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BOUNDARY CONTROLLERS AND OBSERVERS FOR THE LINEARIZED SCHRÖDINGER EQUATION*

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Abstract. We consider a problem of stabilization of the linearized Schrödinger equation using boundary actuation and measurements. We propose two different control designs. First, a simple proportional collocated boundary controller is shown to exponentially stabilize the system. However, the decay rate of the closed-loop system cannot be prescribed. The second, full-state feedback boundary control design, achieves an arbitrary decay rate. We formally view the Schrödinger equation as a heat equation in complex variables and apply the backstepping method recently developed for boundary control of reaction-advection-diffusion equations. The resulting controller is then supplied with the backstepping observer to obtain an output-feedback compensator. The designs are illustrated with simulations.

Key words. distributed parameter systems, backstepping, stabilization

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1. Introduction.

We consider a problem of stabilization of the linearized Schrödinger equation using boundary actuation and measurements. The existing results on the boundary control of the Schrödinger equation usually use the simple "passive damper" feedback [17, 6] most commonly used for stabilization of the wave equation. For exact controllability and observability results, see [10, 19, 16].

We propose two novel control designs. First, a proportional collocated boundary controller is shown to exponentially stabilize the system. The controller is simple and uses only boundary measurements of the state (rather than measurements of the time derivative of the state in case of "passive damper" controllers). However, the decay rate of the closed-loop system cannot be arbitrarily prescribed.

Our second approach relies on the backstepping method recently developed for parabolic PDEs [20, 21]. The idea of the design is to formally consider the Schrödinger equation as a complex-valued heat equation and apply the method developed in [20]. This method uses invertible Volterra integral transformation together with the boundary feedback to convert the unstable plant into a well-damped target system. The kernel of this transformation satisfies a certain PDE, which turns out to be solvable in closed form. The proposed full-state design is explicit and achieves an arbitrary decay rate of the closed-loop system. The interest in achieving arbitrary decay rates in stabilization of distributed parameter systems goes back to at least Theorem 2.1 in [22].

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We also develop dual backstepping observers that require only boundary sensing. These observers are combined with the backstepping controllers to obtain a collocated output-feedback compensator.

2. Collocated boundary controller. Consider the linearized Schrödinger equation

$$\psi_t(x,t) = -j\psi_{xx}(x,t), \qquad 0 < x < 1, \tag{1}$$

where ψ is a complex-valued state and j is the imaginary unit. We assume the following boundary conditions:

$$\psi_x(0,t) = 0, \tag{2}$$
$$\psi(1,t) = u(t)\,, \tag{3}$$

where u is the control input. With $u = 0$, this system displays oscillatory behavior and is not asymptotically stable (all the eigenvalues lie on the imaginary axis).

The idea for our first control design comes from the consideration of the L^2 energy of the system

$$E(t) = \frac{1}{2}\int_0^1 |\psi(x,t)|^2\,dx\,. \tag{4}$$

Differentiating E with respect to time, we get

$$\dot{E}(t) = ju(t)\overline{\psi_x(1,t)}. \tag{5}$$

If we take

$$u(t) = \frac{j}{c_1}\psi_x(1,t), \quad c_1 > 0, \tag{6}$$

then

$$\dot{E}(t) = -\frac{1}{c_1}|\psi_x(1,t)|^2 \le 0\,. \tag{7}$$

The above inequality certainly does not imply that the controller (6) achieves asymptotic stability; it only gives an indication that the proposed controller is potentially useful. We are going to investigate the properties of the closed-loop system

$$\psi_t(x,t) = -j\psi_{xx}(x,t), \tag{8}$$
$$\psi_x(0,t) = 0, \tag{9}$$
$$\psi_x(1,t) = -jc_1\psi(1,t) \tag{10}$$

using the Riesz spectral method.

Define the system operator A_0 in $H = L^2(0,1)$ as

$$\begin{aligned} &A_0\varphi(x) = -j\varphi''(x) \quad \forall\, \varphi \in D(A_0),\\ &D(A_0) = \{\varphi \in H^2(0,1)|\ \varphi'(0) = 0, \varphi'(1) = -jc_1\varphi(1)\}. \end{aligned} \tag{11}$$

Then the system (8)–(10) can be written as an evolutionary equation in H:

$$\frac{d}{dt}\psi(\cdot,t) = A_0\psi(\cdot,t). \tag{12}$$

LEMMA 2.1. *Let the operator A_0 be defined by* (11)*. Then the following hold.*

(i) *A_0^{-1} is compact on H and hence $\sigma(A_0)$, the spectrum of A_0, consists of isolated eigenvalues of finite algebraic multiplicity only.*

(ii) *A_0 is dissipative and generates a C_0-semigroup of contractions on H.*

Proof. The proof of a generalized version of this lemma (for multidimensional systems) can be found in [12]. For completeness, here we give a short proof for the one-dimensional case. A straightforward computation shows that

$$A_0^{-1}f(x) = -\frac{c_1 x}{1+c_1 j}\int_0^1 sf(s)ds + j\int_1^x (x-s)f(s)ds - j\int_0^1 sf(s)ds \quad \forall\, f \in H. \tag{13}$$

Statement (i) follows from the Sobolev embedding theorem. Another direct computation shows that

$$\mathrm{Re}\langle A_0\varphi, \varphi\rangle = -c_1|\varphi(1)|^2 \le 0; \tag{14}$$

therefore (ii) follows from the Lumer–Phillips theorem [18, Theorem 4.3, p. 14]. ∎

Let us compute the eigenvalues of operator A_0:

$$A_0\varphi = \lambda\varphi = \rho^2\varphi\,, \tag{15}$$

i.e.,

$$\varphi''(x) = j\rho^2\varphi(x), \tag{16}$$

$$\varphi'(0) = 0, \tag{17}$$

$$\varphi'(1) = -jc_1\varphi(1)\,. \tag{18}$$

The solution to (16)–(18) is

$$\varphi(x) = \cosh\sqrt{j}\rho x\,, \tag{19}$$

where ρ satisfies

$$e^{2\sqrt{j}\rho} = 1 - \frac{2jc_1}{\sqrt{j}\rho + jc_1}. \tag{20}$$

Suppose that

$$2\sqrt{j}\rho = 2n\pi j + \mathcal{O}(|n|^{-1})\,, \tag{21}$$

where n is a sufficiently large integer. Substituting (21) into (20), we obtain

$$\mathcal{O}(n^{-1}) = -\frac{2c_1}{n\pi + c_1} + \mathcal{O}(n^{-2}). \tag{22}$$

Hence,

$$\sqrt{j}\rho = n\pi j - \frac{k}{n\pi + c_1} + \mathcal{O}(n^{-2}), \tag{23}$$

and therefore

$$\lambda = \lambda_n = -2c_1 + (n\pi)^2 j + \mathcal{O}(n^{-1}). \tag{24}$$

The corresponding eigenfunctions are

$$\varphi(x) = \varphi_n(x) = \cos n\pi x + \mathcal{O}(n^{-1}). \tag{25}$$

Since $\{\cos n\pi x,\ n \geq 0\}$ forms an (orthogonal) Riesz basis for H, we immediately obtain the following result by Theorem 6.3 of [4] (see also [3]).

THEOREM 2.2. *Let the operator A_0 be defined by* (11). *Then, there exists a sequence of generalized eigenfunctions of A_0, which forms a Riesz basis for H. Moreover, the following statements hold:*

(i) *All eigenvalues with sufficiently large modules are algebraically simple.*

(ii) *Eigenvalues have the asymptotic expansion* (24)*, where n's are positive integers.*

(iii) *The spectrum-determined growth condition holds true for the semigroup $e^{A_0 t}$: $S(A_0) = \omega(A_0)$, where $S(A_0)$ is the spectral bound of A_0 and $\omega(A_0)$ is the growth order of $e^{A_0 t}$.*

(iv) *The semigroup $e^{A_0 t}$ is exponentially stable:*

$$\|e^{A_0 t}\| \leq M e^{-\omega t} \quad \forall\, t \geq 0,$$

where M, ω are positive constants. Therefore, the energy of the system (12) *decays exponentially in the following sense:*

$$E(t) \leq M e^{-\omega t} E(0).$$

Proof. The statements (i)–(iii) follow from Theorem 6.3 of [4] and (25). To show (iv), it suffices to show that there are no eigenvalues of A on the imaginary axis. Obviously, zero is not the eigenvalue. Let $\lambda = ja$, where a is real and positive; then $\rho = \sqrt{ja}$. Substituting this into (20), we get

$$e^{2j\sqrt{a}} = 1 - \frac{2c_1}{\sqrt{a} + c_1}.$$

The right-hand side of this equation is real, while the left-hand side is on the unit circle; therefore, it can only be equal to 1 or −1. We get

$$0 = -\frac{2c_1}{\sqrt{a} + c_1} \quad \text{or} \quad -2 = -\frac{2c_1}{\sqrt{a} + c_1}\,. \tag{26}$$

It is clear that neither equation is satisfied when $c_1 > 0$, and therefore there are no eigenvalues on the imaginary axis. ∎

Remark 2.3. We should point out that the result (iv) of Theorem 2.2 has previously been proved in a much more general setting; see [12, 13]. However, Theorem 2.2 shows that for this one-dimensional case the Riesz basis approach provides more profound results than just exponential stability. These results are summarized in statements (i)–(iii).

The control law (6) is simple and requires only boundary observation. However, we should stress that the asymptotic expansion (24), while clearly indicating that high eigenvalues have a negative real part proportional to the feedback gain c_1, does not give us a clue on how the low-number eigenvalues behave when c_1 increases. Only the numerical analysis can provide us with the clear picture.

In Figure 1 the closed-loop eigenvalues are shown for different values of c_1. One can see that by increasing c_1 it is possible to move almost all the eigenvalues arbitrarily

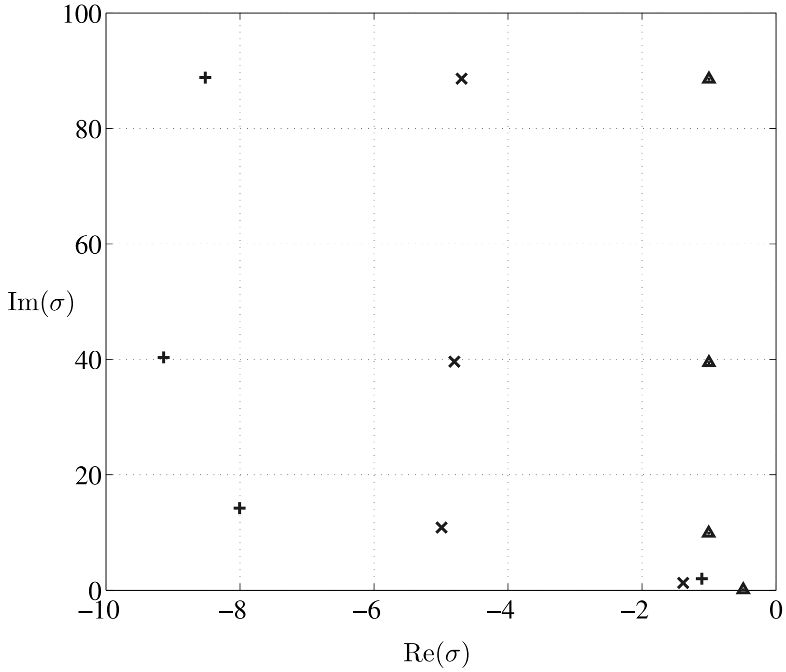

FIG. 1. *The closed-loop eigenvalue map of the Schrödinger equation with "damping" boundary feedback for different values of* c_1*:* $c_1 = 0.5$ ($\triangle$), $c_1 = 2.3$ ($\times$), $c_1 = 4$ ($+$).

to the left, except for the eigenvalue with the lowest imaginary part. One can only move the real part of that eigenvalue to -1.4 (for $c_1 \approx 2.3$), and further increasing c_1 just moves this eigenvalue back to the imaginary axis. Since the mode corresponding to that eigenvalue carries the most energy, it is desirable to have the controller that can move it arbitrarily to the left.

3. Backstepping design. Consider again the plant

$$\psi_t(x,t) = -j\psi_{xx}(x,t), \tag{27}$$

$$\psi_x(0,t) = 0, \tag{28}$$

$$\psi(1,t) = u(t). \tag{29}$$

We propose considering (27)–(29) formally as a heat equation with the imaginary diffusion coefficient and solving the stabilization problem using the backstepping control design for parabolic PDEs [20]. Consider the transformation

$$v(x,t) = \psi(x,t) - \int_0^x k(x,y)\psi(y,t)\,dy\,, \tag{30}$$

where $k(x,y)$ is a complex-valued function that satisfies the PDE

$$k_{xx}(x,y) - k_{yy}(x,y) = cjk(x,y), \tag{31}$$

$$k_y(x,0) = 0, \tag{32}$$

$$k(x,x) = -\frac{cj}{2}x \tag{33}$$

with $c > 0$. It is straightforward to show that this transformation maps (1), (2) into the following target system:

$$v_t(x,t) = -jv_{xx}(x,t) - cv(x,t), \tag{34}$$

$$v_x(0,t) = 0, \tag{35}$$

$$v(1,t) = 0\,. \tag{36}$$

The eigenvalues of this system are

$$\sigma = -c + j\frac{\pi^2(2n+1)^2}{4}, \quad n = 0,1,2,\ldots; \tag{37}$$

therefore, the design parameter c allows us to move them arbitrarily to the left in the complex plane.

The control law is obtained by setting $x = 1$ in (30):

$$\psi(1,t) = \int_0^1 k(1,y)\psi(y,t)\,dy\,, \tag{38}$$

where $k(1,y)$ is obtained from the solution to the PDE (31)–(33), which is [20]

$$\begin{aligned} k(x,y) &= -cjx\frac{I_1\left(\sqrt{cj(x^2-y^2)}\right)}{\sqrt{cj(x^2-y^2)}} = -cjx - cjx\sum_{m=1}^{\infty}\frac{(cj(x^2-y^2))^m}{4^m m!(m+1)!} \\ &= x\sqrt{\frac{c}{2\,(x^2-y^2)}}\left[(j-1)\mathrm{ber}_1\left(\sqrt{c\,(x^2-y^2)}\right) - (1+j)\mathrm{bei}_1\left(\sqrt{c\,(x^2-y^2)}\right)\right]. \end{aligned} \tag{39}$$

Here $I_1(\cdot)$ is the modified Bessel function and $\mathrm{ber}_1(\cdot)$ and $\mathrm{bei}_1(\cdot)$ are the Kelvin functions, which are defined in terms of I_1 as follows [23]:

$$\mathrm{ber}_1(x) = -\mathrm{Im}\left\{I_1\left(\frac{1+j}{\sqrt{2}}x\right)\right\}, \qquad \mathrm{bei}_1(x) = \mathrm{Re}\left\{I_1\left(\frac{1+j}{\sqrt{2}}x\right)\right\}. \tag{40}$$

From the second equality in (39) we see that the kernel k is C^∞ in both variables in the triangle region defined by the inequalities $0 \le y \le x \le 1$ since the series

$$\sum_{m=1}^{\infty}\frac{(cj(x^2-y^2))^m}{4^m m!(m+1)!}$$

and the corresponding series obtained by term-by-term differentiation with respect to x and y is absolutely convergent. Therefore, $k(1,y)$ makes sense in the feedback law (38).

The precise statement of stability of the closed-loop system (1), (2), (38), (39) is given by the following theorem.

THEOREM 3.1. *Consider the system*

$$\frac{d}{dt}\psi(\cdot,t) = A\psi(\cdot,t) \tag{41}$$

in $H = L_2(0,1)$, *where the operator* A *is defined as*

$$\begin{aligned} &A\varphi(x) = -j\varphi''(x) \quad \forall\, \varphi \in D(A), \\ &D(A) = \left\{\varphi \in H^2(0,1) |\ \varphi'(0) = 0, \varphi(1) = \int_0^1 k(1,y)\varphi(y)dy\right\} \end{aligned} \tag{42}$$

and k is given by (39). *Then, there exists a sequence of eigenfunctions of A, which forms a Riesz basis for H. Moreover, the following statements hold:*

(i) *The eigenvalues of A are given by* (37).

(ii) *The spectrum-determined growth condition holds true for the semigroup e^{At}.*

(iii) *The semigroup e^{At} is exponentially stable in the sense*

$$\|e^{At}\| \leq Me^{-ct} \quad \forall\, t \geq 0,$$

where $M > 0$ and c is an arbitrary positive design parameter.

Proof. The main idea of the proof is to first establish well-posedness and stability of the target system (34)–(36) and then to use the fact that the transformation (30) is invertible to get well-posedness and stability of the closed-loop system. One can write (34)–(36) as

$$\frac{d}{dt}v(\cdot,t) = Bv(\cdot,t) \tag{43}$$

on H, where the operator B is defined by

$$\begin{aligned} &B\varphi(x) = -j\varphi''(x) - c\varphi(x) \quad \forall\, \varphi \in D(B),\\ &D(B) = \{\varphi \in H^2(0,1) |\ \varphi'(0) = \varphi(1) = 0\}. \end{aligned} \tag{44}$$

A simple computation shows that eigenvalues of B are

$$\mu_n = -c + j\pi^2 \left(n - \frac{1}{2}\right)^2, \quad n = 1, 2, \ldots, \tag{45}$$

and the corresponding eigenfunctions are

$$f_n(x) = \cos\left(n - \frac{1}{2}\right)\pi x, \quad n = 1, 2, \ldots, \tag{46}$$

which forms an (orthogonal) Riesz basis for H. This shows that the spectrum-determined growth condition holds for (43). Therefore, there exists a constant $M_1 > 0$ such that

$$\|e^{Bt}\| \leq M_1 e^{-ct} \quad \forall\, t \geq 0. \tag{47}$$

One can show by direct substitution that the transformation

$$\psi(x,t) = v(x,t) + \int_0^x l(x,y)v(y,t)\,dy = (I - \mathbb{P})^{-1}v\,, \tag{48}$$

with [20, 23]

$$\begin{aligned} l(x,y) &= -cjx\frac{J_1\left(\sqrt{cj(x^2-y^2)}\right)}{\sqrt{cj(x^2-y^2)}}\\ &= -cjx - cjx\sum_{m=1}^{\infty}\frac{(-1)^m(cj(x^2-y^2))^m}{4^m m!(m+1)!}, \quad 0 \leq y \leq x \leq 1\,, \end{aligned} \tag{49}$$

where J_1 is the Bessel function, is inverse to the transformation (30). Using the same argument as for the kernel k, we conclude from the second equality in (49) that the kernel l is C^∞ in both variables in the triangle region defined by inequalities $0 \le y \le x \le 1$. Therefore, the transformations defined by (30) and (48) are bounded on H. Let us rewrite these transformations in the form

$$v(x) = \psi(x) - \int_0^x k(x,y)\psi(y)\,dy = [(I-\mathbb{P})\psi](x),$$
$$\psi(x) = v(x) + \int_0^x l(x,y)v(y)\,dy = [(I-\mathbb{P})^{-1}v](x),$$

where both $I-\mathbb{P}$ and $(I-\mathbb{P})^{-1}$ are bounded on H. A simple calculation shows that $A(I-\mathbb{P}) = (I-\mathbb{P})B$, that is,

$$A = (I-\mathbb{P})B(I-\mathbb{P})^{-1},$$

where A is defined by (42). Hence, (μ,ϕ) is an eigenpair of B if and only if $(\mu,(I-\mathbb{P})\phi)$ is an eigenpair of A. This fact, together with the spectral properties of B (see (44)–(46)) and the relation $e^{At} = (I-\mathbb{P})e^{Bt}(I-\mathbb{P})^{-1}$, gives the properties of A stated in the theorem. ▯

Remark 3.2. In the proof of Theorem 3.1, the well-known property (47) of the operator B is rederived using the Riesz basis approach so that the Riesz basis property can be established for the operator A. This allows us to strengthen the arbitrary decay property (iii) with properties (i) and (ii). This remark is also relevant for Theorems 3 and 4.

Remark 3.3. It is possible to combine the feedback law (38) with the design presented in section 2. In order to do this, one modifies the target system (34)–(36) by replacing the boundary condition (36) with

$$v_x(1,t) = -jc_1 v(1,t)\,. \tag{50}$$

Then the combined controller for the Schrödinger equation becomes

$$\psi_x(1,t) = -\,j\left(\frac{c}{2}+c_1\right)\psi(1,t) + \int_0^1 (k_x(1,y) + jc_1 k(1,y))\psi(y,t)\,dy\,. \tag{51}$$

4. Observer design. The controller (38) relies on full-state measurements. Let us assume now that only boundary measurements are available (so that $\psi_x(1)$ is measured and $\psi(1)$ is actuated). We are going to design the observer which closely follows the observer design presented in [21] for the heat equation. We denote the estimate of the state by $\hat{\psi}$ and use the observer

$$\hat{\psi}_t(x,t) = -j\hat{\psi}_{xx}(x,t) + p_1(x)[\psi_x(1,t) - \hat{\psi}_x(1,t)], \tag{52}$$
$$\hat{\psi}_x(0,t) = 0, \tag{53}$$
$$\hat{\psi}(1,t) = \psi(1,t), \tag{54}$$

which is in a familiar form of the copy of the plant (27)–(29) plus output injection with the observer gain $p_1(x)$ to be designed.

The observer error $\tilde{\psi} = \psi - \hat{\psi}$ satisfies the PDE

$$\tilde{\psi}_t(x,t) = -j\tilde{\psi}_{xx}(x,t) - p_1(x)\tilde{\psi}_x(1,t), \tag{55}$$
$$\tilde{\psi}_x(0,t) = 0, \tag{56}$$
$$\tilde{\psi}(1,t) = 0\,. \tag{57}$$

Consider the transformation

$$\tilde{\psi}(x,t) = \tilde{w}(x,t) + \int_x^1 p(x,y)\tilde{w}(y,t)\,dy\,. \tag{58}$$

Note that the integral runs from x to 1 here, which is the consequence of the collocated input-output architecture. Let us select the target system for the observer error as

$$\tilde{w}_t(x,t) = -j\tilde{w}_{xx}(x,t) - \tilde{c}\tilde{w}(x,t), \tag{59}$$

$$\tilde{w}_x(0,t) = 0, \tag{60}$$

$$\tilde{w}(1,t) = 0\,. \tag{61}$$

The design parameter $\tilde{c}$ allows to set the desired observer convergence rate (which usually needs to be faster than the closed-loop system decay rate $-c$). Substituting the transformation (58) into (55)–(57) and matching the terms, we get the following conditions on $p(x,y)$ which form a PDE:

$$p_{xx}(x,y) - p_{yy}(x,y) = -j\tilde{c}p(x,y), \tag{62}$$

$$p_x(0,y) = 0, \tag{63}$$

$$p(x,x) = j\frac{\tilde{c}}{2}x\,. \tag{64}$$

The observer gain $p_1(x)$ is computed from $p(x,y)$ as follows:

$$p_1(x) = jp(x,1)\,. \tag{65}$$

The PDE (62)–(64) has the following solution [21]:

$$\begin{aligned} p(x,y) &= j\tilde{c}y\frac{I_1\left(\sqrt{\tilde{c}j(y^2-x^2)}\right)}{\sqrt{\tilde{c}j(y^2-x^2)}} = j\tilde{c}y + j\tilde{c}y\sum_{m=1}^{\infty}\frac{(j\tilde{c}(y^2-x^2))^m}{4^m m!(m+1)!} \\ &= \tilde{c}y\sqrt{\frac{\tilde{c}}{2\,(y^2-x^2)}}\Big[(1-j)\mathrm{ber}_1\Big(\sqrt{\tilde{c}\,(y^2-x^2)}\Big) \\ &\qquad + (1+j)\mathrm{bei}_1\Big(\sqrt{\tilde{c}\,(y^2-x^2)}\Big)\Big]\,. \end{aligned} \tag{66}$$

From the second equality in (66) it follows that p is C^∞ in both variables in the triangle region defined by inequalities $0 \le x \le y \le 1$. Using (65) we get the observer gain

$$\begin{aligned} p_1(x) = \sqrt{\frac{\tilde{c}}{2\,(1-x^2)}}\Big[&(1+j)\mathrm{ber}_1\Big(\sqrt{\tilde{c}\,(1-x^2)}\Big) \\ &+ (j-1)\mathrm{bei}_1\Big(\sqrt{\tilde{c}\,(1-x^2)}\Big)\Big]\,, \end{aligned} \tag{67}$$

which is C^∞ in x. Note that when $\tilde{c} = c$, we have $p_1(x) = jk(1,x)$, which is the well-known duality property between observer and control gains.

The result of this section is summarized in the following theorem.

THEOREM 4.1. *Consider the system*

$$\frac{d}{dt}\tilde{\psi}(\cdot,t) = \tilde{A}\tilde{\psi}(\cdot,t) \tag{68}$$

in $H = L^2(0,1)$*, where the operator* $\tilde{A}$ *is defined as*

$$\begin{aligned} &\tilde{A}\varphi(x) = -j\varphi''(x) - p_1(x)\varphi'(1) \quad \forall\, \varphi \in D(\tilde{A}), \\ &D(\tilde{A}) = \left\{\varphi \in H^2(0,1) |\ \varphi'(0) = 0,\ \varphi(1) = 0\right\} \end{aligned} \tag{69}$$

and $p_1(x)$ *is given by* (67)*. Then, there exists a sequence of eigenfunctions of* $\tilde{A}$*, which forms a Riesz basis for* H*. Moreover, the following statements hold.*

(i) *The eigenvalues of* $\tilde{A}$ *are*

$$\sigma = -\tilde{c} + j\pi^2 \left(n - \frac{1}{2}\right)^2, \quad n = 1, 2, \ldots. \tag{70}$$

(ii) *The spectrum-determined growth condition holds true for the semigroup* $e^{\tilde{A}t}$*.*

(iii) *The semigroup* $e^{\tilde{A}t}$ *is exponentially stable in the sense*

$$\|e^{\tilde{A}t}\| \leq M_2 e^{-\tilde{c}t} \quad \forall\, t \geq 0,$$

where $M_2 > 0$ *and* $\tilde{c}$ *is an arbitrary positive design parameter.*

Proof. The proof closely follows the proof of Theorem 3.1. We define the operator $\tilde{B}$ as

$$\begin{aligned} &\tilde{B}\varphi(x) = -j\varphi''(x) - \tilde{c}\varphi(x) \quad \forall\, \varphi \in D(\tilde{B}), \\ &D(\tilde{B}) = \{\varphi \in H^2(0,1) |\ \varphi'(0) = \varphi(1) = 0\} \end{aligned} \tag{71}$$

and write the system (59)–(61) on H as

$$\frac{d}{dt}\tilde{w}(\cdot, t) = \tilde{B}\tilde{w}(\cdot, t)\,. \tag{72}$$

Since the operator $\tilde{B}$ is the same as B (with c replaced by $\tilde{c}$), using the same arguments as in the proof of Theorem 3.1, we get that there exists a sequence of eigenfunctions of $\tilde{B}$, which forms a Riesz basis for H. Writing the transformation (58) in the form

$$\tilde{\psi}(x) = \tilde{w}(x) + \int_x^1 p(x,y)\tilde{w}(y)\,dy = [(I - \tilde{\mathbb{P}})^{-1}\tilde{w}](x)\,,$$

and using the fact that both $I - \tilde{\mathbb{P}}$ and $(I - \tilde{\mathbb{P}})^{-1}$ are bounded operators on H due to the C^∞ property of $p(x,y)$, we obtain

$$\tilde{A} = (I - \tilde{\mathbb{P}})^{-1}\tilde{B}(I - \tilde{\mathbb{P}})\,.$$

The results (i)–(iii) follow immediately from the spectral properties of $\tilde{B}$ and the relations

$$e^{\tilde{A}t} = (I - \tilde{\mathbb{P}})^{-1}e^{\tilde{B}t}(I - \tilde{\mathbb{P}}), \tag{73}$$

$$\|\tilde{w}(\cdot, t)\| = \|e^{\tilde{B}t}\tilde{w}(\cdot, 0)\| \leq \tilde{M}e^{-\tilde{c}t}\|\tilde{w}(\cdot, 0)\|. \quad \square \tag{74}$$

5. Output-feedback compensator. In this section we combine the observer (52)–(54) with the state feedback controller developed in section 3:

$$\psi(1,t) = \int_0^1 k(1,y)\hat{\psi}(y,t)\,dy\,. \tag{75}$$

The result is an output-feedback compensator which is the alternative to (6).

Under the feedback (75), the plant (27)–(29) becomes

$$\psi_t(x,t) = -j\psi_{xx}(x,t), \tag{76}$$
$$\psi_x(0,t) = 0, \tag{77}$$
$$\psi(1,t) = \int_0^1 k(1,y)\psi(y,t) - \int_0^1 k(1,y)\tilde{\psi}(y,t)\,dy\,, \tag{78}$$

where $\tilde{\psi}$ is the solution of (55)–(57). Applying the transformation (30), we obtain

$$v_t(x,t) = -jv_{xx}(x,t) - cv(x,t), \tag{79}$$
$$v_x(0,t) = 0, \tag{80}$$
$$v(1,t) = -\int_0^1 k(1,y)\tilde{\psi}(y,t)dy\,. \tag{81}$$

Now $\tilde{\psi}$ in (78) can be expressed through $\tilde{w}$ using the transformation (58):

$$v_t(x,t) = -jv_{xx}(x,t) - cv(x,t), \tag{82}$$
$$v_x(0,t) = 0, \tag{83}$$
$$v(1,t) = -\int_0^1 \left[k(1,y) + \int_0^y k(1,s)p(s,y)ds\right]\tilde{w}(y,t)dy\,. \tag{84}$$

The above PDE together with (59)–(61) gives the following system $(v, \tilde{w})$ in $H \times H$, equivalent to the closed-loop system $(\psi, \hat{\psi})$:

$$v_t(x,t) = -jv_{xx} - cv(x,t), \tag{85}$$
$$v_x(0,t) = 0, \tag{86}$$
$$v(1,t) = \int_0^1 L(x)\tilde{w}(x,t)\,dx \triangleq f(t), \tag{87}$$
$$\tilde{w}_t(x,t) = -j\tilde{w}_{xx} - \tilde{c}\tilde{w}(x,t), \tag{88}$$
$$\tilde{w}_x(0,t) = \tilde{w}(1,t) = 0\,, \tag{89}$$

where $L(x) = -k(1,x) - \int_0^x k(1,s)p(s,x)ds \in C^\infty(0,1)$.

In order to discuss the well-posedness of system (85)–(89) in a suitable state space, we introduce an operator

$$\begin{aligned} &\Delta f(x) = -f''(x) \quad \forall\, f \in D(\Delta), \\ &D(\Delta) = \{f \in H^2(0,1) |\ f'(0) = 0, f(1) = 0\}. \end{aligned} \tag{90}$$

The operator Δ is self-adjoint and positive definite in H. We can easily find the analytic expression of $\Delta^{1/2}$:

$$\begin{aligned} &\Delta^{1/2} f(x) = f'(1-x),\ D(\Delta^{1/2}) = \{f \in H^1(0,1) |\ f(1) = 0\}, \\ &\Delta^{-1/2} f(x) = -\int_0^{1-x} f(\tau)d\tau,\ D(\Delta^{-1/2}) = \{f |\ \Delta^{-1/2} f \in H\}. \end{aligned} \tag{91}$$

Since Δ has eigenpairs $\{(\lambda_n = \left(n-\frac{1}{2}\right)^2\pi^2, \sqrt{2}\cos\left(n-\frac{1}{2}\right)\pi x)\}$ and $\{\sqrt{2}\cos\left(n-\frac{1}{2}\right)\pi x)\}$ form an orthonormal basis for H, we can also use Fourier series to characterize $D(\Delta^{-1/2})$:

$$\Delta^{-1/2}f(x) = \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{\lambda_n}}\cos\left(n-\frac{1}{2}\right)\pi x,\ \lambda_n = \left(n-\frac{1}{2}\right)^2\pi^2 \quad \forall\, f\in D(\Delta^{-1/2}),$$

$$D(\Delta^{-1/2}) = \left\{f(x) = \sum_{n=1}^{\infty}a_n\cos\left(n-\frac{1}{2}\right)\pi x\,\middle|\, \sum_{n=1}^{\infty}\frac{|a_n|^2}{\lambda_n}<\infty\right\}, \tag{92}$$

$$\|\Delta^{-1/2}f\|_H^2 = \sum_{n=1}^{\infty}\frac{|a_n|^2}{2\lambda_n} \quad \forall\, f\in D(\Delta^{-1/2}).$$

Let $H_{-1} = \Delta^{-1/2}H = [D(\Delta^{-1/2})]$, where $[D(\Delta^{-1/2})]$ is the graph space of $\Delta^{-1/2}$ in H with norm $\|f\|_{H_{-1}} = \|\Delta^{-1/2}f\|_H$. We consider the system (85)–(89) in the state space $X = H_{-1}\times H$.

Define the system operator $\mathcal{A} : D(\mathcal{A})(\subset X)\to X$ for (85)–(89) as follows:

(93)

$$\mathcal{A}(f,g) = (-jf''-cf, -jg''-\tilde{c}g) \quad \forall\,(f,g)\in D(\mathcal{A}),$$

$$D(\mathcal{A}) = \left\{(f,g)\in(H^2(0,1))^2 \,|\, f'(0)=0, f(1) = \int_0^1 L(x)g(x)dx, g'(0)=g(1)=0\right\}.$$

Then the system (85)–(89) can be written as an evolutionary equation in X:

$$\frac{d}{dt}(v(\cdot,t),\tilde{w}(\cdot,t)) = \mathcal{A}(v(\cdot,t),\tilde{w}(\cdot,t)). \tag{94}$$

THEOREM 5.1. *Let $\mathcal{A}$ be defined by (93) and suppose that $c\neq\tilde{c}$. Then, there exists a sequence of eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X. Moreover, the eigenvalues of $\mathcal{A}$ are*

$$\lambda_{n1} = -c + j\left(n-\frac{1}{2}\right)^2\pi^2, \quad \lambda_{n2} = -\tilde{c}+j\left(n-\frac{1}{2}\right)^2\pi^2,\ n=1,2,\ldots, \tag{95}$$

and the following statements hold:

(i) *The spectral growth condition holds for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ in X.*

(ii) *The solution of (94) satisfies*

$$\|(v(\cdot,t),\tilde{w}(\cdot,t))\|_X \le Ke^{-\min\{c,\tilde{c}\}t}\|(v(\cdot,0),\tilde{w}(\cdot,0))\|_X \tag{96}$$

for some constant $K>0$.

Proof. Consider the eigenvalue problem of $\mathcal{A}$, $\mathcal{A}(f,g) = \lambda(f,g)$, that is,

$$\begin{cases} f''(x) - jcf(x) = j\lambda f(x), \\ f'(0) = 0, f(1) = \displaystyle\int_0^1 L(x)g(x)dx, \\ g''(x) - j\tilde{c}g(x) = j\lambda g(x), \\ g'(0) = g(1) = 0. \end{cases} \tag{97}$$

Since $c \neq \tilde{c}$, we have $\cosh\sqrt{j(c+\lambda_{n2})} \neq 0$ for any $n \geq 1$. Using (44)–(46), we get two families of eigenpairs $\{(\lambda_{n1}, F_{n1}), (\lambda_{n2}, F_{n2})\}_{n=1}^{\infty}$ of $\mathcal{A}$ as follows:

(98)
$$\begin{cases} \lambda_{n1} = -c + j\left(n-\frac{1}{2}\right)^2\pi^2, \ F_{n1}(x) = \left(\sqrt{\lambda_n}\cos\left(n-\frac{1}{2}\right)\pi x, 0\right), \\ \lambda_{n2} = -\tilde{c} + j\left(n-\frac{1}{2}\right)^2\pi^2, \ F_{n2}(x) = \left(\frac{\cosh\sqrt{j(c+\lambda_{n2})}x}{\cosh\sqrt{j(c+\lambda_{n2})}}b_n, \cos\left(n-\frac{1}{2}\right)\pi x\right), \\ b_n = \int_0^1 L(x)\cos\left(n-\frac{1}{2}\right)\pi x dx, \ n = 1, 2, \ldots. \end{cases}$$

Note that since $\{b_n\}$ are the Fourier coefficients of $L(x)$ under the orthonormal basis $\{\sqrt{2}\cos(n-1/2)\pi x\}$ in H, we have

$$\sum_{n=1}^{\infty} |b_n|^2 < \infty. \tag{99}$$

For any $n \geq 1$ we have

$$\begin{aligned} \sqrt{j(c+\lambda_{n2})} &= j\left(n-\frac{1}{2}\right)\pi\left[1 - \frac{j(c-\tilde{c})}{2(n-\frac{1}{2})^2\pi^2} + \mathcal{O}(n^{-4})\right] \\ &= j\left(n-\frac{1}{2}\right)\pi + \frac{c-\tilde{c}}{2\left(n-\frac{1}{2}\right)\pi} + \mathcal{O}(n^{-3}), \end{aligned} \tag{100}$$

$$\cosh\sqrt{j(c+\lambda_{n2})} = j\sin\left(n-\frac{1}{2}\right)\pi\frac{c-\tilde{c}}{2\left(n-\frac{1}{2}\right)\pi} + \mathcal{O}(n^{-2}). \tag{101}$$

From (91) it follows that

$$\Delta^{-1/2}\frac{\cosh\sqrt{j(c+\lambda_{n2})}x}{\cosh\sqrt{j(c+\lambda_{n2})}} = -\frac{1}{\sqrt{j}\sqrt{j(c+\lambda_{n2})}}\frac{\sinh\sqrt{j(c+\lambda_{n2})}(1-x)}{\cosh\sqrt{j(c+\lambda_{n2})}}. \tag{102}$$

Using (100)–(102), we get

(103)
$$\left\|\frac{\cosh\sqrt{j(c+\lambda_{n2})}x}{\cosh\sqrt{j(c+\lambda_{n2})}}b_n\right\|_{H_{-1}} = \left\|\Delta^{-1/2}\frac{\cosh\sqrt{j(c+\lambda_{n2})}x}{\cosh\sqrt{j(c+\lambda_{n2})}}b_n\right\|_H \leq C_0|b_n|, \ n = 1, 2, \ldots,$$

uniformly for $x \in [0, 1]$, where $C_0 > 0$ is a constant independent of n. Define

$$G_{n1}(x) = F_{n1}(x), \ G_{n2}(x) = \left(0, \cos\left(n-\frac{1}{2}\right)\pi x\right), \ n = 1, 2, \ldots. \tag{104}$$

Then $\{G_{n1}, G_{n2}\}_{n=1}^{\infty}$ forms an (orthogonal) Riesz basis for X, and it follows from (98)–(103) that

$$\sum_{n=1}^{\infty}\left[\|F_{n1} - G_{n1}\|_X^2 + \|F_{n2} - G_{n2}\|_X^2\right] \leq C_0^2\sum_{n=1}^{\infty}|b_n|^2 < \infty.$$

By Bari's theorem [2, Theorem 2.3, p. 317], we get that $\{F_{n1}, F_{n2}\}_{n=1}^{\infty}$ forms a Riesz basis for X. This, together with the expression of eigenvalues in (98), gives the required results. □

For $c = \tilde{c}$, it is not easy to find out whether the root subspace of $\mathcal{A}$ is complete in X. For this case, the Riesz basis approach cannot be applied, and we address it differently.

THEOREM 5.2. *Let* $\mathcal{A}$ *be defined by* (93). *Then* $\mathcal{A}$ *generates a* C_0*-semigroup on* X. *Moreover, for any* $0 < \varepsilon < \min\{c, \tilde{c}\}$, *there exists a constant* $K_\varepsilon > 0$, *such that*

$$\|(v(\cdot,t), \tilde{w}(\cdot,t))\|_X \le K_\varepsilon e^{(-\min\{c,\tilde{c}\}+\varepsilon)t} \|(v(\cdot,0), \tilde{w}(\cdot,0))\|_X. \tag{105}$$

Proof. Using the definitions of operators B and $\tilde{B}$ (44), (71), let us write the system (85)–(89) as follows (see, e.g., [6, 11]):

$$\begin{aligned} &v_t(\cdot,t) = Bv(\cdot,t) + bf(t),\ f(t) = \int_0^1 L(x)\tilde{w}(x,t)dx, \\ &\tilde{w}_t(\cdot,t) = \tilde{B}\tilde{w}(\cdot,t), \\ &\langle b^*, u\rangle = i\int_0^1 u(x)\,dx \quad \forall\ u \in H. \end{aligned} \tag{106}$$

It can be easily shown that $b^* B^{*-1}$ is a bounded operator from H_{-1} to $\mathbb{C}$, the complex number field. Consider the adjoint system

$$\begin{aligned} &z_t(\cdot,t) = B^* v(\cdot,t), \\ &y(t) = \langle b^*, z(\cdot,t)\rangle = i\int_0^1 z(x,t)dx, \end{aligned}$$

that is,

$$\begin{aligned} &z_t(x,t) = jz_{xx} - cz(x,t), \\ &z_x(0,t) = z(1,t) = 0, \\ &y(t) = i\int_0^1 z(x,t)dx. \end{aligned} \tag{107}$$

We can write the solution z explicitly using the orthonormal basis $\{\frac{\sqrt{2}}{\sqrt{\lambda_n}}\cos(n - 1/2)\pi x\}$ in H_{-1}:

$$\begin{aligned} &z(x,t) = \sum_{n=1}^{\infty} e^{\nu_n t} d_n \cos\left(n - \frac{1}{2}\right)\pi x, \quad \mu_n = -c - j\left(n - \frac{1}{2}\right)^2 \pi^2, \\ &z(x,0) = \sum_{n=1}^{\infty} d_n \cos\left(n - \frac{1}{2}\right)\pi x\,. \end{aligned}$$

Therefore,

$$y(t) = i\sum_{n=1}^{\infty} \frac{\sin\left(n - \frac{1}{2}\right)\pi}{\left(n - \frac{1}{2}\right)\pi} a_n e^{\mu_n t}.$$

By Ingham's theorem [7, Theorem 4.3, p. 59] and (92), for any $T > 0$, there exists a constant $C_T > 0$ such that

$$\int_0^1 |y(t)|^2 dt \le C_T \sum_{n=1}^{\infty} \frac{|d_n|^2}{2\left(n - \frac{1}{2}\right)^2 \pi^2} = C_T \|z(\cdot,0)\|_{H_{-1}}^2.$$

This shows that b is admissible for the semigroup e^{Bt} generated by B on H_{-1} (see [25]). Therefore, for any $(v(\cdot,0), \tilde{w}(\cdot,0)) \in X$, there exists a unique solution $(v, \tilde{w}) \in C(0,\infty; X)$ to system (106) such that

$$v(\cdot,t) = e^{Bt}v(\cdot,0) + \int_0^t e^{B(t-s)}bf(s)ds, \quad \tilde{w}(\cdot,t) = e^{\tilde{B}t}\tilde{w}(\cdot,0),$$

where e^{Bt} is now understood to be the C_0-semigroup generated by B on H_{-1}. Therefore, $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on X. For any $0 < \varepsilon < \min\{c, \tilde{c}\}$,

(108)
$$e^{\varepsilon t}v(\cdot,t) = e^{(B+\varepsilon)t}v(\cdot,0) + \int_0^t e^{(B+\varepsilon)(t-s)}be^{\varepsilon s}f(s)ds, \quad e^{\varepsilon t}\tilde{w}(\cdot,t) = e^{(\tilde{B}+\varepsilon)t}\tilde{w}(\cdot,0).$$

Since $-c+\varepsilon < 0$, $-\tilde{c}+\varepsilon < 0$, and $e^{\varepsilon s}f(s) \in L^2(0,\infty;\mathbb{C})$, it follows from (47), (73), and (7.4.1.2) of [14, Theorem 7.4.1.1, p. 654] (note that we use (7.4.2) of [14] only for the $\epsilon = 0$ there) that

$$\int_0^t e^{(B+\varepsilon)(t-s)}be^{\varepsilon s}f(s)ds \in L^2(0,\infty; H_{-1}).$$

This, together with (108) and the assumptions $-c+\varepsilon < 0$, $-\tilde{c}+\varepsilon < 0$, proves that

$$\int_0^\infty \|e^{\varepsilon t}e^{\mathcal{A}t}(v_0, \tilde{w}_0)\|_X^2 dt < \infty \quad \forall\, (v_0, \tilde{w}_0) \in X.$$

By Theorem 4.1 in [18], there exist constants $K_0, \delta > 0$ such that

$$\|e^{\varepsilon t}e^{\mathcal{A}t}\|_X \le K_0 e^{-\delta t}.$$

The proof is completed. □

Remark 5.3. A comparison of Theorems 5.1 and 5.2 shows that the Riesz basis approach leads to more profound results in the case of $c \neq \tilde{c}$; in particular, the spectral growth condition and the exact decay rate are established.

6. Simulation. The results of numerical simulation of two designs proposed in the paper (using finite-difference approximations) are presented in Figures 2–4. We present only state-feedback simulations since an observer design with (typically desirable) $\tilde{c} \gg c$ would result in an output-feedback controller performing essentially the same as a full-state controller.

In Figure 2 (top) one can see the uncontrolled oscillations of the open-loop system. The closed-loop response for the backstepping control design with $c = 6$ is shown in the bottom picture of Figure 2 (only for the real part of the state; the imaginary part is similar).

In Figures 3 and 4 we compare the performance of two different control designs (6) and (38) for $c_1 = 3$ and $c = 6$, respectively. The coefficients c_1 and c are chosen so that all the modes of the closed-loop system except for first one have approximately the same decay rate in both designs. From the eigenvalue formulae (24) and (37), it follows that the real parts of all eigenvalue pairs except for the first one are ≈ -6. The difference is in the first eigenvalue pair: in the backstepping design, its real part is -6, whereas in design (6), it has a real part of about -1.4 (as discussed in section 2, it cannot be moved further than that to the left in the complex plane; see Figure 1).

In Figure 3 (right) the real part of the control input $\psi(1,t)$ is shown. One can see that the backstepping controller is much less aggressive; its peak value is about three

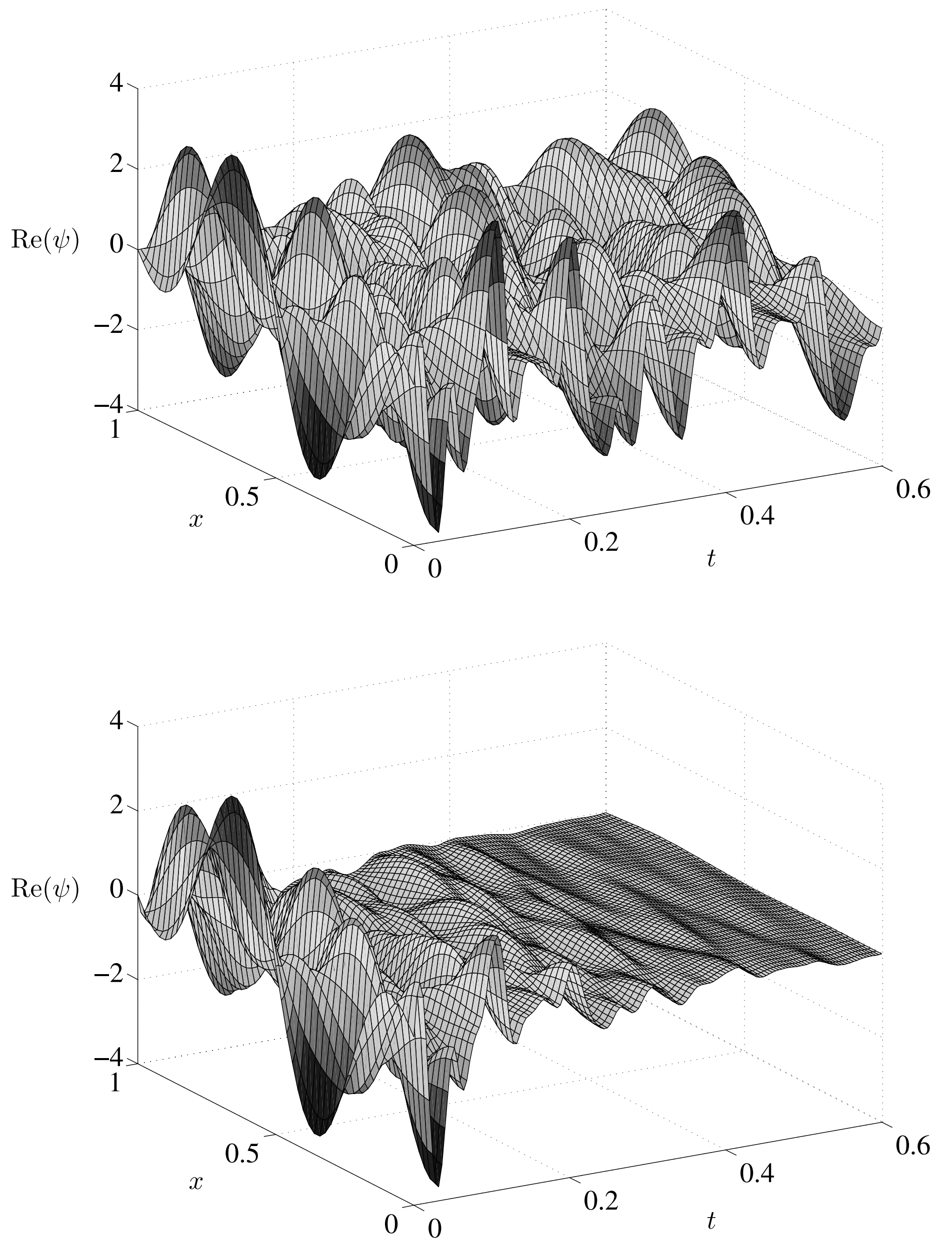

FIG. 2. *The open-loop (top) and closed-loop (bottom) response of the Schrödinger equation. Only the real part of the state is shown.*

times less than that of the control (6). From Figure 3 (left), we see that the output performance is approximately the same in both designs; the backstepping controller achieves slightly faster convergence to zero.

In Figure 4 we plot the logarithm of the L_2-norm of the closed-loop state. Up to

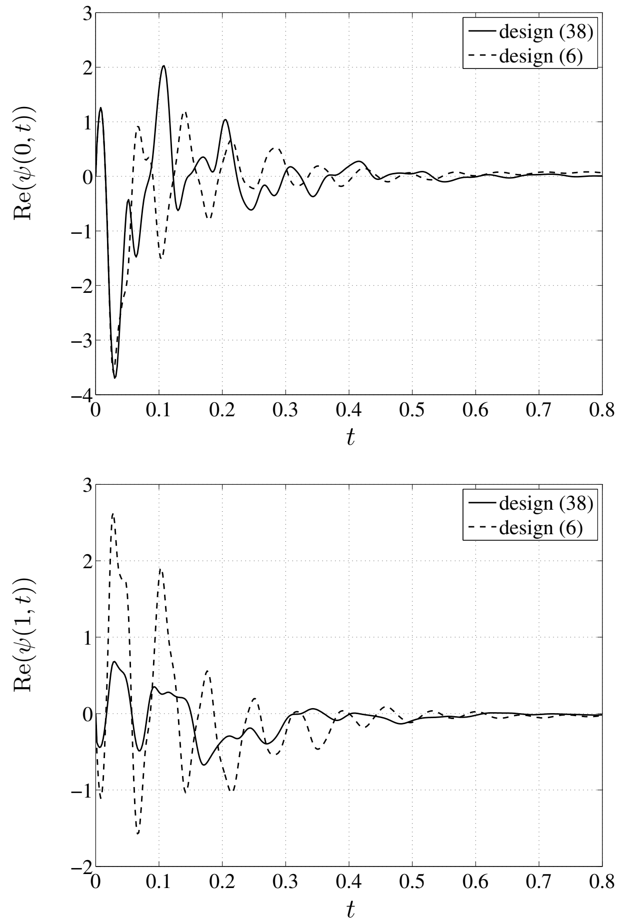

FIG. 3. *Top: The real part of the output* $\psi(0,t)$*. Bottom: The real part of the control effort* $\psi(1,t)$*.*

some point, the energy decays with the same rate in both designs (slope of ≈ -6). At $t \approx 0.45$, higher modes die out, and the difference in the decay rate of the first mode in the two designs becomes obvious.

To summarize, both control designs stabilize the plant successfully; however, the more complicated backstepping controller achieves slightly faster convergence with a much smaller control effort.

7. Future work. We have presented two boundary control designs for the Schrödinger equation. It is well known [17] that the linearized Schrödinger equation is equivalent to the Euler–Bernoulli beam model (both real and imaginary parts of the solution to the Schrödinger equation satisfy separate Euler–Bernoulli beam equations). Therefore, the next natural step is to design the stabilizing controllers for the Euler–Bernoulli beam based on the controllers presented in this paper. We should note, however, that the design does not carry over trivially from one system to another because the boundary conditions of the particular setup of the beam (free, hinged, clamped, etc.) do not directly correspond to the boundary conditions of the

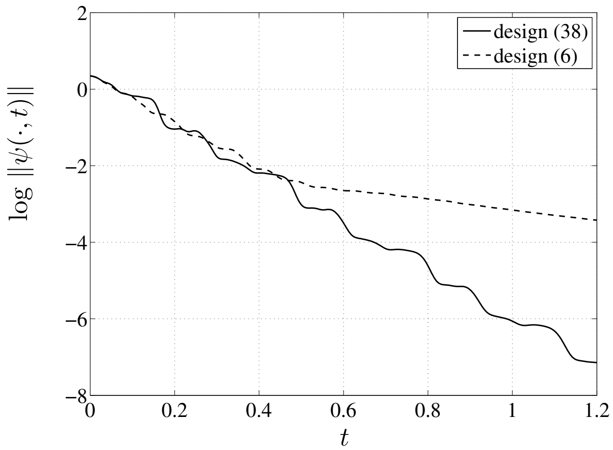

FIG. 4. *Logarithm of the energy of the closed-loop system.*

Schrödinger equation considered here.

This result would be significant since even though there is an extensive literature on the control of the Euler–Bernoulli beam, including, among others, [1, 9, 24, 4, 3, 5, 15], to the best of our knowledge, there are no boundary stabilization results that achieve a prescribed decay rate of the closed-loop system.

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