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Brief paper

New unknown input observer and output feedback stabilization for uncertain heat equation☆

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ABSTRACT

In this paper, we propose a new method, by designing an unknown input type state observer, to stabilize an unstable 1-d heat equation with boundary uncertainty and external disturbance. The state observer is designed in terms of a disturbance estimator. A stabilizing state feedback control is designed for the observer by the backstepping transformation, which is an observer based output feedback stabilizing control for the original system. The well-posedness and stability of the closed-loop system are concluded. The numerical simulations show that the proposed scheme is quite effectively. This is a first result on active disturbance rejection control for a PDE with both boundary uncertainty and external disturbance.

1. Introduction

Disturbance attenuation or rejection is one of the major concerns in modern control theory. Since from the 1970s, there are many methods developed to cope with uncertainty in control systems and most of these methods are generalized to systems described by partial differential equations (PDEs). Among them, internal model principle for special type of disturbances (Rebarber & Weiss, 2003) and adaptive control for unknown parameters (Krstic, 2010) are earlier active disturbance rejection methods in dealing with uncertainty by exploiting estimation/cancellation strategy. Other popular methods include sliding mode control (Guo & Jin, 2013) and robust control method (Christofides, 2001) where the completely unknown uncertainty is passively attenuated.

The idea of estimation/cancellation from internal model principle and adaptive rejection control is later developed in large scale as active disturbance rejection control (ADRC) (Han, 2009) where not only external disturbance but also internal uncertainty are estimated in terms of input and output. The uncertainties dealt with by ADRC are much more complicated. It can be the coupling between unknown internal system dynamics, the external disturbance, and the superadded unknown part of control input, or even if whatever the part that is hardly dealt with by practitioners (Guo & Zhao, 2015). ADRC has been applied to state feedback stabilization for PDEs with external disturbance (Guo & Jin, 2013). The output feedback stabilization for PDEs by ADRC is, however, very complicated. In Guo and Jin (2015), an unknown input observer is first designed for stabilization of 1-d wave equation with external disturbance. However, the observer in Guo and Jin (2015) was designed by variable structure control method, which is very technical and brings many mathematical difficulties. In addition, the extended state observer (ESO) used in ADRC utilizes usually the high gain which is very restrictive from engineering control point of view. So there are several challenges in applying ADRC to PDEs in following typical situations: (a) the total disturbance contains not only external disturbance but also internal uncertainty; (b) output feedback instead of state feedback; (c) the high gain problem in ESO; (d) a finite order derivative of total disturbance is required to be bounded.

In this paper, we meet these challenges by considering unknown type state observer and output feedback stabilization for the following one-dimensional heat equation with boundary unknown nonlinear uncertainty and external disturbance:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \ x\in(0,1), \ t>0, \\ w_x(0,t) = -qw(0,t), t\geq 0, \\ w_x(1,t) = f(w(\cdot,t)) + d(t) + u(t), \ t\geq 0, \\ w(x,0) = w_0(x), \ 0\leq x\leq 1, \\ y(t) = (w(0,t), w(1,t)), \ t\geq 0, \end{cases} \tag{1}$$

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where $q \in \mathbb{R}$, $y(t)$ is the output (measurement), $u(t)$ is the input (control), $w_0(x)$ is the initial value, $f(\cdot)$ is an unknown nonlinear function that represents the boundary uncertainty, and $d(t)$ is the external disturbance. The "$f(w(\cdot, t)) + d(t)$" is called the "total disturbance" in active disturbance rejection control. When $q > 0$, the uncontrolled system (1) may become unstable. For the sake of simplicity, we drop the obvious time and spatial domains in the rest of the paper.

The model (1) is a general 1-d heat equation with boundary convection. Let k be the thermal conductivity of a solid rod, and let h be the convection heat transfer coefficient which varies with the type of flow, the geometry of the body and flow passage area, the physical properties of the fluid, the average surface and fluid temperatures, and many other parameters. The pure convection boundary condition, physically meaning that the temperature gradient within the solid at the surface is coupled to the convective flux at the solid–fluid interface, is prescribed by

$$-kw_x(\partial, t) = \pm h(w(\partial, t) - w_\infty(t)),$$

where $\partial = 0$ or 1 represents the boundary and $w_\infty(t)$ is the ambient fluid temperature. The special case of zero fluid temperature $w_\infty(t) = 0$, given by

$$-kw_x(\partial, t) = \pm hw(\partial, t),$$

represents convection into a fluid medium at zero temperature, noting that a common practice is to redefine or shift the temperature scale such that the fluid temperature is now zero. When k, h, and $w_\infty(t)$ are not known, the convection boundary condition at $x = 1$ leads to the boundary condition of system (1) at $x = 1$. For more details of physical modeling of heat equation, we refer to Hahn and Özisik (2012).

To illustrate the physical model, we give a sketch of (1) with $q = 0$ in Fig. 1 which depicts flow of heat in a rod that is insulated everywhere except the two ends, where the heat of the right end is controlled by a steam chest with placement of a thermometer and the left end is insulated.

Heat equation with unstable term or source term has been extensively studied by the method of backstepping. Examples can be found in Baccoli, Pisano, and Orlov (2015), Meurer (2012), Krstic (2006), Krstic and Smyshlyaev (2008) and Smyshlyaev and Krstic (2007), to name just a few. The backstepping approach is powerful and is still valid to other distributed parameter systems that are corrupted by disturbance or unknown parameters (Aamo, 2013; Krstic, 2010). There are many other works for the parabolic systems control. For classical output regulation theory for distributed parameter system, we refer to (Aulisa & Gilliam, 2016). Recently, the backstepping-based robust output regulation for boundary controlled parabolic PDEs was discussed in Deutscher (2016). In addition, there exist other methods to cope with disturbance or unknown parameters such as the sliding mode control (Orlov, Pisano & Usai, 2011), unknown input observer based control (Chauvin, 2012), and the internal model principle (Rebarber & Weiss, 2003). Our work, however, is different from the existing ones. The main objective of this paper is to propose a new method to cope with the control-matched disturbance that consists of not only external disturbance but also boundary uncertainty. The approach is inspired by the method of ADRC and is different from the existing results in papers for instance (Aamo, 2013; Chauvin, 2012; Guo & Jin, 2015) where the unknown input observers for distributed parameter systems have been designed.

We proceed as follows. In Section 2, we first present a target system as a preliminary for the design of state observer. An unknown input type infinite-dimensional state observer is proposed in Section 3, where the estimation/cancellation strategy in ADRC is used without invoking high gain. The observer could lead immediately to a total disturbance estimator. A state feedback stabilizing control

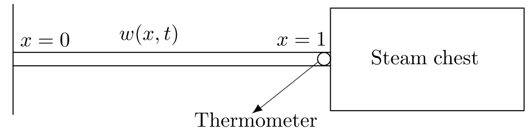

Fig. 1. One-dimensional heated rod.

for the observer is designed in Section 4, which is an observer based feedback control for original system. To do this, the backstepping transformation is applied. Section 5 is devoted to well-posedness and asymptotic stability for the closed-loop system. Numerical simulations are presented in Section 6 to validate the theoretical results, followed by the concluding remarks in Section 7.

2. Preliminary: target system for observer

We first consider a stable heat equation:

$$\begin{cases} \hat{z}_t(x, t) = \hat{z}_{xx}(x, t), \\ \hat{z}_x(0, t) = c_0\hat{z}(0, t),\ \hat{z}_x(1, t) = \mathcal{G}(t), \\ \hat{z}(x, 0) = \hat{z}_0(x), \end{cases} \tag{2}$$

where $c_0 > 0$ is a constant, $\hat{z}_0(x)$ is the initial value, and $\mathcal{G} \in L^2_{loc}(0, \infty)$ is a given function. System (2) can be written as an evolution equation in $\mathcal{H} := L^2(0, 1)$:

$$\frac{d}{dt}\hat{z}(\cdot, t) = A\hat{z}(\cdot, t) + B\mathcal{G}(t), \tag{3}$$

where $B = \delta(x-1)$ with $\delta(\cdot)$ the Dirac distribution, and the operator A is given by

$$\begin{cases} [Af](x) = f''(x), \ \forall f \in D(A), \\ D(A) = \left\{f \in H^2(0, 1) \,|\, f'(0) = c_0 f(0), f'(1) = 0\right\}. \end{cases} \tag{4}$$

Lemma 2.1. *For any $\hat{z}_0 \in \mathcal{H}$ and $\mathcal{G} \in L^2_{loc}(0, \infty)$, there exists a unique solution $\hat{z} \in C(0, \infty; \mathcal{H})$ to system (2) such that the following statements hold:*
(i) *If we assume further that $\mathcal{G} \in L^\infty(0, \infty)$, then there exists a positive constant L_B, independent of t, such that*

$$\sup_{t\in[0,\infty)} \|\hat{z}(\cdot, t)\|_{\mathcal{H}} \le \|\hat{z}_0\|_{\mathcal{H}} + L_B\|\mathcal{G}\|_{L^\infty(0,\infty)} < +\infty; \tag{5}$$

(ii) *If $\mathcal{G}(t) \to 0$ as $t \to \infty$, then*

$$\|\hat{z}(\cdot, t)\|_{\mathcal{H}} \to 0 \ \text{ as } \ t \to \infty. \tag{6}$$

Proof. Inequality (5) is straightforward by noticing that A generates a C_0-semigroup e^{At} of contractions on $\mathcal{H}$ and B is admissible for e^{At} by invoking Remark 2.6 of Weiss (1989). The convergence (6) is a direct result in Feng and Guo (2014) or Guo and Jin (2013) where the admissibility of B and Remark 2.6 of Weiss (1989) are also used. □

Next, consider the following coupled heat system

$$\begin{cases} \varepsilon_t(x, t) = \varepsilon_{xx}(x, t), \\ \varepsilon_x(0, t) = c_0\varepsilon(0, t), \varepsilon_x(1, t) = \tilde{d}_x(1, t), \\ \tilde{d}_t(x, t) = \tilde{d}_{xx}(x, t), \\ \tilde{d}_x(0, t) = c_0\tilde{d}(0, t), \tilde{d}(1, t) = 0, \\ \varepsilon(x, 0) = \varepsilon_0(x), \tilde{d}(x, 0) = \tilde{d}_0(x), \end{cases} \tag{7}$$

where $(\varepsilon_0(x), \tilde{d}_0(x))$ is the initial value. System (7) will serve as a target system for the observer design in next section. We consider system (7) in the Hilbert space $X = \mathcal{H}^2$ with the inner product

$$\langle (f_1, g_1), (f_2, g_2)\rangle_X = \int_0^1 f_1(x)\overline{f_2(x)}dx + \int_0^1 [2g_1(x)\overline{g_2(x)} - g_1(x)\overline{f_2(x)} - f_1(x)\overline{g_2(x)}]dx \tag{8}$$

for any $(f_i, g_i) \in X, i = 1, 2$. It is easy to see that the above inner product is well-defined. Indeed, for any $(f, g) \in X$,

$$\|(f, g)\|_X^2 = \int_0^1 [|f(x)|^2dx - g(x)\overline{f(x)} - f(x)\overline{g(x)}]dx + 2\int_0^1 |g(x)|^2dx \geq \frac{1}{3}\int_0^1 [|f^2(x)| + |g(x)|^2]dx. \tag{9}$$

System (7) can be written as an evolution equation in X:

$$\frac{d}{dt}(\varepsilon(\cdot, t), \tilde{d}(\cdot, t)) = \mathscr{A}(\varepsilon(\cdot, t), \tilde{d}(\cdot, t)) \tag{10}$$

where the operator $\mathscr{A}$ is given by

$$\begin{cases} [\mathscr{A}(f, g)](x) = (f''(x), g''(x)), \ \forall (f, g) \in D(\mathscr{A}), \\ D(\mathscr{A}) = \{(f, g) \in (H^2(0, 1))^2 \mid f'(0) = c_0f(0), \\ g'(0) = c_0g(0), f'(1) = g'(1), g(1) = 0\}. \end{cases} \tag{11}$$

Lemma 2.2. *For any $(\varepsilon_0, \tilde{d}_0) \in X$, system (7) admits a unique solution $(\varepsilon, \tilde{d}) \in C(0, \infty; X)$. Moreover, there exist two positive constants $L_{\mathscr{A}}$ and $\omega_{\mathscr{A}}$ such that*

$$\|\varepsilon(\cdot, t)\|_{\mathcal{H}} + \|\tilde{d}(\cdot, t)\|_{\mathcal{H}} \leq L_{\mathscr{A}}e^{-\omega_{\mathscr{A}}t}, \ \forall t \geq 0. \tag{12}$$

Proof. For any $(f, g) \in D(\mathscr{A})$,

$$\begin{aligned} &\mathrm{Re}\langle \mathscr{A}(f, g), (f, g)\rangle_X \\ &= \mathrm{Re}\left[\langle f'', f\rangle_{\mathcal{H}} + 2\langle g'', g\rangle_{\mathcal{H}} - \langle g'', f\rangle_{\mathcal{H}} - \langle f'', g\rangle_{\mathcal{H}}\right] \\ &= \mathrm{Re}\left[-c_0|f(0)|^2 - \|f'\|_{\mathcal{H}}^2 - 2c_0|g(0)|^2 \right.\\ &\quad \left. - 2\|g'\|_{\mathcal{H}}^2 + 2c_0g(0)\overline{f(0)} + 2\langle f', g'\rangle_{\mathcal{H}}\right] \\ &\leq -\frac{c_0}{3}|f(0)|^2 - \frac{c_0}{2}|g(0)|^2 - \frac{1}{3}\|f'\|_{\mathcal{H}}^2 - \frac{1}{2}\|g'\|_{\mathcal{H}}^2 \leq 0, \end{aligned} \tag{13}$$

which shows that $\mathscr{A}$ is dissipative in X. On the other hand, for any $(\hat{f}, \hat{g}) \in X$, we solve $\mathscr{A}(f, g) = (\hat{f}, \hat{g})$ to obtain

$$\begin{cases} g(x) = c_0(x - 1)g(0) - \int_x^1 \int_0^\alpha \hat{g}(s)dsd\alpha, \\ f(x) = f(0) + xg'(1) - \int_0^x \int_\alpha^1 \hat{f}(s)dsd\alpha, \end{cases} \tag{14}$$

where

$$\begin{cases} g(0) = -\dfrac{1}{1 + c_0}\int_0^1 \int_0^\alpha \hat{g}(s)dsd\alpha, \\ g'(1) = c_0g(0) + \int_0^1 \hat{g}(s)ds, \\ f(0) = \dfrac{1}{c_0}g'(1) - \dfrac{1}{c_0}\int_0^1 \hat{f}(s)ds. \end{cases} \tag{15}$$

This shows that $\mathscr{A}^{-1} \in \mathcal{L}(X)$ is compact on X. By the Lumer–Phillips theorem Pazy (1983, Theorem 1.4.3), $\mathscr{A}$ generates a C_0-semigroup of contractions on X. The remaining proof is for (12). To this purpose, define the Lyapunov functional:

$$L(t) = \frac{1}{2}\|\varepsilon(\cdot, t)\|_{\mathcal{H}}^2 + \|\tilde{d}(\cdot, t)\|_{\mathcal{H}}^2 - \frac{1}{2}\int_0^1 \varepsilon(x, t)\overline{\tilde{d}(x, t)}dx - \frac{1}{2}\int_0^1 \overline{\varepsilon(x, t)}\tilde{d}(x, t)dx. \tag{16}$$

By Hölder's inequality,

$$\begin{aligned} |\varepsilon(x, t)| &\leq |\varepsilon(0, t)| + \int_0^1 |\varepsilon_x(x, t)|dx \\ &\leq |\varepsilon(0, t)| + \left(\int_0^1 |\varepsilon_x(x, t)|^2dx\right)^{1/2}, \end{aligned} \tag{17}$$

which implies that

$$\|\varepsilon(\cdot, t)\|_{\mathcal{H}}^2 \leq 2|\varepsilon(0, t)| + 2\|\varepsilon_x(\cdot, t)\|_{\mathcal{H}}^2. \tag{18}$$

Since $\tilde{d}(1, t) = 0$, we have $\|\tilde{d}(\cdot, t)\|_{\mathcal{H}}^2 \leq \|\tilde{d}_x(\cdot, t)\|_{\mathcal{H}}^2$. Therefore, by a simple computation, it follows that

$$\begin{aligned} &\frac{1}{6}\left[\|\varepsilon(\cdot, t)\|_{\mathcal{H}}^2 + \|\tilde{d}(\cdot, t)\|_{\mathcal{H}}^2\right] \leq L(t) \\ &\leq 4\left[\|\varepsilon_x(\cdot, t)\|_{\mathcal{H}}^2 + \|\tilde{d}_x(\cdot, t)\|_{\mathcal{H}}^2 + \varepsilon^2(0, t)\right]. \end{aligned} \tag{19}$$

Find the derivative of $L(t)$ along the solution of system (7) to obtain

$$\begin{aligned} \dot{L}(t) &= -c_0\varepsilon^2(0, t) - 2c_0\tilde{d}^2(0, t) + c_0\tilde{d}(0, t)\overline{\varepsilon(0, t)} \\ &\quad + c_0\overline{\tilde{d}(0, t)}\varepsilon(0, t) - \|\varepsilon_x(\cdot, t)\|_{\mathcal{H}}^2 - 2\|\tilde{d}_x(\cdot, t)\|_{\mathcal{H}}^2 \\ &\quad + \langle \overline{\tilde{d}_x(\cdot, t)}, \varepsilon_x(\cdot, t)\rangle_{\mathcal{H}} + \langle \tilde{d}_x(\cdot, t), \overline{\varepsilon_x(\cdot, t)}\rangle_{\mathcal{H}} \\ &\leq -\frac{c_0}{3}\varepsilon^2(0, t) - \frac{c_0}{2}\tilde{d}^2(0, t) \\ &\quad - \frac{1}{3}\|\varepsilon_x(\cdot, t)\|_{\mathcal{H}}^2 - \frac{1}{2}\|\tilde{d}_x(\cdot, t)\|_{\mathcal{H}}^2 \\ &\leq -\beta\left[\|\varepsilon_x(\cdot, t)\|_{\mathcal{H}}^2 + \|\tilde{d}_x(\cdot, t)\|_{\mathcal{H}}^2 + \varepsilon^2(0, t)\right] \\ &\leq -\frac{\beta}{4}L(t), \end{aligned} \tag{20}$$

where $\beta = \min\{\frac{1}{2}, \frac{1}{3}, c_0\}$. Finally, (20) together with (19), yields (12). □

3. Unknown input state observer

In this section, we design a state observer in terms of input and output $(u(t), y(t))$ for system (1) in the presence of unknown input. There are three important steps. First, the "total disturbance" is separated from the control and is brought into an ("relatively good") exponentially stable system where the total disturbance is considered as somehow an inhomogeneous term; second, the total disturbance is estimated in the framework of this well stable system; and the last, the total disturbance is compensated leading to an unknown type observer which is actually an alternative extended state observer (ESO). More specially, we split the whole process into the following three steps.

Step 1: Construct an auxiliary system which brings the total disturbance into an exponentially stable system.

The auxiliary system is designed as follows:

$$\begin{cases} z_t(x, t) = z_{xx}(x, t), \\ z_x(0, t) = -qw(0, t) - c_0[w(0, t) - z(0, t)], \\ z_x(1, t) = u(t), \\ z(x, 0) = z_0(x), \end{cases} \tag{21}$$

where $c_0 > 0$ is a constant, $z_0(x)$ is the initial value. System (21) is completely determined by input and output of the original system (1) and hence is known. However, the error between uncertain system (1) and known system (21) is independent of control and satisfies (2) with $\mathcal{G}(t) = f(w(\cdot, t)) + d(t)$. Precisely, $\hat{z}(x, t) = w(x, t) - z(x, t)$ satisfies

$$\begin{cases} \hat{z}_t(x, t) = \hat{z}_{xx}(x, t), \\ \hat{z}_x(0, t) = c_0\hat{z}(0, t), \ \hat{z}_x(1, t) = f(w(\cdot, t)) + d(t), \\ \hat{z}(x, 0) = \hat{z}_0(x) = w_0(x) - z_0(x). \end{cases} \tag{22}$$

It is seen that although the known system (21) and the original uncertain system (1) are different, their error is a "relatively good" system (22): The linear part is exponentially stable and the total disturbance $f(w(\cdot,t)) + d(t)$ is an inhomogeneous term of (22). Moreover, the control does not appear in system (22) so that we only need to concentrate on estimation of the total disturbance from (22) only without taking care of the control. The exponential stability of (22) guarantees that all subsystems involved in estimation of disturbance are uniformly bounded as claimed in Lemma 2.1. In this sense, we say that system (21) separates the total disturbance $f(w(\cdot,t))+d(t)$ from the control $u(t)$ and introduces the total disturbance into a "relatively good" system (22) which is our starting point of estimating $f(w(\cdot,t)) + d(t)$.

Step 2: Design an disturbance estimator for (22).

Since (22) obtained from Step 1 is independent of control $u(t)$, we are able to design an observer for (22), which turns out a disturbance estimator. This is realized by designing the following system:

$$\begin{cases} \hat{d}_t(x,t) = \hat{d}_{xx}(x,t), \\ \hat{d}_x(0,t) = c_0\hat{d}(0,t), \quad \hat{d}(1,t) = \hat{z}(1,t), \\ \hat{d}(x,0) = \hat{d}_0(x), \end{cases} \tag{23}$$

where $\hat{d}_0(x)$ is the initial value that can be chosen arbitrarily. Since $\hat{z}(1,t) = w(1,t) - z(1,t)$ is known, system (23) is completely determined by input and output of the original system (1). The construction of system (23) is based on "relatively good" system (22) only. We shall see that system (23) serves as a disturbance estimator. In fact, if we let

$$\tilde{d}(x,t) = \hat{z}(x,t) - \hat{d}(x,t), \tag{24}$$

then, a simple computation shows that the error $\tilde{d}(x,t)$ is governed by

$$\begin{cases} \tilde{d}_t(x,t) = \tilde{d}_{xx}(x,t), \\ \tilde{d}_x(0,t) = c_0\tilde{d}(0,t), \\ \tilde{d}(1,t) = 0, \end{cases} \tag{25}$$

which is an exponentially stable system. In fact, system (25) has the system operator as follows:

$$\begin{cases} [\mathcal{A}f](x) = f''(x), \ \forall f \in D(\mathcal{A}), \\ D(\mathcal{A}) = \{f \in H^2(0,1) \mid \\ \qquad f'(0) = c_0 f(0), f(1) = 0\}, \end{cases} \tag{26}$$

which generates an exponentially stable C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. Hence, there exist two positive constants $L_{\mathcal{A}}$ and $\omega_{\mathcal{A}} > 0$ such that

$$\left\| e^{\mathcal{A}t} \right\| \le L_{\mathcal{A}} e^{-\omega_{\mathcal{A}} t}, \quad t \ge 0. \tag{27}$$

It is easy to show that $\omega_{\mathcal{A}} \ge 1$ (by direct energy differentiation). Moreover, we have Lemma 3.1.

Lemma 3.1. *For any initial value $\tilde{d}(\cdot,0) \in D(\mathcal{A})$, system (25) admits a unique classical solution $\tilde{d}(\cdot,t) \in C(0,\infty; D(\mathcal{A}))$ such that $\tilde{d}_x(1,\cdot) \in C(0,\infty)$ and*

$$|\tilde{d}_x(1,t)| \le C_0 L_{\mathcal{A}} e^{-\omega_{\mathcal{A}} t}, \quad t \ge 0, \tag{28}$$

where $C_0 > 0$ is independent of t.

Proof. Since $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$, for $\tilde{d}(\cdot,0) \in D(\mathcal{A})$, system (25) admits a unique classical solution $\tilde{d} \in C(0,\infty; D(\mathcal{A}))$ such that

$$\|\tilde{d}_{xx}(\cdot,t)\|_{\mathcal{H}} \le \|\tilde{d}_{xx}(\cdot,0)\|_{\mathcal{H}} L_{\mathcal{A}} e^{-\omega_{\mathcal{A}} t}, \quad t \ge 0. \tag{29}$$

Define

$$\phi(t) = \int_0^1 (x-1)\tilde{d}(x,t)dx. \tag{30}$$

Finding the derivative $\phi(t)$ along the solution of (25) yields

$$\dot{\phi}(t) = \tilde{d}_x(0,t) + \tilde{d}(0,t) = \left(1 + \frac{1}{c_0}\right)\tilde{d}_x(0,t). \tag{31}$$

On the other hand,

$$\dot{\phi}(t) = \int_0^1 (x-1)\tilde{d}_{xx}(x,t)dx \le \left(\int_0^1 |\tilde{d}_{xx}(x,t)|^2 dx\right)^{\frac{1}{2}}, \tag{32}$$

which, together with (31) and (29), leads to

$$\begin{aligned} |\tilde{d}_x(0,t)| &\le \frac{c_0}{1+c_0}\dot{\phi}(t) \le \frac{c_0}{1+c_0}\|\tilde{d}_{xx}(\cdot,t)\|_{\mathcal{H}} \\ &\le \frac{c_0 L_{\mathcal{A}}}{1+c_0}\|\tilde{d}_{xx}(\cdot,0)\|_{\mathcal{H}} e^{-\omega_{\mathcal{A}} t}, \quad t \ge 0. \end{aligned} \tag{33}$$

By Hölder's inequality,

$$\begin{aligned} |\tilde{d}_x(1,t)| &\le |\tilde{d}_x(0,t)| + \left|\int_0^1 \tilde{d}_{xx}(x,t)dx\right| \\ &\le |\tilde{d}_x(0,t)| + \|\tilde{d}_{xx}(\cdot,t)\|_{\mathcal{H}}. \end{aligned} \tag{34}$$

Therefore, (28) follows from (34), (33), and (29) with

$$C_0 = \frac{c_0 L_{\mathcal{A}}}{1+c_0}\|\tilde{d}_{xx}(\cdot,0)\|_{\mathcal{H}} + L_{\mathcal{A}}\|\tilde{d}(\cdot,0)\|_{\mathcal{H}}. \quad \square \tag{35}$$

On the other hand, it follows from (22) and (24) that

$$\tilde{d}_x(1,t) = [f(w(\cdot,t)) + d(t)] - \hat{d}_x(1,t), \tag{36}$$

which, together with (28), implies that $\hat{d}_x(1,t)$ can be regarded as an approximate of the total disturbance $f(w(\cdot,t))+d(t)$ as $t \to \infty$. Very importantly, $\hat{d}_x(1,t)$ gives sufficient estimation of $f(w(\cdot,t)) + d(t)$ in the sense that the error $[f(w(\cdot,t)) + d(t)] - \hat{d}_x(1,t)$ is independent of the total disturbance by (25), (28), and (36). This is a remarkable advantage of this design.

Step 3: Compensate the total disturbance by its estimate to obtain estimator based observer.

The observer can be designed as

$$\begin{cases} \hat{w}_t(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -qw(0,t) - c_0[w(0,t) - \hat{w}(0,t)], \\ \hat{w}_x(1,t) = \hat{d}_x(1,t) + u(t), \\ \hat{w}(x,0) = \hat{w}_0(x), \end{cases} \tag{37}$$

where $\hat{w}_0(x)$ is the initial value that can be chosen arbitrarily. It is seen that the observer (37) is the same as the corresponding observer for disturbance free system designed, except the term $\hat{d}_x(1,t)$ which is used to compensate the unknown total disturbance. Notice that system (37) is completely determined by input and output of system (1).

Combining Steps 1–3, we obtain an unknown input observer for system (1) as follows:

$$\begin{cases} \hat{w}_t(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -qw(0,t) - c_0[w(0,t) - \hat{w}(0,t)], \\ \hat{w}_x(1,t) = \hat{d}_x(1,t) + u(t), \\ \hat{d}_t(x,t) = \hat{d}_{xx}(x,t), \\ \hat{d}_x(0,t) = c_0\hat{d}(0,t), \\ \hat{d}(1,t) = w(1,t) - z(1,t), \\ z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = -qw(0,t) - c_0[w(0,t) - z(0,t)], \\ z_x(1,t) = u(t), \\ \hat{w}(x,0) = \hat{w}_0(x), \hat{d}(x,0) = \hat{d}_0(x), z(x,0) = z_0(x), \end{cases} \tag{38}$$

where $z(x,t)$ is an auxiliary variable, $\hat{d}(x,t)$ is used for disturbance estimation, and $\hat{w}(x,t)$ is considered as an approximate of $w(x,t)$ as $t \to \infty$. Once again, system (38) is completely determined by input and output of original system (1). In addition, we see that the unknown input observer (38) is actually a linear system, which is an interesting fact for nonlinear system (1).

Theorem 3.1. *For any $u \in L^2_{loc}(0,\infty)$ and $w_0 \in \mathcal{H} = L^2(0,1)$, suppose that system (1) admits a unique solution $w \in C(0,\infty;\mathcal{H})$ such that $[f(w)+d] \in L^2_{loc}(0,\infty)$ which is satisfied when f satisfies the global Lipschitz condition like in Remark 3.1. Then, the state observer (38) is well-posed and for any $(\hat{w}_0, \hat{d}_0, z_0) \in \mathcal{H}^3$, (38) admits a unique solution $(\hat{w}, \hat{d}, z) \in C(0,\infty;\mathcal{H}^3)$ satisfying*

$$\|w(\cdot,t) - \hat{w}(\cdot,t)\|_{\mathcal{H}} \to 0 \text{ as } t \to \infty. \tag{39}$$

Proof. By the invertible transformation:

$$\begin{pmatrix} \hat{w} \\ \hat{d} \\ z \\ \hat{z} \end{pmatrix} = \begin{pmatrix} I & -I & 0 & 0 \\ 0 & 0 & -I & I \\ I & 0 & 0 & -I \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} w \\ \varepsilon \\ \tilde{d} \\ \hat{z} \end{pmatrix}, \tag{40}$$

the solution of observer (38) is well defined if and only if the solution of the following system is well defined:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \ x \in (0,1), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = f(w(\cdot,t)) + d(t) + u(t), \\ \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = c_0\hat{z}(0,t), \ \hat{z}_x(1,t) = f(w(\cdot,t)) + d(t), \\ \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = c_0\varepsilon(0,t), \varepsilon_x(1,t) = \tilde{d}_x(1,t), \\ \tilde{d}_t(x,t) = \tilde{d}_{xx}(x,t), \\ \tilde{d}_x(0,t) = c_0\tilde{d}(0,t), \tilde{d}(1,t) = 0. \end{cases} \tag{41}$$

It is seen that the "$(\varepsilon,\tilde{d})$-part" is independent of the "$(w,\hat{z})$-part" and happens to be system (7). By Lemma 2.2, the "$(\varepsilon,\tilde{d})$-part" with the initial values

$$\begin{cases} \varepsilon_0(x) = w_0(x) - \hat{w}_0(x), \\ \tilde{d}_0(x) = w_0(x) - z_0(x) - \hat{d}_0(x) \end{cases} \tag{42}$$

admits a unique solution $(\varepsilon,\tilde{d}) \in C(0,\infty;\mathcal{H}^2)$ such that

$$\|\varepsilon(\cdot,t)\|_{\mathcal{H}} + \|\tilde{d}(\cdot,t)\|_{\mathcal{H}} \to 0 \text{ as } t \to \infty. \tag{43}$$

Next, since "$(w,\hat{z})$-part" is a cascade system of "w-subsystem" and "$\hat{z}$-subsystem", we can solve "w-subsystem" first. Under the assumptions, the "w-part" of system (41) admits a unique solution $w \in C(0,\infty;\mathcal{H})$ such that $[f(w)+d] \in L^2_{loc}(0,\infty)$. Using the cascaded structure of "$(w,\hat{z})$-part", the "$\hat{z}$-part" of system (41) is a linear system with an inhomogeneous term $[f(w(\cdot,t))+d(t)] \in L^2_{loc}(0,\infty)$. By Lemma 2.1, the "$\hat{z}$-part" of system (41) with the initial value $\hat{z}_0(x) = w_0(x) - z_0(x)$ admits a unique solution $\hat{z} \in C(0,\infty;\mathcal{H})$. Finally, $(w,\varepsilon,\tilde{d},\hat{z}) \in C(0,\infty;\mathcal{H}^4)$ is well defined. Owing to the equivalent transformation (40), a simple computation shows that $(\hat{w},\hat{d},z,\hat{z}) \in C(0,\infty;\mathcal{H}^4)$ is well defined and $(\hat{w},\hat{d},z) \in C(0,\infty;\mathcal{H}^3)$ is a solution of system (38). Using the transformation (40) again, it follows that

$$\varepsilon(x,t) = w(x,t) - \hat{w}(x,t), \tag{44}$$

and the convergence (39) follows from (43). □

Remark 3.1. Since we are only interested in the control design and the closed-loop system, the well-posedness of the open-loop system (1) is not touched in Theorem 3.1. In Theorem A.1 of Appendix, we present the existence of solution to (1) under global Lipschitz condition on f from which the condition of $f(w(\cdot,t)) \in L^2_{loc}(0,\infty)$ required in Theorem 3.1 is satisfied.

Remark 3.2. It follows from the three steps in the beginning of the section that we propose a very different scheme of the observer design with control-matched disturbance and collocated observation. From asymptotic stabilization point of view however, the disturbance and control should be matched since otherwise, there is no channel to cancel the disturbance even if we know about the disturbance. From observer design point of view, control matched or not does not matter because our observer design process does not rely on the position of the control and our observation signal is not always in the control end. For example, consider the following problem:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \ x \in (0,1), \ t > 0, \\ w_x(0,t) = -qw(0,t) + u(t), t \ge 0, \\ w_x(1,t) = f(w(\cdot,t)) + d(t), \ t \ge 0, \\ w(x,0) = w_0(x), \ 0 \le x \le 1, \\ y(t) = (w(0,t), w(1,t)), \ t \ge 0. \end{cases} \tag{45}$$

It is seen that the control and the disturbance are unmatched. However, by exploiting our approach, we can also design an unknown input observer for system (45). Indeed, construct the following auxiliary system to bring the disturbance into an exponentially stable system:

$$\begin{cases} z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = -qw(0,t) - c_0[w(0,t) - z(0,t)] + u(t), \\ z_x(1,t) = 0. \end{cases} \tag{46}$$

If we let $\hat{z}(x,t) = w(x,t) - z(x,t)$, then $\hat{z}(x,t)$ is governed by (22). The rest of the observer design is the same as for system (1) in this section.

Corollary 3.1. *In addition to the assumptions in Theorem 3.1, we assume further that the initial state $(w_0, \hat{d}_0, z_0)$ satisfies the compatibility condition:*

$$(w_0 - z_0 - \hat{d}_0) \in D(\mathcal{A}). \tag{47}$$

Then there exists a unique solution $(\hat{d}, z) \in C(0,\infty;\mathcal{H}^2)$ to the "$(\hat{d},z)$-part" of (38) such that

$$\left|\hat{d}_x(1,t) - [f(w(\cdot,t)) + d(t)]\right| \to 0 \text{ as } t \to \infty. \tag{48}$$

Proof. Let $(\varepsilon,\tilde{d}) \in C(0,\infty;\mathcal{H}^2)$ be the solution of (7) with the initial values

$$\begin{cases} \varepsilon_0(x) = w_0(x) - \hat{w}_0(x), \\ \tilde{d}_0(x) = w_0(x) - z_0(x) - \hat{d}_0(x), \end{cases} \tag{49}$$

and let $\hat{z} \in C(0,\infty;\mathcal{H})$ be the solution of (2) with the initial value $\hat{z}_0(x) = w_0(x) - z_0(x)$ and $\mathcal{G}(t) = f(w(\cdot,t)) + d(t)$. Then, $(\hat{w},\hat{d},z) \in C(0,\infty;\mathcal{H}^3)$ is well defined in terms of (40). By compatibility condition (47), $\tilde{d}(\cdot,0) \in D(\mathcal{A})$. The convergence (48) then follows from (28) and (36). □

4. Observer based feedback design

To begin with, we notice that any state feedback for system (37) is an output feedback for original system (1). So in this section, we consider feedback stabilization for system (37). First we use a backstepping transformation to deal with the unstable term in system (37). Let (see e.g. Krstic, 2010)

$$\begin{aligned} \tilde{w}(x,t) &= [(I+\mathbb{P})\hat{w}](x,t) \\ &= \hat{w}(x,t) + q\int_0^x e^{q(x-s)}\hat{w}(s,t)ds, \end{aligned} \tag{50}$$

which is invertible and its inverse is given by

$$\begin{aligned} \hat{w}(x,t) &= [(I+\mathbb{P})^{-1}\tilde{w}](x,t) \\ &= \tilde{w}(x,t) - q\int_0^x \tilde{w}(s,t)ds. \end{aligned} \tag{51}$$

A direct computation shows that $\tilde{w}(x,t)$ is governed by

$$\begin{cases} \tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + q(q+c_0)e^{qx}\varepsilon(0,t), \\ \tilde{w}_x(0,t) = -(q+c_0)\varepsilon(0,t), \\ \tilde{w}_x(1,t) = \hat{d}_x(1,t) + u(t) + q\tilde{w}(1,t), \end{cases} \tag{52}$$

where $\varepsilon(x,t) = w(x,t) - \hat{w}(x,t)$ is given by (44). A stabilizing feedback control can then be designed as

$$u(t) = -\hat{d}_x(1,t) - (q+c_1)\tilde{w}(1,t), \tag{53}$$

where c_1 is a positive tuning parameter, to make system (52) as

$$\begin{cases} \tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + q(q+c_0)e^{qx}\varepsilon(0,t), \\ \tilde{w}_x(0,t) = -(q+c_0)\varepsilon(0,t), \\ \tilde{w}_x(1,t) = -c_1\tilde{w}(1,t). \end{cases} \tag{54}$$

It is seen from (54) that the "unstable" boundary condition $\hat{w}_x(0,t) = -qw(0,t)$ in observer (37) is replaced by the "stable" inhomogeneous term in (54). By (53) and (50), it follows that

$$\begin{aligned} u(t) = & -\hat{d}_x(1,t) - (c_1+q)\hat{w}(1,t) \\ & -q(c_1+q)\int_0^1 e^{q(1-s)}\hat{w}(s,t)ds. \end{aligned} \tag{55}$$

This is a stabilizing state feedback control for observer (37) and consequently an observer based output feedback control for original system (1). The closed-loop of system (1) now reads

$$\begin{cases} w_t(x,t) = w_{xx}(x,t), \\ w_x(0,t) = -qw(0,t), \\ w_x(1,t) = f(w(\cdot,t)) + d(t) - \hat{d}_x(1,t) \\ -(c_1+q)\hat{w}(1,t) - q(c_1+q)\int_0^1 e^{q(1-s)}\hat{w}(s,t)ds, \\ \hat{d}_t(x,t) = \hat{d}_{xx}(x,t), \\ \hat{d}_x(0,t) = c_0\hat{d}(0,t),\ \hat{d}(1,t) = w(1,t) - z(1,t), \\ z_t(x,t) = z_{xx}(x,t), \\ z_x(0,t) = -qw(0,t) - c_0[w(0,t) - z(0,t)], \\ z_x(1,t) = -\hat{d}_x(1,t) - (c_1+q)\hat{w}(1,t) \\ \qquad - q(c_1+q)\int_0^1 e^{q(1-s)}\hat{w}(s,t)ds, \\ \hat{w}_t(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -qw(0,t) - c_0[w(0,t) - \hat{w}(0,t)], \\ \hat{w}_x(1,t) = -(c_1+q)\hat{w}(1,t) \\ \qquad - q(c_1+q)\int_0^1 e^{q(1-s)}\hat{w}(s,t)ds. \end{cases} \tag{56}$$

Now we state our main result of this paper.

Theorem 4.1. *Suppose that $d \in L^\infty(0,\infty)$ and $f \in C(\mathcal{H};\mathbb{R})$. Then, for any $(w_0, \hat{w}_0, \hat{d}_0, z_0) \in \mathcal{H}^4$, the closed-loop system (56) admits a unique solution $(w, \hat{w}, \hat{d}, z) \in C(0,\infty;\mathcal{H}^4)$ which satisfies:*

$$\|(w(\cdot,t), \hat{w}(\cdot,t))\|_{\mathcal{H}^2} \to 0 \text{ as } t \to \infty \tag{57}$$

and

$$\sup_{t\in[0,\infty)} \left\|(\hat{d}(\cdot,t), z(\cdot,t))\right\|_{\mathcal{H}^2} < +\infty. \tag{58}$$

If we assume further that $d \equiv 0$ and $f(0) = 0$, then system is stable in the sense

$$\left\|(w(\cdot,t), \hat{w}(\cdot,t), \hat{d}(\cdot,t), z(\cdot,t))\right\|_{\mathcal{H}^4} \to 0 \text{ as } t \to \infty. \tag{59}$$

In other words, when the external disturbance is disconnected to the system, the closed-loop system is internally asymptotically stable.

5. Stability of closed-loop

We first consider the following linear system:

$$\begin{cases} \varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = c_0\varepsilon(0,t),\ \varepsilon_x(1,t) = \tilde{d}_x(1,t), \\ \tilde{d}_t(x,t) = \tilde{d}_{xx}(x,t), \\ \tilde{d}_x(0,t) = c_0\tilde{d}(0,t),\ \tilde{d}(1,t) = 0, \\ \tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + q(q+c_0)e^{qx}\varepsilon(0,t), \\ \tilde{w}_x(0,t) = -(q+c_0)\varepsilon(0,t), \\ \tilde{w}_x(1,t) = -c_1\tilde{w}(1,t), \\ \varepsilon(x,0) = \varepsilon_0(x),\ \tilde{d}(x,0) = \tilde{d}_0(x),\ \tilde{w}(x,0) = \tilde{w}_0(x), \end{cases} \tag{60}$$

which can be rewritten as an evolutionary equation in $\mathcal{X} = \mathcal{H}^3$:

$$\frac{d}{dt}(\varepsilon(\cdot,t), \tilde{d}(\cdot,t), \tilde{w}(\cdot,t)) = \mathbb{A}(\varepsilon(\cdot,t), \tilde{d}(\cdot,t), \tilde{w}(\cdot,t)), \tag{61}$$

where the operator $\mathbb{A}$ is given by

$$\begin{cases} [\mathbb{A}(f,g,h)](x) = \Big\{f''(x), g''(x), \\ h''(x) + q(q+c_0)e^{qx}f(0)\Big\},\ \forall (f,g,h) \in D(\mathbb{A}), \\ D(\mathbb{A}) = \Big\{(f,g,h) \in (H^2(0,1))^3 \mid f'(0) = c_0 f(0), \\ g'(0) = c_0 g(0), f'(1) = g'(1), g(1) = 0, \\ h'(0) = -(q+c_0)f(0), h'(1) = -c_1 h(1)\Big\}. \end{cases} \tag{62}$$

The inner product of $\mathcal{X}$ is defined by:

$$\begin{aligned} &\langle (f_1,g_1,h_1), (f_2,g_2,h_2)\rangle_{\mathcal{X}} = \langle (f_1,g_1), (f_2,g_2)\rangle_X \\ &+ \gamma\int_0^1 h_1(x)\overline{h_2(x)}dx, \forall\, (f_i,g_i,h_i) \in \mathcal{X}, i = 1,2, \end{aligned} \tag{63}$$

where γ is small enough so that $\mathbb{A}$ is dissipative in $\mathcal{X}$ and will be determined later. It is easy to see that the above inner product is well-defined.

Proposition 5.1. *The operator $\mathbb{A}$ generates an exponentially stable C_0-semigroup on $\mathcal{X}$. That is, for any $(\varepsilon_0, \tilde{d}_0, \tilde{w}_0) \in \mathcal{X}$, system (60) admits a unique solution $(\varepsilon, \tilde{d}, \tilde{w}) \in C(0,\infty;\mathcal{X})$ and there exist $L_1, \omega_1 > 0$ such that*

$$\left\|(\varepsilon(\cdot,t), \tilde{d}(\cdot,t), \tilde{w}(\cdot,t))\right\|_{\mathcal{X}} \le L_1 e^{-\omega_1 t},\ t \ge 0. \tag{64}$$

Proof. Define the Lyapunov functional for system (60):

$$G(t) = L(t) + \frac{\gamma}{2}\|\tilde{w}(\cdot,t)\|^2_{\mathcal{H}}, \tag{65}$$

where $L(t)$ is defined by (16) and $\gamma > 0$. By (19), there exist two positive constants μ_1 and μ_2, independent of time t, such that

$$\begin{aligned} &\mu_1\left[\|\varepsilon(\cdot,t)\|^2_{\mathcal{H}} + \|\tilde{d}(\cdot,t)\|^2_{\mathcal{H}} + \|\tilde{w}(\cdot,t)\|^2_{\mathcal{H}}\right] \le G(t) \\ &\quad \le \mu_2\left[\|\varepsilon_x(\cdot,t)\|^2_{\mathcal{H}} + \|\tilde{d}_x(\cdot,t)\|^2_{\mathcal{H}} + \|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}} + \tilde{w}^2(1,t)\right]. \end{aligned} \tag{66}$$

We find the derivative of $G(t)$ along the solution of system (60) to obtain

$$\begin{aligned} &\dot{G}(t) \le -\frac{c_0}{3}\varepsilon^2(0,t) - \frac{c_0}{2}\tilde{d}^2(0,t) - \frac{1}{3}\|\varepsilon_x(\cdot,t)\|^2_{\mathcal{H}} \\ &-\frac{1}{2}\|\tilde{d}_x(\cdot,t)\|^2_{\mathcal{H}} - c_1\gamma\tilde{w}^2(1,t) + (q+c_0)\gamma|\tilde{w}(0,t)||\varepsilon(0,t)| \\ &-\gamma\|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}} + q(q+c_0)\gamma|\varepsilon(0,t)\langle e^{qx}, \tilde{w}(\cdot,t)\rangle_{\mathcal{H}}|. \end{aligned} \tag{67}$$

By Hölder's inequality and Young's inequality, it follows that

$$\begin{aligned} &\left|\varepsilon(0,t)\langle e^{qx}, \tilde{w}(\cdot,t)\rangle_{\mathcal{H}}\right| \le |\varepsilon(0,t)|\, e^q\int_0^1 |\tilde{w}(x,t)|\,dx \\ &\le e^q\,|\varepsilon(0,t)|\,\|\tilde{w}(\cdot,t)\|_{\mathcal{H}} \le \frac{\kappa e^{2q}}{2}|\varepsilon(0,t)|^2 + \frac{1}{2\kappa}\,\|\tilde{w}(\cdot,t)\|^2_{\mathcal{H}}, \end{aligned} \tag{68}$$

and

$$|\tilde{w}(0,t)\varepsilon(0,t)| \le \frac{\kappa}{2}\varepsilon^2(0,t) + \frac{1}{2\kappa}\tilde{w}^2(0,t), \tag{69}$$

where κ is a positive constant so that

$$\kappa > \max\left\{\left(1+\frac{4}{\pi^2}\right)\frac{q+c_0}{2} + \frac{2q(q+c_0)}{\pi^2}, \frac{(2+q)(q+c_0)}{2c_1}\right\}. \tag{70}$$

We combine (67), (68), and (69) to obtain

$$\begin{aligned}\dot{G}(t) \le & -\left[\frac{c_0}{3} - \frac{1}{2}\gamma\kappa(q+c_0) - \frac{1}{2}\kappa\gamma q(q+c_0)e^{2q}\right]\varepsilon^2(0,t) \\ & - \frac{c_0}{2}\tilde{d}^2(0,t) - \frac{1}{3}\|\varepsilon_x(\cdot,t)\|^2_{\mathcal{H}} - \frac{1}{2}\|\tilde{d}_x(\cdot,t)\|^2_{\mathcal{H}} \\ & - c_1\gamma\tilde{w}^2(1,t) + \frac{\gamma(q+c_0)}{2\kappa}\tilde{w}^2(0,t) \\ & - \gamma\|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}} + \frac{\gamma q(q+c_0)}{2\kappa}\|\tilde{w}(\cdot,t)\|^2_{\mathcal{H}}.\end{aligned} \tag{71}$$

By Poincaré inequality,

$$\|\tilde{w}(\cdot,t)\|^2_{\mathcal{H}} \le \tilde{w}^2(1,t) + \frac{4}{\pi^2}\|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}}, \tag{72}$$

and by Agmon's inequality,

$$\begin{aligned}|\tilde{w}(0,t)|^2 & \le |\tilde{w}(1,t)|^2 + 2\|\tilde{w}(\cdot,t)\|_{\mathcal{H}}\|\tilde{w}_x(\cdot,t)\|_{\mathcal{H}} \\ & \le 2\tilde{w}^2(1,t) + \left(\frac{4}{\pi^2}+1\right)\|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}}.\end{aligned} \tag{73}$$

We combine (71), (20), and (73) to obtain

$$\begin{aligned}\dot{G}(t) \le & -\left[\frac{c_0}{3} - \frac{1}{2}\gamma\kappa(q+c_0) - \frac{1}{2}\kappa\gamma q(q+c_0)e^{2q}\right]\varepsilon^2(0,t) \\ & -\frac{c_0}{2}\tilde{d}^2(0,t) - \frac{1}{3}\|\varepsilon_x(\cdot,t)\|^2_{\mathcal{H}} - \frac{1}{2}\|\tilde{d}_x(\cdot,t)\|^2_{\mathcal{H}} \\ & -\left[c_1 - \frac{q+c_0}{\kappa} - \frac{q(q+c_0)}{2\kappa}\right]\gamma\tilde{w}^2(1,t) \\ & -\left[1-\left(1+\frac{4}{\pi^2}\right)\frac{q+c_0}{2\kappa} - \frac{2q(q+c_0)}{\kappa\pi^2}\right]\gamma\|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}}.\end{aligned} \tag{74}$$

Set

$$\begin{cases}\vartheta_0 = c_1 - \dfrac{q+c_0}{\kappa} - \dfrac{q(q+c_0)}{2\kappa}, \\ \vartheta = 1 - \left(1+\dfrac{4}{\pi^2}\right)\dfrac{q+c_0}{2\kappa} - \dfrac{2q(q+c_0)}{\kappa\pi^2}.\end{cases}$$

Then $\vartheta, \vartheta_0 > 0$ thanks to (70). If we pick γ small enough so that

$$\left(\frac{c_0}{3} - \frac{1}{2}\gamma\kappa(q+c_0) - \frac{1}{2}\kappa\gamma q(q+c_0)e^{2q}\right) > 0, \tag{75}$$

then (74) becomes

$$\begin{aligned}\dot{G}(t) & \le -\frac{1}{3}\|\varepsilon_x(\cdot,t)\|^2_{\mathcal{H}} - \frac{1}{2}\|\tilde{d}_x(\cdot,t)\|^2_{\mathcal{H}} \\ & \quad -\vartheta_0\gamma\tilde{w}^2(1,t) - \vartheta\gamma\|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}} \\ & \le -\theta\Big[\|\varepsilon_x(\cdot,t)\|^2_{\mathcal{H}} + \|\tilde{d}_x(\cdot,t)\|^2_{\mathcal{H}} \\ & \quad + \|\tilde{w}_x(\cdot,t)\|^2_{\mathcal{H}} + \tilde{w}^2(1,t)\Big],\end{aligned} \tag{76}$$

where

$$\theta = \min\left\{\frac{1}{3}, \frac{1}{2}, \vartheta\gamma, \vartheta_0\gamma\right\}. \tag{77}$$

This together with (66) yields

$$\dot{G}(t) \le -\frac{\theta}{\mu_2}G(t). \tag{78}$$

Finally, the above process could also be applied to compute

$$\langle \mathbb{A}(f,g,h),(f,g,h)\rangle_{\mathcal{X}} \le 0, \ \forall (f,g,h) \in D(\mathbb{A}). \tag{79}$$

So $\mathbb{A}$ is dissipative in $\mathcal{X}$. This together with (78) shows that the C_0-semigroup generated by $\mathbb{A}$ is exponentially stable. This completes the proof of the proposition. □

Proof of Theorem 4.1. Since the closed-loop system (56) is a still a nonlinear system, direct treatment seems difficult. Hence, we first convert the closed-loop system (56) into an almost linear one of the following:

$$\begin{cases}\varepsilon_t(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = c_0\varepsilon(0,t), \ \varepsilon_x(1,t) = \tilde{d}_x(1,t), \\ \tilde{d}_t(x,t) = \tilde{d}_{xx}(x,t), \\ \tilde{d}_x(0,t) = c_0\tilde{d}(0,t), \ \tilde{d}(1,t) = 0, \\ \tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + q(q+c_0)e^{qx}\varepsilon(0,t), \\ \tilde{w}_x(0,t) = -(q+c_0)\varepsilon(0,t), \\ \tilde{w}_x(1,t) = -c_1\tilde{w}(1,t), \\ \hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = c_0\hat{z}(0,t), \\ \hat{z}_x(1,t) = f(\varepsilon(\cdot,t) + (I+\mathbb{P})^{-1}\tilde{w}(\cdot,t)) + d(t), \\ \varepsilon(x,0) = \varepsilon_0(x), \tilde{d}(x,0) = \tilde{d}_0(x), \tilde{w}(x,0) = \tilde{w}_0(x), \\ \hat{z}(x,0) = \hat{z}_0(x)\end{cases} \tag{80}$$

by the following invertible transformation:

$$\begin{pmatrix} w \\ \hat{d} \\ z \\ \hat{w}\end{pmatrix} = \begin{pmatrix} I & 0 & (I+\mathbb{P})^{-1} & 0 \\ 0 & -I & 0 & I \\ I & 0 & (I+\mathbb{P})^{-1} & -I \\ 0 & 0 & (I+\mathbb{P})^{-1} & 0\end{pmatrix}\begin{pmatrix}\varepsilon \\ \tilde{d} \\ \tilde{w} \\ \hat{z}\end{pmatrix}, \tag{81}$$

where $(I+\mathbb{P})^{-1}$ is defined as (51). Although system (80) is still a nonlinear one but it can be divided into two linear subsystems. The "$(\varepsilon, \tilde{d}, \tilde{w})$-part", which happens to be system (60), is independent of the "$\hat{z}$-part". Therefore, we can first solve the linear "$(\varepsilon, \tilde{d}, \tilde{w})$-part", and then solve "$\hat{z}$-part" which has linear principal and nonlinear inhomogeneous term $f(\varepsilon(\cdot,t) + (I+\mathbb{P})^{-1}\tilde{w}(\cdot,t)) + d(t)$ which has been obtained from the linear part. This is a remarkable mathematical merit of this design.

Define

$$\begin{cases}\varepsilon_0(x) = w_0(x) - \hat{w}_0(x), \\ \tilde{d}_0(x) = w_0(x) - z_0(x) - \hat{d}_0(x), \\ \tilde{w}_0(x) = \hat{w}_0(x) + q\displaystyle\int_0^x e^{q(x-s)}\hat{w}_0(s)ds.\end{cases} \tag{82}$$

Then $(\varepsilon_0, \tilde{d}_0, \tilde{w}_0) \in \mathcal{X}$. By Proposition 5.1, system (60) with the initial conditions (82) admits a unique solution $(\varepsilon, \tilde{d}, \tilde{w}) \in C(0,\infty;\mathcal{X})$ such that (64) holds.

The continuity of the function $f(\cdot)$ implies that

$$\sup_{t\in[0,\infty)}|f(\varepsilon(\cdot,t) + (I+\mathbb{P})^{-1}\tilde{w}(\cdot,t))| < +\infty. \tag{83}$$

Since $(\varepsilon, \tilde{d}, \tilde{w}) \in C(0,\infty;\mathcal{X})$ is well defined, the "$\hat{z}$-part" of system (80) now becomes a linear system with an inhomogeneous term $f(\varepsilon(\cdot,t) + (I+\mathbb{P})^{-1}\tilde{w}(\cdot,t)) + d(t)$. That is

$$\begin{cases}\hat{z}_t(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = c_0\hat{z}(0,t), \\ \hat{z}_x(1,t) = f(\varepsilon(\cdot,t) + (I+\mathbb{P})^{-1}\tilde{w}(\cdot,t)) + d(t), \\ \hat{z}(x,0) = \hat{z}_0(x),\end{cases} \tag{84}$$

with the initial value $\hat{z}_0(x) = w_0(x) - z_0(x)$. Owing to $d \in L^\infty(0,\infty)$, it follows from (83) that $[f(w(\cdot,t)) + d(t)] \in L^\infty(0,\infty)$. Therefore, by Lemma 2.1, the solution $\hat{z} \in C(0,\infty;\mathcal{H})$ is well defined and

$$\sup_{t\in[0,\infty)}\|\hat{z}(\cdot,t)\|_{\mathcal{H}} < +\infty. \tag{85}$$

Now, we have obtained that $(\varepsilon, \tilde{d}, \tilde{w}, \hat{z}) \in C(0, \infty; \mathcal{H}^4)$ is a solution of system (80). Owing to the equivalent transformation (81), it is easy to find that $(w, \hat{w}, \hat{d}, z) \in C(0, \infty; \mathcal{H}^4)$ solves closed-loop system (56) and satisfies (58).

When $d \equiv 0$ and $f(0) = 0$, it follows from (64) and the continuity of $f(\cdot)$ that

$$|f(\varepsilon(\cdot, t) + (I + \mathbb{P})^{-1}\tilde{w}(\cdot, t))| \to 0 \text{ as } t \to \infty. \tag{86}$$

By Lemma 2.1, the solution $\hat{z} \in C(0, \infty; \mathcal{H})$ satisfies

$$\|\hat{z}(\cdot, t)\|_{\mathcal{H}} \to 0 \text{ as } t \to \infty, \tag{87}$$

which, together with (64) and (81), leads to (59) easily. This completes the proof of the theorem. □

Corollary 5.1. *In addition to the assumptions in Theorem 4.1, if we assume further that the initial state $(w_0, \hat{w}_0, \hat{d}_0, z_0)$ satisfies the compatibility condition (47), then convergence (48) holds, which implies that $\hat{d}_x(1, t)$ can be regarded as an approximate of the total disturbance $f(w(\cdot, t)) + d(t)$ as $t \to \infty$.*

Proof. By Proposition 5.1, system (60) with the initial conditions (82) admits a unique solution $(\varepsilon, \tilde{d}, \tilde{w}) \in C(0, \infty; \mathcal{X})$. By compatibility condition (47), $\tilde{d}(\cdot, 0) \in D(\mathcal{A})$. Since the "$\tilde{d}$-part" in system (60) is independent of the other part, the convergence (48) then follows from (28) and (36). □

Remark 5.1. When there is no total disturbance, system (80) becomes a linear coupled system. By Proposition 5.1, it is easy to find that system (80) is exponentially stable and so is for the closed-loop (56) due to (81). This property is not only important itself in applications but is also crucial to asymptotic stability of the closed-loop system in the presence of total disturbance because the total disturbance is only approximately (not completely) canceled in real time.

6. Numerical simulation

To demonstrate our controller visually, we present in this section some numerical simulations for system (56). The finite difference scheme is adopted in discretization. The numerical results are programmed in Matlab. The time step and the space step are taken as $dt = 0.001$ and $dx = 0.05$, respectively. Obviously, the total disturbance like $f(w(\cdot, t)) = \int_0^1 w^2(x, t)dx$ satisfies condition of Theorem 4.1 but our class may cover more. To showcase numerically, we choose total disturbance to be $f(w(\cdot, t)) + d(t) = \int_0^1 w^3(x, t)dx + 0.1\xi(t)$ which does not satisfy condition of Theorem 4.1 but the closed-loop is still convergent as shown below, where $\xi(t)$ is the sawtooth disturbance generated by Matlab. The heat equation with an integral condition can model various physical phenomena in the context of chemical engineering, thermoelasticity, population dynamics, heat conduction process, control theory, medical science, life sciences, and so forth. The initial values and the tuning parameters are chosen as $w(x, 0) = 1 - x$, $\hat{w}(x, 0) = \hat{d}(x, 0) = z(x, 0) = 0$, $q = 1$, $c_0 = c_1 = 10$. The solution of system (56) is plotted in Fig. 2. The feedback control, and the total disturbance and its estimate $\hat{d}_x(1, t)$ are plotted in Fig. 3(a), and the uncontrolled system is plotted in Fig. 3(b). It is seen that the disturbance is estimated effectively. The convergence is very fast and smooth.

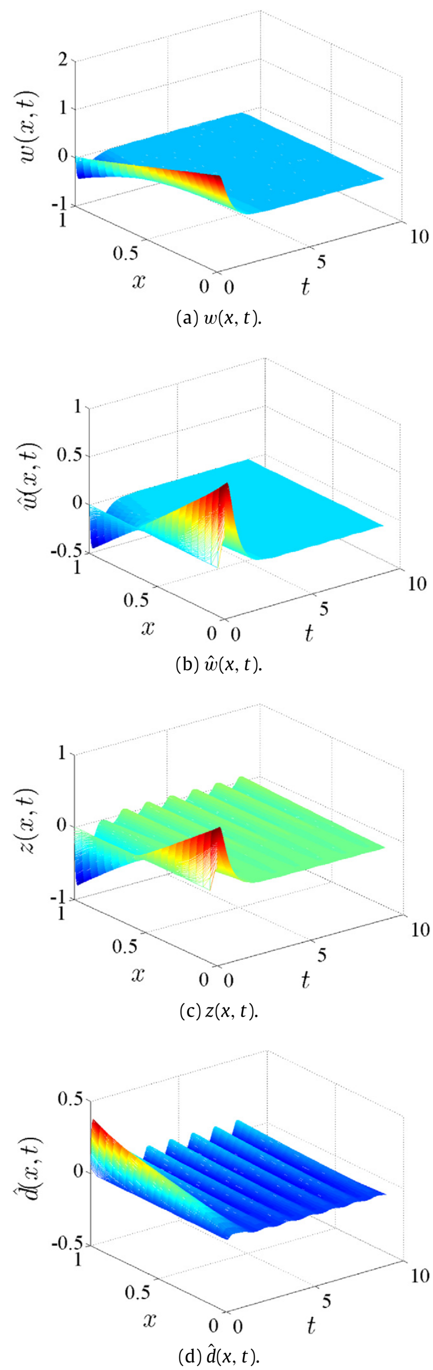

Fig. 2. Simulation for system (56).

7. Concluding remarks

In this paper, we propose a new method to stabilize 1-d unstable heat equation with unknown boundary uncertainty and external disturbance on control boundary. A new type of unknown input state observer is designed with essentially estimation of unknown input, which is very different to variable structure method in Guo and Jin (2015). It is remarkable that the convergence rate of disturbance estimation is independent of the disturbance itself, which implies that the uncertainty is sufficiently estimated in real time. In addition, the proposed disturbance estimator does not use high gain that is very different from the existing results (Guo & Zhao, 2015). By exploiting backstepping method, we are able to design a stabilizing feedback control for the observer and hence

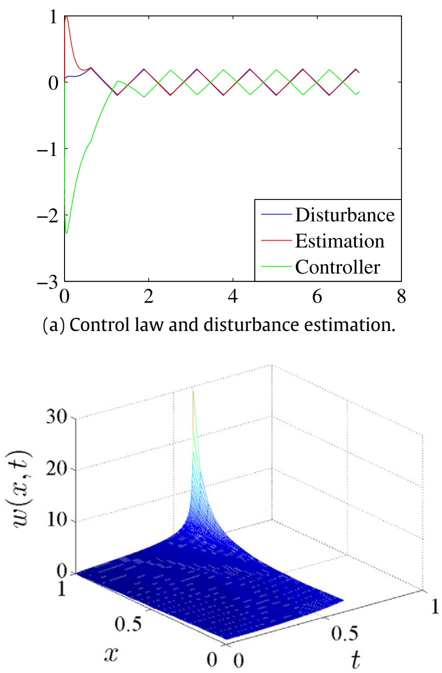

Fig. 3. Feedback control, disturbance estimation, and uncontrolled system.

an observer based output feedback for original unstable system with unknown input. This is a first effort not only estimating the external disturbance but also internal uncertainty for a PDE system. The approach is easy to be applied for other PDEs.

The approach used in this paper seems difficult to extend to disturbances acting in-domain (unless the control is also in-domain) and at the unactuated boundary because we need the control to compensate the disturbance (unless we do not pursue the asymptotic stability).

Finally, we indicate a possible problem that needs further investigation. This is about the infinite-dimensional nature of the extended state observer. The convergence of the discrete scheme for infinite-dimensional observer makes the feedback controller practically applicable and hence is an important issue to be addressed in the future work.

Appendix

For any $q > 0$, we define the operator A_q as the following:

$$\begin{cases} [A_q f](x) = f''(x), \ \forall f \in D(A_q), \\ D(A_q) = \left\{ f \in H^2(0,1) | f'(0) = -qf(0), f'(1) = 0 \right\}. \end{cases}$$

Then A_q generates a C_0-semigroup on $\mathcal{H}$. In addition, it follows from Liu and Wang (2015, Lemma 2.2) that the operator $B = \delta(x-1)$ is admissible to the C_0-semigroup $e^{A_q t}$. Now, we write system (1) into the following abstract form:

$$\frac{d}{dt} w(\cdot,t) = A_q w(\cdot,t) + B[f(w(\cdot,t)) + d(t) + u(t)]. \tag{88}$$

Theorem A.1. *Suppose that the initial state $w_0 \in \mathcal{H}$, $d, u \in L^2_{loc}(0,\infty)$ and function $f \in C(\mathcal{H};\mathbb{R})$ satisfies the Lipschitz condition:*

$$|f(v_1) - f(v_2)| \le L\|v_1 - v_2\|_{\mathcal{H}}, \ \forall \ v_1, v_2 \in \mathcal{H} \tag{89}$$

for some positive constant L. Then, system (1) admits a unique solution $w \in C(0,\infty;\mathcal{H})$, which can be written as

$$w(\cdot,t) = e^{A_q t} w_0 + \int_0^t e^{A_q(t-s)} B[f(w(\cdot,s)) + d(s) + u(s)]ds, \ \forall t \ge 0. \tag{90}$$

Proof. The proof is very similar to Pazy (1983, Theorem 1.2, p.184) where the Banach contraction principle is utilized to treat the nonlinear term. Here we only give a sketch of the proof. For initial state $w_0(x)$ and $T > 0$, we define a mapping $\mathscr{P} : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ by

$$\mathscr{P}v(\cdot,t) := e^{A_q t} w_0 + \int_0^t e^{A_q(t-s)} B[f(v(\cdot,s)) + d(s) + u(s)]ds. \tag{91}$$

Owing to the admissibility of the operator B, this mapping is well defined. Define the norm on $C(0,T;\mathcal{H})$ by $\|v\|_\infty = \sup_{t\in[0,T]} \|v(\cdot,t)\|$ for $\forall \ v \in C(0,T;\mathcal{H})$. Then, a simple computation gives

$$\begin{aligned} &\|\mathscr{P}v_1(\cdot,t) - \mathscr{P}v_2(\cdot,t)\|_{\mathcal{H}} \\ &= \left\| \int_0^t e^{A_q(t-s)} B[f(v_1(\cdot,s)) - f(v_2(\cdot,s))]ds \right\|_{\mathcal{H}} \\ &\le L_B T |f(v_1(\cdot,s)) - f(v_2(\cdot,s))|_{L^\infty(0,T)} \\ &\le L_B T L \|v_1 - v_2\|_\infty, \end{aligned} \tag{92}$$

where L_B is a positive constant independent of T. Hence,

$$\|\mathscr{P}v_1(\cdot,t) - \mathscr{P}v_2(\cdot,t)\|_\infty \le L_B T L \|v_1 - v_2\|_\infty. \tag{93}$$

For T small enough $T < \frac{1}{L_B L}$ and by a well known Banach contraction principle $\mathscr{P}$ has a unique fixed point w in $C(0,T;\mathcal{H})$. This fixed point is the desired solution of integral equation (90) on $[0,T]$. Similarly, this local solution can be extended to $[0,2T]$. As a consequence, the global solution can be obtained by induction. □

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