} \dfrac{d^4 h_i^*(x)}{dx^4} = -\mu_i^2 h_i^*(x) + f(x) q_{d_1 i}^*, \\ \dfrac{d^2 h_i^*}{dx^2}\Big|_{x=0} = -q_{d_2 i}^*, \dfrac{d^3 h_i^*}{dx^3}\Big|_{x=0} = -q_{d_3 i}^*, h_i^*(1) = 0, \\ \dfrac{d^2 h_i^*}{dx^2}\Big|_{x=1} + c_1 \mu_i \dfrac{d h_i^*}{dx}\Big|_{x=1} + c_2 \dfrac{d h_i^*}{dx}\Big|_{x=1} = 0, \end{cases} \tag{2.2}$$

for $i=1,2,\cdots,n_1$, where $h_i^*(x)=h^\top(x)w_i, q_{d_1i}^*=q_{d_1}^\top w_i, q_{d_2i}^*=q_{d_2}^\top w_i$.

There are two cases for $\mu_i (i=1,2,\cdots,n_1)$:

Case 1: $\mu_i\neq 0$. In this case, $\overline{\mu}_i$ is also the eigenvalue of S_d. Let $\alpha_i^4=-\mu_i^2$. Then $\mu_i=j\alpha_i^2, \alpha_i\in\mathbb{R}^+$ or $\mu_i=-j\alpha_i^2, \alpha_i\in\mathbb{R}^+$. We only consider the case of $\mu_i=j\alpha_i^2$ since the later case can be discussed in the same manner. The general solution of (2.2) can be written as

$$\begin{aligned} h_i^*(x) &= C_{i5}\sinh(\alpha_i x)+C_{i6}\cosh(\alpha_i x)+C_{i7}\sin(\alpha_i x)\\ &+C_{i8}\cos(\alpha_i x)+\frac{1}{2\alpha_i^3}\int_1^x[\sinh(\alpha_i(x-y))-\sin(\alpha_i(x-y))]f(y)dyq_{d_1i}^*. \end{aligned} \tag{2.3}$$

The first and second boundary conditions of (2.2) give

$$\begin{cases} C_{i8}=C_{i6}+N_{1i}, C_{i7}=C_{i5}-N_{2i},\\ N_{1i}=\dfrac{q_{d_2i}^*}{\alpha_i^2}+\dfrac{1}{2\alpha_i^3}\displaystyle\int_0^1[\sinh(\alpha_i y)+\sin(\alpha_i y)]f(y)dyq_{d_1i}^*,\\ N_{2i}=-\dfrac{q_{d_3i}^*}{\alpha_i^3}+\dfrac{1}{2\alpha_i^3}\displaystyle\int_0^1[\cosh(\alpha_i y)+\cos(\alpha_i y)]f(y)dyq_{d_1i}^*. \end{cases} \tag{2.4}$$

Taking the last two boundary conditions of (2.2) into account, and with (2.4), we obtain

$$\begin{cases} C_{i5}(\sinh\alpha_i+\sin\alpha_i)+C_{i6}(\cosh\alpha_i+\cos\alpha_i)=N_{2i}\sin\alpha_i-N_{1i}\cos\alpha_i,\\ C_{i5}[2\alpha_i\sinh\alpha_i+(jc_1\alpha_i^2+c_2)(\cosh\alpha_i+\cos\alpha_i)]\\ +C_{i6}[2\alpha_i\cosh\alpha_i+(jc_1\alpha_i^2+c_2)(\sinh\alpha_i-\sin\alpha_i)]\\ =(jc_1\alpha_i^2+c_2)(N_{1i}\sin\alpha_i+N_{2i}\cos\alpha_i). \end{cases} \tag{2.5}$$

Eliminating C_{i5} in (2.5) yields

$$\begin{cases} C_{i5}=-\dfrac{\cosh\alpha_i+\cos\alpha_i}{\sinh\alpha_i+\sin\alpha_i}C_{i6}+\dfrac{N_{2i}\sin\alpha_i-N_{1i}\cos\alpha_i}{\sinh\alpha_i+\sin\alpha_i},\\ C_{i6}[2\alpha_i(\cosh\alpha_i\sin\alpha_i-\sinh\alpha_i\cos\alpha_i)-2(jc_1\alpha_i^2+c_2)(1+\cosh\alpha_i\cos\alpha_i)]\\ =(\sinh\alpha_i+\sin\alpha_i)(jc_1\alpha_i^2+c_2)(N_{2i}\cos\alpha_i+N_{1i}\sin\alpha_i)\\ \quad-(N_{2i}\sin\alpha_i-N_{1i}\cos\alpha_i)[2\alpha_i\sinh\alpha_i+(jc_1\alpha_i^2+c_2)(\cosh\alpha_i+\cos\alpha_i)]. \end{cases} \tag{2.6}$$

Now we show that the coefficient of C_{i6} in (2.6) is not vanishing for any $\alpha_i\in\mathbb{R}^+$. Otherwise if the real and imaginary parts of coefficient are zero, then we have

$$\begin{cases} \cosh\alpha_i\sin\alpha_i-\sinh\alpha_i\cos\alpha_i=0,\\ 1+\cosh\alpha_i\cos\alpha_i=0, \end{cases} \tag{2.7}$$

which implies that

$$\begin{cases} \cosh^2\alpha_i\sin^2\alpha_i=\sinh^2\alpha_i\cos^2\alpha_i,\\ \cosh^2\alpha_i\cos^2\alpha_i=1. \end{cases} \tag{2.8}$$

Adding two equations on both sides in (2.8), we obtain

$$\sinh^2\alpha_i(1-\cos^2\alpha_i)=0, \tag{2.9}$$

which gives $\cos\alpha_i=\pm 1$. Going back to the second equation of (2.8), we find $\cosh\alpha_i=\mp 1$ which is a contradiction for $\alpha_i\neq 0$. Hence the coefficient of C_{i6} in (2.6) is not vanishing for any $\alpha_i\in\mathbb{R}$. The C_{i6} is solvable in (2.6). This determines C_{i5}, C_{i7}, C_{i8} and so does $h_i^*(x)$ in (2.3).

Case 2: $\mu_i=0$. In this case, the BVP (2.2) becomes

$$\begin{cases} \dfrac{d^4h_i^*(x)}{dx^4}=f(x)q_{d_1i}^*,\\ \left.\dfrac{d^2h_i^*}{dx^2}\right|_{x=0}=-q_{d_2i}^*, \left.\dfrac{d^3h_i^*}{dx^3}\right|_{x=0}=-q_{d_3i}^*,\\ h_i^*(1)=0, \left.\dfrac{d^2h_i^*}{dx^2}\right|_{x=1}+c_2\left.\dfrac{dh_i^*}{dx}\right|_{x=1}=0. \end{cases} \tag{2.10}$$

A straightforward computation gives the solution of (2.10):

$$h_i^*(x) = a_0 + a_1 x - \frac{q_{d_2 i}^*}{2} x^2 - \frac{q_{d_3 i}^*}{6} x^3 + \int_0^x \frac{(x-y)^3}{6} f(y) dy q_{d_1 i}^*, \tag{2.11}$$

where

$$\begin{cases} a_1 = q_{d_2 i}^* + \dfrac{q_{d_3 i}^*}{2} - \displaystyle\int_0^1 \frac{(1-y)^2}{2} f(y) dy q_{d_1 i}^* + \frac{1}{c_2}\left[q_{d_2 i}^* + q_{d_3 i}^* - \int_0^1 (1-y) f(y) dy q_{d_1 i}^* \right], \\ a_0 = -a_1 + \dfrac{q_{d_2 i}^*}{2} + \dfrac{q_{d_3 i}^*}{6} - \displaystyle\int_0^1 \frac{(1-y)^3}{6} f(y) dy q_{d_1 i}^*. \end{cases} \tag{2.12}$$

We therefore obtain the solution to BVP (31)

$$h^\top(x) = [h_1^*(x), h_2^*(x), \cdots, h_{n_1}^*(x)] W^{-1}. \tag{2.13}$$

□

3 Lemma 3.2

Lemma 3.2. *The numerator of the transfer matrix $T_d^\top(s) = \frac{N_d^\top(s)}{D_d(s)}$ of (1) from $(d_1, d_2, d_3)^\top$ to one of output signal $u_x(1,t)$ is*

$$N_d(s) = \begin{pmatrix} \displaystyle\int_0^1 R(s,y) f(y) dy \\ -r_2\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) \\ -2r_1\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{cases} R(s,y) = r_1\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) r_2\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, y\right) \\ \qquad\qquad - r_2\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) r_1\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, y\right), \\ r_1(s,y) = \dfrac{\sinh(sy) + \sin(sy)}{2s}, \\ r_2(s,y) = \cosh(sy) + \cos(sy). \end{cases} \tag{3.2}$$

The pair $(\frac{dh^\top}{dx}\big|_{x=1}, S_d)$ is observable if and only if

$$N_d^\top(\mu_i) \begin{pmatrix} q_{d_1}^\top w_i \\ q_{d_2}^\top w_i \\ q_{d_3}^\top w_i \end{pmatrix} \neq 0, i = 1, 2, \cdots, n_1, \tag{3.3}$$

where w_i is the eigenvector of S_d corresponding to the eigenvalue μ_i of S_d.

Proof. The transfer matrix can be obtained by a straightforward computation. The pair $(\frac{dh^\top}{dx}|_{x=1}, S_d)$ is observable if and only if $\frac{dh_i^*}{dx}|_{x=1} = \frac{dh^\top}{dx}|_{x=1} w_i \neq 0, n = 1, 2, \cdots, n_1$. This is because the eigenvalues of S_d are distinct ([1]). Similar to the computation in Lemma 3.1., there are two cases for μ_i:

Case 1: $\mu_i \neq 0$. In this case, it follows from (2.3) that

$$\left.\frac{dh_i^*}{dx}\right|_{x=1} = \alpha_i [C_{i5} \cosh \alpha_i + C_{i6} \sinh \alpha_i + C_{i7} \cos \alpha_i - C_{i8} \sin \alpha_i].$$

Substituting $C_{ip}, p = 5, 6, 7, 8$ defined by (2.4) and (2.6) into $\frac{dh_i^*}{dx}|_{x=1}$, we obtain

$$\left.\frac{dh_i^*}{dx}\right|_{x=1} = \frac{\alpha_i^2 [N_2(\sinh \alpha_i + \sin \alpha_i) - N_1(\cosh \alpha_i + \cos \alpha_i)]}{\alpha_i^2 D_{1\alpha_i} - \alpha_i (j c_1 \alpha_i^2 + c_2) D_{2\alpha_i}}, \tag{3.4}$$

where N_1, N_2 are defined by (2.4) and

$$\begin{aligned} D_{1\alpha_i} &= \cosh\alpha_i \sin\alpha_i - \sinh\alpha_i \cos\alpha_i, \\ D_{2\alpha_i} &= 1 + \cosh\alpha_i \cos\alpha_i. \end{aligned} \tag{3.5}$$

Case 2: $\mu_i = 0$. In this case, we have

$$\left.\frac{dh_i^*}{dx}\right|_{x=1} = \frac{1}{c_2}\left[q_{d_2 i}^* + q_{d_3 i}^* - \int_0^1 (1-y) f(y) dy q_{d_1 i}^*\right].$$

Hence $\left.\frac{dh_i^*}{dx}\right|_{x=1} \neq 0$ is equivalent to the conditions of the lemma. □

References

[1] T. Kailath, *Linear Systems*, Englewood Cliffs, NJ: Prentice Hall, 1980.