Automatica 109 (2019) 108507

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Boundary output tracking for an Euler–Bernoulli beam equation with unmatched perturbations from a known exosystem ☆

Feng-Fei Jin [a,*], Bao-Zhu Guo [b,c,d]

[a] School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, PR China
[b] Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, PR China
[c] Academy of Mathematics and Systems Science, Academia Sinica, Beijing, 100190, PR China
[d] School of Mathematics and Big Data, Foshan University, Foshan 528000, PR China

ARTICLE INFO

Article history:
Received 7 September 2018
Received in revised form 29 May 2019
Accepted 13 July 2019
Available online xxxx

Keywords:
Output feedback regulator
Boundary control
Euler–Bernoulli beam
Exosystem

ABSTRACT

In this paper, we consider boundary output regulation for an Euler–Bernoulli beam equation which can describe typically the flexible arm of robots. The reference signal and disturbance are generated by a finite-dimensional exosystem. The measurements are angular and angular velocity of the right end where the control is imposed. However, the performance output is on the left end which is non-collocated with control, a difficult case in practice where the control takes time to perform its force from the right end to the left. The objective is to design an output feedback controller to regulate the displacement of the left end to track the reference signal. We first design a state feedback regulator to make the performance output track the reference signal exponentially. An observer is then constructed to recover the state, with which, an output feedback regulator is designed by replacing state feedback with its estimation. The closed-loop system is shown to admit a unique bounded solution and the tracking error converges to zero exponentially. Some numerical simulations are presented to illustrate the effectiveness of the proposed output feedback regulator.

1. Introduction

Flexible structures such as flexible robot arms are widely used in aerospace technologies, satellites, flexible manipulators, and other industry applications. In many situations, the Euler–Bernoulli beam equation can well describe the flexible arms. Because of light weight and high speed, the vibration of flexible arms is inevitable, which reduces accuracy in industrial streamlines. Vibration control for Euler–Bernoulli beam has therefore been considered by many researchers. Some of works can be found in Chen, Delfour, Krall, and Payre (1987), Chen, Krantz, Ma, Wayne, and West (1988), Conrad and Pierre (1990), Luo and Guo (1997), Smyshlyaev, Guo, and Krstic (2009) and Xu and Sallet (1992) and the references therein. Most of these works are related to stabilization problem. Recently, the disturbance rejection problem has been addressed for Euler–Bernoulli beam systems in Ge, Zhang, and He (2011), He, Zhang, and Ge (2013) and Jin and Guo (2015).

On the other hand, output regulation is one of the central issues in control theory. Many industry processes can be formulated as regulation problem like the military field. The output regulation was started initially for finite-dimensional systems like (Byrnes, Priscoli, & Isidori, 1997; Callier & Desoer, 1980; Desoer & Lin, 1985) from which many results have been extended to infinite-dimensional systems and some examples can be found in Byrnes, Laukó, Gilliam, and Shubov (2000), Deutscher (2015), Deutscher (2017), Paunonen and Pohjolainen (2010), Paunonen and Pohjolainen (2014), Xu and Dubljevic (2017a), Xu and Dubljevic (2017b) and Xu, Pohjolainen, and Dubljevic (2017). The most profound result in this regard is the internal model principal which has been generalized from lumped parameter systems into distributed parameter systems. For bounded control and observation, we refer to Paunonen and Pohjolainen (2010) and unbounded control and observation can be found in Paunonen and Pohjolainen (2014). In Xu and Dubljevic (2017a), an observer-based output regulator was designed for an abstract infinite-dimensional system with bounded control and unbounded observation. Adaptive control method has also been applied to regulation problem for a wave equation in Guo and Guo (2016) where the unknown constant coefficients in harmonic disturbance were identified. Recently, in Deutscher (2015), an output regulation problem for a heat equation was discussed by means of backstepping approach (Krstic & Smyshlyaev, 2008), where

☆ This work was supported by the National Natural Science Foundation of China. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Thomas Meurer under the direction of Editor Miroslav Krstic.

* Corresponding author.
E-mail addresses: jinfengfei@amss.ac.cn (F.-F. Jin), bzguo@iss.ac.cn (B.-Z. Guo).

https://doi.org/10.1016/j.automatica.2019.108507
0005-1098/

the control and observation were allowed to be unbounded and the disturbance and reference signals are generated by a finite-dimensional exosystem. In addition, the output can be either bounded or unbounded. This method was generalized to regulation problem for a coupled wave equation with unbound control and observation in Gu, Wang, and Guo (2018), where the output was supposed to be bounded. In Jin and Guo (2018), we designed an output regulator for a heat equation with unbounded control and unbounded observation. The main idea of Jin and Guo (2018) is that an extended state observer can be constructed to estimate the state and the general external disturbance, but the control and performance output are matched. There are also some other works related to motion planning of beam equation. In Shifman (1990), a tracking problem for an Euler–Bernoulli beam was discussed, where the reference trajectory is the same as the Euler–Bernoulli beam except the control channel, and the displacement and velocity of the beam are supposed to be known in control design. In Meurer, Thull, and Kugi (2008) and Schröck, Meurer, and Kugi (2011), the differential flatness method, a powerful tool for tracking problem, was applied to flexible beam motion planning. For these beam works, no disturbance was considered.

In this paper, we consider output regulation problem for an Euler–Bernoulli beam equation described by

$$\begin{cases} u_{tt}(x,t)+u_{xxxx}(x,t)+f(x)d_1(t)=0, x\in(0,1), t>0,\\ u_{xx}(0,t)=d_2(t), t\geq 0,\\ u_{xxx}(0,t)=d_3(t),\ t\geq 0,\\ u(1,t)=0,\ u_{xx}(1,t)=U(t),\ t\geq 0,\\ y_m(t)=\{u_x(1,t),u_{xt}(1,t)\},\ t\geq 0,\\ y_c(t)=u(0,t),\ t\geq 0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in[0,1], \end{cases} \tag{1}$$

where $u(t)$ is the state, $U(t)$ the control (input), $y_m(t)$ the measured output, and $y_c(t)$ is the performance output to be regulated. The $f\in C[0,1]$ represents the intensity of unknown spatial disturbance $d_1\in C(0,\infty)$, and $d_2, d_3\in C(0,\infty)$ are boundary disturbances. Please note that for notational simplicity, we assume all uniform linear mass density, the uniform flexural rigidity, and the length of the beam to be one (by space and time scaling), without loss of generality.

The disturbances and reference signal are supposed to be generated by an exosystem as follows:

$$\begin{cases} \dot v(t)=Sv(t),\ t>0,\\ d_1(t)=p_{d_1}^\top v(t),\ t\geq 0,\\ d_2(t)=p_{d_2}^\top v(t),\ t\geq 0,\\ d_3(t)=p_{d_3}^\top v(t),\ t\geq 0,\\ y_{ref}(t)=p_r^\top v(t),\ t\geq 0,\\ v(0)=v_0\in\mathbb{C}^n. \end{cases} \tag{2}$$

Here $S=\operatorname{diag}(S_d,S_r)$ is a block diagonalizable matrix with all eigenvalues on the imaginary axis which is standard because the disturbance represented by the eigenvalues on the left plane diminishes itself as time involves and those represented by the eigenvalues on the right plane requires infinite control force, that is, the v-subsystem can be divided into two decoupled subsystems:

$$\begin{cases} \dot v_d(t)=S_d v_d(t),\ t>0,\\ d_1(t)=q_{d_1}^\top v_d(t),\ t\geq 0,\\ d_2(t)=q_{d_2}^\top v_d(t),\ t\geq 0,\\ d_3(t)=q_{d_3}^\top v_d(t),\ t\geq 0,\\ v_d(0)=v_{d0}\in\mathbb{C}^{n_1}, \end{cases} \tag{3}$$

and

$$\begin{cases} \dot v_r(t)=S_r v_r(t),\ t>0,\\ y_{ref}(t)=q_r^\top v_r(t),\ t\geq 0,\\ v_r(0)=v_{r0}\in\mathbb{C}^{n_2}, \end{cases} \tag{4}$$

where $(v_d^\top, v_r^\top)^\top=v$ and $n=n_1+n_2$. In addition, we assume that the eigenvalues of S_d are distinct and $(q_r^\top, S_r)$ is observable. The reference signal $y_{ref}(t)$ is also measurable. Our target is to design an output feedback controller $U(t)$ such that

$$\lim_{t\to\infty}[y_c(t)-y_{ref}(t)]=\lim_{t\to\infty}[u(0,t)-y_{ref}(t)]=0, \tag{5}$$

in the presence of disturbances. Meanwhile, the state of the closed-loop system is required to be bounded. From now on, we omit initial value and domain for all systems when there is no confusion for simplicity.

The system (1) describes well the movement of the flexible robot arms. The left end is in free movement yet both bending moment and share force are affected by unknown disturbances, and the displacement is required to be regulated to track a reference signal. The control end is pinned which is a typical case in applications yet the bending moment is actuated. The internal disturbance is not uniform by the introduction of the intensity function $f(x)$.

We point out that there are very limited literature on output regulation for PDEs with non-collocated control and regulated output. The heat equation (parabolic type) was considered in Deutscher (2015). A special output regulation with reference (set point) zero for wave equation (hyperbolic type) was developed in Guo, Shao, and Krstic (2017). This paper is the first paper for beam equation (Petrovsky type) with non-collocated control and regulated output.

We proceed as follows. In Section 2, we design a state feedback controller for tracking control system (1) where the disturbances are assumed to be known. In Section 3, we construct a state observer for system (1) and exosystem (2) in terms of the measured output and the reference signal. It is shown that the observer error decays exponentially as time goes to infinity. An output regulator is designed by replacing the state with its estimation obtained in Section 4. By means of C_0-semigroup and admissibility theory for linear infinite-dimensional systems, the closed-loop system is shown to admit a unique bounded solution and the displacement of the left end tracks reference signal exponentially. Some numerical simulations are presented in Section 5 to illustrate the effect of the proposed control law, following up concluding remarks in Section 6.

2. State feedback regulator

To design an output feedback, we need state feedback first. In this section, we assume that all states of systems (1) and (2) are known. We propose a state feedback controller for tracking control system (1) as follows:

$$U(t)=-k_1u_{xt}(1,t)-k_2u_x(1,t)+m_v^\top v(t), \tag{6}$$

where $k_1,k_2>0$ are tuning parameters, $m_v^\top$ is an n-dimensional row vector to be determined later. It is noted that the first two terms are used to stabilize internally exponentially the system (disturbance free system), and the third term is for output tracking. The closed-loop of systems (1) and (2) under the controller

(6) is

$$\begin{cases}\dot{v}(t)=Sv(t),\\ u_{tt}(x,t)+u_{xxxx}(x,t)+f(x)p_{d_1}^{\top}v(t)=0,\\ u_{xx}(0,t)=p_{d_2}^{\top}v(t),\\ u_{xxx}(0,t)=p_{d_3}^{\top}v(t),\\ u(1,t)=0,\\ u_{xx}(1,t)=-k_1u_{xt}(1,t)-k_2u_x(1,t)+m_v^{\top}v(t).\end{cases} \tag{7}$$

Now we find $m_v^{\top}$ so that the tracking condition (5) holds. To this end, we introduce a transformation

$$e(x,t)=u(x,t)-g^{\top}(x)v(t), \tag{8}$$

so that the tracking error $e(0,t)=u(0,t)-y_{ref}(t)\to 0$ by proper choice of $g(x)$. In this way, $v(t)$ and $e(x,t)$ are governed by

$$\begin{cases}\dot{v}(t)=Sv(t),\\ e_{tt}(x,t)+e_{xxxx}(x,t)=0,\\ e_{xx}(0,t)=e_{xxx}(0,t)=e(1,t)=0,\\ e_{xx}(1,t)=-k_1e_{xt}(1,t)-k_2e_x(1,t),\end{cases} \tag{9}$$

provided that $g^{\top}(x)$ is chosen to satisfy

$$\begin{cases}g^{(4)}(x)+(S^{\top})^2g(x)+f(x)p_{d_1}=0,\\ g''(0)=p_{d_2},\ \ g'''(0)=p_{d_3}, g(1)=0,\\ g''(1)+k_1S^{\top}g'(1)+k_2g'(1)=m_v.\end{cases} \tag{10}$$

It is well known that the e-part of system (9) is well-posed and exponentially stable (Gnedin, 1992) in the state space

$$\mathcal{H}_1=\{(f,g)^{\top}\in H^2(0,1)\times L^2(0,1)|f(1)=0\} \tag{11}$$

with the norm

$$\|(f,g)^{\top}\|_{\mathcal{H}_1}^2=\int_0^1[|f''(x)|^2+|g(x)|^2]dx \\ +k_2|f'(1)|^2, \forall\,(f,g)^{\top}\in\mathcal{H}_1. \tag{12}$$

In particular, $e(0,t)\to 0$ exponentially as $t\to\infty$. Next, the tracking error

$$\begin{aligned}u(0,t)-y_{ref}(t)&=e(0,t)+g^{\top}(0)v(t)-p_r^{\top}v(t)\\ &=e(0,t)\to 0,\end{aligned} \tag{13}$$

if we choose

$$g(0)=p_r. \tag{14}$$

Substituting (14) into (10) brings us to solve the following boundary value problem (BVP):

$$\begin{cases}g^{(4)}(x)+(S^{\top})^2g(x)+f(x)p_{d_1}=0,\\ g''(0)=p_{d_2},\ g'''(0)=p_{d_3},\\ g(1)=0,\ g(0)=p_r.\end{cases} \tag{15}$$

The solvability of BVP (15) is guaranteed by succeeding Lemma 2.1 and the proof is arranged in Appendix.

Lemma 2.1. *Assume that S has distinct eigenvalues on the imaginary axis only. Then, BVP (15) admits a unique solution.*

The m_v in (6) can be determined by

$$m_v^{\top}=\left.\frac{d^2g^{\top}}{dx^2}\right|_{x=1}+k_1\left.\frac{dg^{\top}}{dx}\right|_{x=1}S+k_2\left.\frac{dg^{\top}}{dx}\right|_{x=1}. \tag{16}$$

Theorem 2.1. *Let $g(x)$ be the solution of (15) and $m_v=g''(1)+k_1S^{\top}g'(1)+k_2g'(1)$. For any initial value $(v_0^{\top},u_0,u_1)^{\top}\in\mathbb{C}^n\times\mathcal{H}_1$, the closed-loop system (7) admits a unique bounded solution $(v^{\top}(t),u(\cdot,t),u_t(\cdot,t))^{\top}\in C(0,\infty;\mathbb{C}^n\times\mathcal{H}_1)$. Moreover,*

$u(0,t)-y_{ref}(t)\to 0$ exponentially as $t\to\infty$.

Proof. From transformation (8), we find the solution to (7) as $(v^{\top}(t),u(x,t),u_t(x,t))^{\top}=(v^{\top}(t),e(x,t)+g^{\top}(x)v(t),e_t(x,t)+g^{\top}(x)Sv(t))^{\top}$ and

$$\begin{aligned}&\|(v^{\top}(t),u(\cdot,t),u_t(\cdot,t))^{\top}\|_{\mathbb{C}^n\times\mathcal{H}_1}^2\\ =\;&\|(v^{\top}(\cdot))\|_{\mathbb{C}^n}^2+\|(u(\cdot,t),u_t(\cdot,t))^{\top}\|_{\mathcal{H}_1}^2\\ \le\;&\|(v^{\top}(\cdot))\|_{\mathbb{C}^n}^2+2\|(e(\cdot,t),e_t(\cdot,t))^{\top}\|_{\mathcal{H}_1}^2\\ &+2\|(g^{\top}(\cdot)v(t),g^{\top}(\cdot)Sv(t))^{\top}\|_{\mathcal{H}_1}^2\\ =\;&\|(v^{\top}(\cdot))\|_{\mathbb{C}^n}^2+2\|(e(\cdot,t),e_t(\cdot,t))^{\top}\|_{\mathcal{H}_1}^2\\ &+2\int_0^1\left[\left|(g^{\top})''(x)v(t)\right|^2+\left|g^{\top}(x)Sv(t)\right|^2\right]dx\\ &+2k_2|(g^{\top})'(1)v(t)|^2.\end{aligned} \tag{17}$$

Since all eigenvalues of S are located on the imaginary axis, $v(t)$ and $Sv(t)$ are bounded as $t\to\infty$. Since $g(x)$ is the solution of (15) with $g^{\top}\in(C^2[0,1])^n$, both $\|(g^{\top}(\cdot)v(t),g^{\top}(\cdot)Sv(t))^{\top}\|_{\mathcal{H}_1}^2$ $\|(v^{\top}(\cdot))\|_{\mathbb{C}^n}^2$ are uniformly bounded, and $\|(e(\cdot,t),e_t(\cdot,t))^{\top}\|_{\mathcal{H}_1}^2$ decays exponentially, there exists constant $L>0$ depending on initial value $(v_0^{\top},u_0,u_1)$ such that

$$\|(v^{\top}(t),u(\cdot,t),u_t(\cdot,t))^{\top}\|_{\mathbb{C}^n\times\mathcal{H}_1}\le L,\ \ \forall\ t\ge 0. \tag{18}$$

Now we consider the tracking performance. By (13), it follows that the tracking error $e(0,t)$ has the estimation:

$$\begin{aligned}|e(0,t)|^2\ &\le\left|\int_0^1e_x(x,t)dx\right|^2\\ &\le L_1\|(e(\cdot,t),e_t(\cdot,t))^{\top}\|_{\mathcal{H}_1}^2, L_1>0.\end{aligned} \tag{19}$$

Since $\|(e(\cdot,t),e_t(\cdot,t))^{\top}\|_{\mathcal{H}_1}$ decays exponentially, so does $e(0,t)\to 0$ exponentially as $t\to\infty$. □

3. Observer design

In this section, we assume that the disturbance is unknown yet reference is known. Then, $v(t)$ in state feedback controller (6) needs to be estimated. We design a state observer as follows:

$$\begin{cases}\dot{\hat{v}}_d(t)=S_d\hat{v}_d(t)+k_d[\hat{u}_x(1,t)-u_x(1,t)],\\ \dot{\hat{v}}_r(t)=S_r\hat{v}_r(t)+k_r[q_r^{\top}\hat{v}_r(t)-y_{ref}(t)],\\ \hat{u}_{tt}(x,t)+\hat{u}_{xxxx}(x,t)+f(x)q_{d_1}^{\top}\hat{v}_d(t)\\ \quad+\gamma_1(x)[\hat{u}_{xt}(1,t)-u_{xt}(1,t)]\\ \quad+\gamma_2(x)[\hat{u}_x(1,t)-u_x(1,t)]=0,\\ \hat{u}_{xx}(0,t)=q_{d_2}^{\top}\hat{v}_d(t),\ \ \hat{u}_{xxx}(0,t)=q_{d_3}^{\top}\hat{v}_d(t),\\ \hat{u}(1,t)=0,\\ \hat{u}_{xx}(1,t)=-c_1[\hat{u}_{xt}(1,t)-u_{xt}(1,t)]\\ \quad-c_2[\hat{u}_x(1,t)-u_x(1,t)]+U(t),\end{cases} \tag{20}$$

where $\gamma_1,\gamma_2\in C[0,1]$, $c_1>0, c_2>0$ are the tuning constant parameters, k_r is chosen such that $S_r+k_rq_r^{\top}$ is Hurwitz which follows from the observability of $(q_r^{\top},S_r)$. Then, the error variables

$\tilde{v}_d = \hat{v}_d - v_d$, $\tilde{v}_r = \hat{v}_r - v_r$, $\tilde{u} = \hat{u} - u$ are governed by

$$\begin{cases} \dot{\tilde{v}}_d(t) = S_d\tilde{v}_d(t) + k_d\tilde{u}_x(1,t), \\ \dot{\tilde{v}}_r(t) = (S_r + k_r q_r^\top)\tilde{v}_r(t), \\ \tilde{u}_{tt}(x,t) + \tilde{u}_{xxxx}(x,t) + f(x)q_{d_1}^\top\tilde{v}_d(t) \\ \quad +\gamma_1(x)\tilde{u}_{xt}(1,t) + \gamma_2(x)\tilde{u}_x(1,t) = 0, \\ \tilde{u}_{xx}(0,t) = q_{d_2}^\top\tilde{v}_d(t),\ \tilde{u}_{xxx}(0,t) = q_{d_3}^\top\tilde{v}_d(t), \\ \tilde{u}(1,t) = 0, \\ \tilde{u}_{xx}(1,t) = -c_1\tilde{u}_{xt}(1,t) - c_2\tilde{u}_x(1,t). \end{cases} \tag{21}$$

We need to prove that the error system (21) is stable. To this purpose, since $S_r + k_r q_r^\top$ is Hurwitz, the variable $\tilde{v}_r$ is decoupled from others and is exponentially stable. For $\tilde{v}_d$ and $\tilde{u}$, we introduce a transformation

$$\begin{cases} \tilde{v}_d(t) = \tilde{v}_d(t), \tilde{v}_r(t) = \tilde{v}_r(t), \\ \varepsilon(x,t) = \tilde{u}(x,t) + h^\top(x)\tilde{v}_d(t). \end{cases} \tag{22}$$

to make (21) slightly simpler. In this way, $\varepsilon(x,t)$ satisfies

$$\begin{aligned} &\varepsilon_{tt}(x,t) + \varepsilon_{xxxx}(x,t) \\ &= \tilde{u}_{tt}(x,t) + \tilde{u}_{xxxx}(x,t) + h^\top(x)\ddot{\tilde{v}}_d(t) + (h^\top(x))^{(4)}\tilde{v}_d(t) \\ &= -f(x)q_{d_1}^\top\tilde{v}_d(t) - \gamma_1(x)\tilde{u}_{xt}(1,t) - \gamma_2(x)\tilde{u}_x(1,t) \\ &\quad +h^\top(x)[S_d^2\tilde{v}_d(t) + S_d k_d\tilde{u}_x(1,t) + k_d\tilde{u}_{xt}(1,t)] \\ &\quad +(h^\top(x))^{(4)}\tilde{v}_d(t) = 0, \end{aligned} \tag{23}$$

provided that we choose

$$\begin{cases} \dfrac{d^4h^\top(x)}{dx^4} = -h^\top(x)S_d^2 + f(x)q_{d_1}^\top, \\ \gamma_1(x) = h^\top(x)k_d,\ \gamma_2(x) = h^\top(x)S_d k_d. \end{cases} \tag{24}$$

The boundary condition of $\varepsilon(x,t)$ at $x = 0$ gives

$$\begin{cases} \varepsilon_{xx}(0,t) = q_{d_2}^\top\tilde{v}_d(t) + (h^\top)''(0)\tilde{v}_d(t) = 0, \\ \varepsilon_{xxx}(0,t) = q_{d_3}^\top\tilde{v}_d(t) + (h^\top)'''(0)\tilde{v}_d(t) = 0, \end{cases} \tag{25}$$

provided

$$(h^\top)''(0) = -q_{d_2},\ (h^\top)'''(0) = -q_{d_3}. \tag{26}$$

Similarly, the boundary condition at $x = 1$ gives

$$\begin{cases} \varepsilon(1,t) = h^\top(1)\tilde{v}_d(t) = 0, \\ \varepsilon_{xx}(1,t) + c_1\varepsilon_{xt}(1,t) + c_2\varepsilon_x(1,t) \\ = (h^\top)''(1)\tilde{v}_d(t) + c_1\tilde{v}_d^\top(t)S_d^\top(h^\top)'(1) \\ +c_2\tilde{v}_d^\top(t)(h^\top)'(1) = 0, \end{cases} \tag{27}$$

provided

$$\begin{aligned} &h(1) = 0, \\ &(h^\top)''(1) + c_1 S_d^\top(h^\top)'(1) + c_2(h^\top)'(1) = 0. \end{aligned} \tag{28}$$

For $\tilde{v}_d(t)$, we have

$$\dot{\tilde{v}}_d(t) = \left(S_d - k_d(h^\top)'(1)\right)\tilde{v}_d(t) + k_d\varepsilon_x(1,t). \tag{29}$$

Combining (23), (25), (27) and (29), we obtain

$$\begin{cases} \dot{\tilde{v}}_d(t) = \left[S_d - k_d(h^\top)'(1)\right]\tilde{v}_d(t) + k_d\varepsilon_x(1,t), \\ \dot{\tilde{v}}_r(t) = (S_r + k_r q_r^\top)\tilde{v}_r(t), \\ \varepsilon_{tt}(x,t) + \varepsilon_{xxxx}(x,t) = 0, \\ \varepsilon_{xx}(0,t) = \varepsilon_{xxx}(0,t) = \varepsilon(1,t) = 0, \\ \varepsilon_{xx}(1,t) = -c_1\varepsilon_{xt}(1,t) - c_2\varepsilon_x(1,t), \end{cases} \tag{30}$$

where $h(x)$ satisfies the boundary value problem (BVP) of the following:

$$\begin{cases} h^{(4)}(x) = -(S_d^2)^\top h(x) + f(x)q_{d_1}, \\ h''(0) = -q_{d_2}, h'''(0) = -q_{d_3}, h(1) = 0, \\ h''(1) + c_1 S_d^\top h'(1) + c_2 h'(1) = 0. \end{cases} \tag{31}$$

Lemma 3.1. *Assume that all eigenvalues of S_d are distinct and are located on the imaginary axis. Then, the BVP (31) admits a unique solution.*

The proofs for Lemmas 3.1 and 3.2 are arranged in Appendix.

Now we are in a position to show that there exists k_d such that $S_d - k_d \frac{dh^\top}{dx}\big|_{x=1}$ is Hurwitz.

Lemma 3.2. *The numerator of the transfer matrix $T_d^\top(s) = \frac{N_d^\top(s)}{D_d(s)}$ of (1) from $(d_1, d_2, d_3)^\top$ to one of output signal $u_x(1,t)$ is*

$$N_d(s) = \begin{pmatrix} \displaystyle\int_0^1 R(s,y)f(y)dy \\ -r_2\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) \\ -2r_1\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) \end{pmatrix}, \tag{32}$$

where

$$\begin{cases} R(s,y) = r_1\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) r_2\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, y\right) \\ \qquad -r_2\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, 1\right) r_1\left(\dfrac{1-j}{\sqrt{2}}\sqrt{s}, y\right), \\ r_1(s,y) = \dfrac{\sinh(sy) + \sin(sy)}{2s}, \\ r_2(s,y) = \cosh(sy) + \cos(sy). \end{cases} \tag{33}$$

The pair $(\frac{dh^\top}{dx}\big|_{x=1}, S_d)$ is observable if and only if

$$N_d^\top(\mu_i)\begin{pmatrix} q_{d_1}^\top w_i \\ q_{d_2}^\top w_i \\ q_{d_3}^\top w_i \end{pmatrix} \neq 0, i = 1, 2, \ldots, n_1, \tag{34}$$

where w_i is the eigenvector of S_d corresponding to the eigenvalue μ_i of S_d.

Note that $\frac{dh_i^*}{dx}\big|_{x=1} = \frac{dh^\top}{dx}|_{x=1}w_i \neq 0, i = 1, 2, \ldots, n_1$ as long as the eigenvalues of S_d satisfy the conditions in Lemma 3.2. When this is satisfied, $(\frac{dh^\top}{dx}|_{x=1}, S_d)$ is observable. Then, there exists k_d such that $S_d - k_d\frac{dh^\top}{dx}|_{x=1}$ is Hurwitz. Going back to system (30), the PDE-part is exponentially stable. This together with $S_d - k_d\frac{dh^\top}{dx}\big|_{x=1}$ and $S_r + k_r q_r^\top$ being Hurwitz shows that system (30) is exponentially stable. By the inverse transformation of (22),

$$\tilde{u}(x,t) = \varepsilon(x,t) - h^\top(x)\tilde{v}_d(t), \tag{35}$$

where $h(x)$ is the classical solution of BVP (31).

Theorem 3.1. *Assume that $N_d^\top(\mu_i)\left(q_{d_1}^\top w_i, q_{d_2}^\top w_i, q_{d_3}^\top w_i\right)^\top \neq 0 (i = 1, 2, \ldots, n_1)$, where $N_d(s)$ is defined in Lemma 3.2. Let $h(x)$ be the classical solution of (31). Choose k_d such that $S_d - k_d\frac{dh^\top}{dx}\big|_{x=1}$ is Hurwitz, and let $c_1 > 0, c_2 > 0$, $\gamma_1(x) = h^\top(x)k_d, \gamma_2(x) = h^\top(x)S_d k_d$. Then, the observer (20) converges to system (3), (4) and (1) exponentially, that is, the observer error system (21) is well-posed and exponentially stable.*

Proof. This is a consequence of the equivalence between (21) and (30) by invertible transformation (22). The PDE-part of system (30) is

$$\begin{cases}\varepsilon_{tt}(x,t)+\varepsilon_{xxxx}(x,t)=0,\\ \varepsilon_{xx}(0,t)=\varepsilon_{xxx}(0,t)=\varepsilon(1,t)=0,\\ \varepsilon_{xx}(1,t)=-c_1\varepsilon_{xt}(1,t)-c_2\varepsilon_x(1,t).\end{cases} \tag{36}$$

Consider system (36) in the state space $\mathcal{H}_1$, which is well known exponentially stable in $\mathcal{H}_1$ (Gnedin, 1992): There exist $L_1,\delta_1>0$ such that

$$\begin{aligned}&\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top\|_{\mathcal{H}_1}\\ &\le L_1e^{-\delta_1 t}\|(\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top\|_{\mathcal{H}_1},\ \ t>0,\end{aligned} \tag{37}$$

which implies that

$$\begin{cases}|\varepsilon(0,t)|\le L'e^{-\delta_1 t}\|(\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top\|_{\mathcal{H}_1},t>0,\\ |\varepsilon_x(1,t)|\le L''e^{-\delta_1 t}\|(\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top\|_{\mathcal{H}_1},t>0,\end{cases}$$

for some $L'>0,L''>0$.

For the ODE part, a straightforward computation gives

$$\begin{cases}\tilde{v}_r(t)=e^{(S_r+k_rq_r^\top)t}\tilde{v}_r(0),\\ \tilde{v}_d(t)=e^{(S_d-k_dh^\top(0))t}\tilde{v}_d(0)\\ \qquad+\displaystyle\int_0^t e^{(S_d-k_dh^\top)(t-s)}k_d\varepsilon_x(1,s)ds,\end{cases} \tag{38}$$

from which we can estimate that

$$\begin{aligned}\|\tilde{v}_r(t)\|_{\mathbb{C}^{n_2}}&=\|e^{(S_r+k_rq_r^\top)t}\tilde{v}_r(0)\|_{\mathbb{C}^{n_2}}\\ &\le L_re^{-\delta_r t}\|\tilde{v}_r(0)\|_{\mathbb{C}^{n_2}},\end{aligned} \tag{39}$$

$$\begin{aligned}\|\tilde{v}_d(t)\|_{\mathbb{C}^{n_1}}&\le\|e^{(S_d-k_dh^\top)t}\tilde{v}_d(0)\|_{\mathbb{C}^{n_2}}\\ &\quad+\Big\|\int_0^t e^{(S_d-k_dh^\top)(t-s)}k_d\varepsilon_x(1,s)ds\Big\|_{\mathbb{C}^{n_2}}\\ &\le L_de^{-\delta_d t}\|\tilde{v}_d(0)\|_{\mathbb{C}^{n_2}}\\ &\quad+L_2e^{-\delta_3 t}\|(\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top\|_{\mathcal{H}_1},\end{aligned} \tag{40}$$

for some $L_r>0,L_d>0,L_2>0,\delta_r>0,\delta_d>0,\delta_3>0$. Combining (37), (39) and (40), we obtain

$$\begin{aligned}&\|(\tilde{v}_d^\top(t),\tilde{v}_r^\top(t),\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top\|_{\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\mathcal{H}_1}\\ &\le L_4e^{-\delta_4 t}\|(\tilde{v}_d^\top(0),\tilde{v}_r^\top(0),\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top\|_{\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\mathcal{H}_1},\end{aligned}$$

for some $L_4>0,\delta_4>0$. By (22), we define a bounded invertible operator $P_0:\mathbb{C}^n\times\mathcal{H}_1\to\mathbb{C}^n\times\mathcal{H}_1$ by

$$\begin{aligned}&(\tilde{v}_d^\top(t),\tilde{v}_r^\top(t),\tilde{u}(x,t),\tilde{u}_t(x,t))^\top\\ &=P_0(\tilde{v}_d^\top(t),\tilde{v}_r^\top(t),\varepsilon(x,t),\varepsilon_t(x,t))^\top\\ &=(\tilde{v}_d^\top(t),\tilde{v}_r^\top(t),\varepsilon(x,t)-h^\top(x)\tilde{v}_d(t),\varepsilon_t(x,t)\\ &\quad-h^\top(x)(S_d-k_dh^\top(0))\tilde{v}_d(t)-h^\top(x)k_d\varepsilon(0))^\top.\end{aligned}$$

Then,

$$\begin{aligned}&\|(\tilde{v}_d^\top(t),\tilde{v}_r^\top(t),\tilde{u}(\cdot,t),\tilde{u}_t(\cdot,t))^\top\|_{\mathbb{C}^n\times\mathcal{H}_1}\\ &\le L_5e^{-\delta_5 t}\|(\tilde{v}_d^\top(0),\tilde{v}_r^\top(0),\tilde{u}(x,0),\tilde{u}_t(x,0))^\top\|_{\mathbb{C}^n\times\mathcal{H}_1}.\end{aligned} \tag{41}$$

We thus complete the proof of the theorem. □

4. Output feedback regulator

In this section we design an output feedback regulator for system (1). In light of state feedback regulator in Section 2, we design an output feedback regulator as follows

$$U(t)=-k_1u_{xt}(1,t)-k_2u_x(1,t)+m_v^\top\hat{v}(t), \tag{42}$$

where $\hat{v}(t)=(\hat{v}_r(t),\hat{v}_d(t))^\top$, $m_v=(g^\top)''(1)+k_1(g^\top)'(1)S+k_2(g^\top)'(1)$, and $g(x)$ is the solution of (15). The difference between the output feedback regulator and the state feedback is only the state of the exosystem. In the output feedback, we just replace $v(t)$ in state feedback controller (6) by its estimation $\hat{v}(t)$. The closed-loop of system (1), (3), (4), and (20) under the controller (42) is

$$\begin{cases}\dot{v}_d(t)=S_dv_d(t),\ \dot{v}_r(t)=S_rv_r(t),\\ d_1(t)=q_{d_1}^\top v_d(t),\ d_2(t)=q_{d_2}^\top v_d(t),\\ d_3(t)=q_{d_3}^\top v_d(t),\ y_{ref}(t)=q_r^\top v_r(t),\\ u_{tt}(x,t)+u_{xxxx}(x,t)+f(x)q_{d_1}^\top v_d(t)=0,\\ u_{xx}(0,t)=q_{d_2}^\top v_d(t),\ \ u_{xxx}(0,t)=q_{d_3}^\top v_d(t),\\ u(1,t)=0,\\ u_{xx}(1,t)=-k_1u_{xt}(1,t)-k_2u_x(1,t)+m_v^\top\hat{v}(t),\\ \dot{\hat{v}}_d(t)=S_d\hat{v}_d(t)+k_d[\hat{u}_x(1,t)-u_x(1,t)],\\ \dot{\hat{v}}_r(t)=S_r\hat{v}_r(t)+k_r[q_r^\top\hat{v}_r(t)-y_{ref}(t)],\\ \hat{u}_{tt}(x,t)+\hat{u}_{xxxx}(x,t)+f(x)q_{d_1}^\top\hat{v}_d(t)\\ \qquad+\gamma_1(x)[\hat{u}_{xt}(1,t)-u_{xt}(1,t)]\\ \qquad+\gamma_2(x)[\hat{u}_x(1,t)-u_x(1,t)]=0,\\ \hat{u}_{xx}(0,t)=q_{d_2}^\top\hat{v}_d(t),\ \hat{u}_{xxx}(0,t)=q_{d_3}^\top\hat{v}_d(t),\\ \hat{u}(1,t)=0,\\ \hat{u}_{xx}(1,t)=-c_1\hat{u}_{xt}(1,t)+(c_1-k_1)u_{xt}(1,t)\\ -c_2\hat{u}_x(1,t)+(c_2-k_2)u_x(1,t)+m_v^\top\hat{v}(t),\end{cases} \tag{43}$$

where $h(x)$ is the classical solution of (31), k_d,k_r are chosen so that $S_d-k_d\frac{dh^\top}{dx}|_{x=1}$ and $S_r+k_rq_r^\top$ are Hurwitz, $c_1>0,c_2>0$, $k_1>0,k_2>0$, $\gamma_1(x)=h^\top(x)k_d$, $\gamma_2(x)=h^\top(x)S_dk_d$.

We consider system (43) in the state space $\mathbb{H}=(\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\mathcal{H}_1)^2$ equipped with the inner product induced norm

$$\begin{aligned}&\|(\xi_1,\xi_2,f_1,g_1,\eta_1,\eta_2,f_2,g_2)^\top\|_{\mathbb{H}}^2\\ &=\|\xi_1\|_{\mathbb{C}^{n_1}}^2+\|\xi_2\|_{\mathbb{C}^{n_2}}^2+\int_0^1[|f_1''(x)|^2+|g_1(x)|^2]dx\\ &+\|\eta_1\|_{\mathbb{C}^{n_1}}^2+\|\eta_2\|_{\mathbb{C}^{n_2}}^2+\int_0^1[|f_2''(x)|^2+|g_2(x)|^2]dx\\ &+k_2|f_1'(1)|^2+c_2|f_2'(1)|^2,\\ &\forall\ (\xi_1,\xi_2,f_1,g_1,\eta_1,\eta_2,f_2,g_2)^\top\in\mathbb{H}.\end{aligned} \tag{44}$$

Define the system operator $\mathbb{A}$ of (43) by

$$\begin{aligned}&\mathbb{A}(\xi_1,\xi_2,f_1,g_1,\eta_1,\eta_2,f_2,g_2)^\top\\ &=(S_d\xi_1,S_r\xi_2,g_1,-f_1^{(4)}-fq_{d_1}^\top\xi_1,S_d\eta_1\\ &\quad+k_d(f_2'(1)-f_1'(1)),S_r\eta_2+k_r(q_r^\top\eta_2-q_r^\top\xi_2),\\ &\quad g_2,-f_2^{(4)}-fq_{d_1}^\top\eta_1-\gamma_1(g_2'(1)-g_1'(1))\\ &\quad-\gamma_2(f_2'(1)-f_1'(1))^\top,\\ &\quad\forall\ (\xi_1,\xi_2,f_1,g_1,\eta_1,\eta_2,f_2,g_2)^\top\in D(\mathbb{A}),\end{aligned} \tag{45}$$

with

$$\begin{aligned}D(\mathbb{A})=\{&(\xi_1,\xi_2,f_1,g_1,\eta_1,\eta_2,f_2,g_2)^\top\in\mathbb{H}|\\ &\mathbb{A}(\xi_1,\xi_2,f_1,g_1,\eta_1,\eta_2,f_2,g_2)^\top\in\mathbb{H},\\ &f_1''(0)=q_{d_2}^\top\xi_1,f_1'''(0)=q_{d_3}^\top\xi_1,\\ &f_1''(1)=-k_1g_1'(1)-k_2f_1'(1,t)+m^\top(\eta_1^\top,\eta_2^\top)^\top,\\ &f_2''(0)=q_{d_2}^\top\eta_1,f_2'''(0)=q_{d_3}^\top\eta_1,\\ &f_2''(1)=-c_1g_2'(1)-c_2f_2'(1,t)+(c_1-k_1)g_1'(1)\\ &+(c_2-k_2)f_1'(1)+m^\top(\eta_1^\top,\eta_2^\top)^\top\}.\end{aligned}$$

Then, system (43) can be written as an abstract evolutionary equation in $\mathbb{H}$:

$$\begin{aligned}&\frac{d}{dt}(v_d(t), v_r(t), u(\cdot,t), u_t(\cdot,t),\\ &\qquad \hat{v}_d(t), \hat{v}_r(t), \hat{u}(\cdot,t), \hat{u}_t(\cdot,t))^\top\\ &= \mathbb{A}(v_d(t), v_r(t), u(\cdot,t), u_t(\cdot,t),\\ &\qquad \hat{v}_d(t), \hat{v}_r(t), \hat{u}(\cdot,t), \hat{u}_t(\cdot,t))^\top.\end{aligned} \tag{46}$$

Now we state the main result of this paper.

Theorem 4.1. *Assume $N_d^\top(\mu_i)\left(q_{d_1}^\top w_i, q_{d_2}^\top w_i, q_{d_3}^\top w_i\right)^\top \neq 0 (i = 1, 2, \ldots, n_1)$, where $N_d(s)$ is defined in Lemma 3.2. Let $h(x)$ be the solution of (31). The k_d is chosen so that $S_d - k_d(h^\top)'(1)$ is Hurwitz, and $c_1 > 0, c_2 > 0, k_1 > 0, k_2 > 0$, $\gamma_1(x) = h^\top(x)k_d$, $\gamma_2(x) = h^\top(x)S_dk_d$. For any initial value $(v_d(0), v_r(0), u(\cdot,0), u_t(\cdot,0), \hat{v}_d(0), \hat{v}_r(0), \hat{u}(\cdot,0), \hat{u}_t(\cdot,0))^\top \in \mathbb{H}$, system (46) admits a unique bounded solution $(v_d(t), v_r(t), u(\cdot,t), u_t(\cdot,t), \hat{v}_d(t), \hat{v}_r(t), \hat{u}(\cdot,t), \hat{u}_t(\cdot,t))^\top \in C(0,\infty;\mathbb{H})$. Moreover, the tracking error $u(0,t) - y_{ref}(t) \to 0$ exponentially as $t \to \infty$: There exist constants $M > 0, \omega > 0$ such that*

$$|u(0,t) - y_{ref}(t)| \le Me^{-\omega t}.$$

Proof. Define an invertible operator $P : \mathbb{H} \to \mathbb{H}$ by

$$\begin{aligned}&P(v_d(t), v_r(t), u(\cdot,t), u_t(\cdot,t), \hat{v}_d(t), \hat{v}_r(t), \hat{u}(\cdot,t), \hat{u}_t(\cdot,t))^\top\\ &= (v_d(t), v_r(t), u(\cdot,t), u_t(\cdot,t), \hat{v}_d(t) - v_d(t),\\ &\quad \hat{v}_r(t) - v_r(t), \hat{u}(\cdot,t) - u(\cdot,t), \hat{u}_t(\cdot,t) - u_t(\cdot,t))^\top.\end{aligned} \tag{47}$$

By the observer error $(\tilde{v}_d, \tilde{v}_r, \tilde{u}, \tilde{u}_t)$, we obtain an equivalent system of (43):

$$\begin{cases}\dot{v}_d(t) = S_dv_d(t), \dot{v}_r(t) = S_rv_r(t),\\ d_1(t) = q_{d_1}^\top v_d(t), d_2(t) = q_{d_2}^\top v_d(t),\\ d_3(t) = q_{d_3}^\top v_d(t), y_{ref}(t) = q_r^\top v_r(t),\\ u_{tt}(x,t) + u_{xxxx}(x,t) + f(x)q_{d_1}^\top v_d(t) = 0,\\ u_{xx}(0,t) = q_{d_2}^\top v_d(t),\ u_{xxx}(0,t) = q_{d_3}^\top v_d(t),\\ u_{xx}(1,t) = -k_1u_{xt}(1,t) - k_2u_x(1,t)\\ \quad +m_v^\top v(t) + m^\top \tilde{v}(t), u(1,t) = 0,\\ \dot{\tilde{v}}_d(t) = S_d\tilde{v}_d(t) + k_d\tilde{u}_x(1,t),\\ \dot{\tilde{v}}_r(t) = (S_r + k_rq_r^\top)\tilde{v}_r(t),\\ \tilde{u}_{tt}(x,t) + \tilde{u}_{xxxx}(x,t) + f(x)q_{d_1}^\top \tilde{v}_d(t)\\ \quad +\gamma_1(x)\tilde{u}_{xt}(1,t) + \gamma_2(x)\tilde{u}_x(1,t) = 0,\\ \tilde{u}_{xx}(0,t) = q_{d_2}^\top \tilde{v}_d(t),\ \tilde{u}_{xxx}(0,t) = q_{d_3}^\top \tilde{v}_d(t),\\ \tilde{u}(1,t) = 0, \tilde{u}_{xx}(1,t) = -c_1\tilde{u}_{xt}(1,t) - c_2\tilde{u}_x(1,t).\end{cases} \tag{48}$$

From Theorem 3.1, we know that the decoupled $(\tilde{v}_d, \tilde{v}_r, \tilde{u}, \tilde{u}_t)^\top$-part is exponentially stable. In particular, $\tilde{v}(t)$ decays exponentially. The $(v_d, v_r, u, u_t)^\top$-part which is the same as system (7) except one more term $m^\top\tilde{v}(t)$ in $u_{xx}(1,t)$ is governed by

$$\begin{cases}\dot{v}_d(t) = S_dv_d(t), \dot{v}_r(t) = S_rv_r(t),\\ d_1(t) = q_{d_1}^\top v_d(t), d_2(t) = q_{d_2}^\top v_d(t),\\ d_3(t) = q_{d_3}^\top v_d(t), y_{ref}(t) = q_r^\top v_r(t),\\ u_{tt}(x,t) + u_{xxxx}(x,t) + f(x)q_{d_1}^\top v_d(t) = 0,\\ u_{xx}(0,t) = q_{d_2}^\top v_d(t),\ u_{xxx}(0,t) = q_{d_3}^\top v_d(t),\\ u_{xx}(1,t) = -k_1u_{xt}(1,t) - k_2u_x(1,t)\\ \quad +m_v^\top v(t) + m^\top\tilde{v}(t), u(1,t) = 0.\end{cases} \tag{49}$$

By the same transformation (8), we can find that $e(x,t)$ is governed by

$$\begin{cases}e_{tt}(x,t) + e_{xxxx}(x,t) = 0,\\ e_{xx}(0,t) = e_{xxx}(0,t) = e(1,t) = 0,\\ e_{xx}(1,t) = -k_1e_{xt}(1,t) - k_2e_x(1,t) + m^\top\tilde{v}(t).\end{cases} \tag{50}$$

We claim that system (50) is also exponentially stable. Indeed, consider system (50) in the state space $\mathcal{H}_1$. Define the system operator $\mathcal{A}_2$ by

$$\begin{cases}\mathcal{A}_2(f,g)^\top = (g, -f^{(4)})^\top,\ \forall (f,g)^\top \in D(\mathcal{A}_2),\\ D(\mathcal{A}_2) = \{(f,g)^\top \in H^4(0,1)\times H^2(0,1)|\\ \qquad \mathcal{A}_2(f,g)^\top \in \mathcal{H}_1, f''(0) = f'''(0) = 0,\\ \qquad f''(1) = -k_1g'(1) - k_2f'(1)\}.\end{cases} \tag{51}$$

Then, system (50) can be written as an abstract evolutionary equation in $\mathcal{H}_1$:

$$\begin{aligned}&\frac{d}{dt}(e(\cdot,t), e_t(\cdot,t))^\top\\ &= \mathcal{A}_2(e(\cdot,t), e_t(\cdot,t))^\top + \mathcal{B}_2m^\top\tilde{v}(t),\end{aligned} \tag{52}$$

where $\mathcal{B}_2 = (0, -\delta'(x-1))^\top$. It is well known $\mathcal{A}_2$ generates an exponentially stable C_0-semigroup $e^{\mathcal{A}_2t}$ on $\mathcal{H}_1$ (Gnedin, 1992). Direct computations show that the adjoint operator $\mathcal{A}_2^*$ of $\mathcal{A}_2$ is given by

$$\begin{cases}\mathcal{A}_2^*(f,g)^\top = (-g, f^{(4)})^\top,\ \forall (f,g)^\top \in D(\mathcal{A}_2^*),\\ D(\mathcal{A}_2^*) = \{(f,g)^\top \in H^4(0,1)\times H^2(0,1)|\\ \qquad \mathcal{A}_2^*(f,g)^\top \in \mathcal{H}_1, f''(0) = f'''(0) = 0,\\ \qquad f''(1) = k_1g'(1) - k_2f'(1)\}.\end{cases} \tag{53}$$

For any $(f,g)^\top \in \mathcal{H}_1$, $\mathcal{B}^*\mathcal{A}^{*-1}(f,g)^\top = f'(1)$ which indicates that $\mathcal{B}^*\mathcal{A}^{*-1}$ is bounded. The dual system is governed by

$$\begin{cases}p_{tt}(x,t) + p_{xxxx}(x,t) = 0,\\ p_{xx}(0,t) = p_{xxx}(0,t) = p(1,t) = 0,\\ p_{xx}(1,t) = -k_1p_{xt}(1,t) - k_2p_x(1,t),\\ y_o^*(t) = -p_{xt}(1,t),\end{cases} \tag{54}$$

which is exponentially stable. Define the Lyapunov function

$$E^*(t) = \frac{1}{2}\int_0^1 [p_t^2(x,t) + p_{xx}^2(x,t)]dx + \frac{k_2}{2}p_x^2(1,t).$$

Differentiate $E^*(t)$ along the solution of system (54) to yield

$$\dot{E}^*(t) = -k_1p_{xt}^2(1,t) \le 0. \tag{55}$$

Integrating (55) with respect to t from 0 to T, we obtain

$$\int_0^T p_{xt}^2(1,t)dt \le \frac{1}{k_1}E(0), \tag{56}$$

which together with the boundedness of $\mathcal{B}^*\mathcal{A}^{*-1}$ implies that $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^*t}$. As a result, $\mathcal{B}_2$ is admissible for $e^{\mathcal{A}_2t}$ (Weiss, 1989). Therefore, the solution of (52) can be written as

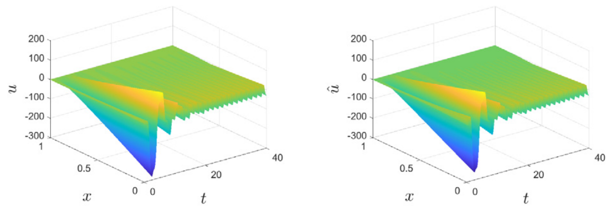
$$\begin{aligned}(e(\cdot,t), e_t(\cdot,t))^\top &= e^{\mathcal{A}_2t}(e(\cdot,0), e_t(\cdot,0))^\top\\ &\quad + \int_0^t e^{\mathcal{A}_2(t-s)}\mathcal{B}_2m^\top\tilde{v}(s)ds,\end{aligned} \tag{57}$$

which is also exponentially stable on $\mathcal{H}_1$ by the method in Theorem 2 and Eq.(60) of Su, Guo, Wang, and Krstic (2017) because $\tilde{v}(t)$ decays exponentially.

By the inverse transformation of (8), similar to the proof of Theorem 2.1, system (49) admits a unique bounded solution in $\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}\times\mathcal{H}_1$. The tracking error is

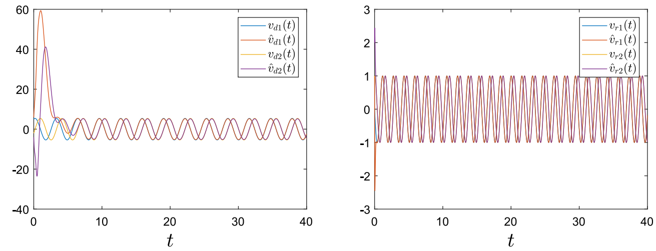
$$u(0,t) - y_{ref}(t) = e(0,t) \to 0 \tag{58}$$

exponentially because system (50) decays exponentially. □



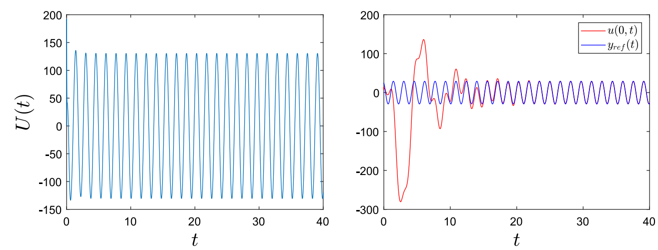
(a) Displacement of u (b) Displacement of $\hat{u}$

Fig. 1. Displacements of PDEs' part of closed-loop system (43).



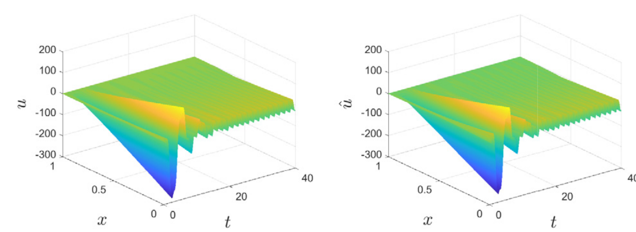
(a) Estimation performance of $\hat{v}_d$ (b) Estimation performance of $\hat{v}_r$

Fig. 2. Trajectory of ODEs' part of closed-loop system (43).



(a) Controller trajectory (b) Boundary tracking performance

Fig. 3. Control trajectory and boundary tracking performance in closed-loop system (43).



(a) Displacement of u (b) Displacement of $\hat{u}$

Fig. 4. Displacements of PDEs' part of closed-loop system (43) with output noise.

5. Numerical simulation

In this section, we present some numerical simulations to illustrate the effectiveness of the proposed feedback control. In the closed-loop system (43), we choose the matrix of the exosystem as

$$S_d = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, S_r = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}, q_{d_1}^\top = (1, 2), \tag{59}$$
$$q_{d_2}^\top = (0, 2), q_{d_3}^\top = (2, 0), q_r = (15, 25),$$

and the parameters are chosen to be $c_1 = 0.5, c_2 = 1, k_1 = 0.2, k_2 = 0.5$. For simplicity, set $f(x) = 1, x \in (0, 1)$ and $k_r = (-5, 1)^\top$ which makes $S_r + k_r q_r$ stable. From Lemmas 2.1 and 3.1, $g(x)$ and $h(x)$ can then be obtained by (15) and (31) respectively, and we choose $k_d = (-4, 8)^\top$ which makes $S_d - k_d(h^\top)'(1)$ stable and the $\gamma_1(x)$ and $\gamma_2(x)$ can be obtained via boundary condition of (24). The initial values are taken as:

$$\begin{cases} u_0(x) = 3x + 5\cos(\pi x) + 2, \ u_1(x) = 3 + 3\cos(\pi x), \\ \hat{u}_0(x) = 4\sin(\pi x), \ \hat{u}_1(x) = 3(x-1) - 4\sin(\pi x), \\ v_d(0) = (5, -2)^\top, \ v_r(0) = (0, 1)^\top, \\ \hat{v}_d(0) = (6, -4)^\top, \ \hat{v}_r(0) = (1, 2)^\top. \end{cases} \tag{60}$$

The finite element method is applied to compute the solution numerically. Let $h = 0.1$ and $N = 1/h$. We give $[0, 1]$ an N equipartition with nodes $x_i = ih, i = 0, 1, 2, \ldots, N$. On the node x_i, two finite element basis functions are selected as

$$\phi_{2i+1}(x) = \begin{cases} (x - x_i)(x - x_{i-1})^2/h^2, & x \in [x_{i-1}, x_i], \\ (x - x_i)(x - x_{i+1})^2/h^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{others}, \end{cases}$$

$$\phi_{2i+2}(x) = \begin{cases} \dfrac{(h - x_i + x)^2(2(x_i - x) + h)}{h^3}, & x \in [x_{i-1}, x_i], \\ \dfrac{(h + x_i - x)^2(2(x - x_i) + h)}{h^3}, & x \in [x_i, x_{i+1}], \\ 0, & \text{others}. \end{cases}$$

For node $x_N = 1$, only one basis function ϕ_{2N+1} is selected. There are totally $2N + 1$ basis functions which satisfy the natural boundary conditions $\phi_i(1) = 0, i = 1, 2 \cdots, 2N + 1$. We consider the Galerkin approximation solution of the system (43) in the finite dimensional space generated by these basis functions, which takes the form

$$u^N(x, t) = \sum_{i=1}^{2N+1} a_i(t)\phi_i(x), \hat{u}^N(x, t) = \sum_{i=1}^{2N+1} b_i(t)\phi_i(x),$$

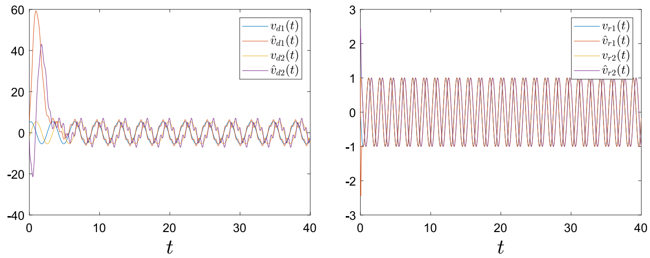
where $a_i(t)$ and $b_i(t)$ are determined by standard finite element Galerkin method to satisfy some ODEs. To solve the ODEs, the step of time is set to be 0.01.

Figs. 1(a) and 1(b) display the solutions of PDE part $u(x, t)$ and $\hat{u}(x, t)$ in the closed-loop system (43). It is seen that the displacement of $u(x, t)$ and $\hat{u}(x, t)$ is bounded and $\hat{u}(x, t)$ can track $u(x, t)$ well after $t = 20$. Fig. 2(a) indicates that the estimation $\hat{v}_d(t)$ can track $v_d(t)$ after $t = 7$. Similarly from Fig. 2(b), $\hat{v}_r(t)$ converges to $v_r(t)$ rapidly because this part is decoupled from others. Fig. 3(a) presents the trajectory of the controller. Fig. 3(b) displays the tracking performance of the boundary displacement $u(0, t)$. After $t = 20$, $u(0, t)$ tracks the reference signal $y_{ref}(t)$ satisfactorily which illustrates the effectiveness of the proposed output regulator.

When the measured output signal is contaminated by external noise, we consider the case where the first component of the measurement becomes $u_x(1, t) + 2\cos(10t)$ with high frequency noise $2\cos(10t)$. The counterparts of Figs. 1–3 become Figs. 4–6 respectively. It is seen from Fig. 5(a) that the tracking performance of exosystem is affected by output noise apparently. But the output tracking works well from Fig. 6(b).

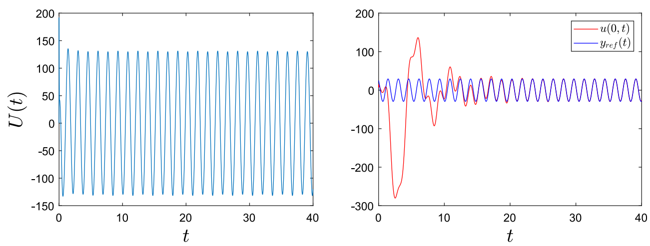
6. Concluding remarks

In this paper, we develop an output feedback control scheme to solve output regulation for an Euler–Bernoulli beam equation, where the performance output is on the left end which



(a) Estimation performance of $\hat{v}_d$ (b) Estimation performance of $\hat{v}_r$

Fig. 5. Trajectory of ODEs' part of closed-loop system (43) with output noise.



(a) Controller trajectory (b) Boundary tracking performance

Fig. 6. Control trajectory and boundary tracking performance in closed-loop system (43) with output noise.

is non-collocated with the control at the right end. This is a difficult case because the boundary control at boundary $x = 1$ must go through the entire interval [0, 1] to reach the regulated boundary $x = 0$ to take effect. The basic idea is to design an observer to recover the states of the plant and the disturbance. We suppose all disturbance and reference signals are produced by a finite-dimensional exosystem which covers general harmonic disturbances, and the harmonic disturbance can be considered as an approximation of periodic signal. This is also the general formulation for disturbance in the framework of the internal model principle.

As indicated in introduction, a regulation problem for a one-dimensional wave equation suffered from harmonic boundary disturbance with unknown amplitudes was consider recently in Guo et al. (2017). The method adopted there is the adaptive control method to estimate all amplitudes. Compared with (Guo et al., 2017), there are some advantages by the approach developed in this paper: (a) the order of controller is much lower than (Guo et al., 2017) because (Guo et al., 2017) needs to estimate all amplitudes of harmonic disturbance; (b) the exponentially convergence is much faster than (Guo et al., 2017) where only asymptotically convergence was possible. This can even be seen from numerical simulations in both papers because in Guo et al. (2017), it must wait all the amplitudes to be convergent before the output being convergent; (c) our disturbance allows in-domain and boundary disturbances, which is much complicated than a single disturbance discussed in Guo et al. (2017). Finally, the reference signal of Guo et al. (2017) is zero (output regulation), while our reference signal is usually not zero (output tracking).

More interesting problem is the output regulation for general bounded reference signal and disturbances like paper (Jin & Guo, 2018), which is considered as future work.

Appendix. Proof of three lemmas

Due to the page limitation, we put the proofs of Lemmas 2.1, 3.1 and 3.2 into "http://lsc.amss.ac.cn/~bzguo/papers/Jinguobeam.pdf".

References

Byrnes, C. I., Laukó, I. G., Gilliam, D. S., & Shubov, V. I. (2000). Output regulation problem for linear distributed parameter systems. *IEEE Transactions on Automatic Control*, *45*, 2236–2252.

Byrnes, C. I., Priscoli, F. D., & Isidori, A. (1997). *Output regulation for uncertain nonlinear systems*. Boston: Birkäuser.

Callier, F. M., & Desoer, C. A. (1980). Stabilization, tracking and disturbance rejection in multivariable convolution systems. *Annales de la Société scientifique de Bruxelles*, *94*, 7–51.

Chen, G., Delfour, M. C., Krall, A. M., & Payre, G. (1987). Modeling, stabilization and control of serially connected beams. *SIAM Journal on Control and Optimzation*, *25*, 526–546.

Chen, G., Krantz, S. G., Ma, D. W., Wayne, C. E., & West, H. H. (1988). The Euler–Bernoulli beam equation with boundary energy dissipation. In S. J. Lee (Ed.), *Operator methods for optimal control problems* (pp. 67–96). NY: Marcel Dekker.

Conrad, F., & Pierre, M. (1990). Stabilization of Euler–Bernoulli beam by nonlinear boundary feedback. Research report INRIA, (1235) Univ. of Nancy 1, Dept. of Math. France.

Desoer, C. A., & Lin, C. A. (1985). Tracking and disturbance rejection of MIMO nonlinear systems with PI controller. *IEEE Transactions on Automatic Control*, *30*, 861–867.

Deutscher, J. (2015). A backstepping approach to the output regulation of boundary controlled parabolic PDEs. *Automatica*, *57*, 56–64.

Deutscher, J. (2017). Finite-time output regulation for linear 2 ×2 hyperbolic systems using backstepping. *Automatica*, *54*, 54–62.

Ge, S. S., Zhang, S., & He, W. (2011). Modeling and control of an Euler–Bernoulli beam under unknown spatiotemporally varying disturbance. *International Journal of Control*, *84*, 947–960.

Gnedin, A. V. (1992). Stabilization of Euler–Bernoulli beam by a boundary control. *Results in Mathematics*, *22*, 528–544.

Gu, J., Wang, J. M., & Guo, Y. (2018). Output regulation of anti-stable coupled wave equations via the backstepping technique. *IET Control Theory & Applications*, *12*, 431–445.

Guo, W., & Guo, B. Z. (2016). Performance output tracking for a wave equation subject to unmatched general boundary harmonic disturbance. *Automatica*, *68*, 194–202.

Guo, W., Shao, Z. C., & Krstic, M. (2017). Adaptive rejection of harmonic disturbance anticollocated with control in 1D wave equation. *Automatica*, *79*, 17–26.

He, W., Zhang, S., & Ge, S. S. (2013). Boundary output-feedback stabilization of a Timoshenko beam using disturbance observer. *IEEE Transactions on Industrial Electronics*, *60*, 5186–5194.

Jin, F. F., & Guo, B. Z. (2015). Lyapunov Approach to output feedback stabilization for Euler–Bernoulli beam equation with boundary. *Automatica*, *52*, 95–102.

Jin, F. F., & Guo, B. Z. (2018). Performance boundary output tracking for one-dimensional heat equation with boundary unmatched disturbance. *Automatica*, *96*, 1–10.

Krstic, M., & Smyshlyaev, A. (2008). *Boundary control of PDEs: A course on backstepping designs*. Philadelphia, USA: Society for Industrial and Applied Mathematics.

Luo, Z. H., & Guo, B. Z. (1997). Shear force feedback control of a single link flexible robot with revolute joint. *IEEE Transactions on Automatic Control*, *42*, 53–65.

Meurer, T., Thull, D., & Kugi, A. (2008). Flatness-based tracking control of a piezoactuated Euler–Bernoulli beam with non-collocated output feedback: theory and experiments. *International Journal of Control*, *81*, 475–493.

Paunonen, L., & Pohjolainen, S. (2010). Internal model theory for distributed parameter systems. *SIAM Journal on Control and Optimization*, *48*, 4753–4775.

Paunonen, L., & Pohjolainen, S. (2014). The internal model principle for systems with unbounded control and observation. *SIAM Journal on Control and Optimization*, *52*, 3967–4000.

Schröck, J., Meurer, T., & Kugi, A. (2011). Control of a flexible beam actuated by macro-fiber composite patches: I. Modeling and feedforward trajectory control. *Smart Materials and Structures*, *20*, 1–7.

Shifman, J. J. (1990). A tracking controller for the Euler–Bernoulli beam. In *IEEE international conference on robotics and automation* (pp. 928–933). Cincinnati, OH.

Smyshlyaev, A., Guo, B. Z., & Krstic, M. (2009). Arbitrary decay rate for Euler–Bernoulli beam by backstepping boundary feedback. *IEEE Transactions on Automatic Control*, *54*, 1134–1140.

Su, L., Guo, W., Wang, J. M., & Krstic, M. (2017). Boundary stabilization of wave equation with velocity recirculation. *IEEE Transactions on Automatic Control*, *62*, 4760–4767.

Weiss, G. (1989). Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, *27*, 527–545.

Xu, X. D., & Dubljevic, S. (2017a). Output and error feedback regulator designs for linear infinite-dimensional systems. *Automatica*, *83*, 170–178.

Xu, X. D., & Dubljevic, S. (2017b). Output regulation for a class of linear boundary controlled first-order hyperbolic PIDE systems. *Automatica*, *85*, 43–52.

Xu, X. D., Pohjolainen, S., & Dubljevic, S. (2017). Finite-dimensional regulators for a class of regular hyperbolic PDE systems. *International Journal of Control*, *6*, 1–34.

Xu, C. Z., & Sallet, G. (1992). In R. F. Curtain, A. Bensoussan, & J. L. Lions (Eds.), *Lecture notes in control and information sciences*, *Boundary stabilization of rotating flexible systems* (pp. 347–365). New York: Springer-Verlag.

Feng-Fei Jin received the M.Sc. degree from the Shandong Normal University, Jinan, China in 2008, and the Ph.D. degree from the Academy of Mathematics and Systems Science, Academia Sinica, Beijing, China in 2011. He was a postdoctoral fellow at the University of the Witwatersrand, South Africa during 2011–2012 and 2013–2015. He is currently with the Shandong Normal University, Jinan, China. His research interests focus on distributed parameter systems control.

Bao-Zhu Guo received the Ph.D. degree from the Chinese University of Hong Kong in applied mathematics in 1991. From 1985 to 1987, he was a Research Assistant at the Beijing Institute of Information and Control, China. During the period 1993–2000, he was with the Beijing Institute of Technology, first as an associate professor (1993–1998) and subsequently a professor (1998–2000). Since 2000, he has been with the Academy of Mathematics and Systems Science, the Chinese Academy of Sciences, where he is a research professor in mathematical system theory. From 2004–2019, he had been with School of Computer Science and Applied Mathematics, the University of the Witwatersrand, South Africa as a chair professor. His research interests include the theory of control and application of infinite-dimensional systems.