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Brief paper

The active disturbance rejection and sliding mode control approach to the stabilization of the Euler–Bernoulli beam equation with boundary input disturbance[☆]

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ABSTRACT

In this paper, we are concerned with the boundary feedback stabilization of a one-dimensional Euler–Bernoulli beam equation with the external disturbance flowing to the control end. The active disturbance rejection control (ADRC) and sliding mode control (SMC) are adopted in investigation. By the ADRC approach, the disturbance is estimated through an extended state observer and canceled online by the approximated one in the closed-loop. It is shown that the external disturbance can be attenuated in the sense that the resulting closed-loop system under the extended state feedback tends to any arbitrary given vicinity of zero as the time goes to infinity. In the second part, we use the SMC to reject the disturbance by removing the condition in ADRC that the derivative of the disturbance is supposed to be bounded. The existence and uniqueness of the solution for the closed-loop via SMC are proved, and the monotonicity of the "reaching condition" is presented without the differentiation of the sliding mode function, for which it may not always exist for the weak solution of the closed-loop system. The numerical simulations validate the effectiveness of both methods.

1. Introduction

In the past three decades, the Euler–Bernoulli beam equation has been a representative model for the control of systems governed by partial differential equations (PDEs). This is spurred by the outer space applications, such as flexible link manipulators and antennas, for which the suppression of the vibration by the boundary feedback is a central issue. We refer Han, Benaroya, and Wei (1999) for engineering interpretation of the beam equations.

There are many works contributed to the stabilization of the beam equation. The examples can be found in Chen, Delfour, Krall, and Payre (1987), Guo and Yu (2001), He, Ge, How, Choo, and Hong (2011), He, Zhang, and Ge (2012), Luo, Guo, and Morgul (1999), Luo, Kitamura, and Guo (1995), Nguyen and Hong (2012) and the references therein. However, most of the control designs for the beam equation are collocated control based on the passive principle and do not take the disturbance into account. The earlier non-collocated control design for the beam equation is Luo and Guo (1997). Recently, a powerful backstepping method is introduced to stabilize the Euler–Bernoulli beam equation via completely non-collocated control (Smyshlyaev, Guo, & Krstic, 2009). Once again, the external disturbance is not considered in these works.

There are several different approaches to deal with the uncertainties in system control. The sliding mode control (SMC) that is inherently robust is the most popular one that has been studied widely for both finite-dimensional systems and infinite-dimensional counterparts. For the latter, many works require the input and output operators are to be bounded (Pisano, Orlov, & Usai, 2011). Recently, a boundary SMC controller for a one-dimensional heat equation with boundary input disturbance is designed in Cheng, Radisavljevic, and Su (2011). In Guo, Guo, and Shao (2011), Krstic (2010), the adaptive controls are designed for one-dimensional wave equations in which the uncertainties are the unknown parameters in disturbance. Another powerful method in dealing with uncertainties is based on Lyapunov functional approach. In Ge, Zhang, and He (2001), a boundary control is designed by the Lyapunov method for an Euler–Bernoulli

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beam equation with spatial and boundary disturbance. Generally speaking, there are not so many works, to the best of our knowledge, to the stabilization of the beam equation with disturbance.

The active disturbance rejection control (ADRC), as an unconventional design strategy, was first proposed by Han in 1990s (Han, 2009). It has been now acknowledged to be an effective control strategy for lumped parameter systems in the absence of proper models and in the presence of model uncertainty. Its power has been demonstrated by many engineering practices such as motion control, tension control in web transport and strip precessing systems, DC–DC power converts in power electronics, continuous stirred tank reactor in chemical and process control, micro-electro-mechanical systems gyroscope (Gao, 2006; Guo & Zhao, 2011; Han, 2009). For more details on practical perspectives, we refer to a nice recent review paper Zheng and Gao (2010). The main idea of the ADRC is using the estimation/cancellation strategy in dealing with the uncertainties. Its convergence has been proved for finite-dimensional systems in Guo and Zhao (2011). Very recently, this approach is successfully applied to the attenuation of disturbance for a one-dimensional anti-stable wave equation in Guo and Jin (2013).

In this paper, we are concerned with the stabilization of a one-dimensional Euler–Bernoulli beam equation with uncertainty at the input boundary via both SMC and ADRC approaches. The system is governed by the following PDEs:

$$\begin{cases} u_{tt}(x,t)+u_{xxxx}(x,t)=0, \quad x\in(0,1),\ t>0,\\ u(0,t)=u_x(0,t)=0, \quad t\geqslant 0,\\ u_{xx}(1,t)=0, \quad t\geqslant 0,\\ u_{xxx}(1,t)=U(t)+d(t), \quad t\geqslant 0, \end{cases} \tag{1}$$

where $u(x,t)$ is the transverse displacement of the beam at time t and position x, U is the control input through shear force, d is the external disturbance at the control end. This one end fixed and another end free beam equation (1) models typically the vibration control of a single link flexible robot arm with the external disturbance in the free (working) end (see e.g., Han et al., 1999, Luo & Guo, 1997).

It is well-known that when there is no disturbance, the collocated feedback control $U(t)=ku_t(1,t)$, $k>0$ will stabilize exponentially the system (1) (Chen et al., 1987). However, this stabilizer is not robust to the external disturbance. For instance, when $d(t)=d$ is a constant, the system (1) under the feedback $U(t)=ku_t(1,t)$ has a solution $(u,u_t)=\left(-\frac{d}{2}x^2+\frac{d}{6}x^3,0\right)$. Therefore, in the presence of the disturbance, the control must be re-designed.

We proceed as follows. In Section 2, we use the ADRC approach to attenuate the disturbance by designing an estimator to estimate the disturbance. After canceling the disturbance by the approximated one, we design the collocated like feedback controller. The closed-loop system is shown to tend any arbitrary given vicinity of zero as the time goes to infinity. Section 3 is devoted to the disturbance rejection by the SMC approach, in which the boundedness of the disturbance required in ADRC is removed. The existence and uniqueness of the solution are proved, and the monotonicity of the "reaching condition" is presented without the differentiation of the sliding mode function, for which it does not always exist for the weak solution of the closed-loop system. The numerical simulations are presented in Section 4 for illustration of the effectiveness of both methods.

2. Feedback via active disturbance rejection control

In this section, we suppose that the unknown disturbance d and its derivative $\dot{d}$ are bounded measurable. That is, $|d(t)|\leqslant M$, $|\dot{d}(t)|\leqslant M$ for some $M>0$ and all $t\geqslant 0$. The ADRC approach is used to attenuate the disturbance, which is an estimation/cancellation strategy. Let

$$y_1(t)=\int_0^1 x^2u_t(x,t)dx, \qquad y_2(t)=u_x(1,t), \quad t\geqslant 0. \tag{2}$$

We consider the system (1) in the energy Hilbert state space defined by

$$\mathcal{H}=\{(f,g)^\top\in H^2(0,1)\times L^2(0,1)|f(0)=f'(0)=0\}, \tag{3}$$

where the inner product induced norm is given by

$$\|(f,g)^\top\|^2=\int_0^1[|f''(x)|^2+|g(x)|^2]dx, \quad \forall (f,g)^\top\in\mathcal{H}.$$

Define the operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}(f,g)^\top=(g,-f^{(4)})^\top, \quad \forall (f,g)^\top\in D(\mathcal{A}),\\ D(\mathcal{A})=\{(f,g)^\top\in\mathcal{H}\cap(H^4(0,1)\times H^2(0,1))| \\ \qquad g(0)=g'(0)=f''(1)=f'''(1)=0\}. \end{cases} \tag{4}$$

Then it is easy to verify that $\mathcal{A}^*=-\mathcal{A}$ in $\mathcal{H}$. The solution of (1) is equivalent to

$$\frac{d}{dt}\begin{pmatrix}u\\u_t\end{pmatrix}=\mathcal{A}\begin{pmatrix}u\\u_t\end{pmatrix}+\mathcal{B}[U(t)+d(t)],$$
$$\mathcal{B}=(0,-\delta(x-1))^\top.$$

It is well-known that $\mathcal{B}$ is admissible to the semigroup generated by $\mathcal{A}$ (Weiss, 1989). So for any initial value $(u(\cdot,0),u_t(\cdot,0))^\top\in\mathcal{H}$, $U\in L^2_{\rm loc}(0,\infty)$, there exists a unique (weak) solution $(u,u_t)^\top\in\mathcal{H}$ to (1), which satisfies

$$\frac{d}{dt}\left\langle\begin{pmatrix}u\\u_t\end{pmatrix},\begin{pmatrix}f\\g\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}u\\u_t\end{pmatrix},\mathcal{A}^*\begin{pmatrix}f\\g\end{pmatrix}\right\rangle-g(1)[U(t)+d(t)],$$
$$\forall (f,g)^\top\in D(\mathcal{A}^*),$$

that is,

$$\begin{aligned}\frac{d}{dt}&\int_0^1[u_{xx}(x,t)f''(x)+u_t(x,t)g(x)]dx\\ &=\int_0^1[-u_{xx}(x,t)g''(x)+u_t(x,t)f^{(4)}(x)]dx\\ &\quad-g(1)[U(t)+d(t)], \quad \forall\ (f,g)^\top\in D(\mathcal{A}^*).\end{aligned}$$

Let $(f,g)^\top=(0,x^2)^\top\in D(\mathcal{A}^*)=D(\mathcal{A})$. By the equalities above, it follows that

$$\begin{aligned}\dot{y}_1(t)&=\frac{d}{dt}\int_0^1 x^2u_t(x,t)\\ &=-\int_0^1 u_{xx}(x,t)g''(x)dx-g(1)[U(t)+d(t)]\\ &=-[U(t)+d(t)]-2u_x(1,t).\end{aligned} \tag{5}$$

Design the high gain estimator as follows (Guo & Zhao, 2011):

$$\begin{cases}\dot{\hat{y}}(t)=-(U(t)+\hat{d}(t))-2y_2(t)-\dfrac{1}{\varepsilon}(\hat{y}(t)-y_1(t)),\\ \dot{\hat{d}}(t)=\dfrac{1}{\varepsilon^2}(\hat{y}(t)-y_1(t)),\end{cases} \tag{6}$$

where $\varepsilon>0$ is the design small parameter, and $\hat{d}$ is regarded as an approximation of d. Let

$$\tilde{y}(t)=\hat{y}(t)-y_1(t), \qquad \tilde{d}(t)=\hat{d}(t)-d(t) \tag{7}$$

be the errors. Then $\tilde{y}, \tilde{d}$ satisfy

$$\frac{d}{dt}\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix} = \begin{pmatrix}-\frac{1}{\varepsilon} & -1\\ \frac{1}{\varepsilon^2} & 0\end{pmatrix}\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix} + \begin{pmatrix}0\\ -1\end{pmatrix}\dot{d}(t) = A\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix} + B\dot{d}(t). \tag{8}$$

The collocated like state feedback controller to (1) is designed as follows:

$$U(t) = ku_t(1,t) - \hat{d}(t), \quad k > 0. \tag{9}$$

It is clearly seen that the control design in (9) is an estimation/cancellation strategy. The second term in (9) is used to eliminate, in real time, the effect of the disturbance. The first term is the usual control that makes the closed-loop system (1) exponentially stable without the disturbance (Chen et al., 1987). Under the feedback (9), the closed-loop system of (1) becomes

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0,\\ u(0,t) = u_x(0,t) = 0, \quad t \geqslant 0,\\ u_{xx}(1,t) = 0, \quad t \geqslant 0,\\ u_{xxx}(1,t) = ku_t(1,t) - \hat{d}(t) + d(t), \quad t \geqslant 0,\\ \dot{\hat{y}}(t) = -ku_t(1,t) - 2y_2(t) - \frac{1}{\varepsilon}(\hat{y}(t) - y_1(t)), \quad t \geqslant 0,\\ \dot{\hat{d}}(t) = \frac{1}{\varepsilon^2}(\hat{y}(t) - y_1(t)), \quad t \geqslant 0. \end{cases} \tag{10}$$

Remark 2.1. The choice of feedback again k in (9) is a complicated problem. From numerical simulation, there is an optimal feedback gain k, but it is very hard to give an analytic analysis. We refer the numerical result to Wang and Yao (2000).

Using the error variables $(\tilde{y}, \tilde{d})$ defined in (7), we can write the equivalent system of (10) as follows:

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0,\\ u(0,t) = u_x(0,t) = 0, \quad t \geqslant 0,\\ u_{xx}(1,t) = 0, \quad t \geqslant 0,\\ u_{xxx}(1,t) = ku_t(1,t) - \tilde{d}(t), \quad t \geqslant 0,\\ \dot{\tilde{y}}(t) = -\frac{1}{\varepsilon}\tilde{y}(t) - \tilde{d}(t), \quad t \geqslant 0,\\ \dot{\tilde{d}}(t) = \frac{1}{\varepsilon^2}\tilde{y}(t) - \dot{d}(t), \quad t \geqslant 0. \end{cases} \tag{11}$$

We can solve $(\tilde{y}, \tilde{d})$, the ODE part of (11) separately:

$$\begin{pmatrix}\tilde{y}(t)\\ \tilde{d}(t)\end{pmatrix} = e^{At}\begin{pmatrix}\tilde{y}(0)\\ \tilde{d}(0)\end{pmatrix} + \int_0^t e^{A(t-s)}B\dot{d}(s)ds, \tag{12}$$

where A, B are defined in (8). A simple exercise shows that

$$e^{At}B = \left(\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\varepsilon^2(e^{\lambda_1 t} - e^{\lambda_2 t}),\ \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right)^\top,$$

where

$$\lambda_1 = \frac{1}{2\varepsilon}[1+\sqrt{3}i], \qquad \lambda_2 = \frac{1}{2\varepsilon}[1-\sqrt{3}i]$$

are eigenvalues of A. By this fact, we see that the solution $(\tilde{y}, \tilde{d})$ of (12) satisfies (taking the uniform boundedness of $\dot{d}$ into account)

$$(\tilde{y}(t), \tilde{d}(t)) \to 0 \quad \text{as } t \to \infty,\ \varepsilon \to 0. \tag{13}$$

Remark 2.2. It is seen from (11) that $(\tilde{y}, \tilde{d})$ is an external model for the "u part" of the system. The approach is very similar to the external model principle in Medvedev and Hillerström (1995) but it is different to the internal model principle in Immonen and Pohjolainen (2006) where the disturbance is produced from an exogenous system, and the dynamical behavior of the disturbances must be similar to that of the reference signals.

Now we consider the "u part" of the system (11) which is rewritten as

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0,\\ u(0,t) = u_x(0,t) = 0, \quad t \geqslant 0,\\ u_{xx}(1,t) = 0, \quad t \geqslant 0,\\ u_{xxx}(1,t) = ku_t(1,t) - \tilde{d}(t), \quad t \geqslant 0. \end{cases} \tag{14}$$

Define operator $\mathbb{A}$ as

$$\begin{aligned} &\mathbb{A}(f,g)^\top = (g, -f^{(4)})^\top, \quad \forall (f,g)^\top \in D(\mathbb{A}),\\ &D(\mathbb{A}) = \{(f,g)^\top \in \mathcal{H} \cap (H^4(0,1) \times H^2(0,1)) | g(0)\\ &\qquad = g'(0) = 0, f''(1) = 0, f'''(1) = kg(1)\}. \end{aligned} \tag{15}$$

With the operator $\mathbb{A}$ at hand, we can write system (14) into an evolutionary equation in $\mathcal{H}$:

$$\frac{d}{dt}\begin{pmatrix}u\\ u_t\end{pmatrix} = \mathbb{A}\begin{pmatrix}u\\ u_t\end{pmatrix} + \mathbb{B}\tilde{d}, \quad \mathbb{B} = \begin{pmatrix}0\\ \delta(x-1)\end{pmatrix}. \tag{16}$$

It is well-known that $\mathbb{A}$ generates an exponential stable C_0-semigroup $e^{\mathbb{A}t}$ on $\mathcal{H}$ (Chen et al., 1987; Guo & Yu, 2001). Now we show that $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$ (Weiss, 1989). Actually, a straightforward computation gives

$$\begin{cases} \mathbb{A}^*(\varphi,\psi)^\top = (-\psi, \varphi^{(4)})^\top, \quad \forall (\varphi,\psi)^\top \in D(\mathbb{A}^*),\\ D(\mathbb{A}^*) = \{(\varphi,\psi)^\top \in \mathcal{H} \cap (H^4(0,1) \times H^2(0,1)) | \psi(0)\\ \quad = \psi'(0) = 0, \varphi''(1) = 0, \varphi'''(1) = -k\psi(1)\}. \end{cases} \tag{17}$$

The dual system to (14) is

$$\begin{cases} u^*_{tt}(x,t) + u^*_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0,\\ u^*(0,t) = u^*_x(0,t) = 0, \quad t \geqslant 0,\\ u^*_{xx}(1,t) = 0, \quad t \geqslant 0,\\ u^*_{xxx}(1,t) = ku^*_t(1,t), \quad t \geqslant 0,\\ y_o(t) = u^*_t(1,t), \quad t \geqslant 0. \end{cases} \tag{18}$$

Since $\mathbb{A}$ generates a C_0-semigroup solution, and so does for $\mathbb{A}^*$. Hence system (18) associates with a C_0-semigroup solution. Define the energy function to (18) as

$$E(t) = \frac{1}{2}\int_0^1 [u^{*2}_{xx}(x,t) + u^{*2}_t(x,t)]dx.$$

Differentiate $E(t)$ with respect to t along the solution to (18) to obtain

$$\dot{E}(t) = -ku^{*2}_t(1,t). \tag{19}$$

Integrating from 0 to T with respect to t in the above equation, we have

$$\int_0^T u^{*2}_t(1,t)dt = \frac{1}{k}(E(0) - E(T)) \leqslant \frac{1}{k}E(0), \quad \forall T \geqslant 0. \tag{20}$$

On the other hand,

$$\begin{aligned} &\mathbb{A}^{*-1}\begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix} = \begin{pmatrix}G_1(x) + G_2(x)\\ -\varphi(x)\end{pmatrix},\\ &G_1(x) = \frac{k(x^3 - 3x^2)\varphi(1)}{6},\\ &G_2(x) = \int_0^x\int_0^y\int_z^1\int_\xi^1 \psi(\tau)d\tau d\xi dz dy,\\ &\mathbb{B}^*\mathbb{A}^{*-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = -\varphi(1). \end{aligned} \tag{21}$$

So $\mathbb{B}^*\mathbb{A}^{*-1}$ is bounded from $\mathcal{H}$ to $\mathbb{C}$. This together with (20) shows that $\mathbb{B}^*$ is admissible for $e^{\mathbb{A}^*t}$, and so is $\mathbb{B}$ for $e^{\mathbb{A}t}$. By Weiss (1989), it follows that for any initial value $(u(\cdot,0),u_t(\cdot,0))^\top\in\mathcal{H}$, there exists a unique solution $(u(\cdot,t),u_t(\cdot,t))^\top\in\mathcal{H}$ provided that $\tilde{d}\in L^2_{\rm loc}(0,\infty)$. The solution can be written as

$$\begin{pmatrix}u(\cdot,t)\\u_t(\cdot,t)\end{pmatrix}=e^{\mathbb{A}t}\begin{pmatrix}u(\cdot,0)\\u_t(\cdot,0)\end{pmatrix}+\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds. \tag{22}$$

By (13), for any given $\varepsilon_0>0$, there exist $t_0>0$ and $\varepsilon_1>0$ such that $|\tilde{d}(t)|<\varepsilon_0$ for all $t>t_0$ and $0<\varepsilon<\varepsilon_1$. We rewrite the solution of (22) as

$$\begin{aligned}\begin{pmatrix}u(\cdot,t)\\u_t(\cdot,t)\end{pmatrix}&=e^{\mathbb{A}t}\begin{pmatrix}u(\cdot,0)\\u_t(\cdot,0)\end{pmatrix}+e^{\mathbb{A}(t-t_0)}\\&\quad\times\int_0^{t_0}e^{\mathbb{A}(t_0-s)}\mathbb{B}\tilde{d}(s)ds+\int_{t_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds.\end{aligned} \tag{23}$$

The admissibility of B implies that

$$\begin{aligned}\left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds\right\|^2_{\mathcal{H}}&\leqslant C_t\|\tilde{d}\|^2_{L^2_{\rm loc}(0,t)}\\&\leqslant t^2C_t\|\tilde{d}\|^2_{L^\infty(0,t)},\quad\forall\tilde{d}\in L^\infty(0,\infty)\end{aligned} \tag{24}$$

for some constant C_t that is independent of $\tilde{d}$. Since $e^{\mathbb{A}t}$ is exponentially stable, it follows from Proposition 2.5 of Weiss (1989) that

$$\begin{aligned}\left\|\int_{t_0}^t e^{\mathbb{A}(t-s)}\mathbb{B}\tilde{d}(s)ds\right\|&=\left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}(0\underset{t_0}{\diamond}\tilde{d})(s)ds\right\|\\&\leqslant L\|\tilde{d}\|_{L^\infty(0,\infty)}\leqslant L\varepsilon_0,\end{aligned} \tag{25}$$

where L is a constant that is independent of $\tilde{d}$, and

$$(u\underset{\tau}{\diamond}v)(t)=\begin{cases}u(t),&0\leqslant t\leqslant\tau,\\v(t-\tau),&t>\tau.\end{cases} \tag{26}$$

Suppose that $\|e^{\mathbb{A}t}\|\leqslant L_0e^{-\omega t}$ for some $L_0,\omega>0$. By (23), (24), and (25), we have

$$\begin{aligned}\left\|\begin{pmatrix}u(\cdot,t)\\u_t(\cdot,t)\end{pmatrix}\right\|&\leqslant L_0e^{-\omega t}\left\|\begin{pmatrix}u(\cdot,0)\\u_t(\cdot,0)\end{pmatrix}\right\|+L_0C_{t_0}\\&\quad\times e^{-\omega(t-t_0)}\|\tilde{d}\|_{L^\infty(0,t_0)}+L\varepsilon_0.\end{aligned} \tag{27}$$

As $t\to\infty$, the first two terms of (27) tend to zero. The result is then proved by the arbitrariness of ε_0.

We summarize the above as the following theorem.

Theorem 2.1. *Suppose that both d and $\dot{d}$ are bounded measurable. Then for any initial value $(u(\cdot,0),u_t(\cdot,0))^\top\in\mathcal{H}$, the closed-loop system (28) of (1) following*

$$\begin{cases}u_{tt}(x,t)+u_{xxxx}(x,t)=0,\quad x\in(0,1),\ t>0,\\u(0,t)=u_x(0,t)=0,\quad t\geqslant0,\\u_{xx}(1,t)=0,\quad t\geqslant0,\\u_{xxx}(1,t)=ku_t(1,t)-\hat{d}(t)+d(t),\quad t\geqslant0,\\\dot{\hat{y}}(t)=-ku_t(1,t)-2u_x(1,t)\\\qquad-\dfrac{1}{\varepsilon}\left(\hat{y}(t)-\displaystyle\int_0^1x^2u_t(x,t)dx\right),\quad t\geqslant0,\\\dot{\hat{d}}(t)=\dfrac{1}{\varepsilon^2}\left(\hat{y}(t)-\displaystyle\int_0^1x^2u_t(x,t)dx\right),\quad t\geqslant0,\end{cases} \tag{28}$$

admits a unique solution $(u,u_t)^\top\in C(0,\infty;\mathcal{H})$. Moreover, the solution of system (28) tends to any arbitrary given vicinity of zero as $t\to\infty$, $\varepsilon\to0$.

3. Feedback via sliding mode control

In this section, we use the SMC to reject the disturbance by removing the condition that $\dot{d}$ is supposed to be bounded in ADRC. That is, in this section, the disturbance is assumed to satisfy $|d(t)|\leqslant M$ for all $t\geqslant0$ for some $M>0$ only.

Choose the sliding surface

$$\begin{aligned}S=\Bigg\{&(f,g)^\top\in\mathcal{H}\,|\,f(1)-\int_0^1k(x)g(x)dx\\&+2\int_0^1k(x)f(x)dx=0\Bigg\},\end{aligned} \tag{29}$$

which is a closed-subspace in $\mathcal{H}$, where

$$\begin{cases}k^{(4)}(x)+4k(x)=0,\quad 0<x<1,\\k(0)=k'(0)=k''(1)=0,\\k'''(1)=2,\quad k(1)<0.\end{cases} \tag{30}$$

The solution to (30) is found explicitly as (Polyanin & Zaitsev, 1995, p. 615)

$$\begin{cases}k(x)=C_1[\cosh x\sin x-\sinh x\cos x]+C_2\sinh x\sin x,\\C_1=\dfrac{\cosh1\cos1}{\cos^21+\cosh^21},\\C_2=-\dfrac{(\sinh1\cos1+\cosh1\sin1)}{\cos^21+\cosh^21},\\k(1)=\dfrac{\sin2-\sinh2}{2(\cos^21+\cosh^21)}<0.\end{cases} \tag{31}$$

The corresponding sliding mode function is then

$$S(t)=u(1,t)-\int_0^1k(x)u_t(x,t)dx+2\int_0^1k(x)u(x,t)dx. \tag{32}$$

On the sliding surface $S(t)=0$, the system (1) becomes

$$\begin{cases}u_{tt}(x,t)+u_{xxxx}(x,t)=0,\quad x\in(0,1),\ t>0,\\u(0,t)=u_x(0,t)=0,\quad t\geqslant0,\\u_{xx}(1,t)=0,\quad t\geqslant0,\\u(1,t)=\displaystyle\int_0^1k(x)u_t(x,t)dx-2\int_0^1k(x)u(x,t)dx,\quad t\geqslant0.\end{cases} \tag{33}$$

It is a trivial exercise to show that in the sliding surface S, system (33) associates with a C_0-semigroup of contractions solution, which is displayed by the dissipativity of the following function $E_0(t)$:

$$\begin{aligned}E_0(t)&=\frac12\int_0^1[u_t^2(x,t)+u_{xx}^2(x,t)]dx,\\\rho_0(t)&=\int_0^1xu_t(x,t)u_x(x,t)dx.\end{aligned} \tag{34}$$

Actually, differentiating $E_0(t)$ and $\rho_0(t)$ along the solution of system (33) gives

$$\begin{aligned}\dot{E}_0(t)&=-u_t(1,t)u_{xxx}(1,t)=k(1)u_{xxx}^2(1,t)\leqslant0,\\\dot{\rho}_0(t)&=-u_{xxx}(1,t)u_x(1,t)-\frac32\int_0^1u_{xx}^2(x,t)dx\\&\quad+\frac12u_t^2(1,t)-\frac12\int_0^1u_t^2(x,t)dx.\end{aligned} \tag{35}$$

Then for sufficiently small $\delta > 0$, it has

$$\frac{d}{dt}(E_0(t) + \delta\rho_0(t)) \leqslant -\gamma E_0(t), \tag{36}$$

for some $\gamma > 0$. This shows that system (33) is exponentially stable in S.

Next we seek the finite "reaching condition" by designing the sliding mode feedback. Differentiating the sliding surface function formally gives

$$\begin{aligned}\dot{S}(t) &= u_t(1,t) - \int_0^1 k(x)u_{tt}(x,t)dx + 2\int_0^1 k(x)u_t(x,t)dx \\ &= u_t(1,t) + k(1)u_{xxx}(1,t) - 2S(t) \\ &= u_t(1,t) + k(1)(U(t) + d(t)) - 2S(t).\end{aligned} \tag{37}$$

It is seen from above that if we choose the controller

$$\begin{aligned}U(t) &= -k^{-1}(1)[u_t(1,t) - 2S(t)] \\ &\quad + (M+\eta)\text{sign}(S(t)), \quad \eta > 0,\end{aligned} \tag{38}$$

then it has

$$S(t)\dot{S}(t) \leqslant k(1)\eta|S(t)|, \tag{39}$$

which is just the finite "reaching condition" since $k(1)\eta < 0$. However, we do not know if $\dot{S}$ always exists which is remarkably different to the ordinary differential equation systems. Also, comparing the controller (38) with (9) in ADRC, we see that the energy of the controller in SMC is much higher than that in ADRC because in any circumstances, the controller (38) copes with the worst case of the disturbance. In other words, the controller for $d(t)$ is the same as that for $d(t) = M$.

Under the state feedback controller (38), the closed-loop system of (1) is

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ u(0,t) = u_x(0,t) = 0, \quad t \geqslant 0, \\ u_{xx}(1,t) = 0, \quad t \geqslant 0, \\ u_{xxx}(1,t) = -k^{-1}(1)[u_t(1,t) - 2S(t)] \\ \qquad + (M+\eta)\text{sign}(S(t)) + d(t) \\ \quad = -k^{-1}(1)u_t(1,t) + 2k^{-1}(1)S(t) + \tilde{d}(t), \quad t \geqslant 0, \end{cases} \tag{40}$$

where $S(t)$ is defined by (32), and

$$\tilde{d}(t) = (M+\eta)\text{sign}(S(t)) + d(t). \tag{41}$$

Define the operator $\mathcal{A}_0$ as follows:

$$\begin{aligned} &\mathcal{A}_0(f,g)^\top = (g, -f^{(4)})^\top, \quad \forall (f,g)^\top \in D(\mathcal{A}_0), \\ &D(\mathcal{A}_0) = \Big\{(f,g)^\top \in \mathcal{H} \cap (H^4(0,1) \times H^2(0,1)) \,|\, g(0) \\ &\quad = g'(0) = 0, f''(1) = 0, f'''(1) = -k^{-1}(1)g(1) \\ &\quad + 2k^{-1}(1)\left[f(1) - \int_0^1 k(x)g(x)dx \right.\\ &\quad \left.+ 2\int_0^1 k(x)f(x)dx\right]\Big\}. \end{aligned} \tag{42}$$

Then system (40) can be written as

$$\begin{aligned}&\frac{d}{dt}(u(\cdot,t), u_t(\cdot,t))^\top = \mathcal{A}_0(u(\cdot,t), u_t(\cdot,t))^\top + \mathcal{B}_0\tilde{d}(t), \\ &\mathcal{B}_0 = (0, -\delta(x-1))^\top.\end{aligned} \tag{43}$$

Lemma 3.1. *Let $\mathcal{A}_0$ be defined by (42). Then $\mathcal{A}_0$ generates a C_0-semigroup on $\mathcal{H}$.*

Proof. For any $(f,g)^\top \in D(\mathcal{A}_0)$, it is computed that

$$\begin{aligned} &\text{Re}\left\langle \mathcal{A}_0(f,g)^\top, (f,g)^\top \right\rangle \\ &= -\text{Re} f'''(1)\overline{g(1)} = k^{-1}(1)|g(1)|^2 - 2k^{-1}(1) \\ &\quad \times \text{Re}\left[f(1) - \int_0^1 k(x)g(x)dx + 2\int_0^1 k(x)f(x)dx\right]\overline{g(1)} \\ &\leqslant -k^{-1}(1)\left|f(1) - \int_0^1 k(x)g(x)dx + 2\int_0^1 k(x)f(x)dx\right|^2 \\ &\leqslant -4k^{-1}(1)\left[\int_0^1 |f''(x)|^2dx + \int_0^1 k^2(x)dx \int_0^1 |g(x)|^2dx \right.\\ &\quad \left. + 4\int_0^1 k^2(x)dx \int_0^1 |f''(x)|^2dx\right] \leqslant L_1\|(f,g)\|^2 \end{aligned}$$

for some $L_1 > 0$ that is independent of $(f,g)^\top$, where we used the fact $|f(x)| \leqslant \int_0^1 |f''(x)|^2dx$ for any $x \in [0,1]$. So for any $M_1 \geqslant L_1$, $\mathcal{A}_0 - M_1$ is dissipative in $\mathcal{H}$.

A direct computation shows that $\mathcal{A}_0^*$, the adjoint of $\mathcal{A}_0$, is given by

$$\begin{cases} \mathcal{A}_0^*(f(x), g(x))^\top = (-g(x) + k^{-1}(1)g(1)k(x), \\ \qquad f^{(4)}(x) + 2k^{-1}(1)g(1)k(x))^\top, \\ D(\mathcal{A}_0^*) = \{(f,g)^\top \in \mathcal{H} \cap (H^4(0,1) \times H^2(0,1)) | g(0) \\ \qquad = g'(0) = 0,\ f''(1) = 0,\ f'''(1) = k^{-1}(1)g(1)\}. \end{cases} \tag{44}$$

Then for any $(f,g)^\top \in D(\mathcal{A}_0^*)$,

$$\begin{aligned} &\text{Re}\left\langle \mathcal{A}_0^*(f,g)^\top, (f,g)^\top \right\rangle = k^{-1}(1)|g(1)|^2 - 2k^{-1}(1) \\ &\quad \times \text{Re}\left[\overline{f(1)} - \int_0^1 k(x)\overline{g(x)}dx + 2\int_0^1 k(x)\overline{f(x)}dx\right]g(1) \\ &\leqslant L_1\|(f,g)\|^2. \end{aligned}$$

So, $\mathcal{A}_0 - M_1$ and $(\mathcal{A}_0 - M_1)^*$ are dissipative in $\mathcal{H}$. By Lemma 3.2 below $\mathcal{A}_0$ is a closed operator. This together with Corollary 4.4 of Pazy (1983) on p. 15 shows that $\mathcal{A}_0 - M_1$ generates a C_0-semigroup of contractions on $\mathcal{H}$. Therefore, $\mathcal{A}_0$ generates a C_0-semigroup on $\mathcal{H}$. ■

Lemma 3.2. *Let $\mathcal{A}_0$, $\mathcal{B}_0$ be defined by (42) and (43) respectively. Then $\mathcal{B}_0$ is admissible to the semigroup generated by $\mathcal{A}_0$.*

Proof. For any $(\varphi,\psi)^\top \in \mathcal{H}$, find that $(f,g)^\top \in D(\mathcal{A}_0^*)$ such that

$$(\mathcal{A}_0^* - 2M_2^2)(f,g)^\top = (\varphi,\psi)^\top, \tag{45}$$

where $M_2 > 0$ is a constant. Then from the definition of $\mathcal{A}_0^*$ and the boundary conditions on the left hand in (44), we can get $g(x) = k^{-1}(1)g(1)k(x) - 2M_2^2 f(x) - \varphi(x)$, and (Polyanin & Zaitsev, 1995, p. 654)

$$\begin{aligned} f(x) &= C_3[\cosh(M_2x)\sin(M_2x) - \sinh(M_2x)\cos(M_2x)] \\ &\quad + C_4\sinh(M_2x)\sin(M_2x) \\ &\quad - \frac{1}{4M_2^3}\int_0^x [\sinh(M_2(x-y))\cos(M_2(x-y)) \\ &\quad - \cosh(M_2(x-y))\sin(M_2(x-y))]h(y)dy, \end{aligned} \tag{46}$$

where

$$h(x) = \psi(x) - 2M_2^2\varphi(x) + 2(M_2^2 - 1)k^{-1}(1)g(1)k(x). \tag{47}$$

For simplicity, we choose $2M_2^2 > M_1$ such that $\cos M_2 = 0$, $\sin M_2 = 1$. Taking the boundary condition on the right hand in (44) and

$f(1) = -\frac{1}{2M_2^2}\varphi(1)$ into account, we can get

$$\begin{aligned} g(1) &= L_2\bigg[-\frac{1}{M_2}\int_0^1 \sinh(M_2(1-y)) \\ &\quad \times \cos(M_2(1-y))(\psi(y) - 2M_2^2\varphi(y))dy \\ &\quad + \tanh M_2 \int_0^1 \cosh(M_2(1-y))\cos(M_2(1-y)) \\ &\quad \times (\psi(y) - 2M_2^2\varphi(y))dy + \varphi(1)\bigg], \end{aligned} \tag{48}$$

$$\begin{aligned} C_3 &= -\frac{1}{4M_2^3\cosh M_2}\int_0^1 [\cosh(M_2(1-y))\sin(M_2(1-y)) \\ &\quad + \sinh(M_2(1-y))\cos(M_2(1-y))]h(y)dy, \\ C_4 &= \frac{1}{2M_2^2\cosh M_2}\bigg[\int_0^1 \cosh(M_2(1-y)) \\ &\quad \times \cos(M_2(1-y))h(y)dy - k^{-1}(1)g(1)\bigg], \end{aligned}$$

where

$$\begin{aligned} L_2 &= \bigg[\frac{2(M_2^2-1)k^{-1}(1)}{M_2}\int_0^1 \sinh(M_2(1-y)) \\ &\quad \times \cos(M_2(1-y))k(y)dy - 2(M_2^2-1)k^{-1}(1) \\ &\quad \times \tanh M_2 \int_0^1 \cosh(M_2(1-y))\cos(M_2(1-y)) \\ &\quad \times k(y)dy + \tanh M_2 k^{-1}(1)\bigg]^{-1}. \end{aligned} \tag{49}$$

Therefore, $(\mathcal{A}_0^* - 2M_2^2)^{-1}$ exists and is bounded, and

$$\mathcal{B}_0^*(\mathcal{A}_0^* - 2M_2^2)^{-1}(\varphi, \psi)^\top = -g(1),$$

which is bounded from $\mathcal{H}$ to $\mathbb{C}$. Consider the dual system:

$$\frac{d}{dt}(p(\cdot,t), q(\cdot,t))^\top = \mathcal{A}_0^*(p(\cdot,t), q(\cdot,t))^\top.$$

Then we have

$$\begin{cases} p_t(x,t) = -q(x,t) + k^{-1}(1)q(1,t)k(x), \\ q_t(x,t) = p^{(4)}(x,t) + 2k^{-1}(1)q(1,t)k(x), \\ p(0,t) = p'(0,t) = q(0,t) = q'(0,t) = 0, \\ p''(1,t) = 0, \qquad p'''(1,t) = k^{-1}(1)q(1,t), \\ y_0(t) = -q(1,t). \end{cases} \tag{50}$$

Define

$$F(t) = \frac{1}{2}\int_0^1 [q^2(x,t) + p_{xx}^2(x,t)]dx.$$

Since $\mathcal{A}_0$ generates a C_0-semigroup on $\mathcal{H}$, and so does $\mathcal{A}_0^*$, there exist constants ω, M_ω such that $F(t) \leqslant M_\omega e^{\omega t}F(0)$ for all $t \geqslant 0$. Finding the derivative of $F(t)$ along the solution of (50) gives

$$\begin{aligned} k^{-1}(1)q^2(1,t) &= \dot{F}(t) - 2k^{-1}(1)\int_0^1 k(x)q(x,t)dx \\ &\quad \times q(1,t) - k^{-1}(1)\int_0^1 k''(x)p''(x,t)dx q(1,t), \end{aligned}$$

which implies

$$\begin{aligned} q^2(1,t) &\leqslant k(1)\dot{F}(t) + 4\left(\int_0^1 k(x)q(x,t)dx\right)^2 + \frac{1}{4}q^2(1,t) \\ &\quad + \left(\int_0^1 k''(x)p''(x,t)dx\right)^2 + \frac{1}{4}q^2(1,t) \\ &\leqslant k(1)\dot{F}(t) + 4\int_0^1 k^2(x)dx\int_0^1 q^2(x,t)dx \\ &\quad + \int_0^1 k''^2(x)dx\int_0^1 p''^2(x,t)dx + \frac{1}{2}q^2(1,t). \end{aligned}$$

Hence for any given $T > 0$, we have

$$\begin{aligned} \frac{1}{2}\int_0^T q^2(1,t)dt &\leqslant k(1)[F(T) - F(0)] \\ &\quad + 4\int_0^T\int_0^1 k^2(x)dx\int_0^1 q^2(x,t)dxdt \\ &\quad + \int_0^T\int_0^1 k''^2(x)dx\int_0^1 p''^2(x,t)dxdt \\ &\leqslant D_T F(0), \end{aligned}$$

where $D_T > 0$ is independent of $F(0)$. This fact together with the boundedness of $\mathcal{B}_0^*(\mathcal{A}_0^* - 2M_2^2)^{-1}$ completes the proof (Weiss, 1989). ∎

We are now in a position to show the main result of this section.

Theorem 3.1. *Suppose that d is bounded measurable and $S(t)$ is defined by (32). Then for any $(u(\cdot,0), u_t(\cdot,0))^\top \in \mathcal{H}$, $S(0) \neq 0$, there exists a $t_0 > 0$ such that (40) admits a unique solution $(u, u_t)^\top \in C(0, t_0; \mathcal{H})$ and $S(t) = 0$ for all $t \geqslant t_0$. Moreover, $S(t)$ is continuous, monotone in $[0, t_0]$. On the sliding surface $S(t) = 0$, the system (1) becomes (33) which is exponentially stable.*

Proof. We need only to prove that $S(t)$ is continuous, monotone in $[0, t_0]$. Suppose without lost of generality that $S(0) > 0$ since the proof for $S(0) < 0$ is similar.

Since by Lemma 3.2, $\mathcal{B}_0$ is admissible for $e^{\mathcal{A}_0 t}$, the solution to the system (40) can be written as

$$\begin{aligned} (u(\cdot,t), u_t(\cdot,t))^\top &= e^{\mathcal{A}_0 t}(u(\cdot,0), u_t(\cdot,0))^\top \\ &\quad + \int_0^t e^{\mathcal{A}_0(t-s)}\mathcal{B}_0\tilde{d}(s)ds. \end{aligned} \tag{51}$$

(51) simply means, for all $(f,g)^\top \in D(\mathcal{A}_0^*)$, that

$$\begin{aligned} &\frac{d}{dt}\int_0^1 [u_{xx}(x,t)f''(x) + u_t(x,t)g(x)]dx \\ &= \int_0^1 \{u_{xx}(x,t)[-g''(x) + k^{-1}g(1)k''(x)] \\ &\quad + u_t(x,t)[f^{(4)}(x) + 2k^{-1}(1)g(1)k(x)]\}dx. \end{aligned} \tag{52}$$

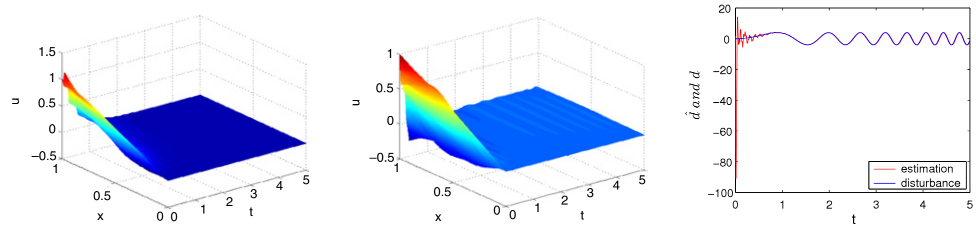
Substitute $(f,g)^\top = (k, 2k)^\top \in D(\mathcal{A}_0^*)$ into (52) to obtain

$$\begin{aligned} &\frac{d}{dt}\left[-2u(1,t) + 2\int_0^1 (k(x)u_t(x,t) - 2k(x)u(x,t))dx\right] \\ &= -2k(1)\tilde{d}(t), \end{aligned} \tag{53}$$

which is

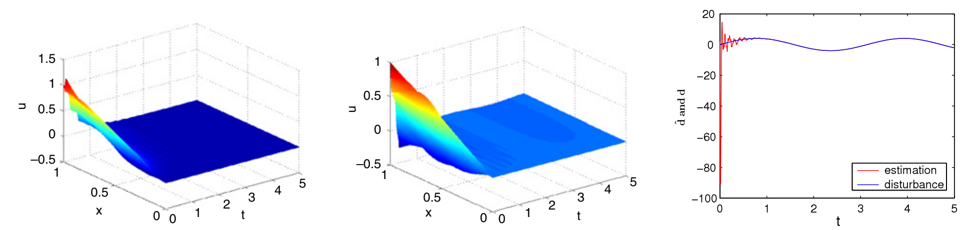
$$\dot{S}(t) = k(1)\tilde{d}(t) \quad \text{for } S(t) \neq 0. \tag{54}$$

This shows that S is continuous in the interval where $S \neq 0$. Moreover, (39) holds true. Therefore, there exists a $t_0 \geqslant 0$ such that $S(t)$ is monotone in $[0, t_0]$ and $S(t) = 0$ for all $t \geqslant t_0$. In particular, if $S(0) = 0$, then $t_0 = 0$. This completes the proof. ∎



(a) SMC. (b) ADRC. (c) Tracking of disturbance.

Fig. 1. Displacements and tracking of disturbance with $d(t) = 4\sin(2t^2)$ (unbounded derivative).



(a) SMC. (b) ADRC. (c) Tracking of disturbance.

Fig. 2. Displacements and tracking of disturbance with $d(t) = 4\sin(2t)$ (bounded derivative).

4. Numerical simulation

In this section, the finite difference method is applied to compute the displacements numerically for both ADRC and SMC to illustrate the effect of the controllers. Fig. 1(a) and (b) show the displacements of system (40) and (10) respectively. Fig. 1(c) plots the disturbance d and its tracked signal $\hat{d}$ by the extended state observer. Here the steps of space and time are taken as 0.02 and 0.0001, respectively. We choose $\varepsilon = 0.01$, $k = 2$, $M = 5$, $d(t) = 4\sin(2t^2)$, and the initial value:

$$u(x, 0) = x, \qquad u_t(x, 0) = -x. \tag{55}$$

Since $|d| < 4$ is bounded but $|\dot{d}|$ is unbounded, it is seen from Fig. 1 that system (40) converges satisfactorily yet system (10) demonstrates oscillation around the equilibrium. So ADRC may not work for the disturbance with unbounded derivative. This is coincidence with the theoretical results.

Fig. 2 displays the displacements of system (40) and (10) respectively, both with the disturbance $d(t) = 4\sin(2t)$. In this case, both d and $\dot{d}$ are uniformly bounded. All parameters and the initial value as well are taken the same as that in Fig. 1. It is seen that in this case, both ADRC and SMC are convergent satisfactorily. In addition, $\hat{d}$ tracks well the true value of disturbance d.

5. Concluding remarks

In this paper, we deal with the stabilization of an Euler–Bernoulli beam system which has disturbance on the input boundary. Both the active disturbance rejection control (ADRC) and the sliding mode control (SMC) approaches are adopted. By the ADRC, we are able to estimate the disturbance and cancel the disturbance in the feedback loop. The most advantage of the ADRC lies in its economy in the controller yet with the price that the disturbance should have the bounded derivative and the disturbance can only be attenuated. By SMC approach, we can remove the restriction of the boundedness of the disturbance and the rejection of the disturbance can be achieved. The existence and uniqueness of the solution for the closed-loop system by SMC are proved. The "reaching condition" is presented without differentiation of the sliding mode function for which it may not exist for the weak solution of the closed-loop system. The price for SMC is that the gain in the controller is high (in the worst case of the disturbance), and possibly there is the chattering problem caused by the discontinuity of the controller. The numerical simulations validate the theoretical results. We also point out that both methods are by the state feedback (similar to Cheng et al., 2011 by the SMC). This causes the practical implementation problem. However, it would be the first step toward the output pointwise feedback control in the further research.

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