**Backstepping approach to the arbitrary decay rate for Euler–Bernoulli beam under boundary feedback**

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(Received 13 May 2010; final version received 27 June 2010)

In this article, we are concerned with the boundary stabilisation of the Euler–Bernoulli beam equation for which all eigenvalues of the (control) free system are located on the imaginary axis of the complex plane. The fourth-order system in spacial variable is transformed into a coupled heat-like system. This enables us to make a natural backstepping transformation in vector form to transform the system into a target system which has arbitrary decay rate. The state feedback is thus designed. It is shown that the original closed-loop system is exponentially stable with the given arbitrary decay rate.

**Keywords:** beam equation; stability; backstepping; boundary control

1. Introduction

In the past decade, the backstepping method has been successfully applied to the stabilisation of one-dimensional parabolic equations and wave equations, see Liu and Krstic (2000), Boskovic, Kristic, and Liu (2001), Boskovic, Balogh, and Kristic (2003), Smyshlyaev and Krstic (2005, 2007, 2009) and Krstic, Guo, Balogh, and Smyshlyaev (2008). A good summary can be found in Krstic and Smyshlyaev (2009). However, there is much difficulty in applying this approach to the stabilisation of Euler–Bernoulli beam equations. One of the reasons behind this is that there are too many possibilities to make the backstepping transformation for Euler–Bernoulli beam which is the fourth order in spacial variable. The first effort was made in Smyshlyaev, Guo, and Kristic (2009), where a new variable was introduced so that it satisfies the Schrödinger equation. By designing a stabilising controller for Schrödinger equation, the authors are able to construct the boundary state feedback for the original Euler–Bernoulli beam with arbitrary decay rate. This approach is not only introducing the interim step for Schrödinger equation but also the decay rate cannot be $-\epsilon = -\xi (2n + 1)^2$, $n \geq 0$.

In this article, we use a different approach to deal with this problem. The Euler–Bernoulli beam equation is first transformed into a coupled heat-like system.

This enables us to make a natural backstepping transformation in a vector form to transform the system into a target system which has arbitrary decay rate. The state feedback is thus designed. We prove that the original closed-loop system is exponentially stable with arbitrary decay rate that has no restrictions like that in Smyshlyaev et al. (2009). This idea can be expected to deal with other boundary control problems for Euler–Bernoulli beam equations.

The system that we are concerned is the Euler–Bernoulli beam equation with the pinned left end and controlled sliding right end, which is described by the following partial differential equation:

$$
\begin{align*}
\frac{\partial}{\partial t}w(x, t) + \frac{\partial^4}{\partial x^4}w(x, t) &= 0, \quad x \in (0, 1), \quad t > 0, \\
\frac{\partial}{\partial t}w(0, t) &= \frac{\partial}{\partial x}w_x(0, t) = 0, \quad t \geq 0, \\
\frac{\partial}{\partial t}w_x(1, t) &= u_1(t), \quad t \geq 0, \\
\frac{\partial}{\partial t}w_{xxx}(1, t) &= u_2(t), \quad t \geq 0, \\
\frac{\partial}{\partial x}w(x, 0) &= \frac{\partial}{\partial t}w(x, 0) = 0, \quad x \in [0, 1], \quad t \geq 0, \\
\frac{\partial}{\partial x}w_t(x, 0) &= w_1(x), \quad x \in [0, 1], \quad t \geq 0,
\end{align*}
$$

(1)

where $w$ is the displacement of the beam, $u_1$ and $u_2$ are the control inputs and $(w_0, w_1)$ is the initial state.

Since the energy of the system (1) is given by

$$
E(t) = \frac{1}{2} \int_0^1 \left( w_{xx}^2(x, t) + w_t^2(x, t) \right) dx,
$$

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naturally, we introduce a new vector variable $v(x,t) = (v_1(x,t), v_2(x,t))^T$ (Chen, Delfour, Kraill, and Payre 1987; Guo, Xie, and Hou 2004):

$$
\begin{align*}
  v_1(x,t) &= \frac{1}{2}[w_{xx}(x,t) - w_x(x,t)], \quad x \in [0,1], \quad t > 0, \\
  v_2(x,t) &= \frac{1}{2}[w_{xx}(x,t) + w_x(x,t)], \quad x \in [0,1], \quad t > 0.
\end{align*}
$$

(2)

By this transformation, the $L^2$-norm of the vector $v$ is just the energy of the system (1) and $v$ satisfies the following heat-like coupled equation:

$$
\begin{align*}
  v_x(x,t) &= Dv_{xx}(x,t), \quad 0 < x < 1, \quad t > 0, \\
  v(0,t) &= v_0, \quad t \geq 0, \\
  v_x(1,t) &= U(t), \quad t \geq 0, \\
  v(x,0) &= v_0(x), \quad 0 \leq x \leq 1, \quad t \geq 0,
\end{align*}
$$

(3)

where $v_0$ is the initial state of system (3), and

$$
D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D^{-1} = -D.
$$

(4)

$U$ is the input for this coupled system.

It should be pointed out that although any kind of boundary conditions of system (1) can be transformed into heat-like system (3) under the transformation (2), only some special boundary conditions like that in (1) and Smyshlyaev et al. (2009) can make the boundary conditions of system (3) be expressed in vector form, which leads to the easy treatment of system (3). Otherwise, the boundary conditions of system (3) could be complicated in the sense that the boundary conditions can only be expressed by the components of $v$ separately. This brings the difficulty for the control design of system (3).

We proceed as follows. In Section 2, we present the whole process of the design of the stabilising state-feedback controller for the system (3) by backstepping method. Section 3 is devoted to the design of the state feedback for the system (1).

2. State feedback for coupled heat-like equation

In this section, we find the state feedback to stabilise the system (3). We consider system (3) in the state Hilbert space $\mathcal{H}_v = (L^2(0,1))^2$.

**Lemma 2.1:** Let $c > 0$, $d \neq 0$, $d \neq 1$, $\mathcal{H}_v = (L^2(0,1))^2$. Then the closed-loop system (3) under the state-feedback control

$$
U(t) = \left( dI + \frac{cD}{2} \right) v(1,t) + \int_0^t [k_x(1,y) - dk(1,y)]v(y,t)dy,
$$

(5)

is equivalent to the following system:

$$
\begin{align*}
  \tilde{v}_d(x,t) &= D\tilde{v}_{xx}(x,t) - c\tilde{v}(x,t), \quad 0 < x < 1, \quad t > 0, \\
  \tilde{v}(0,t) &= 0, \quad t \geq 0, \\
  \tilde{v}_x(1,t) &= d\tilde{v}(1,t), \quad t \geq 0, \\
  \tilde{v}(x,0) &= \tilde{v}_0(x), \quad 0 \leq x \leq 1, \quad t \geq 0,
\end{align*}
$$

(7)

in the state space $\mathcal{H}_v$ provided that both the classical solutions of the closed-loop system (3) with (5) and target system (7) exist in $\mathcal{H}_v$, where $c$ and $d$ are design parameters, $\tilde{v}(x,t) = (\tilde{v}_1(x,t), \tilde{v}_2(x,t))^T$, $\tilde{v}_0$ is the initial state.

**Proof:** Since the state space $\mathcal{H}_v = (L^2(0,1))^2$, we make a natural invertible transformation that we expect to transfer the system (3) into the target system (7):

$$
\tilde{v}(x,t) = v(x,t) - \int_0^x k(x,y)v(y,t)dy,
$$

(8)

where the kernel $k(x,y) = \{k_{ij}(x,y)\}_{i,j=1,2}$ is a $2 \times 2$ matrix-valued function.

Differentiate (8) with respect to $x$, to give

$$
\begin{align*}
  \tilde{v}_x(x,t) &= v_x(x,t) - k(x,x)v(x,t) - \int_0^x k_{xx}(x,y)v(y,t)dy, \\
  \tilde{v}_{xx}(x,t) &= v_{xx}(x,t) - \frac{d}{dx}(k(x,x)v(x,t)) - k_x(x,x)v(x,t) \\
  &\quad - \int_0^x k_{xx}(x,y)v(y,t)dy \\
  &= v_{xx}(x,t) - \frac{d}{dx}k(x,x)v(x,t) - k(x,x)v_x(x,t) \\
  &\quad - k_x(x,x)v(x,t) - \int_0^x k_{xx}(x,y)v(y,t)dy.
\end{align*}
$$

(9)

Differentiate (8) with respect to $t$, to get

$$
\begin{align*}
  \tilde{v}_t(x,t) &= v_t(x,t) - \int_0^x k(x,y)v(y,t)dy = Dv_{xx}(x,t) \\
  &\quad - \int_0^x k(x,y)Dv_{yy}(y,t)dy \\
  &= Dv_{xx}(x,t) - k(x,x)v_{xx}(x,t) \\
  &\quad + k(x,0)v_{yy}(0,t) + \tilde{k}_x(x,x)v_{xx}(x,t) \\
  &\quad - k_x(x,0)v_t(0,t) - \int_0^x \tilde{k}_{yy}(x,y)v_{yy}(y,t)dy.
\end{align*}
$$

(10)
Substitute the second equality of (9) into (10) to obtain
\[
\tilde{v}(x, t) = \tilde{Dv}_{xx}(x, t) + c\tilde{v}(x, t)
\]
\[
= \left[ 2D\frac{d}{dx}k(x, x) + k_y(x, x)D - Dk_y(x, x) + c \right] v(x, t)
+ [Dk(x, x) - k(x, x)D]v_x(x, t) + k(x, 0)Dv_x(0, t)
+ \int_0^t [Dk_{xx}(x, y) - k_{xx}(x, y)D - ck(x, y)]v(y, t)dy.
\]
(11)

We choose \(k(x, y)\) so that \(k(x, y)D = Dk(x, y)\) for any \(0 \leq y \leq x \leq 1\). Setting the right-hand side of (11) to be zero, we obtain the following partial differential equation satisfied by the kernel:
\[
k(x, y) = \begin{cases} 
  k_{xx}(x, y) = k_{yy}(x, y) + cD^{-1}k(x, y), & 0 < y \leq x < 1, \\
  k(x, x) = -\frac{c}{2}D^{-1}x, & 0 \leq x \leq 1, \\
  k(x, 0) = 0, & 0 \leq x \leq 1.
\end{cases}
\]
(12)

The solution of (12) is found to be (Krstic and Smyshlyaev 2009, pp. 32–35)
\[
k(x, y) = -cD^{-1}y \sum_{m=0}^{\infty} \frac{c^mD^{-m}(x^2 - y^2)^m}{m!(m + 1)!2^{2m+1}}
= cDy \sum_{m=0}^{\infty} \frac{c^mD^m(y^2 - x^2)^m}{m!(m + 1)!2^{2m+1}},
\]
(13)
\[
k_{11}(x, y) = k_{22}(x, y), k_{12}(x, y) = -k_{21}(x, y),
\forall 0 \leq y \leq x \leq 1.
\]

By the boundary condition \(\tilde{v}(1, t) = d\tilde{v}(1, t)\) of (7), and the first equality of (9), we get the control
\[
U(t) = v_x(1, t) = \left( dl + \frac{cD}{2} \right)v(1, t)
+ \int_0^1 [k_x(1, y) - dk(1, y)]v(y, t)dy.
\]
(14)

Since the kernel \(k\) is continuous, it is well known that in \(H_2\), the Voterra transform \((8)\) is invertible. Now we find the inverse of this transform. To do it, let
\[
v(x, t) = \tilde{v}(x, t) + \int_0^x s(x, y)v(y, t)dy,
\]
(15)
where \(s(x, y)\) is a \(2 \times 2\) matrix-valued function satisfying \(s(x, y)D = Ds(x, y)\) for any \(0 \leq y \leq x \leq 1\).

Similar to the above process, we find, from (15), that
\[
0 = v_x(x, t) - Dv_{xx}(x, t)
= -\left( 2D\frac{d}{dx}s(x, x) + c \right)\tilde{v}(x, t)
- \int_0^t [Ds_{xx}(x, y) - s_{yy}(x, y)D + cs(x, y)]\tilde{v}(y, t)dy
- s(x, 0)D\tilde{v}_x(0, t).
\]
(16)

Since \(s(x, y)D = Ds(x, y)\), \(s(x, y)\) satisfies
\[
\begin{align*}
  s_{xx}(x, y) &= s_{yy}(x, y) - cD^{-1}s(x, y), & 0 < y \leq x < 1, \\
  s(x, 0) &= 0, & 0 \leq x \leq 1, \\
  s(x, x) &= -\frac{c}{2}D^{-1}x, & 0 \leq x \leq 1,
\end{align*}
\]
(17)
which, similar to (12), has solution of the form:
\[
s(x, y) = -cD^{-1}y \sum_{m=0}^{\infty} \frac{(-1)^m c^m D^{-m}(x^2 - y^2)^m}{m!(m + 1)!2^{2m+1}}
= cDy \sum_{m=0}^{\infty} \frac{c^m D^m(x^2 - y^2)^m}{m!(m + 1)!2^{2m+1}},
\]
(18)
\[
s_{11}(x, y) = s_{22}(x, y), s_{12}(x, y) = -s_{21}(x, y),
\forall 0 \leq y \leq x \leq 1.
\]

By this kernel, it is easy to check that the boundary conditions
\[
\begin{align*}
  v(0, t) &= 0, \\
  v_x(1, t) &= \left( dl + \frac{cD}{2} \right)v(1, t)
  + \int_0^1 [k_x(1, y) - dk(1, y)]v(y, t)dy
\end{align*}
\]
are also satisfied. The proof is complete. \(\square\)

For the sake of convenience, we write the closed-loop system (3) under the feedback (14) as follows:
\[
\begin{align*}
  v_x(x, t) &= Dv_{xx}(x, t), & 0 < x < 1, t > 0, \\
  v(0, t) &= 0, & t \geq 0, \\
  v_x(1, t) &= \left( dl + \frac{cD}{2} \right)v(1, t)
  + \int_0^1 [k_x(1, y) \notag \\
  - dk(1, y)]v(y, t)dy, & t \geq 0,
\end{align*}
\]
(19)

where \(k(x, y)\) is given by (13).

Define the system operator \(A : D(A) \subset \mathcal{H}_t \to \mathcal{H}_t\) for system (19) as follows:
\[
A(f, g)^T = D(f'', g'')^T,
\]
\[
D(A) = \left\{ (f, g)^T \in \mathcal{H}_t \cap (H^2(0, 1))^2 \mid A(f, g)^T \in \mathcal{H}_t, \right. \\
\left. f(0) = g(0) = 0, \quad \left( f'(1), g'(1) \right)^T = \left( dl + \frac{cD}{2} \right)(f(1), g(1))^T
  + \int_0^1 [k_x(1, y) \notag \\
  - dk(1, y)](f(y), g(y))^Tdy. \right\}
\]
(20)
Then (19) can be written as an evolutionary equation in $\mathcal{H}_r$:

$$
\frac{d}{dt}(v_1(\cdot, t), v_2(\cdot, t))^T = A(v_1(\cdot, t), v_2(\cdot, t))^T.
$$

(21)

Now, we are in a position to prove the existence of solutions for both closed-loop system and target system.

**Theorem 2.2:** Suppose that $c > 0$, $d \neq 0$, $d \neq 1$. Then the system (19) is well posed: the operator $A$ defined by (20) generates a $C_0$-semigroup $e^{At}$ on $\mathcal{H}_r$. Therefore, we have the following existence results of classical and mild solutions:

$$
(v_1(\cdot, t), v_2(\cdot, t))^T \in C^1(0, \infty; D(A))
$$

$$
\forall (v_1(\cdot, 0), v_2(\cdot, 0))^T \in D(A)
$$

$$
(v_1(\cdot, t), v_2(\cdot, t))^T \in C(0, \infty; \mathcal{H}_r),
$$

$$
\forall (v_1(\cdot, 0), v_2(\cdot, 0))^T \in \mathcal{H}_r.
$$

Moreover, $e^{At}$ is exponentially stable: There is a positive constant $M > 0$ such that

$$
\|e^{At}\| \leq Me^{-ct}.
$$

(22)

**Proof:** This is the direct consequence of the equivalence between systems (19) and (7) claimed by Lemma 2.1. Actually, system (7) can be written as an evolutionary equation in $\mathcal{H}_r$:

$$
\frac{d}{dt}(\tilde{v}_1(\cdot, t), \tilde{v}_2(\cdot, t))^T = B(\tilde{v}_1(\cdot, t), \tilde{v}_2(\cdot, t))^T + C(\tilde{v}_1(\cdot, t), \tilde{v}_2(\cdot, t))^T,
$$

(23)

where $B$ is defined by

$$
B(f, g)^T = D(f', g''),
$$

(24)

and $C = -cI$, $I$ is the identity of $\mathcal{H}_r$.

It is easy to show that $B$ is conservative:

$$
\text{Re}(B(f, g)^T, (f, g)^T) = 0
$$

for any $(f, g)^T \in D(B)$ and

$$
B^{-1}(f, g)^T = \left(\frac{dx}{1-\alpha} \int_0^1 sg(s)ds + \int_0^\infty \int_y^1 s\gamma(s)dsdy, \quad \frac{dx}{1-\beta} \int_0^1 sf(s)ds + \int_0^\infty \int_y^1 s\gamma(s)dsdy\right)^T
$$

for any $(f, g) \in \mathcal{H}_r$. By the Lumer–Phillips theorem (Pazy 1983, p. 14), $B$ generates a $C_0$-semigroup $e^{Bt}$ on $\mathcal{H}$, and so does $B + C$ by the bounded perturbation of $C_0$-semigroups (Pazy 1983, p. 76). Thus system (7) has a mild solution $\tilde{v}(x, t) = e^{B+Ct}\tilde{v}(x, 0) \in C(0, \infty; \mathcal{H}_r)$ for any $\tilde{v}(\cdot, 0) \in \mathcal{H}_r$, and classical solution $\tilde{v}(x, t) \in C^1(0, \infty; D(B))$ provided that $\tilde{v}(\cdot, 0) \in D(B)$.

The latter is the existence of the classical solution for target system (7).

Define the Lyapunov-like function:

$$
\tilde{E}(t) = \frac{1}{2} \int_0^1 \tilde{v}^T(x, t)\tilde{v}(x, t)dx = \frac{1}{2} \|\tilde{v}\|^2_{\mathcal{H}_r}.
$$

(25)

A straightforward computation shows that

$$
\dot{\tilde{E}}(t) = -2c\tilde{E}(t),
$$

and hence

$$
\tilde{E}(t) = e^{-ct}\tilde{E}(0).
$$

That is

$$
\|e^{B+Ct}\| = e^{-ct} \quad \forall t \geq 0.
$$

(26)

Now define the mapping $P : \mathcal{H}_r \to \mathcal{H}_r$ by

$$
P(f(x), g(x))^T = (f(x), g(x))^T - \int_0^\infty k(x, y)(f(y), g(y))^T dy,
$$

(27)

$$
\forall (f, g)^T \in \mathcal{H}_r,
$$

where $k(x, y)$ is given by (13). The inverse of $P$ is given by

$$
P^{-1}(f(x), g(x))^T = (f(x), g(x))^T + \int_0^\infty s(x, y)(f(y), g(y))^T dy,
$$

(28)

$$
\forall (f, g)^T \in \mathcal{H}_r,
$$

where $s(x, y)$ is given by (18).

Lemma 2.1 tells us that $A = P^{-1}(B + C)P$. Since both $P$ and $P^{-1}$ are bounded, $A$ generates a $C_0$-semigroup $e^{At} = P^{-1}e^{(B+C)t}P$. It follows from (26) that

$$
\|e^{At}\| \leq \|P^{-1}\|\|P\|e^{-ct} = Me^{-ct},
$$

(29)

This is (22). The proof is complete.

3. Stability of closed-loop system of beam equation

In this section, we transform the coupled heat-like equation (19) into the beam system (1) with boundary feedback through the transformation (2). By (2), (13) and

$$
(w_{xx}, w_{xt})^T(1, t) = F^{-1}\left(D\frac{c}{2} + D\right)F(w_{xx}, w_t)^T(1, t)
$$

$$
+ \int_0^1 F^{-1}(k_x - dk)(1, y)F(w_{yy}, w_t)^T(y, t)dy
$$

$$
= F^{-1}\left(\begin{array}{ccc} 1 & 1 \\
-1 & 1 \end{array}\right),
$$

we get
it has

\[ v_{1x}(1, t) + v_{2x}(1, t) = w_{xx}(1, t) = dw_{xx}(1, t) + \frac{c}{2} w_{x}(1, t) \]

\[ + \int_0^1 [(k_{11x}(1, y) - dk_{11}(1, y))w_{yy}(y, t)] \] 

\[ + (k_{12x}(1, y) - dk_{12}(1, y))w_{y}(y, t) \] 

\[ dy, \quad (29) \]

\[ v_{2x}(1, t) - v_{1x}(1, t) = w_{xx}(1, t) = -\frac{c}{2} w_{xx}(1, t) + dw_{x}(1, t) \]

\[ + \int_0^1 [(k_{21x}(1, y) - dk_{21}(1, y))w_{yy}(y, t)] 

\[ + (k_{22x}(1, y) - dk_{22}(1, y))w_{y}(y, t) \] 

\[ dy. \quad (30) \]

Equation (29) is the second state-feedback control \( u_2 \) for system (1). Now from (1), we have

\[ w_{xx}(x, t) = xw_{xx}(1, t) + \int_x^1 \int_y^1 w_{xx}(\xi, \eta) d\xi dy. \quad (31) \]

Substitute (31) into (29) and (30) to obtain

\[ \left[ 1 - \int_0^1 k_{11x}(1, y)dy - d\left(1 - \int_0^1 k_{11}(1, y)dy \right) \right] w_{xx}(1, t) \]

\[ = d \int_0^1 \int_y^1 w_{x}(\xi, \eta) d\xi dy + \frac{c}{2} w_{x}(1, t) \]

\[ + \int_0^1 [(k_{11x}(1, y) - dk_{11}(1, y))] \int_0^y \int_\eta^1 w_{y}(\xi, \eta) d\xi d\eta dy \]

\[ + \int_0^1 [(k_{12x}(1, y) - dk_{22}(1, y))w_{y}(y, t)dy \] 

\[ (32) \]

and

\[ w_{x}(1, t) = \left[-\frac{c}{2} + \int_0^1 (k_{21x}(1, y) - dk_{21}(1, y))dy \right] w_{xx}(1, t) \]

\[ + dw_{x}(1, t) - \frac{c}{2} \int_0^1 w_{x}(\xi, \eta) d\xi dy \]

\[ + \int_0^1 [(k_{21x}(1, y) - dk_{21}(1, y))] \]

\[ \cdot \int_0^y \int_\eta^1 w_{y}(\xi, \eta) d\xi d\eta dy + \int_0^1 (k_{22x}(1, y) \]

\[ - dk_{22}(1, y))w_{y}(y, t)dy \] 

\[ (33) \]

In the Appendix, we have shown that for any \( c > 0, \)

\[ 1 - \int_0^1 k_{11x}(1, y)dy \] 

\[ 1 - \int_0^1 k_{11}(1, y)dy \] 

cannot be identical to zero simultaneously. Thus we can choose \( d \) such that

\[ 1 - \int_0^1 k_{11x}(1, y)dy - d \left(1 - \int_0^1 k_{11}(1, y)dy \right) \neq 0. \]

\[ (34) \]

Under this assumption, we have

\[ w_{xx}(1, t) = \frac{1}{1 - d - \int_0^1 [k_{11x}(1, y) - dk_{11}(1, y)]y dy} \]

\[ \times \left[ d \int_0^1 \int_y^1 w_{x}(\xi, \eta) d\xi dy + \frac{c}{2} w_{x}(1, t) \right. \]

\[ + \int_0^1 [k_{11x}(1, y) - dk_{11}(1, y)] \]

\[ \cdot \int_0^y \int_\eta^1 w_{y}(\xi, \eta) d\xi d\eta dy + \int_0^1 [k_{12x}(1, y) \]

\[ - dk_{12}(1, y))w_{y}(y, t)dy \] 

\[ (35) \]

and hence

\[ w_{x}(1, t) = \frac{-\frac{c}{2} + \int_0^1 [k_{21x}(1, y) - dk_{21}(1, y)]y dy \]

\[ 1 - d - \int_0^1 [k_{11x}(1, y) - dk_{11}(1, y)]y dy \]

\[ \times \left[ d \int_0^1 \int_y^1 w_{x}(\xi, \eta) d\xi dy + \frac{c}{2} w_{x}(1, t) \right. \]

\[ + \int_0^1 [k_{11x}(1, y) - dk_{11}(1, y)] \]

\[ \cdot \int_0^y \int_\eta^1 w_{y}(\xi, \eta) d\xi d\eta dy + \int_0^1 [k_{21x}(1, y) \]

\[ - dk_{21}(1, y))\int_0^1 \int_\eta^1 w_{y}(\xi, \eta) d\xi d\eta dy \]

\[ + \int_0^1 [k_{22x}(1, y) - dk_{22}(1, y)]w_{y}(y, \eta)dy \] 

\[ (36) \]

Integrating the above equality with respect to \( t \) to get

(by removing a constant)

\[ w_{x}(1, t) = \frac{-\frac{c}{2} + \int_0^1 [k_{21x}(1, y) - dk_{21}(1, y)]y dy \]

\[ 1 - d - \int_0^1 [k_{11x}(1, y) - dk_{11}(1, y)]y dy \]

\[ \times \left[ d \int_0^1 \int_y^1 w_{x}(\xi, \eta) d\xi dy + \frac{c}{2} w_{x}(1, t) \right. \]

\[ + \int_0^1 [k_{11x}(1, y) - dk_{11}(1, y)] \]

\[ \cdot \int_0^y \int_\eta^1 w_{y}(\xi, \eta) d\xi d\eta dy + \int_0^1 (k_{21x}(1, y) \]

\[ - dk_{21}(1, y))w_{y}(y, \eta)dy \] 

\[ + dw(1, t) \]
which is the first-state feedback control

By (13), we finally write the closed-loop system (1) under the state feedback (29) and (37):

\[
\begin{align*}
\frac{\partial w_{xt}}{\partial t} + w_{xxx}(x, t) &= 0, \\
w(0, t) &= w_0(x, t) = 0, \\
w_{xt}(1, t) &= \frac{-z - \int_0^1 [k_{12x}(1, y) - d k_{12}(1, y)] y \, dy}{1 - d - \int_0^1 [k_{11x}(1, y) - d k_{11}(1, y)] y \, dy} \\
&\times \left[ \int_0^1 \int_0^y w_{x}(\xi, t) \, d\xi \, dy + \frac{c}{2} w_{x}(1, t) \\
&+ \int_0^1 [k_{11x}(1, y) - d k_{11}(1, y)] w(y, t) \, dy \\
&- d k_{12}(1, y)] w_y(y, t) + [k_{12x}(1, y)] w(t, t) \, dy \right] + d w(1, t) \\
&- \frac{c}{2} \int_0^1 \int_0^y w_{x}(\xi, t) \, d\xi \, dy + \int_0^1 [k_{21x}(1, y)] w(y, t) \, dy \\
&- d k_{21}(1, y)] w(y, t) + \int_0^1 \int_0^y g(\xi) \, d\xi \, dy \\
&- d k_{11}(1, y)] w(y, t) \, dy + \int_0^1 [k_{12x}(1, y)] w(y, t) \, dy.
\end{align*}
\]

(38)

Consider the system (38) in the state Hilbert space \( \mathcal{H} \) given by

\[
\mathcal{H} = \{ (f, g)^T \in H^2(0, 1) \times L^2(0, 1) | f(0) = 0, \}
\]

\[
f(1) = \frac{-z - \int_0^1 [k_{12x}(1, y) - d k_{12}(1, y)] y \, dy}{1 - d - \int_0^1 [k_{11x}(1, y) - d k_{11}(1, y)] y \, dy} \\
&\times \left[ \int_0^1 \int_0^y g(\xi) \, d\xi \, dy + \frac{c}{2} f(1) + \int_0^1 [k_{11x}(1, y)] \\
&- d k_{11}(1, y)] \int_0^y \int_0^y g(\xi) \, d\xi \, dy \right. \\
&\left. + \int_0^1 \int_0^y [k_{12x}(1, y)] \int_0^y \int_0^y [k_{12x}(1, y)] \right].
\]

(41)

System (38) can then be written as an evolution equation in \( \mathcal{H} \):

\[
\frac{d}{dt} (w, w_t)^T = A_T(w, w_t)^T.
\]

(42)
Theorem 3.1: Suppose that $c > 0$ and $d \neq 0$, $d \neq 1$ satisfies (34). Then the system (38) is well posed: the operator $A_T$ defined by (41) generates a $C_0$-semigroup $e^{A_T t}$ on $\mathcal{H}_T$. Moreover, $e^{A_T t}$ is exponentially stable: there is a positive constant $K > 0$ such that

$$\|e^{A_T t}\| \leq Ke^{-ct}. \quad (43)$$

Proof: Define a mapping $\mathbb{S}: \mathcal{H}_T \rightarrow \mathcal{H}_T$ as follows:

$$\begin{align*}
(\varphi, \psi)^T &= \mathbb{S}(f, g)^T \quad \forall (f, g)^T \in \mathcal{H}_T, \\
\varphi(x) &= \frac{1}{2}[f''(x) - g(x)], \\
\psi(x) &= \frac{1}{2}[f''(x) + g(x)],
\end{align*}$$

which comes from (2). Obviously, $\mathbb{S}$ is bounded. A straightforward computation gives $\mathbb{S}^{-1}: \mathcal{H}_T \rightarrow \mathcal{H}_T$

$$\begin{align*}
(f, g)^T &= \mathbb{S}^{-1}(\varphi, \psi)^T, \quad \forall (\varphi, \psi)^T \in \mathcal{H}_T, \\
f(x) &= Qx - \int_0^x \int_y^1 (\varphi(\xi) + \psi(\xi))d\xi dy, \\
g(x) &= \psi(x) - \varphi(x),
\end{align*}$$

where

$$Q = \frac{\beta}{\beta^2 + \gamma^2} \left[ d \int_0^1 \int_y^1 (\varphi(\xi) - \varphi(\xi))d\xi dy - \frac{c}{2} \int_0^1 \int_y^1 (\psi(\xi))d\xi dy + \int_0^1 \int_1^y [k_{11}(x, y) - dk_{11}(x, y)] \\
+ \int_0^1 \int_0^1 (\varphi(\xi) - \varphi(\xi))d\xi dy - \frac{c}{2} \int_0^1 \int_y^1 (\psi(\xi))d\xi dy + \int_0^1 \int_1^y [k_{21}(x, y) - dk_{21}(x, y)] \\
- \frac{\gamma}{\beta^2 + \gamma^2} \left[ d \int_0^1 \int_y^1 (\varphi(\xi) + \psi(\xi))d\xi dy + \frac{c}{2} \int_0^1 \int_y^1 (\psi(\xi))d\xi dy - \int_0^1 [k_{22}(x, y) - dk_{22}(x, y)] \\
- \int_0^1 \int_0^1 (\varphi(\xi) + \psi(\xi))d\xi dy - \frac{c}{2} \int_0^1 \int_y^1 (\psi(\xi))d\xi dy - \int_0^1 [k_{21}(x, y) - dk_{21}(x, y)] \\
+ \int_0^1 \int_0^1 (\varphi(\xi) + \psi(\xi))d\xi dy - \frac{c}{2} \int_0^1 \int_y^1 (\psi(\xi))d\xi dy - \int_0^1 [k_{22}(x, y) - dk_{22}(x, y)] \right].
\]$$

Then we get $\mathbb{S}^{-1}$ is also bounded. By (20) and (41), we claim that

$$A_T = \mathbb{S}^{-1}A\mathbb{S}. \quad (47)$$

Now, we look at the boundary conditions. At $x = 1$, one has

$$g'(1) = \frac{\beta}{\gamma} \left[ d \int_0^1 \int_y^1 (-f''(\xi))d\xi dy + \frac{c}{2} g(1) \\
+ \int_0^1 [k_{11}(x, y) - dk_{11}(x, y)] \\
- k_{11}(x, y)]g(y)dy + \int_0^1 [k_{12}(x, y) \\
- dk_{21}(x, y)]ydy - \int_0^1 [k_{11}(x, y) - dk_{11}(x, y)]ydy, \quad (48)$$

by which, we have

$$g'(1) = -\frac{c}{2} f''(1) + dg(1) + \int_0^1 [k_{21}(x, y) \\
- dk_{21}(x, y)]ydy - \int_0^1 [k_{11}(x, y) \\
- dk_{11}(x, y)]ydy. \quad (49)$$

In the definition (41), it has

$$f''(1) = df''(1) + \frac{c}{2} g(1) + \int_0^1 [k_{11}(x, y) \\
- dk_{11}(x, y)]f''(y)dy + \int_0^1 [k_{12}(x, y) \\
- dk_{21}(x, y)]g(y)dy.$$

This together with (49) and definitions of (44) and (20) shows that $(f, g) \in D(A\mathbb{S})$.

On the other hand, for any $(f, g) \in D(A\mathbb{S})$, that is $\frac{1}{2}(f'' - g, f'' + g) \in D(A)$, it has $(f, g)^T \in H^4(0, 1) \times H^2(0, 1)$, $f'(0) = 0$, and

$$f''(1) = df''(1) + \frac{c}{2} g(1) + \int_0^1 [k_{11}(x, y) - dk_{11}(x, y)]f''(y) + \int_0^1 [k_{12}(x, y) - dk_{21}(x, y)]g(y)dy. \quad (50)$$

Thus $(f, g) \in D(A_T)$. Therefore, $D(A_T) = D(A\mathbb{S})$.

Secondly, we prove that $A_T(f, g)^T = \mathbb{S}^{-1} A\mathbb{S}(f, g)^T$ for any $(f, g)^T \in D(A_T)$. Actually,

$$\mathbb{S}^{-1} A\mathbb{S}(f, g)^T = \frac{1}{2} \mathbb{S}^{-1} A(f'' - g, f'' + g)^T, \quad (51)$$

$$= \frac{1}{2} \mathbb{S}^{-1} (f'' + g'', - f'' + g'')^T \quad (51)$$

$$= \left(Ry - \int_0^y \int_y^1 (\psi(\xi))d\xi dy - \int_0^y \int_y^1 (\psi(\xi))d\xi dy \right).
where

\[
R = \frac{\beta}{\beta^2 + \gamma^2} \left[ d \int_0^1 \int_{y}^1 (-f''(\xi))d\xi dy - \frac{c}{2} \int_0^1 g''(\xi) d\xi dy \\
+ \int_0^1 \left[ k_{11x}(1,y) - \frac{d k_{111}(1,y)}{d\xi} \right] \int_0^y \int_{\tau}^1 (-f''(\xi))d\xi d\tau dy \\
- \int_0^1 \left[ k_{121}(1,y) - \frac{d k_{1211}(1,y)}{d\xi} \right] \int_0^y \int_{\tau}^1 g''(\xi) d\xi d\tau dy \right] \\
- \frac{\gamma}{\beta^2 + \gamma^2} \left[ \int_0^1 \int_{y}^1 g''(\xi) d\xi dy \\
+ \frac{c}{2} \int_0^1 \int_{y}^1 (-f''(\xi))d\xi dy + \int_0^1 \left[ k_{221}(1,y) \right] \int_0^y \int_{\tau}^1 g''(\xi) d\xi d\tau dy - \int_0^1 \left[ k_{211}(1,y) \right] \int_0^y \int_{\tau}^1 (-f''(\xi))d\xi d\tau dy \right] \right]. \tag{52}
\]

Notice that \(f''''(1)\) satisfies (50). By \(A_T(f,g)^T \in \mathcal{H}_T\), it follows from (41) that \(f'(0) = g(0) = 0\), and

\[
g'(1) = \frac{\beta}{\gamma} \left[ -d \int_0^1 \int_{y}^1 f''(\xi)d\xi dy + \frac{c}{2} g(1) \\
- \int_0^1 \left[ k_{11x}(1,y) - \frac{d k_{111}(1,y)}{d\xi} \right] \int_0^y \int_{\tau}^1 f''(\xi)d\xi d\tau dy \\
+ \int_0^1 \left[ k_{121}(1,y) - \frac{d k_{1211}(1,y)}{d\xi} \right] g(y) dy \right] + dg(1) \\
+ \frac{c}{2} \int_0^1 \int_{y}^1 f''(\xi)d\xi dy - \int_0^1 \left[ k_{211}(1,y) \right] \int_0^y \int_{\tau}^1 (-f''(\xi))d\xi d\tau dy \\
- \frac{d k_{211}(1,y)}{d\xi} \int_0^y \int_{\tau}^1 (-f''(\xi))d\xi d\tau dy \\
+ \int_0^1 \left[ k_{211}(1,y) - \frac{d k_{2111}(1,y)}{d\xi} \right] g(y) dy. \tag{53}
\]

Now by virtue of (50) and (53), after a tedious but straightforward computation, we can show that the right term in (52) is just

\[
R = g'(1). \tag{54}
\]

Therefore, \(\mathbb{S}^{-1} A \mathbb{S} (f, g)^T = (g, -f''''(1)^T) = A_T(f, g)^T\).

Since by Theorem 2.2, \(A\) generates a \(C_0\)-semigroup in \(\mathcal{H}_r\), the relation (47) shows that \(A_T\) generates a \(C_0\)-semigroup \(\mathbb{S}^{-1} e^{A_T t} \mathbb{S}\) on \(\mathcal{H}_T\). Hence

\[
\|e^{A_T t}\| \leq ||\mathbb{S}^{-1}\| ||e^{A t}|| ||\mathbb{S}|| \leq K e^{-ct}, \quad K = M ||\mathbb{S}^{-1}\| ||\mathbb{S}||,
\]

where \(M\) is the constant in (43). \(\Box\)

**Acknowledgements**

This work was carried out with the support of the National Natural Science Foundation of China and the National Research Foundation of South Africa.

**References**


**Appendix**

In Section 3, we need that \(1 - \int_0^1 k_{11x}(1,y) dy\) and \(1 - \int_0^1 k_{11x}(1,y) dy\) should not be identical to zero simultaneously for \(c > 0\). Now we prove this fact.

Since \(k(x,y)\) is defined by (13), \(k_{11x}(x,y)\) can be calculated as

\[
k_{11x}(x,y) = -\frac{y}{\sqrt{2m}} \sum_{m=1}^{\infty} (-1)^m e^{2m} \left(x^2 - y^2\right)^{2m-1} \tag{55}
\]

\[
(2m-1)! \left(2m\right)^{2m-1}.
\]
Thus
\[
E(c) = 1 - \int_0^1 k_{11}(1,y) y \, dy = 1 + \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m c^{2m}(1-y^2)^{2m-1} y^2}{(2m-1)! (2m)! (2m+1)!} \, dy,
\]
\[
F(c) = 1 - \int_0^1 k_{11x}(1,y) y \, dy = 1 + \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m c^{2m}(1-y^2)^{2m-2} y^2}{(2m-2)! (2m)! (2m+2)!} \, dy.
\]

From Krstic and Smyshlyaev (2009) and Smyshlyaev et al. (2009), we have
\[
G(c) = 1 + \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m c^{2m}(1-y^2)^{2m-1}}{(2m-1)! (2m)! (2m+1)!} \, dy = \cosh \sqrt{\frac{c}{2}} \cos \sqrt{\frac{c}{2}},
\]
\[
H(c) = \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m c^{2m+1}(1-y^2)^{2m}}{(2m)! (2m+1)! (2m+2)!} \, dy = \sinh \sqrt{\frac{c}{2}} \sin \sqrt{\frac{c}{2}}.
\]

Hence
\[
E(c) = G(c) - \frac{4 \pi F(c)}{C_0^2} = \frac{1}{\sqrt{2c}} \left[ \cosh \sqrt{\frac{c}{2}} \sin \sqrt{\frac{c}{2}} + \sinh \sqrt{\frac{c}{2}} \cos \sqrt{\frac{c}{2}} \right],
\]
\[
F(c) = -1 - \frac{1}{2} \int_0^1 H(\tau) d\tau - 2G(c)c + 2G(c)
= -2 \left[ \sqrt{\frac{c}{2}} \cosh \sqrt{\frac{c}{2}} \sin \sqrt{\frac{c}{2}} - \sqrt{\frac{c}{2}} \sinh \sqrt{\frac{c}{2}} \cos \sqrt{\frac{c}{2}} \right]
+ \cosh \sqrt{\frac{c}{2}} \cos \sqrt{\frac{c}{2}}.
\]

Setting \( E(c) = F(c) = 0 \), we obtain
\[
\begin{align*}
\cosh \sqrt{\frac{c}{2}} \cos \sqrt{\frac{c}{2}} - 4 \sqrt{\frac{c}{2}} \sin \sqrt{\frac{c}{2}} &= 0, \\
\cosh \sqrt{\frac{c}{2}} \sin \sqrt{\frac{c}{2}} + \sinh \sqrt{\frac{c}{2}} \cos \sqrt{\frac{c}{2}} &= 0,
\end{align*}
\]
which has no solution for \( c \geq 0 \).