Performance boundary output tracking for a wave equation with control unmatched disturbance

Feng-Fei Jin, Bao-Zhu Guo

Department of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China
Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
School of Mathematics and Big Data, Foshan University, Foshan 528000, China

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In this paper, we consider boundary output regulation for a one-dimensional wave equation with anti-stable term at control unmatched end. The reference signal and (in-domain and boundary) disturbances are supposed to be generated from a finite-dimensional exosystem. We first design a state feedback regulator by the backstepping approach to make the performance output track the reference signal exponentially. Next, with the measured output, we construct an observer for both wave PDE and exosystem. Based on the observer, an output feedback regulator is developed by replacing the states with their estimations. The closed-loop system is shown to admit a unique bounded solution and the tracking error is shown to decay exponentially. Finally, some numerical simulations are presented to illustrate the effectiveness of the proposed output feedback regulator.

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1. Introduction

Output regulation is one of the major concerns in control theory. Many results have been developed from finite-dimensional systems [2–4] to infinite-dimensional systems [1,5,6,16,17,22–24]. Several approaches have been developed to solve regulation problem for distributed parameter systems, among them the internal model principal is the major approach, which has been extended from lumped parameter systems to distributed parameter systems [16,17]. In [16], an abstract framework of linear infinite-dimensional systems with bounded control and observation was developed. The results were then extended to the systems with unbounded control and observation in [17]. In [22], an observer-based output regulator for an abstract linear infinite dimensional system was designed, where the control operator is bounded yet the observation operator is unbounded. The adaptive control method was applied to regulation problem for a wave equation in [10] where the unknown constant parameters in harmonic disturbance were identified. In [12], we designed an output regulator for a heat equation with unbounded control and observation, where an infinite-dimensional extended state observer was constructed to estimate both state and general external disturbance. In [25], output regulation for a wave equation with general boundary disturbance was considered. However, both Jin and Guo [12] and Zhou and Guo [25] considered collocated case where the control and regulated output are on the same boundary, which is very different to the non-collocated case considered in this paper.

The backstepping method for systems described by partial differential equations (PDEs) turns out to be a powerful tool in dealing with stabilization problem for systems like parabolic equations, wave equations, special beam equations, Schrödinger equations, and linearized KdV equations, among many others. A nice summary was made in the book [15]. This method has been also applied to observer design. Recently, in [5], an output regulator for a heat equation was designed by backstepping approach, where both control and observation are supposed to be unbounded and
the disturbance and reference signals are generated by a finite-dimensional exosystem. In addition, the performance output could be bounded or unbounded either. This result has been generalized to regulation problem for a coupled wave equation with unbound control and observation in [8], where the performance output is supposed to be bounded. In this paper, we extend the backstepping method to regulation problem for a wave equation where the control operator and observation operator can be unbounded and the control and regulated output are non-collocated. The system is governed by the following wave equation:

$$\begin{align*}
\ddot{u}_1(x, t) &= u_{\text{ref}}(x, t) + f(x) \Pi_1(t), \ x \in (0, 1), t > 0, \\
\dot{u}_0(0, t) &= -q_1 \dot{u}_0(t) + d_2(t), \ t \geq 0, \\
\dot{u}_1(0, t) &= U(t), \ t \geq 0, \\
\gamma_0(t) &= \{u(0, t), \ u(0, t)\}, \ t \geq 0, \\
\dot{u}(x, 0) &= u_0(x), \ u_1(x, 0) = u_1(x), \ x \in [0, 1],
\end{align*}$$

where \(u(x, t)\) is the state at position \(x\) and time \(t\), \(U(t)\) is the control (input), \(\gamma_0(t)\) is the output (measurement), \((u_0(x), \ u_1(x))\) is the initial state, \(q_1 \in \mathbb{R}\) is a known constant and \(q_1 \neq 1, f \in C[0, 1]\) represents the intensity of the unknown spatial disturbance \(d_1(t)\). The system is also disturbed by a boundary unknown disturbance \(d_2(t)\). For any \(U \in L_2^{\infty}(0, \infty)\), and \((u_0, u_1) \in H^1(0, 1) \times L^2(0, 1)\), through the backstepping transformation (2.3) later presented in [19], one can easily show that there exists a unique solution to (1.1) belonging to the state space \(H^1(0, 1) \times L^2(0, 1)\).

We emphasize that the assumption \(q_1 \neq 1\) is necessary. This is because one of the requirements for output regulation is that whenever the reference and disturbances are zero, the system (1.1) should be asymptotically stabilized (internal stability). However, when \(q_1 = 1\), as indicated in [14] that “the real part of all the plant eigenvalues is \(+\infty\), requiring infinite control gains for stability”. A physical explanation for anti-stable wave system has been well explained in [14].

For brevity in notation, we omit the initial value in the rest of this paper whenever there is no confusion. In addition, we omit the obvious domains of variables. We suppose that the disturbances and reference signal are generated by an exosystem as follows:

$$\begin{align*}
\dot{v}(t) &= S \dot{v}(t), \ t > 0, \\
\dot{d}_1(t) &= p_1^2 \dot{v}(t), \ t \geq 0, \\
\dot{d}_2(t) &= p_2^2 \dot{v}(t), \ t \geq 0, \\
\gamma_{\text{ref}}(t) &= \mathbf{p}_d^T \dot{v}(t), \ t \geq 0, \\
\mathbf{v}(0) &= \mathbf{v}_0 \in \mathbb{C}^n,
\end{align*}$$

where \(S = \text{diag}(S_\alpha, S_\beta)\) is a block diagonalizable matrix with all eigenvalues on the imaginary axis, that is, the \(v\)-system can be divided into two decoupled subsystems of the following:

$$\begin{align*}
\dot{v}_d(t) &= S_d \dot{v}_d(t), \ t > 0, \\
\dot{d}_1(t) &= q_{1d}^2 \dot{v}_d(t), \ t \geq 0, \\
\dot{d}_2(t) &= q_{2d}^2 \dot{v}_d(t), \ t \geq 0, \\
\mathbf{v}_d(0) &= \mathbf{v}_{d0} \in \mathbb{C}^{n_d},
\end{align*}$$

where the initial value \(\mathbf{v}_{d0}\) is unknown which makes both \(\dot{d}_1(t)\) and \(\dot{d}_2(t)\) unknown, and

$$\begin{align*}
\dot{v}_r(t) &= S_r \dot{v}_r(t), \ t > 0, \\
\gamma_{\text{ref}}(t) &= q_{1r}^2 \dot{v}_r(t), \ t \geq 0, \\
\mathbf{v}_r(0) &= \mathbf{v}_{r0} \in \mathbb{C}^{n_r},
\end{align*}$$

where \((\dot{v}_r^T, \dot{v}_d^T)^T = \mathbf{v}\) and \(n = n_1 + n_2\). In addition, we assume that the eigenvalues of \(S_d\) are distinct and \(\Sigma(q_{1d}^2, S_d)\) is observable. The reference signal \(\gamma_{\text{ref}}(t)\) is supposed to be measurable. For any give \(x_0 \in [0, 1]\), our target is to design an output feedback controller \(U(t)\) such that

$$\lim_{t \to \infty} |u(x_0, t) - \gamma_{\text{ref}}(t)| = 0,$$

in the presence of disturbance and in the same time, the state of the closed-loop system keeps bounded. The signal transmissions are illustrated in block diagram Fig. 1.

We point out that there are very limited literature on output regulation for PDEs with non-collocated control and regulated output. The heat equation (parabolic type) was considered in [5]. A special output regulation with reference (point) zero for wave equation (hyperbolic type) was developed in [11]. Our formulation (1.1) and (1.5) can be considered as a broad generalization of paper Guo et al. [11] with much more complicated disturbances both in domain and boundaries. From methodology point of view, this paper is a generalization of Deutscher [6] from \(2 \times 2\) hyperbolic system to wave equation. A recent interesting result was developed in [7] for wave equation on output regulation at also span point yet the feedback there is the full state feedback. Another point is that the system of Gabriel and Deutscher [7] is not generally anti-stable.

We proceed as follows. In Section 2, we first design, with Backstepping transformation, a state feedback controller for system (1.1). Through a new transformation, we are able to show the output tracking. Section 3 is devoted to the design of an observer which is used to recover the states of both wave PDE and exosystem. In Section 4, we design an output feedback regulator by replacing the states in state feedback with their estimations obtained from the observer. Some numerical simulations are presented in Section 5 to illustrate the effect of the proposed control law, following up concluding remarks in Section 6.

### 2. State feedback regulator

In this section, we assume that the states of system (1.1) and exosystem (1.2) are known. We propose a state feedback controller for system (1.1) as follows ([19]):

$$U(t) = -\frac{q_1 + c}{1 + q_1 c} \dot{u}(1, t) - c_0 \dot{u}(1, t) + \frac{q_0 c_q (q_1 + c)}{1 + q_1 c} u(0, t) - c_0 \frac{q_1 + c}{1 + q_1 c} \int_0^t u(y, t) d\tau - \frac{q_1^2 - 1}{1 + q_1 c} m_0^T \mathbf{v}(t),$$

where \(c > 0, \ c_0 > 0\) are tuning parameters and \(m_0^T\) is an \(n\)-dimensional row vector to be determined. The closed-loop system
(1.1) and (1.2) under the controller (2.1) therefore becomes
\[
\begin{align*}
\dot{v}(t) &= Sv(t), \\
\mu(t,x,t) &= u_0(x,t) + f(x)p_{d_t}^T v(t), \\
\mu_1(0,t) &= -q_1 u_1(0,t) + q_2 p_{d_t}^T v(t), \\
\mu_2(1,t) &= -q_1 + c \frac{q_1 c_0(q_1 + c)}{q_1^2 - 1} u_0(0,t) + q_1 c_0(q_1 + c) u_0(1,t) + \frac{q_2}{q_1^2 - 1} \int_0^1 u_t(y,t) dy - \frac{q_2 - 1}{q_1^2 - 1} m_t^e v(t), \\
v(0) &= v_0 \in C^0, u(x,0) = u_0(x), u_t(x,0) = u_1(x).
\end{align*}
\]

(2.2)

Introducing an invertible backstepping transformation (2.8) of (19):
\[
w(x,t) = -\frac{1 + q_1 c}{q_1^2 - 1} u(x,t) + \frac{q_1 (q_1 + c)}{q_1^2 - 1} u(0,t) - \frac{q_1 + c}{q_1^2 - 1} \int_0^1 u_t(y,t) dy,
\]
we can transform (2.2) into the following system
\[
\begin{align*}
\dot{v}(t) &= Sw(t), \\
\dot{w}_r(t) &= w_{xx}(x,t) + f^T(x)v(t), \\
\dot{w}_x(0,t) &= cw_0(0,t) - \frac{1 + q_1 c}{q_1^2 - 1} q_2 p_{d_t}^T v(t), \\
\dot{w}_x(1,t) &= -c_0 w(1,t) + m_t^e v(t), \\
w(0) &= w_0(x), \quad w_t(x,0) = w_1(x),
\end{align*}
\]
where \(c \neq 1\) is the design parameter and
\[
\dot{f}^T(x) = -\frac{1 + q_1 c}{q_1^2 - 1} f(x)p_{d_t}^T + \frac{q_2 (q_1 + c)}{q_1^2 - 1} p_{d_t}^T S - \frac{q_1 + c}{q_1^2 - 1} \int_0^x f(y) dp_{d_t}^T S.
\]

It is well known that when \(v(t) \equiv 0\), the \(w\)-part of system (2.4) is exponentially stable [19].

In order to make tracking result (1.5) hold true, we have to find \(m_t^e\) for which we follow the method presented in [6] for \(2 \times 2\) hyperbolic systems. To this end, we introduce a transformation
\[
\begin{align*}
\dot{v}(t) &= v(t), \\
\dot{e}(x,t) &= w(x,t) - g^T(x)v(t),
\end{align*}
\]
where \(g^T(x)\) is to be determined later. Then \(v(t)\) and \(e(x,t)\) are governed by
\[
\begin{align*}
\dot{v}(t) &= Sv(t), \\
\dot{e}_{xt}(x,t) &= e_{xx}(x,t), \\
\dot{e}_x(0,t) &= ce_0(0,t), \\
\dot{e}_x(1,t) &= -c_0 e(1,t), \\
v(0) &= v_0 \in C^0, e(x,0) = e_0(x), e_t(x,0) = e_1(x),
\end{align*}
\]
provided that \(g^T(x)\) is chosen to satisfy
\[
\begin{align*}
\frac{d^2 g^T(x)}{dx^2} &= g^T(x)S^2 - \dot{f}(x), \\
\frac{dg^T}{dx} \bigg|_{x=0} &= cg^T(0)S - \frac{1 + q_1 c}{q_2} q_2 p_{d_t}^T, \\
m_t^e &= \frac{dg^T}{dx} \bigg|_{x=1} + cg^T(1).
\end{align*}
\]

(2.7)

It is well known that the \(e\)-subsystem of (2.6) is well-posed and is exponentially stable in the state space \(\mathbb{H}_1 = H^1(0,1) \times L^2(0,1)\) with the norm
\[
\|f,g\| \mathbb{H}_1 = \int_0^1 \left( |f(x)|^2 + |g(x)|^2 dx + c_0 |f(1)|^2 \right) \forall (f,g) \mathbb{H}_1.
\]

(2.8)

As a result, \((e_{0}(t), e_t(0,t)) \mathbb{H}_1 \rightarrow 0\) when \(t \rightarrow \infty\). Next, in order to make the tracking error
\[
e(t) = u(x_0,t) - y_{ref}(t)
\]
\[
= -\frac{1 + q_1 c}{c_2 - 1} e(x_0,t) + \frac{c_0 q_1 + c}{c_2 - 1} e(0,t) + \frac{q_1 + c}{c_2 - 1} \int_0^x e(x,t) dy S v(t) + \frac{q_1 + c}{c_2 - 1} \int_0^x g(x,y) dy S v(t) - p_t^e v(t) \rightarrow 0,
\]
we choose
\[
-\frac{1 + q_1 c}{c_2 - 1} g^T(x_0) + \frac{c(q_1 + c)}{c_2 - 1} g^T(0) + q_1 + c \int_0^x g(x,y) dy S v(t) = p_t^e.
\]

(2.9)

Going back to (2.7), we need to prove that
\[
\begin{align*}
\frac{d^2 g^T(x)}{dx^2} &= g^T(x)S^2 - \dot{f}(x), \\
\frac{dg^T}{dx} \bigg|_{x=0} &= cg^T(0)S - \frac{1 + q_1 c}{q_2} q_2 p_{d_t}^T, \\
-\frac{1 + q_1 c}{c_2 - 1} g^T(x_0) + \frac{c(q_1 + c)}{c_2 - 1} g^T(0) + q_1 + c \int_0^x g(x,y) dy S v(t) & = p_t^e,
\end{align*}
\]

(2.10)

which is the regulator equation, is solvable.

Lemma 2.1. Assume that \(S\) is diagonalizable. Then (2.11) admits a unique solution.

Proof. Since \(S\) is diagonalizable, there exists an invertible matrix \(V = (v_1, v_2, \ldots, v_n)\) such that
\[
V^{-1} S V = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),
\]
where \(v_i\) is the eigenvector of \(S\) corresponding to the eigenvalue \(\lambda_i\) of \(S\) for \(i = 1, 2, \ldots, n\). Postmultiplying (2.11) by \(v_i\), \(i = 1, 2, \ldots, n\), respectively, we obtain \(n\) ODEs as follows:
\[
\begin{align*}
\frac{d^2 g_i^T(x)}{dx^2} &= \lambda_i g_i^T(x) - \dot{f}(x)v_i, \\
\frac{dg_i^T}{dx} \bigg|_{x=0} &= cg_i^T(0)S - \frac{1 + q_1 c}{q_2} q_2 p_{d_t}^T v_i, \\
-\frac{1 + q_1 c}{c_2 - 1} g_i^T(x_0) + \frac{c(q_1 + c)}{c_2 - 1} g_i^T(0) + q_1 + c \lambda_i \int_0^x g_i(x,y) dy S v_i & = p_t^e v_i,
\end{align*}
\]

(2.13)

for \(i = 1, 2, \ldots, n\), where \(g_i^T(x) = g^T(x)v_i\). There are two cases for \(\lambda_i (i = 1, 2, \ldots, n)\).

Case 1: \(\lambda_i \neq 0\). In this case, the general solution of (2.13) can be written as
\[
g_i^T(x) = C_0 \sinh(\lambda_i x) + C_2 \cosh(\lambda_i x)
\]
\[
- \frac{1}{\lambda_i} \int_0^x \sinh \lambda_i (x - y) \dot{f}(y) v_i dy.
\]

(2.14)
The last two equalities in (2.13) imply that
\[
\lambda_i C_{i1} = C_{i2} - \frac{1 + q_1 c}{c^2 - 1} \left\{ \int_0^\infty \sinh(\lambda_i (x_0 - y)) f^+(y)v dy \right\} + \frac{c(1 + q_1 c)}{c^2 - 1} C_{i2}
\]
\[
- \frac{1 + q_1 c}{c^2 - 1} \left\{ \int_0^\infty \sinh(\lambda_i (x_0 - y)) f^+(y)v dy \right\} + \frac{q_1 + c}{c^2 - 1} \lambda_i \int_0^\infty \sinh(\lambda_i y) + C_{i2} \cosh(\lambda_i y)
\]
\[
- \frac{1}{\lambda_i} \int_0^\infty \sinh(\lambda_i (y - z)) f^+(z)v dz dy = p_i^+ v_i.
\]
Solving above equation, we obtain $C_{i1}$, $C_{i2}$ as
\[
C_{i2} = \left[ \frac{c + 1}{1 - q_1} \right] \left( \frac{c \cosh(\lambda_i x_0)}{\lambda_i} \right) - \frac{1 + q_1 c}{(c^2 - 1)} \left\{ \int_0^\infty \sinh(\lambda_i (x_0 - y)) f^+(y)v dy \right\} + \frac{(q_1 + c)(1 + q_1 c)}{(c^2 - 1)(q_i - 1)} q_2 p_i^+ v_i
\]
\[
+ \frac{q_i^+}{1 - q_1^+ v_i} - \frac{1 + q_1 c}{(c^2 - 1)} \left\{ \int_0^\infty \sinh(\lambda_i (x_0 - y)) f^+(y)v dy \right\} + \frac{(q_1 + c)(1 + q_1 c)}{(c^2 - 1)(q_i - 1)} q_2 p_i^+ v_i
\]
\[
- \frac{q_i^+}{1 - q_1^+ v_i} - \frac{1 + q_1 c}{(c^2 - 1)} \left\{ \int_0^\infty \sinh(\lambda_i (x_0 - y)) f^+(y)v dy \right\} + \frac{(q_1 + c)(1 + q_1 c)}{(c^2 - 1)(q_i - 1)} q_2 p_i^+ v_i.
\]
Substituting (2.15) into (2.14), we obtain the solution of (2.13).

Case 2: $\lambda_i = 0$. In this case, (2.13) becomes
\[
\begin{align*}
\frac{d^2 g_i^+(x)}{dx^2} &= f^+(x) v_i, \\
\frac{dg_i^+(x)}{dx} \bigg|_{x=0} &= \frac{1 + q_1 c}{c^2 - 1} q_2 p_i^+ v_i, \\
- \frac{1 + q_1 c}{(c^2 - 1)} g_i^+(x) + \frac{c(1 + q_1 c)}{c^2 - 1} g_i^+(0) &= p_i^+ v_i.
\end{align*}
\]
A straightforward computation gives the solution of (2.16)
\[
g_i^+(x) = \left[ \frac{1 + q_1 c}{c^2 - 1} \right] \left( \frac{q_2 p_i^+ v_i x_0}{c^2 - 1} \right) \left\{ \int_0^x (x - y) f^+(y)v dy \right\} + (\lambda_i - q_i^+) v_i.
\]
We thus obtain the unique solution to (2.11):
\[
ge^+(x) = \left[ g_i^+(x), g_2^+(x), \ldots, g_n^+(x) \right] v_i^{-1}.
\]
\[
\text{Theorem 2.1. Let } g(x) \text{ be the solution of (2.11) and } m_i^+(x) = \frac{d g_i^+(x)}{dx} |_{x=1} + c_0 g(1). \text{ For any initial value } (y_0^i, u_0, u_1) \in C^n \times H_1, \text{ the closed-loop system (2.2) admits a unique bounded solution } (v^i(t), u_i(t), y_0(t)) \in C(0, \infty; C^n \times H_1). \text{ Moreover, the tracking error } e_y(t) = u(x_0, t) - y_{ref}(t) \to 0 \text{ exponentially as } t \to \infty.\\
\text{Proof. The results follow from the relation between system (2.2) and (2.6). From transformation (2.5), we find the solution to (2.4) to be } (w(x,t), t_w(x,t)) \in C(0, \infty; C^n \times H_1), \text{ the vector } (w(x,t), t_w(x,t)) \text{ is bounded and hence there exists constant } L > 0 \text{ depending on initial value only such that}
\]
\[
\text{for some constant } C > 0. \text{ Since } \|e(t, t, e(t) \to 0 \text{ exponentially as } t \to \infty, \text{ so does } e_y(t) \to 0(t \to \infty).}
\]

3. Observer design

In this section, we assume that only $y_0(t)$ in (1.1) is measurable. The observer is designed as follows:
\[
\hat{v}_t(t) = S_d \hat{v}_t(t) + k_2 (\hat{u}_t(0, t) - u(0, t)), \\
\hat{y}_t(t) = \hat{u}_t(t) + k_2 (\hat{p}_t^+(0, t) - y_{ref}(t)), \\
\hat{\bar{u}}_t(x, t) = \hat{u}_t(x, t) + f(x) q_2 \hat{v}_t(t) + k_1 (\hat{u}(0, t) - u(0, t)), \\
\hat{u}_d(t) = -q_1 u_0(t) + c_3 (\hat{u}(0, t) - u(0, t)) + q_2 \hat{v}_t(t) + k_2 \hat{u}_d(t), \\
\hat{\bar{u}}_d(t) = U(t), \\
\hat{\bar{u}}_d(0) = \hat{\bar{u}}_d(0) = \hat{u}_d = \hat{u}_d(0) = \bar{u}(0, t) = \bar{u}_d(0, t).
\]
\[
\text{where } k_1(x), k_2(x) \in C(0, \infty), c_2 > 0, c_3 > 0 \text{ are tuning constant parameters and } c_4 \in R \text{ is } \text{to be determined later, } k_2 \text{ is chosen such that } S_1 + k_2 q_i^+. \text{ Hurwitz from the observability of } (q_i^+, S_i). \text{ Then, the}
errors $\tilde{\nu}_d = \tilde{\nu}_d - \nu_d, \, \tilde{\nu}_t = \tilde{\nu}_t - \nu_t, \, \tilde{u} = \tilde{u} - u$ are governed by

\[
\begin{align*}
\tilde{\nu}_d(t) &= S_d \tilde{\nu}_d(t) + k_d \tilde{u}(0, t), \\
\tilde{\nu}_t(t) &= (S_r + k_d q_d^*) \tilde{\nu}_t(t), \\
\tilde{u}_t(x, t) &= \tilde{u}_{ax}(x, t) + f(x) q_{d1}^* \tilde{\nu}_d(t) + k_2(x) \tilde{u}(0, t) + k_2(x) \tilde{u}_t(0, t), \\
\tilde{u}_d(0, t) &= c_2 \tilde{u}(0, t) + (c_3 + c_4) \tilde{u}(0, t) + q_d q_d^* \tilde{\nu}_d(t), \\
\tilde{u}_t(1, t) &= 0, \\
\tilde{\nu}_d(0) &= \tilde{\nu}_{d0}, \, \tilde{\nu}_t(0) = \tilde{\nu}_{t0}, \, \tilde{u}(x, 0) = \tilde{u}_0(x), \\
\tilde{u}_t(0) &= \tilde{u}_1(x), \\
\end{align*}
\]

(3.2)

where for mathematical reason in (3.6), we write $c_3 + c_4$ instead of one single constant. Since $S_r + k_d q_d^*$ is Hurwitz, the $\tilde{\nu}_t(t)$ is decoupled from others and decays exponentially as time evolving. For $\tilde{\nu}_d(t)$ and $\tilde{u}(t)$, we introduce a transformation

\[
\begin{align*}
\text{which is followed from the approach used in [6]. Then, } \varepsilon(x, t) \text{ satisfies}
\end{align*}
\]

\[
\varepsilon_t(x, t) - \varepsilon_{xx}(x, t) = \tilde{u}_t(x, t) - \tilde{u}_{ax}(x, t) + h^+(x) \tilde{\nu}_d(t)
\]

\[
- \frac{d^2 h^+(x)}{dx^2} \tilde{u}_d(t)
\]

\[
\begin{align*}
&= f(x) q_{d1}^* \tilde{\nu}_d(t) + k_1(x) \tilde{u}(0, t) + k_2(x) \tilde{u}_t(0, t) + h^+(x) S_d^2 \tilde{\nu}_d(t) + S_d k_2 \tilde{u}(0, t) + k_2 \tilde{u}_t(0, t) - \frac{d^2 h^+(x)}{dx^2} \tilde{u}_d(t) = 0,
\end{align*}
\]

provided that

\[
\begin{align*}
\frac{d^2 h^+(x)}{dx^2} &= h^+(x) S_d^2 + f(x) q_{d1}^*, \\
k_1(x) = h^+(x) S_d k_d, \quad k_2(x) = h^+(x) k_d.
\end{align*}
\]

The boundary condition at $x = 0$ of $\varepsilon(x, t)$ gives

\[
\varepsilon_x(0, t) = c_3 \varepsilon_t(0, t) - c_1 \varepsilon(0, t) = q_d q_d^* \tilde{\nu}_d(t) + c_4 \tilde{u}(0, t)
\]

\[
+ \frac{d h^+}{dx} \bigg|_{x=0} \tilde{u}_d(t) - c_2 h^+(0) S_d \tilde{\nu}_d(t) - c_2 h^+(0) k_d \tilde{u}(0, t) - c_2 h^+(0) \tilde{u}_t(0, t) = 0,
\]

provided that

\[
\begin{align*}
\frac{d h^+}{dx} &= c_2 h^+(0) S_d + c_1 h^+(0) - q_d q_d^*, \\
c_4 &= c_2 h^+(0) k_d.
\end{align*}
\]

Similarly, the boundary condition at $x = 1$ gives

\[
\varepsilon_x(1, t) = \frac{d h^+}{dx} \bigg|_{x=0} \tilde{\nu}_d(t) = 0,
\]

provided that

\[
\frac{d h^+}{dx} \bigg|_{x=0} = 0.
\]

For $\tilde{\nu}_d(t)$, we have

\[
\text{Combining (3.4), (3.6), (3.8) and (3.10), we obtain}
\]

\[
\varepsilon_t(x, t) = \varepsilon_{xx}(x, t), \, x \in (0, 1), \, t > 0.
\]

\[
\varepsilon_t(x, t) = \varepsilon_{xx}(x, t), \, x \in (0, 1), \, t > 0.
\]

\[
\begin{align*}
\varepsilon_t(x, 0) &= 0, \, t \geq 0, \\
\varepsilon(x, 0) &= 0, \, x \in [0, 1],
\end{align*}
\]

where $h^+(x)$ satisfies boundary value problem (BVP):

\[
\begin{align*}
\frac{d^2 h^+(x)}{dx^2} &= h^+(x) S_d^2 + f(x) q_{d1}^*, \\
\frac{dh^+}{dx} \bigg|_{x=0} &= c_2 h^+(0) S_d + c_3 h^+(0) - q_d q_d^*, \\
\frac{dh^+}{dx} \bigg|_{x=1} &= 0,
\end{align*}
\]

which is the regulator equation.

Lemma 3.1. The BVP (3.12) has a unique solution.

Proof. The proof is very similar to the proof of Lemma 2.1. Since $S$ is a diagonalizable and so is $S_d$. There exists an invertible matrix $W = (w_1, w_2, \ldots, w_m)$ such that

\[
W^{-1} S_d W = diag(\mu_1, \mu_2, \ldots, \mu_n),
\]

where $w_i$ is the eigenvector of $S_d$ corresponding to the eigenvalue $\mu_i$ of $S_d$ for $i = 1, 2, \ldots, n$. Postmultiplying (3.12) by $w_i, \, i = 1, 2, \ldots, n$, respectively, we obtain $n$ ODEs as follows:

\[
\begin{align*}
\frac{d^2 h^+(x)}{dx^2} &= \mu_i^2 h^+_i(x) + f(x) q_{d1}^*, \\
\frac{dh^+_i}{dx} \bigg|_{x=0} &= c_2 \mu_i h^+_i(0) + c_3 h^+_i(0) - q_d q_{d1}^*, \\
\frac{dh^+_i}{dx} \bigg|_{x=1} &= 0,
\end{align*}
\]

for $i = 1, 2, \ldots, n$, where $h^+_i(x) = h^+(x) w_i, q_{d1}^* w_i = q_{d1}^* w_i, q_{d1}^* w_i$. There are two cases for $\mu_i (i = 1, 2, \ldots, n)$. Case 1: $\mu_i \neq 0$. In this case, the general solution of (3.14) can be written as

\[
\begin{align*}
h^+_i(x) &= C_{i3} \sinh(\mu_i x) + C_{i4} \cosh(\mu_i x)
\end{align*}
\]

\[
+ \frac{1}{\mu_i} \int_0^x \sinh(\mu_i (x - y)) f(y) dy q_{d1}^*.
\]

(3.15)

The first boundary condition of (3.14) implies that

\[
\begin{align*}
\frac{dh^+_i}{dx} \bigg|_{x=0} &= \mu_i C_{i3} \cosh(\mu_i t) + \mu_i C_{i4} \sinh(\mu_i t)
\end{align*}
\]

which gives $C_{i3} = \frac{1}{\mu_i^2} [(c_2 \mu_i + c_3) C_{i4} - q_d q_{d1}^*]$. By the second boundary condition of (3.14), we obtain

\[
\begin{align*}
\frac{dh^+_i}{dx} \bigg|_{x=1} &= \mu_i C_{i3} \cosh(\mu_i t) + \mu_i C_{i4} \sinh(\mu_i t)
\end{align*}
\]

\[
+ \int_0^1 \cosh(\mu_i (1 - y)) f(y) dy q_{d1}^*.
\]

(3.16)

The $C_{i4}$ is uniquely determined in (3.16) if $\mu_i \sinh(\mu_i t) + (c_2 \mu_i + c_3) \cosh(\mu_i t) \neq 0$ for all $\mu_i \in \sigma(S_d), \, \mu_i \neq 0$. Since all eigenvalues of $S_d$, 

\[
\begin{align*}
\end{align*}
\]
are located on the imaginary axis, and so are the eigenvalues of $S_d$. For $\mu_i = i\alpha$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we have

$$\mu_i \sinh \mu_i + (c_2 \mu_i + c_3) \cosh \mu_i = -\alpha \sin \alpha + (i c_2 \alpha + c_3) \cos \alpha = c_3 \cos \alpha - \alpha \sin \alpha + i c_2 \alpha \cos \alpha \neq 0, \quad \forall \alpha \in \mathbb{R}, \quad \alpha \neq 0.$$  \hfill (3.17)

**Case 2:** $\mu_i = 0$. In this case, the BVP (3.14) becomes

$$\begin{align*}
\left. \frac{d^2}{dx^2} h_i^*(x) \right|_{x=0} &= \frac{d}{dx} h_i^*(0) + q_2 q_{d,i}^*, \\
\left. \frac{dh_i^*}{dx} \right|_{x=0} &= c_3 h_i^*(0) - q_2 q_{d,i}^*, \\
\left. \frac{dh_i^*}{dx} \right|_{x=1} &= 0.
\end{align*} \hfill (3.18)$$

A straightforward computation gives the general solution of (3.18)

$$h_i^*(0) = \frac{d}{dx} h_i^*(0) = \int_0^x (x-y) f(y) dy. \hfill (3.19)$$

By the boundary conditions in (3.18), it follows that

$$\begin{align*}
\left. \frac{dh_i^*}{dx} \right|_{x=0} &= c_3 h_i^*(0) - q_2 q_{d,i}, \\
\left. \frac{dh_i^*}{dx} \right|_{x=1} &= \int_0^1 f(y) dy = 0,
\end{align*} \hfill (3.20)$$

which gives

$$\left. \frac{dh_i^*}{dx} \right|_{x=0} = -q_{d,i} c_3 \int_0^1 f(y) dy. \quad h_i^*(0) = -\frac{q_{d,i} c_3}{c_3} \int_0^1 f(y) dy + \frac{q_2 q_{d,i}^*}{c_3}. \hfill (3.21)$$

We thus obtain the unique solution to BVP (3.12):

$$h_i^*(x) = [h_i^*(0), h_i^*(x), \ldots, h_i^n(x)] \mathbb{J}^{-1}. \hfill (3.22)$$

We next show that there exists a vector $k_d$ such that $S_d - k_d h^*(0)$ is Hurwitz.

**Lemma 3.2.** The numerator of the transfer matrix $T_d^*(s) = \frac{N_d^*(s)}{D_d(s)}$ of (1.1) from $(d_1, d_2)^T$ to $u(0, t)$ is

$$N_d^*(s) = \left( - \int_0^s \cosh(s(1-y)) f(y) dy, q_2 \cosh s \right). \hfill (3.23)$$

The pair $(h^*(0), S_d)$ is observable if and only if

$$N_d^*(\mu_i) q_{d,i}^* w_i \neq 0, \quad i = 1, 2, \ldots, n_1, \hfill (3.24)$$

where $w_i$ is the eigenvector of $S_d$ corresponding to the eigenvalue $\mu_i$ of $S_d$.

**Proof.** Take the Laplace transform to (1.1) to obtain

$$\begin{align*}
s^2 \tilde{u}(x, s) &= \tilde{u}(x, s) + f(x) \tilde{d}_1(s), \\
\tilde{u}(0, s) &= -q_1 \tilde{u}(0, s) + q_2 \tilde{d}_2(s), \\
\tilde{d}_2(1, s) &= 0,
\end{align*} \hfill (3.25)$$

where $\tilde{p}(s)$ represents the Laplace transform of $p(t)$. The general solution for above equation is

$$\tilde{u}(x, s) = D_1 \sinh(sx) + D_2 \cosh(sx) - \tilde{d}_1(s) \int_0^x \frac{\sinh(s(x-y))}{s} f(y) dy.$$ 

Taking boundary conditions into account, we obtain

$$\tilde{u}(0, s) = D_2 = \left| \frac{s}{2} \cosh(s(1-y)) f(y) dy \right|_{y=0} + q_2 \cosh s \tilde{d}_2(s).$$

The pair $(h^*(0), S_d)$ is observable if and only if $h_i^*(0) = h_i^*(0) w_i \neq 0, n = 1, 2, \ldots, n_1$, because the eigenvalues of $S_d$ are distinct [13]. From (3.15), (3.16) and (3.21) in Lemma 3.1, we have

- $h_i^*(0) = C_d = (\mu_i \sinh \mu_i + (c_2 \mu_i + c_3) \cosh \mu_i)^{-1} N_d^*(\mu_i) q_{d,i}^* w_i q_d^* w_i^T$ for $\mu_i \neq 0$;
- $h_i^*(0) = C_d^* N_d^*(0) (q_{d,i}^* w_i q_d^* w_i^T)^T$ for $\mu_i = 0$.

Then, $N_d^*(\mu_i) (q_{d,i}^* w_i q_d^* w_i^T)^T$ is equivalent to $h_i^*(0) \neq 0$ for $i = 1, 2, \ldots, n_1$ and the result follows.

Suppose that the eigenvalues of $S_d$ satisfy the conditions of Lemma 3.2. Then $h_i^*(0) = h_i^*(0) w_i \neq 0$ for all $n = 1, 2, \ldots, n_1$ and hence $(h^*(0), S_d)$ is observable. Therefore, there exists a vector $k_d$ such that $S_d - k_d h^*(0)$ is Hurwitz. Going back to system (3.11), we see that the PDE-part is exponentially stable. This together with $S_d - k_d h^*(0)$ and $S_d + k_d q_d^*$ being Hurwitz shows that system (3.11) is exponentially stable. By the inverse transformation of (3.3),

$$\tilde{u}(x, t) = \varepsilon(x, t) - h_i^*(0) \tilde{d}_d(t) \hfill (3.26)$$

where $h_i(0)$ is the solution of BVP (3.12). We then have the following

**Theorem 3.1.** Assume $N_d^*(\mu_i) (q_{d,i}^* w_i q_d^* w_i^T)^T \neq 0$ for all $(i = 1, 2, \ldots, n_1)$.

**Proof.** The PDE-part of system (3.11) is

$$\begin{align*}
\varepsilon(t) &= \varepsilon(0) + \int_0^t f(c_0 + c_1 \varepsilon(s), f(0, t)) ds, \\
\varepsilon(0) &= 0, \\
\varepsilon(1, t) &= 0,
\end{align*} \hfill (3.27)$$

It is well known that $A_1$ generates an exponentially stable $C_0$-semigroup $e^{t A_1}$ on $H_1 [19]$: There exist $L_1, \delta_1 > 0$ such that

$$\|\varepsilon(\varepsilon, t)\|^2_{H_1} \leq L_1 e^{-\delta_1 t} \|\varepsilon(\varepsilon, 0)\|^2_{H_1}, \quad t > 0, \hfill (3.28)$$

which implies that $\|\varepsilon(\varepsilon, t)\|^2_{H_1} \leq \int_0^t e^{-\delta_1 t} \|\varepsilon(\varepsilon, t)\|^2_{H_1} dt, \quad t > 0$. For the ODE part, a straightforward computation gives

$$\begin{align*}
\tilde{p}_d(t) &= e^{S_k h(0)} \tilde{p}_d(0), \\
\tilde{p}_d(t) &= e^{S_k h(0)} \tilde{p}_d(0) + \int_0^t e^{S_k h(0)}(t-s) k_d \varepsilon(0, s) ds, \hfill (3.29)
\end{align*}$$

which has estimations:

$$\begin{align*}
\|\tilde{p}_d(t)\|_{C_2} &\leq \|e^{S_k h(0)}\|_{C_2} \tilde{p}_d(0) \leq L_2 e^{-\delta_2 t} \|\tilde{p}_d(0)\|_{C_2}, \\
\|\tilde{p}_d(t)\|_{C_2} &\leq \|e^{S_k h(0)}\|_{C_2} \tilde{p}_d(0) + \int_0^t e^{S_k h(0)}(t-s) k_d \varepsilon(0, s) ds \leq L_4 e^{-\delta_3 t} \|\tilde{p}_d(0)\|_{C_2} + L_2 e^{-\delta_3 t} \|\varepsilon(\varepsilon, 0)\|^2_{H_1}, \hfill (3.30)
\end{align*}$$
for some \( L_0 > 0, L_2 > 0, \delta_1 > 0, \delta_2 > 0, \delta_3 > 0 \). Combining (3.28) and (3.30), we have

\[
\| (\hat{D}_{1}^T(t), \hat{D}_{1}^T(t), \varepsilon(t), \varepsilon(t), \varepsilon(t)) \|_{\mathcal{C}^{x_1} \times \mathcal{C}^{x_1} \times \mathcal{H}_t} \leq L_4 e^{-5d_t} \| (\hat{D}_{1}^T(0), \\
\hat{D}_{1}^T(0), \varepsilon(0), \varepsilon(0), \varepsilon(0)) \|_{\mathcal{C}^{x_1} \times \mathcal{C}^{x_1} \times \mathcal{H}_t},
\]

(3.31)

for some \( L_4 > 0, \delta_4 > 0 \). By (3.3), we define a bounded invertible operator \( \mathcal{P}_0 : \mathcal{C}^{x_1} \times \mathcal{H}_t \to \mathcal{C}^{x_1} \times \mathcal{H}_t \) by

\[
(\hat{D}_1^T(t), \hat{D}_1^T(t), \varepsilon(t), \varepsilon(t), \varepsilon(t))^T = \mathcal{P}_0 (\hat{D}_1^T(t), \hat{D}_1^T(t), \varepsilon(t), \varepsilon(t))^T,
\]

(3.32)

Then, there are some \( L_5, \delta_5 > 0 \) which are independent of initial value such that

\[
\| (\hat{D}_1^T(t), \hat{D}_1^T(t), \varepsilon(t), \varepsilon(t), \varepsilon(t)) \|_{\mathcal{C}^{x_1} \times \mathcal{H}_t} \leq L_5 e^{-5d_t} \| (\hat{D}_{1}^T(0), \\
\hat{D}_{1}^T(0), \varepsilon(0), \varepsilon(0), \varepsilon(0)) \|_{\mathcal{C}^{x_1} \times \mathcal{H}_t}.
\]

(3.33)

This completes the proof of the theorem. □

4. Output feedback regulator

In this section, we turn to design an output feedback regulator for system (1.1). In light of the state feedback design result (2.1), we design the output feedback regulator as follows:

\[
U(t) = -q_1 c_1 c + q_1 c_1 c + \frac{c_0 c_1 + c_1 c}{1 + q_1 c} \hat{u}(0, t)
\]

\[
- \frac{c_0 c_1 + c_1 c}{1 + q_1 c} \int_0^1 \hat{u}_0(t) dy - \frac{q_1 c_1 + c_1 c}{1 + q_1 c} m_0^T \hat{v}(t),
\]

(4.1)

where \( \hat{v}(t) = (\hat{D}_1^T(0), \hat{D}_1^T(0))^T \). It is seen that the difference between (2.1) and (4.1) is that we replace the states in (2.1) by their estimates obtained from observer (3.1). The closed-loop of systems (1.1), (1.3), (1.4), and (1.1) under the controller (4.1) is

\[
\dot{\hat{u}}_1(t) = S_d \hat{u}_2(t), \dot{v}_1(t) = S_r \hat{u}_1(t), \dot{t}_0 \geq 0,
\]

\[
d_1(t) = q_1 c_1 c + \frac{c_0 c_1 + c_1 c}{1 + q_1 c} \hat{u}(0, t) + \frac{c_0 c_1 + c_1 c}{1 + q_1 c} \int_0^1 \hat{u}_0(t) dy - \frac{q_1 c_1 + c_1 c}{1 + q_1 c} m_0^T \hat{v}(t),
\]

(4.2)

We consider system (4.2) in the state space \( \mathbb{H} = (\mathbb{C}^{x_1} \times \mathbb{C}^{x_2} \times \mathbb{H}^{(0, 1)} \times \mathbb{L}^2(0, 1))^2 \) equipped with the inner product induced norm

\[
\| (\xi_1, \xi_2, f_1, g_1, \eta_1, f_2, g_2) \|_\mathbb{H} = \left( \| \xi_1 \|^2_\mathcal{C} + \| \xi_2 \|^2_\mathcal{C} \right)
\]

\[
+ \frac{1}{2} \int_0^1 \left( \| f_1(x) \|^2 + \| g_1(x) \|^2 \right) dx + c_0 f_1(0)^2 + \| \eta_1 \|^2_\mathcal{C} + \| \eta_2 \|^2_\mathcal{C} \}
\]

\[
+ \frac{1}{2} \int_0^1 \left( \| f_2(x) \|^2 + \| g_2(x) \|^2 \right) dx + c_1 f_2(0)^2 \}
\]

\[
\forall (\xi_1, \xi_2, f_1, g_1, \eta_1, f_2, g_2) \in \mathbb{H}.
\]

(4.3)

Define system operator \( A \) for (4.2) as

\[
A(\xi_1, \xi_2, f_1, g_1, \eta_1, f_2, g_2) = (S_d \xi_1, S_r \xi_2, f_1 + f_q \xi_1, \eta_1, f_2 + f_q \xi_2, g_2, f_2 + f_q \xi_2, g_2, g_2 - g_1(0) - g_1(0))
\]

\[
\forall (\xi_1, \xi_2, f_1, g_1, \eta_1, f_2, g_2) \in \mathbb{H}.
\]

(4.4)

Then, system (4.2) can be written as an evolutionary equation in \( \mathbb{H} \):

\[
\frac{d}{dt} (v_1(t), v_1(t), u(t), u(t), u(t), u(t), u(t), u(t))^T = A(v_1(t), v_1(t), u(t), u(t), u(t), u(t), u(t), u(t))^T.
\]

(4.5)

Now we state the main result of this paper.

**Theorem 4.1.** Assume \( N_d^T (\mu_\eta (q_1^2 w_1), q_1^2 w_1)^T \neq 0 \) for all (distinct) eigen-pairs \((\mu_\eta, \omega_\eta)\) of \( S_d \), where \( N_d(s) \) is defined in Lemma 3.2. Let \( h(x) \) be the solution of (3.12), \( k_d \) be chosen such that \( S_d - k_d h^T(0) \) is Hurwitz, \( c_2 > 0, c_3 > 0, c_4 = c_2 h^T(0) k_d \), \( k_1(x) = h^T(x) S_{d_2} k_d \), \( k_2(x) = h^T(x) k_d \). Then for any initial value \((v_\eta(0), v_\eta(0), u_\eta(0), u_\eta(0), \tilde{v}_\eta(0), \tilde{v}_\eta(0), \tilde{u}_\eta(0), \tilde{u}_\eta(0))^T \in \mathbb{H} \), system (4.5) admits a unique bounded solution \((v_\eta(t), v_\eta(t), u_\eta(t), u_\eta(t), \tilde{v}_\eta(t), \tilde{v}_\eta(t), \tilde{u}_\eta(t), \tilde{u}_\eta(t))^T \in \mathbb{C}(0, \infty) \). Moreover, the tracking error \( u(x_0, t) - y_{\eta \eta}(t) \) does not such that

\[
|u(x_0, t) - y_{\eta \eta}(t)| \leq M e^{-\omega t},
\]

(4.2)

where \( M, \omega > 0 \).

**Proof.** Define an invertible operator \( P : \mathbb{H} \to \mathbb{H} \) by

\[
P(v(t), v(t), u(t), u(t), u(t), u(t), u(t), u(t))^T = (v(t), v(t), u(t), u(t), \tilde{v}_\eta(t), \tilde{v}_\eta(t), \tilde{u}_\eta(t), \tilde{u}_\eta(t))^T.
\]

(4.6)
By the observer error \((\hat{v}_d, \hat{v}_r, \hat{u}_1)\), we obtain an equivalent system of (4.2):

\[
\begin{align*}
\dot{\hat{v}}_d(t) &= S_d\dot{v}_d(t), \quad \hat{v}_r(t) = S_r v_r(t), \\
\dot{d}_1(t) &= q_1^2\dot{v}_d(t), \quad d_2(t) = q_1^2\dot{v}_d(t), \quad y_{ref}(t) = q_1^2 v_r(t), \\
u_{ct}(x, t) &= u_{ac}(x, t) + f(x)d_1(t), \\
u_u(0, t) &= -q_1u_1(0, t) + q_2d_2(t), \\
u_1(t, 1) &= -\frac{q_1 + c}{1 + q_1c}u_1(1, t) - c_0u(1, t) + \frac{q_1c_0(q_1 + c)}{1 + q_1c}u(0, t) - \frac{c_0(q_1 + c)}{1 + q_1c}\int_0^1 u_1(y, t) dy - \frac{q_1^2 - 1}{1 + q_1c}\int_0^1 \dot{v}_d(t) dy - \frac{q_1^2 - 1}{1 + q_1c}\tilde{f}(t), \\
\hat{\dot{u}}_1(t) &= \hat{\nu}_2(0, t) + q_1\hat{u}_1(0, t), \\
\hat{\dot{u}}_r(t) &= \hat{\nu}_3(0, t) + (c_1 + c_4)\hat{u}_3(0, t) + q_2\hat{u}_2(t), \\
\hat{\dot{u}}_3(t) &= 0.
\end{align*}
\]

(4.7)

where

\[
\begin{align*}
\tilde{f}(t) &= \frac{q_1 + c}{q_1^2 - 1}\hat{u}_1(1, t) + \frac{c_0(q_1 + c)}{q_1^2 - 1}\hat{u}(1, t) \\
&+ \frac{c_0(q_1 + c)}{q_1^2 - 1}\int_0^1 \hat{u}_1(y, t) dy + m_\nu\tilde{v}(t).
\end{align*}
\]

(4.8)

From Theorem 3.1, we know that the decoupled \((\hat{v}_d, \hat{v}_r, \hat{u}_1)^T\) - part of system (4.7) is exponentially stable. The \(\hat{f}(t)\) defined by (4.8) satisfies \(\hat{f}(t) - \hat{f}(0)\hat{u}_1(1, t) \to 0\) exponentially as \(t \to \infty\) from the exponential stability of \((\hat{v}_d, \hat{v}_r, \hat{u}_1)^T\). As for \(\hat{u}_1(1, t)\), by transformation (3.3), we have

\[
\hat{u}_1(1, t) = \varepsilon_1(t, 1) - h^T(1)(S_d - k_d h^T(0))\hat{v}_d(t) + k_d\varepsilon(0, 0).
\]

(4.9)

where \(\varepsilon(x, t)\) is governed by (3.11). Define the Lyapunov functions

\[
E_1(t) = \frac{1}{2}\int_0^1 \varepsilon_x^2(x, t) + \varepsilon_2^2(x, t) dx + \frac{c_3}{2}\varepsilon^2(0, t),
\]

\[
\rho_1(t) = \int_0^1 x\varepsilon(t, x)\varepsilon(x, t) dx.
\]

(4.10)

Differentiate (3.11) along the solution of (3.11) to yield

\[
\begin{align*}
\dot{E}_1(t) &= -c_2\varepsilon^2(0, t) \leq 0, \\
\dot{\rho}_1(t) &= \frac{x}{2}[\varepsilon_x^2(x, t) + \varepsilon_2^2(x, t)]|_{x = 0} - \frac{1}{2}\int_0^1 [\varepsilon_x^2(x, t) + \varepsilon_2^2(x, t)] dx \\
&= \frac{1}{2}\varepsilon_x^2(1, t) - \frac{1}{2}\int_0^1 [\varepsilon_x^2(x, t) + \varepsilon_2^2(x, t)] dx.
\end{align*}
\]

(4.11)

Integrating the second equation in (4.11) from 0 to \(T\) and taking (3.28) into account, we obtain

\[
\begin{align*}
\int_0^T \varepsilon_x^2(1, t) dx& \leq \rho_1(T) - \rho_1(0) + \int_0^T E_1(t) dt \leq 2E_1(0) \\
\frac{t^2}{2B_1}E_1(0) &= \left(2 + \frac{t^2}{2B_1}\right)E_1(0).
\end{align*}
\]

(4.12)

which means that \(\varepsilon_1(t, \cdot) \in L^2(0, \infty)\). As a result, \(\hat{u}_1(1, \cdot) \in L^2(0, \infty)\) from (4.9) and thus \(\hat{f} \in L^2(0, \infty)\).

Next, we show that

\[
\begin{align*}
\dot{v}_d(t) &= S_d\dot{v}_d(t), \quad \dot{v}_r(t) = S_r v_r(t), \\
\dot{d}_1(t) &= q_1^2\dot{v}_d(t), \quad d_2(t) = q_1^2\dot{v}_d(t), \quad y_{ref}(t) = q_1^2 v_r(t), \\
u_{ct}(x, t) &= u_{ac}(x, t) + f(x)d_1(t), \\
u_u(0, t) &= -q_1u_1(0, t) + q_2d_2(t), \\
u_1(t, 0) &= -\frac{q_1 + c}{1 + q_1c}u_1(1, t) - c_0u(1, t) - \frac{q_1c_0(q_1 + c)}{1 + q_1c}u(0, t) \\
&+ \frac{c_0(q_1 + c)}{1 + q_1c}\int_0^1 u_1(y, t) dy - \frac{q_1^2 - 1}{1 + q_1c}\int_0^1 \dot{v}_d(t) dy - \frac{q_1^2 - 1}{1 + q_1c}\tilde{f}(t).
\end{align*}
\]

(4.13)

admits a unique bounded solution. It is easy to see that system (4.13) is the same as (2.2) except \(\tilde{f}(t)\) flowing into system through the boundary \(x = 1\). We use the same method in (2.2) to investigate the solution and tracking performance for system (4.13).

By the same transformations (2.3) and (2.5), we can find that \(e(x, t)\) is governed by

\[
\begin{align*}
e_1(x, t) &= e_{ax}(x, t), \\
e_2(x, t) &= ce_1(x, t), \\
e_3(1, t) &= -c_0e(1, t) + \tilde{f}(t),
\end{align*}
\]

(4.14)

which can be shown to be exponentially stable. Actually, consider system (4.14) in the state space \(\mathcal{H}_1\) and define the system operator \(A_2\) by

\[
A_2(f, g)^T = (g, f''(0))\quad \forall (f, g)^T \in D(A_2),
\]

(4.15)

\[
D(A_2) = \{(f, g)^T \in H^2(0, 1) \times H^1(0, 1)|A_2(f, g)^T \in \mathcal{H}_1\}.
\]

Then system (4.14) can be written as an evolutionary equation in \(\mathcal{H}_1\):

\[
\frac{d}{dt}(e(\cdot, t), e(\cdot, t))^T = A_2(e(\cdot, t), e(\cdot, t))^T + B_2\tilde{f}(t),
\]

(4.16)

where \(B_2 = \begin{pmatrix} 0 \\ \delta(x - 1) \end{pmatrix}\). It is well known \(A_2\) generates an exponentially stable \(C_0\)-semigroup \(e^{tA_2}\) on \(\mathcal{H}_1\) ([118]). A direct computations shows that \(B_2\) is admissible for \(e^{tA_2}\) ([21]). Therefore, the solution of (4.16) can be written as

\[
(e(\cdot, t), e(\cdot, t))^T = e^{tA_2}(e(0, \cdot), e(0, \cdot))^T + \int_0^t e^{(t-s)A_2}B_2\tilde{f}(s)ds,
\]

(4.17)

which is also exponentially stable on \(\mathcal{H}_1\) by the method of Theorem 2 and (61) in [20].

Finally, by the inverse transformation of (2.3) and (2.5), similar to the proof of Theorem 2.1, system (4.13) admits a unique bounded solution on \(C^0[0, T] \times C^0[0, T] \times L^2(0, 1) \times L^2(0, 1)\). The tracking error

\[
u(x_0, t) = y_{ref}(t) = -\frac{1 + q_1c}{c_2 - 1}e(x_0, t) + \frac{c_0(q_1 + c)}{c_2 - 1}e(0, t) \\
+ \frac{c_0(q_1 + c)}{c_2 - 1}\int_0^{x_0} e_1(y, t) dy \to 0
\]

(4.18)

exponentially because system (4.14) decays exponentially. □
5. Numerical simulations

In this section, we present some numerical simulations to illustrate the effectiveness of the proposed feedback control. In closed-loop system (4.2), we choose the parameters $q_1 = 3, c_0 = 8, c_2 = 5, c_3 = 10$ and $v_0 = 0, f(x) = x$. The matrices of exosystem are

$$S_d = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, S_r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$q_{d_1}^T = [1, 5], q_{d_2}^T = [0, 10], q_r^T = [-2, 2].$$

Using MATLAB R2017a, we obtain numerical solutions of $g^T(x), h^T(x)$ and $h^T(0)$. Therefore $m_v^c$ can be obtained from (2.19) and so are $c_4, k_1(x), k_2(x)$ from Theorem 3.1. We choose $k_d = [-1, 5]^T, k_r = [1/2, -1/4]^T$ such that $S_d - k_d g^T(x), S_r + k_r d_r$ are Hurwitz.

The finite difference method is applied to compute the solution numerically and the steps of time and space are set as 0.002 and 0.005, respectively. The initial values are taken as:

$$u_0(x) = -3x^2 + 2 \sin(3x), u_1(x) = -2 + 3 \cos(x),$$
$$\dot{u}_0(x) = 4 \sin(\pi x), \dot{u}_1(x) = 1 - 4 \sin(\pi x),$$
$$\hat{v}_d(0) = [2, 4]^T, \hat{v}_r(0) = [1, -5]^T, v_0(0) = [5, -2]^T, v_r(0) = [0, 1]^T.$$

Fig. 2(a) and (b) display the trajectory of controller (4.1) and the solution of PDE-part of closed-loop system (4.2). It is obvious that the solution of PDE-part is bounded. Figs. 3 and 4 present the estimation of exosystem. After $t = 40$, all four signals, $\hat{v}_d(\tau), \hat{v}_d(\tau), v_r(\tau), v_r(\tau)$, are recovered satisfactorily. Fig. 5(a) displays the trajectory of controller (4.1). Fig. 5(b) indicates that the boundary displacement tracks reference well after $t = 50$. It thus illustrates the effectiveness of our output feedback controller (4.1).
6. Concluding remarks

Adopting the method developed in [6], in this paper, a boundary output regulation problem for a one-dimensional wave equation with anti-stable term at control unmatched end has been fully investigated. This represents a difficult situation where the control and regulated output are not in the same boundary (non-collocated). We suppose that there are external disturbances both in domain and boundaries. An observer to cover both wave PDE and exosystem states is constructed. An observer based output feedback control is then designed to achieve output regulation and at the same time to guarantee all subsystems in the closed-loop to be uniformly bounded. This generalizes the approach for heat equation (parabolic type) presented in [5] to wave equation (hyperbolic type). The results generalize the special problem developed in [11].

Declaration of competing interest

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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