Brief paper

Boundary output tracking for an Euler–Bernoulli beam equation with unmatched perturbations from a known exosystem

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A B S T R A C T

In this paper, we consider boundary output regulation for an Euler–Bernoulli beam equation which can describe typically the flexible arm of robots. The reference signal and disturbance are generated by a finite-dimensional exosystem. The measurements are angular and angular velocity of the right end where the control is imposed. However, the performance output is on the left end which is non-collocated with control, a difficult case in practice where the control takes time to perform its force from the right end to the left. The objective is to design an output feedback controller to regulate the displacement of the left end to track the reference signal. We first design a state feedback regulator to make the performance output track the reference signal exponentially. An observer is then constructed to recover the state, with which, an output feedback regulator is designed by replacing state feedback with its estimation. The closed-loop system is shown to admit a unique bounded solution and the tracking error converges to zero exponentially. Some numerical simulations are presented to illustrate the effectiveness of the proposed output feedback regulator.

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1. Introduction

Flexible structures such as flexible robot arms are widely used in aerospace technologies, satellites, flexible manipulators, and other industry applications. In many situations, the Euler–Bernoulli beam equation can well describe the flexible arms. Because of light weight and high speed, the vibration of flexible arms is inevitable, which reduces accuracy in industrial streamlines. Vibration control for Euler–Bernoulli beam has therefore been considered by many researchers. Some of works can be found in Chen, Delfour, Krall, and Payre (1987), Chen, Krantz, Ma, Wayne, and West (1988), Conrad and Pierre (1990), Luo and Guo (1997), Smyshlyaev, Guo, and Krstic (2009) and Xu and Sallet (1992) and the references therein. Most of these works are related to stabilization problem. Recently, the disturbance rejection problem has been addressed for Euler–Bernoulli beam systems in Ge, Zhang, and He (2011), He, Zhang, and Ge (2013) and Jin and Guo (2015).

On the other hand, output regulation is one of the central issues in control theory. Many industry processes can be formulated as regulation problem like the military field. The output regulation was started initially for finite-dimensional systems like (Byrnes, Priscoli, & Isidori, 1997; Callier & Desoer, 1980; Desoer & Lin, 1985) from which many results have been extended to infinite-dimensional systems and some examples can be found in Byrnes, Laukí, Gilliam, and Shubov (2000), Deutscher (2015), Deutscher (2017), Paunonen and Pohjolainen (2010), Paunonen and Pohjolainen (2014), Xu and Dubljevic (2017a), Xu and Dubljevic (2017b) and Xu, Pohjolainen, and Dubljevic (2017). The most profound result in this regard is the internal model principal which has been generalized from lumped parameter systems into distributed parameter systems. For bounded control and observation, we refer to Paunonen and Pohjolainen (2010) and unbounded control and observation can be found in Paunonen and Pohjolainen (2014). In Xu and Dubljevic (2017a), an observer-based output regulator was designed for an abstract infinite-dimensional system with bounded control and unbounded observation. Adaptive control method has also been applied to regulation problem for a wave equation in Guo and Guo (2016) where the unknown constant coefficients in harmonic disturbance were identified. Recently, in Deutscher (2015), an output regulation problem for a heat equation was discussed by means of backstepping approach (Krstic & Smyshlyaev, 2008), where
the control and observation were allowed to be unbounded and the disturbance and reference signals are generated by a finite-dimensional exosystem. In addition, the output can be either bounded or unbounded. This method was generalized to regulation problem for a coupled wave equation with unbounded control and observation in Gu, Wang, and Guo (2018), where the output was supposed to be bounded. In Jin and Guo (2018), we designed an output regulator for a heat equation with unbounded control and unbounded observation. The main idea of Jin and Guo (2018) is that an extended state observer can be constructed to estimate the state and the general external disturbance, but the control and performance output are matched. There are also some other works related to motion planning of beam equation. In Shifman (1990), a tracking problem for an Euler–Bernoulli beam was discussed, where the reference trajectory is the same as the Euler–Bernoulli beam except the control channel, and the displacement and velocity of the beam are supposed to be known in control design. In Meurer, Thull, and Kugi (2008) and Schröck, Meurer, and Kugi (2011), the differential flatness method, a powerful tool for tracking problem, was applied to flexible beam motion planning. For these beam works, no disturbance was considered.

In this paper, we consider output regulation problem for a Euler–Bernoulli beam equation described by

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \\
0 &= g(x(t), u(t)), \\
\dot{p}(t) &= q(x(t), u(t)), \\
\dot{m}(t) &= r(x(t), u(t)),
\end{align*}
\]

where \(f, g, q, r\) are smooth functions, \(x(t) \in \mathbb{R}^n\) is the state, \(u(t) \in \mathbb{R}^m\) is the control input, \(p(t) \in \mathbb{R}^p\) is the bounded disturbance, \(m(t) \in \mathbb{R}^q\) is the bounded noise, \(n, m, p, q\) are positive integers. The system is assumed to be in the presence of disturbances. Meanwhile, the state and the control end are assumed to be known in design. From now on, we omit initial value and domain for all systems when there is no confusion for simplicity.

The system (1) describes well the movement of the flexible robot arms. The left end is in free movement yet both bending moment and share force are affected by unknown disturbances, and the displacement is required to be regulated to track a reference signal. The control end is pinned which is a typical case in applications yet the bending moment is actuated. The internal disturbance is not uniform by the introduction of the intensity function \(f(x)\).

We point out that there are very limited literature on output regulation for PDEs with non-collocated control and regulated output. The heat equation (parabolic type) was considered in Deutzer (2015). A special output regulation with reference (set point) zero for wave equation (hyperbolic type) was developed in Guo, Shao, and Krstic (2017). This paper is the first paper for beam equation (Petrovsky type) with non-collocated control and regulated output.

We proceed as follows. In Section 2, we design a state feedback controller for tracking control system (1) where the disturbances are assumed to be known. In Section 3, we construct a state observer for system (1) and exosystem (2) in terms of the measured output and the reference signal. It is shown that the observer error decays exponentially as time goes to infinity. An output regulator is designed by replacing the state with its estimation obtained in Section 4. By means of \(Q_0\)-semigroup and admissibility theory for linear infinite-dimensional systems, the closed-loop system is shown to admit a unique bounded solution and the displacement of the left end tracks reference signal exponentially. Some numerical simulations are presented in Section 5 to illustrate the effect of the proposed control law, following up concluding remarks in Section 6.

2. State feedback regulator

To design an output feedback, we need state feedback first. In this section, we assume that all states of systems (1) and (2) are known. We propose a state feedback controller for tracking control system (1) as follows:

\[
U(t) = -k_1 u_1(1, t) - k_2 u_2(1, t) + m^T v(t),
\]

where \(k_1, k_2 > 0\) are tuning parameters, \(m^T\) is an \(n\)-dimensional row vector to be determined later. It is noted that the first two terms are used to stabilize internally exponentially the system (disturbance free system), and the third term is for output tracking. The closed-loop of systems (1) and (2) under the controller
Theorem 2.1. Assume that $S$ has distinct eigenvalues on the imaginary axis only. Then, BVP (15) admits a unique solution.

The $m_n$ in (6) can be determined by

\[
m_n = \frac{\partial^2 g}{\partial x^2}
\]

\begin{align*}
\left|_{x=1} + k_1 \frac{\partial g^T}{\partial x}
\right|_{x=1} + k_2 \frac{\partial^2 g^T}{\partial x^2}
\right|_{x=1}.
\end{align*}

Theorem 2.1. Let $g(x)$ be the solution of (15) and $m_n = g''(1) + k_1 S^T g'(1) + k_2 g'(1)$. For any initial value $(v_0, u_0, u_1) \in \mathbb{C}^n \times \mathcal{H}_1$, the closed-loop system (7) admits a unique bounded solution $(v^T(t), u^T(t), u_1^T(t)) \in C(0, \infty; \mathbb{C}^n \times \mathcal{H}_1)$. Moreover,

\[
u(0, t) - y_{ref}(t) \to 0 \text{ exponentially as } t \to \infty.
\]

Proof. From transformation (8), we find the solution to (7) as

\[
(v(t), u(t), u_1(t))^T = (v^T(t), e^T(t) + g^T(x) v(t), e_1(t) + g^T(x) S v(t))^T
\]

and

\[
\|v^T(t), e(t), u_1(t)\|^2_{\mathcal{H}_1} \leq \|v^T(\cdot), e(t), u_1(t)\|^2_{\mathcal{H}_1} + 2\|g^T(x) v(t)\|^2_{\mathcal{H}_1} + 2 \int_0^1 \left[ \left| g^T(\cdot) v(t) \right|^2 + |g^T(x) S v(t)|^2 \right] dt
\]

\[
+ 2k_2 \|g^T(\cdot) \|^2_{\mathcal{H}_1} + 2 \left( \|e(t), e(t), u_1(t)\|^2_{\mathcal{H}_1} + 2 \int_0^1 \left| g^T(\cdot) v(t) \right|^2 + |g^T(x) S v(t)|^2 \right) dt.
\]

Since all eigenvalues of $S$ are located on the imaginary axis, $v(t)$ and $S v(t)$ are bounded as $t \to \infty$. Since $g(x)$ is the solution of (15) with $g^T(\cdot) \in C^1[0, 1])$, both $\|g^T(x) v(t)\|^2_{\mathcal{H}_1}$ and $\|v^T(\cdot), e(t), u_1(t)\|^2_{\mathcal{H}_1}$ are uniformly bounded, and $\|e(t), e(t), u_1(t)\|^2_{\mathcal{H}_1}$ decays exponentially, there exists constant $L > 0$ depending on initial value $(v_0^T, u_0, u_1)$ such that

\[
\|v^T(t), e(t), u_1(t)\|^2_{\mathcal{H}_1} \leq L, \forall t \geq 0.
\]

Now we consider the tracking performance. By (13), it follows that the tracking error $e(0, t)$ has the estimation:

\[
\|e(0, t)\|^2 \leq \left| \int_0^1 e_v(x, t) dx \right|^2 \leq L_1 \|e(0, t), e(t), u_1(t)\|^2_{\mathcal{H}_1}, L_1 > 0.
\]

Since $\|e(0, t), e(t), u_1(t)\|^2_{\mathcal{H}_1}$ decays exponentially, so does $e(0, t) \to 0$ exponentially as $t \to \infty$. □

3. Observer design

In this section, we assume that the disturbance is unknown yet reference is known. Then, $v(t)$ in state feedback controller (6) needs to be estimated. We design a state observer as follows:

\[
\begin{align*}
\dot{\hat{u}}_{a}(t) &= S_a \hat{u}_{a}(t) + k_a \hat{u}_a(t) + k_1 \hat{u}_a(0) + k_2 \hat{u}_a(1), \\
\dot{\hat{u}}_{b}(t) &= S_b \hat{u}_{b}(t) + k_b \hat{u}_b(t) + k_1 \hat{u}_b(0) + k_2 \hat{u}_b(1), \\
\dot{\hat{u}}_{c}(t) &= S_c \hat{u}_{c}(t) + k_c \hat{u}_c(t) + k_1 \hat{u}_c(0) + k_2 \hat{u}_c(1), \\
\dot{\hat{u}}_{d}(t) &= S_d \hat{u}_{d}(t) + k_d \hat{u}_d(t) + k_1 \hat{u}_d(0) + k_2 \hat{u}_d(1), \\
\dot{\hat{u}}_{e}(t) &= S_e \hat{u}_{e}(t) + k_e \hat{u}_e(t) + k_1 \hat{u}_e(0) + k_2 \hat{u}_e(1).
\end{align*}
\]

where $\gamma_1, \gamma_2 \in C[0, 1], c_1 > 0, c_2 > 0$ are the tuning constant parameters, $k_i$ is chosen such that $S_i + k_i q_i^T$ is Hurwitz which follows from the observability of $(q_i^T, S_i)$. Then, the error variables...
\[ \begin{align*}
\dot{v}_d &= \dot{u}_d - v_d, \\
\dot{u}_d &= S_2 \hat{\bar{v}}_d(t) + k_d \bar{u}_d(1, t), \\
\hat{v}_d(t) &= (S_2 + k_d q_d^2) \hat{v}_d(t), \\
\dot{v}_d(0, t) &= q_{d,0}^2 \hat{v}_d(0, t) + f(x)q_d^2 \hat{v}_d(t) \\
&\quad + \gamma_1(x) \bar{u}_d(1, t) + y_2(x) \bar{u}_d(1, t) = 0, \\
\bar{u}_d(0, t) &= q_{d,0}^2 \bar{u}_d(0, t) = q_{d,0}^2 \tilde{v}_d(t), \\
\hat{u}_d(1, t) &= 0, \\
\bar{u}_d(1, t) &= -c_1 \bar{u}_d(1, t) - c_2 \bar{u}_d(1, t).
\end{align*} \]

We need to prove that the error system (21) is stable. To this purpose, since \( S_2 + k_d q_d^2 \) is Hurwitz, the variable \( \hat{v}_d \) is decoupled from others and is exponentially stable. For \( \hat{v}_d \) and \( \bar{v}_d \), we introduce a transformation

\[ \begin{align*}
\hat{v}_d(t) &= \bar{v}_d(t), \\
\hat{u}_d(t) &= \bar{u}_d(t), \\
\hat{u}_d(0, t) &= \bar{u}_d(0, t) + h^T(x) \hat{v}_d(t).
\end{align*} \]

We make (21) slightly simpler. In this way, \( \hat{v}_d(t) \) satisfies

\[ \begin{align*}
\hat{v}_d(t) &= \bar{v}_d(t), \\
\hat{u}_d(t) &= \bar{u}_d(t), \\
\hat{u}_d(0, t) &= \bar{u}_d(0, t) + h^T(x) \hat{v}_d(t), \\
\hat{u}_d(1, t) &= 0, \\
\bar{u}_d(1, t) &= -c_1 \bar{u}_d(1, t) - c_2 \bar{u}_d(1, t).
\end{align*} \]

provided that we choose

\[ \begin{align*}
\begin{cases}
\frac{d^2 h^T(x)}{dx^2} = -h^T(x) S_2^2 + f(x)q_d^2, \\
\gamma_1(x) = h^T(x) k_d, \gamma_2(x) = h^T(x) S_2 k_d.
\end{cases}
\end{align*} \]

The boundary condition of \( \hat{v}_d(t) \) at \( x = 0 \) gives

\[ \begin{align*}
\gamma_1(0) &= h^T(0) k_d, \\
\gamma_2(0) &= h^T(0) S_2 k_d.
\end{align*} \]

The problem (21) is the boundary value problem (BVP) of the following:

\[ \begin{align*}
\begin{cases}
h^T(x) = -(S_2^2)^T h(x) + f(x)q_d, \\
h'^T(x) = -(S_2^2)^T h(x) + f(x)q_d, \\
h''(x) = -q_d, h''(0) = 0, \\
h''(1) = c_1 S_2^2 h'(1) + c_2 h'(1) = 0.
\end{cases}
\end{align*} \]

**Lemma 3.1.** Assume that all eigenvalues of \( S_d \) are distinct and are located on the imaginary axis. Then, the BVP (31) admits a unique solution.

The proofs for Lemmas 3.1 and 3.2 are arranged in Appendix. Now we are in a position to show that there exists \( k_d \) such that \( S_d - k_d \frac{dh}{dx} \bigg|_{x=1} \) is Hurwitz.

**Lemma 3.2.** The numerator of the transfer matrix \( T_d^T(s) = \frac{N_d(s)}{D_d(s)} \) of (1) from \( d_1, d_2, d_3 \) to one output signal \( u_d(1, t) \) is

\[ \begin{align*}
N_d(s) &= \begin{pmatrix}
-2r_2 \left( \frac{1-j}{\sqrt{2}} \right.
\end{pmatrix},
\end{align*} \]

provided that

\[ \begin{align*}
\begin{cases}
R(s, y) &= r_1 \left( \frac{1-j}{\sqrt{2}} \right) \left( \frac{1-j}{\sqrt{2}} \right),
\end{cases}
\end{align*} \]

The pair \( \left( \frac{dh}{dx} \bigg|_{x=1} = S_d \right) \) is observable if and only if

\[ \begin{align*}
\begin{cases}
q_{d,0}^2 w_i, \\
q_{d,0}^2 w_i.
\end{cases}
\end{align*} \]

where \( w_i \) is the eigenvector of \( S_d \) corresponding to the eigenvalue \( \mu_i \) of \( S_d \).

Note that \( \left. \frac{dh}{dx} \right|_{x=1} = \left. \frac{dh}{dx} \right|_{x=1} \neq 0, i = 1, 2, \ldots, n_1 \), as long as the eigenvalues of \( S_d \) satisfy the conditions in Lemma 3.2. When this is satisfied, \( \frac{dh}{dx} \bigg|_{x=1} = S_d \) is observable. Then, there exists \( k_d \) such that \( S_d - k_d \frac{dh}{dx} \bigg|_{x=1} \) is Hurwitz. Going back to system (31), the PDE-part is exponentially stable. This together with \( S_d - k_d \frac{dh}{dx} \bigg|_{x=1} \) and \( S_d + k_d q_d^2 \), being Hurwitz shows that system (30) is exponentially stable. By the inverse transformation of (22),

\[ \begin{align*}
\hat{u}_d(t) = \epsilon(x, t) - h^T(x) \hat{u}_d(t),
\end{align*} \]

where \( h(x) \) is the classical solution of BVP (31).

**Theorem 3.1.** Assume that \( N_d^T(\mu_i) \left( q_{d,0}^2 w_i, q_{d,0}^2 w_i, q_{d,0}^2 w_i \right)^T \neq 0(i = 1, 2, \ldots, n_1) \), where \( N_d(s) \) is defined in Lemma 3.2. Let \( h(x) \) be the classical solution of (31). Choose \( k_d \) such that \( S_d - k_d \frac{dh}{dx} \bigg|_{x=1} \) is Hurwitz, and let \( c_1 > 0, c_2 > 0, \gamma_1(x) = h^T(x) k_d, \gamma_2(x) = h^T(x) S_2 k_d \). Then, the observer (20) converges to system (3), (4), and (1) exponentially, that is, the observer error system (21) is well-posed and exponentially stable.
Proof. This is a consequence of the equivalence between (21) and (30) by invertible transformation (22). The PDE-part of system (30) is
\begin{equation}
\begin{aligned}
\left\{ \begin{array}{l}
\dot{x}_1(t, x) + \dot{x}_2(x, t) = 0, \\
\dot{x}_2(x, 0, t) = \dot{x}_2(0, t) = x(1, t) = 0,
\end{array} \right.
\end{aligned}
\tag{36}
\end{equation}
Consider system (36) in the space state $H_1$, which is well known exponentially stable in $H_1$ (Gnedin, 1992): There exist $L_1, \delta_1 > 0$ such that
\begin{equation}
\begin{aligned}
\|\dot{s}(t, 0, e(0, t), \epsilon(0), t, \epsilon(0)))\|_{H_1} \leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0), \dot{s}(0, t)\|_{H_1}, \quad t > 0,
\end{aligned}
\tag{37}
\end{equation}
which implies that
\begin{equation}
\begin{aligned}
|\epsilon(0, t)| \leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0), \dot{s}(0, t)\|_{H_1}, \quad t > 0,
\end{aligned}
\end{equation}
for some $L' > 0, \varepsilon > 0$.
For the ODE part, a straightforward computation gives
\begin{equation}
\begin{aligned}
\dot{v}_t(t) &= e^{(S_k + kq_k^2)\tilde{t}^2} \tilde{v}_t(0), \\
\dot{u}_d(t) &= e^{S_k k_k^H \tilde{t}^2} \tilde{u}_d(0) \\
&+ \int_0^{\tilde{t}} e^{S_k k_k^H (\tilde{t} - s)} k_k \dot{e}_k(1, s) ds,
\end{aligned}
\tag{38}
\end{equation}
from which we can estimate that
\begin{equation}
\begin{aligned}
\|\dot{v}_t(t)\|_{C^2} &= \|e^{(S_k + kq_k^2)\tilde{t}^2} \tilde{v}_t(0)\|_{C^2} \\
&\leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0)\|_{C^2},
\end{aligned}
\tag{39}
\end{equation}
and
\begin{equation}
\begin{aligned}
\|\dot{u}_d(t)\|_{C^1} &\leq \|e^{S_k k_k^H \tilde{t}^2} \tilde{u}_d(0)\|_{C^1} \\
&+ \int_0^{\tilde{t}} e^{S_k k_k^H (\tilde{t} - s)} k_k \dot{e}_k(1, s) ds \\
&\leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0)\|_{C^1} \\
&+ L_1 e^{-\delta_1 t}\|\epsilon(0, 0)\|_{C^1} \|\dot{e}_k(1, s)\|_{H_1},
\end{aligned}
\tag{40}
\end{equation}
for some $L_1 > 0, L_2 > 0, L_2 > 0, \delta_1 > 0, \delta_2 > 0, \delta_3 > 0$.
Combining (37), (39), and (40), we obtain
\begin{equation}
\begin{aligned}
\|\dot{v}_t(t), \dot{v}_t(0), x(0, t), \epsilon(0, t))\|_{C^1} &\leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0)\|_{C^1} \\
&\leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0)\|_{C^1} + L_1 e^{-\delta_1 t}\|\dot{e}_k(1, s)\|_{H_1},
\end{aligned}
\end{equation}
for some $L_1 > 0, \delta_1 > 0$.
By (22), we define a bounded invertible operator $P_0 : C^0 \times H_1 \rightarrow C^0 \times H_1$ by
\begin{equation}
\begin{aligned}
(P_0)_{(v_0, t), (\dot{v}_0, t), (\dot{u}_d, t), (\dot{u}_d, t)) &= (P_0)_{(v_0, t), (\dot{v}_0, t), (\dot{u}_d, t), (\dot{u}_d, t)) \\
&= (v_0, t), (\dot{v}_0, t), (\dot{u}_d, t), (\dot{u}_d, t)) \\
&= (v_0, t), (\dot{v}_0, t) - h^T(x) \dot{s}_d(t), (\dot{u}_d, t) \\
&- h^T(x) S_d - k_k h(0) \dot{u}_d(t) - h^T(x) k_k e(0)\). \quad \text{Then,}
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
\|\dot{v}_t(t), \dot{v}_t(0), (\dot{u}_d, t)\|_{C^0 \times H_1} \leq L_1 e^{-\delta_1 t}\|\epsilon(0, 0)\|_{C^0 \times H_1},
\end{aligned}
\end{equation}
We thus complete the proof of the theorem. \hfill \square

4. Output feedback regulator
In this section we design an output feedback regulator for system (1). In light of state feedback regulator in Section 2, we design an output feedback regulator as follows
\begin{equation}
U(t) = -k_1 u_a(t, 1, t) - k_2 u_a(1, t) + m^T \hat{v}(t),
\tag{42}
\end{equation}
where $\hat{v}(t) = (\hat{v}_t(t), \hat{v}_d(t))$. $m = (g^T(\gamma(1)) + k_1 g(\gamma(1)) S + k_2 g(\gamma(1)))$, and $g(x)$ is the solution of (15). The difference between the output feedback regulator and the state of the exosystem is in the output feedback, we just replace $v(t)$ in state feedback controller (6) by its estimation $\hat{v}(t)$. The closed-loop of system (1), (3), (4), and (20) under the controller (42) is
\begin{equation}
\begin{aligned}
\hat{u}_d(t) &= S_d \dot{u}_d(t), \quad \hat{v}_t(t) = S_i \hat{v}_t(t), \\
\dot{u}_1(t) &= -k_1 u_a(t, 1, t) - k_2 u_a(1, t) + m^T \hat{v}(t), \\
\dot{\hat{u}}(t) &= S_i \hat{v}_t(t) + k_1 u_a(1, t) - u_c(t)), \\
\dot{u}(t) &= \delta(t) \gamma(t) \gamma(t) - \gamma(t) \gamma(t) S \delta(t),
\end{aligned}
\end{equation}
for some $L' > 0, \varepsilon > 0$. The proof follows along similar lines as before.

We consider system (43) in the space state $H = (C^0(1) \times C^0(1))$ equipped with the inner product defined norm
\begin{equation}
\begin{aligned}
\|\xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta\|_H^2 &= \|\xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta\|_H^2 \\
&= \|\xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta\|_H^2 + \int_0^1 \|f(t)\|_H^2 + \|g(t)\|^2 dx \\
&+ \|\xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta\|_H^2 + \int_0^1 \|f(t)\|_H^2 + \|g(t)\|^2 dx \\
&+ k_2 f^2(t)) + c_2 f^2(t)) \\
\forall (\xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta) \in H.
\end{aligned}
\end{equation}

Define the system operator $A_{(43)}$ by
\begin{equation}
\begin{aligned}
A_{(43)} (\xi, \eta, \xi, \eta, \xi, \eta, \xi, \eta) &= \left(\begin{array}{c}
\dot{\xi}(1) - f_1(1), \\
\dot{\eta}(1) - f_2(1), \\
\dot{\xi}(1) - f_3(1), \\
\dot{\eta}(1) - f_4(1), \\
\dot{\xi}(1) - f_5(1), \\
\dot{\eta}(1) - f_6(1), \\
\dot{\xi}(1) - f_7(1), \\
\dot{\eta}(1) - f_8(1),
\end{array} \right)
\end{aligned}
\tag{45}
\end{equation}
with
\begin{equation}
\begin{aligned}
D(A_{(43)}) &= D(A_{(43)}) = \left(\begin{array}{c}
\dot{\xi}(1) - f_1(1), \\
\dot{\eta}(1) - f_2(1), \\
\dot{\xi}(1) - f_3(1), \\
\dot{\eta}(1) - f_4(1), \\
\dot{\xi}(1) - f_5(1), \\
\dot{\eta}(1) - f_6(1), \\
\dot{\xi}(1) - f_7(1), \\
\dot{\eta}(1) - f_8(1),
\end{array} \right)
\end{aligned}
\tag{45}
\end{equation}
By the same transformation (8), we can find that \( e(x, t) \) is governed by
\[
\begin{cases}
e_{\text{fr}}(x, t) + e_{\text{wcek}}(x, t) = 0, \\
e_{\text{fr}}(0, t) = e_{\text{wcek}}(0, t) = e(1, t) = 0, \\
e_{\text{fr}}(1, t) = -k_{1}e_{\text{fr}}(1, t) - k_{2}e_{\text{fr}}(1, t) + m^{T}\hat{v}(t).
\end{cases}
\] (50)

We claim that system (50) is also exponentially stable. Indeed, consider system (50) in the state space \( \mathcal{H}_{1} \). Define the system operator \( A_{2} \) by
\[
\begin{cases}
A_{2}[f, g] = (g, -f^{(4)})^{T}, \quad &V(f, g) \in D(A), \\
D(A_{2}) = \{f, g\} \in H^{4}(0, 1) \times H^{2}(0, 1)
\end{cases}
\] (51)

Then, system (50) can be written as an abstract evolutionary equation in \( \mathcal{H}_{1} \):
\[
\begin{align*}
&\frac{d}{dt}(e_{\text{fr}}(t), e_{\text{fr}}(t))^{T} \\
&= A_{2}(e_{\text{fr}}(t), e_{\text{fr}}(t))^{T} + B_{2}m^{T}\hat{v}(t),
\end{align*}
\] (52)

where \( B_{2} = \begin{pmatrix} 0 & -\delta(x - 1) \end{pmatrix}^{T} \). It is well known \( A_{2} \) generates an exponentially stable C0-semigroup \( e^{A_{2}t} \) on \( \mathcal{H}_{1} \) (Gnedin, 1992).

Direct computations show that the adjoint operator \( A_{2}^{*} \) of \( A_{2} \) is given by
\[
\begin{align*}
&\begin{cases}
A_{2}^{*}[f, g] = (-f, g^{(4)})^{T}, \\
D(A_{2}^{*}) = \{f, g\} \in H^{2}(0, 1) \times H^{4}(0, 1)
\end{cases}
\end{align*}
\] (53)

For any \( (f, g)^{T} \in \mathcal{H}_{1}, B^{*}A_{2}^{*}f^{(4)}g^{(4)} = f^{(1)}(t) \) which indicates that \( B^{*}A_{2}^{*} \) is bounded. The dual system is governed by
\[
\begin{align*}
&\begin{cases}
p_{\text{fr}}(x, t) + p_{\text{wcek}}(x, t) = 0, \\
p_{\text{fr}}(0, t) = p_{\text{wcek}}(0, t) = p(1, t) = 0, \\
p_{\text{fr}}(1, t) = -k_{1}p_{\text{fr}}(1, t) - k_{2}p_{\text{fr}}(1, t), \\
p_{\text{wcek}}(t) = -p_{\text{wcek}}(1, t)
\end{cases}
\end{align*}
\] (54)

which is exponentially stable. Define the Lyapunov function
\[
E^{*}(t) = \frac{1}{2} \int_{0}^{1} p_{\text{fr}}^{2}(x, t) + p_{\text{wcek}}^{2}(x, t) dx + \frac{k_{2}}{2} p_{\text{fr}}^{2}(1, t).
\]

Differentiate \( E^{*}(t) \) along the solution of system (54) to yield
\[
\dot{E}^{*}(t) = -k_{1}p_{\text{fr}}^{2}(1, t) \leq 0.
\] (55)

Integrating (55) with respect to \( t \) from 0 to \( T \), we obtain
\[
\int_{0}^{T} p_{\text{fr}}^{2}(1, t) dt \leq \frac{1}{k_{1}} E(0),
\] (56)

which together with the boundedness of \( B^{*}A_{2}^{*} \) implies that \( B^{*} \) is admissible for \( e^{A_{2}t} \). As a result, \( B_{2} \) is admissible for \( e^{A_{2}t} \) (Weiss, 1989). Therefore, the solution of (52) can be written as
\[
\begin{align*}
&\begin{cases}
e_{\text{fr}}(t), e_{\text{fr}}(t) \end{cases} \rightarrow e_{\text{fr}}(t), e_{\text{fr}}(t) \end{cases}
\end{align*}
\] (57)

which is also exponentially stable on \( \mathcal{H}_{1} \) by the method in Theorem 2 and Eq.(60) of Su, Guo, Wang, and Krstic (2017) because \( \hat{v}(t) \) decays exponentially.

By the inverse transformation of (8), similar to the proof of Theorem 2.1, system (49) admits a unique bounded solution in \( C^{0}_{1} \times C^{2}_{1} \times \mathcal{H}_{1} \). The tracking error is
\[
\begin{align*}
u(0, t) - y_{\text{ref}}(t) = e(0, t) \rightarrow 0
\end{align*}
\] (58)

exponentially because system (50) decays exponentially. □
5. Numerical simulation

In this section, we present some numerical simulations to illustrate the effectiveness of the proposed feedback control. In the closed-loop system (43), we choose the matrix of the exosystem as

$$
S_d = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad S_r = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}, \quad \phi_{d1}^T = (1, 2),
$$

and the parameters are chosen to be $c_1 = 0.5, c_2 = 1, k_1 = 0.2, k_2 = 0.5$. For simplicity, set $f(x) = 1, x \in (0, 1)$ and $k_r = (−5, 1)^T$ which makes $S_r + k_r q \in S$ stable. From Lemmas 2.1 and 3.1, $g(x)$ and $h(x)$ can then be obtained by (15) and (31) respectively, and we choose $k_h = (−4, 8)^T$ which makes $S_d + k_h q = S$ stable and the $y_2(x)$ and $y_3(x)$ can be obtained via boundary condition of (24). The initial values are taken as:

$$
u_0(x) = 3x + 5 \cos(\pi x) + 2, \quad u_1(x) = 3 + 3 \cos(\pi x),
$$

$$
\dot{\nu}_0(x) = 4 \sin(\pi x), \quad \dot{u}_1(x) = 3(x - 1) - 4 \sin(\pi x),
$$

$$
\nu_0(0) = (5, -2)^T, \quad \nu_1(0) = (0, 1)^T,
$$

$$
\dot{\nu}_0(0) = (6, -4)^T, \quad \dot{\nu}_1(0) = (1, 2)^T.
$$

The finite element method is applied to compute the solution numerically. Let $h = 0.1$ and $N = 1/h$. We give $[0, 1]$ an $N$ equi-partition with nodes $x_i = ih, i = 0, 1, 2, \ldots, N$. On the node $x_i$, two finite element basis functions are selected as

$$
\phi_{2i+1}(x) = \begin{cases} (x - x_i)(x - x_{i-1})^2 / h^2, & x \in [x_{i-1}, x_i], \\ (x - x_i)(x - x_{i+1})^2 / h^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}. \end{cases}
$$

$$
\phi_{2i+2}(x) = \begin{cases} (h - x_i)(x - x_{i})^2 / h^3, & x \in [x_{i-1}, x_i], \\ (h - x_i)(x - x_{i+1})^2 / h^3, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}. \end{cases}
$$

For node $x_N = 1$, only one basis function $\phi_{2N+1}$ is selected. There are totally $2N + 1$ basis functions which satisfy the natural boundary conditions $\phi_i(1) = 0, i = 1, 2 \ldots, 2N + 1$. We consider the Galerkin approximation solution of the system (43) in the finite dimensional space generated by these basis functions, which takes the form

$$
\dot{u}^N(x, t) = \sum_{i=1}^{2N+1} a_i(t) \phi_i(x), \quad \ddot{u}^N(x, t) = \sum_{i=1}^{2N+1} b_i(t) \phi_i(x),
$$

where $a_i(t)$ and $b_i(t)$ are determined by standard finite element Galerkin method to satisfy some ODEs. To solve the ODEs, the step of time is set to be 0.01.

Figs. 1(a) and 1(b) display the solutions of PDE part $u(x, t)$ and $\ddot{u}(x, t)$ in the closed-loop system (43). It is seen that the displacement of $u(x, t)$ and $\ddot{u}(x, t)$ is bounded and $\ddot{u}(x, t)$ can track $u(x, t)$ well after $t = 20$. Fig. 2(a) indicates that the estimation $\hat{v}_d(t)$ can track $v_d(t)$ after $t = 7$. Similarly from Fig. 2(b), $\hat{v}_b(t)$ converges to $v_b(t)$ rapidly because this part is decoupled from others. Fig. 3(a) presents the trajectory of the controller. Fig. 3(b) displays the tracking performance of the boundary displacement $u_0(t)$. After $t = 20$, $u_0(t)$ tracks the reference signal $y_{ref}(t)$ satisfactorily which illustrates the effectiveness of the proposed output regulator.

When the measured output signal is contaminated by external noise, we consider the case where the first component of the measurement becomes $u_0(1, t) + 2 \cos(10t)$ with high frequency noise $2 \cos(10t)$. The counterparts of Figs. 1–3 become Figs. 4–6 respectively. It is seen from Fig. 5(a) that the tracking performance of exosystem is affected by output noise apparently. But the output tracking works well from Fig. 6(b).

6. Concluding remarks

In this paper, we develop an output feedback control scheme to solve output regulation for an Euler–Bernoulli beam equation, where the performance output is on the left end which
is non-collocated with the control at the right end. This is a difficult case because the boundary control at boundary $x = 1$ must go through the entire interval $[0, 1]$ to reach the regulated boundary $x = 0$ to take effect. The basic idea is to design an observer to recover the states of the plant and the disturbance. We suppose all disturbance and reference signals are produced by a finite-dimensional exosystem which covers general harmonic disturbances, and the harmonic disturbance can be considered as an approximation of periodic signal. This is also the general formulation for disturbance in the framework of the internal model principle.

As indicated in introduction, a regulation problem for a one-dimensional wave equation suffered from harmonic boundary disturbance with unknown amplitudes was considered recently in Guo et al. (2017). The method adopted there is the adaptive control method to estimate all amplitudes. Compared with (Guo et al., 2017), there are some advantages by the approach developed in this paper: (a) the order of controller is much lower than (Guo et al., 2017) because (Guo et al., 2017) needs to estimate all amplitudes of harmonic disturbance; (b) the exponentially convergence is much faster than (Guo et al., 2017) where only asymptotically convergence was possible. This can even be seen from numerical simulations in both papers because in Guo et al. (2017), it must wait all the amplitudes to be convergent before the output being convergent; (c) our disturbance allows in-domain and boundary disturbances, which is much complicated than a single disturbance discussed in Guo et al. (2017). Finally, the reference signal of Guo et al. (2017) is zero (output regulation), while our reference signal is usually not zero (output tracking).

More interesting problem is the output regulation for general bounded reference signal and disturbances like paper (Jin & Guo, 2018), which is considered as future work.

Appendix

Due to the page limitation, we put the proofs of Lemmas 2.1, 3.1 and 3.2 into “http://lscamss.ac.cn/~bzguo/papers/jinguobeam.pdf”.

References


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