Trajectory Planning Approach to Output Tracking for a 1-D Wave Equation

Hongyinping Feng, Bao-Zhu Guo, and Xiao-Hui Wu

***Abstract*—In this article, we propose a trajectory planning approach to deal with various of the noncollocated configurations of output tracking through a one-dimensional wave equation. We mainly consider two noncollocated configurations: the performance output is noncollocated to the control input and the disturbance is noncollocated to the measurement output. By proper trajectory planning, the noncollocated configurations can be converted into the collocated ones so that the conventional method can be applied. An error-based feedback is proposed to realize the output tracking. Finally, as an application, the output tracking with general harmonic disturbance and reference signal are exemplified. Numerical simulation shows that the proposed approach is very effective.**

***Index Terms*—Error-based feedback, noncollocated configuration, observer, output tracking, wave equation.**

I. Introduction

OUTPUT tracking is one of the fundamental issues in control theory. In most of the situations, we are only concerned with the performance output tracking for a control system and, importantly, the reference signal is usually not specified due to disturbance. At the same time, we need to guarantee that all the states of internal loops are uniformly bounded and the system is internally asymptotically stable when the disturbance and reference are disconnected to the system. The output tracking has been studied systematically for lumped parameter systems by the internal model principle (IMP) since from [2], [3], [9]. Partial results have been generalized to the infinite-dimensional systems, see, for instance, [1], [4], [5], [21], [24], [28]–[30], among many others. A recent interesting work is [25] where output tracking problem was considered for a general 2×2 system of first-order linear hyperbolic partial differential equations (PDEs) but no uncertainty and disturbance were taken into consideration. Very recently, a noncollocated output tracking problem was considered in [15] by adaptive control method and an early effort by adaptive control can also be found in [16], by rejecting the harmonic disturbance only.

In our previous work [11], we proposed a new active disturbance rejection control to stabilize an antistable wave equation. This method can also be applied to deal with interior unknown dynamics in [34] and output tracking in [33]. The idea of this estimation and cancelation strategy was applied to an output tracking for a heat equation with unbounded control and unbounded observation in [22]. The main idea of [22] is that an extended state observer can be constructed to estimate the state and the general external disturbance, but the control and performance output are collocated. The same idea was applied in [35] to an output regulation problem for a wave equation with general boundary disturbance where the control and regulated output are also collocated.

As aforementioned, the IMP is one of the most powerful systematic approaches in dealing with output regulation and output tracking, which has been extensively studied for abstract infinite-dimensional regular linear systems in [27], [29], and the references therein. However, the applications from abstract theory to PDEs usually need to show existence of solutions to some Sylvester equations, which turns out to be a difficult, see, for instance, a one-dimensional (1-D) heat example presented in [29, sec. 9]. A recent study shows that the regulation problem can also be solved by constructing special solutions for regulator equations from which the controllers can be designed in terms of kernel equations [5], [6]. With a combination of the IMP and backstepping approach, the regulation problem via state feedback control has been solved for some PDEs in [7] and [12]. In [7], second-order hyperbolic PDEs with spatially varying coefficient was discussed and the disturbances can be in-domain and boundary disturbances, and in [12], an n-coupled wave equation with spatially varying coefficients was considered with both in-domain as well as boundary disturbances. An interesting situation of [12] is that the regulated output can be pointwise or distributed in-domain or defined on a boundary.

In this article, we propose a direct method to solve various of noncollocated output tracking problems for a 1-D wave equation via error-based feedback control. We demonstrate the whole process of the control design in terms of the technique of trajectory planning. The problem that we consider is the output

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tracking for the following 1-D wave equation:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), & x \in (0,1), \quad t > 0 \\ w_x(0,t) = d(t), & w_x(1,t) = u(t), \quad t \geq 0 \end{cases} \tag{1.1}$$

where $u(t)$ is the control input and $d(t)$ is the external disturbance. For a given reference signal $y_{\rm ref}(t)$, we aim at designing an error-based feedback control so that

$$w(0,t) \to y_{\rm ref}(t) \text{ as } t \to \infty. \tag{1.2}$$

The only measurement available for the control design is the tracking error between the performance output $w(0,t)$ and the reference signal $y_{\rm ref}(t)$

$$y_{\rm e}(t) = y_{\rm ref}(t) - w(0,t). \tag{1.3}$$

There are a couple of reasons to consider model (1.1). First, it is a suitable PDE to demonstrate the approach without involving much complicated mathematics for our approach is quite general to be applied to other PDEs. Second, the system (1.1) is a hyperbolic PDE, which means that it is truly infinite-dimensional. One cannot truncate it into high-order ordinary differential equations (ODEs) because the high frequencies are just as important as low frequencies. In addition, there are many engineering problems that can be well described by output tracking of the wave equation like wharf gantry cranes carrying cargo in marine industry [20], container cranes in port automation, and flexible links in gantry robots. One of the typical examples is the flexible crane system, which consists of a cable and payload. The actuator is fixed at one end of the cable and the other end is the payload. The payload driven by the controller needs to achieve the tracking of given commands. This physical mechanism can be modeled as an output tracking of wave equation, for details, we refer to [17]. Other engineering problems of output tracking of the wave equation can be found in piezoelectric stack actuators [26] and moving string systems with tip payload [18].

The reference signal $y_{\rm ref}(t)$ and the disturbance $d(t)$ are supposed to be generated by an exosystem of the following:

$$\begin{cases} \dot{v}(t) = Gv(t) \\ d(t) = Qv(t), \quad y_{\rm ref}(t) = Fv(t) \end{cases} \tag{1.4}$$

where $G \in \mathbb{C}^{n\times n}$, $Q \in \mathbb{C}^{1\times n}$, and $F \in \mathbb{C}^{1\times n}$ are known but the initial state $v(0)$ is unknown, which makes both $y_{\rm ref}(t)$ and $d(t)$ unknown to some extent.

When the input $u(t) \equiv 0$, system (1.1) and exosystem (1.4) constitute a cascade observation system

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) \\ w_x(0,t) = Qv(t), \quad w_x(1,t) = 0 \\ \dot{v}(t) = Gv(t) \\ y_{\rm e}(t) = Fv(t) - w(0,t). \end{cases} \tag{1.5}$$

For notational simplicity, we omit in equations hereafter the obvious domains for both time t and spatial variable x when there is no confusion. For a continuous complex function $f : \mathbb{C} \to \mathbb{C}$ defined over complex plane $\mathbb{C}$, the matrix $f(G)$ can be defined, see, for instance [19, Definition 1.2, p. 3]. In the output regulation problems (1.1) and (1.2), we always assume that

$$(G, FG\sinh G + Q\cosh G) \text{ is observable} \tag{1.6}$$

which is almost equivalent to the approximate observability of system (1.5). It is seen from (1.5) that exosystem (1.4) is independent of the control plant (1.1) and the information transmission from the exosystem to the control plant is unidirectional. Precisely, the signal of the exosystem enters the control plant as a disturbance $w_x(0,t) = Qv(t)$ and also through the error-based feedback control due to $y_{\rm e}(t) = Fv(t) - w(0,t)$. Roughly speaking, the assumption (1.6) implies that the whole information of the exosystem is injected into the control plant, and the observer can therefore be possibly designed. When the assumption (1.6) does not hold, there exist some dynamic of exosystem, which cannot be reflected by the measured output. In fact, when the pair $(G, FG\sinh G + Q\cosh G)$ is not observable, by Hautus's lemma, there exist $\lambda \in \sigma(G)$ and $0 \neq h \in \mathbb{C}^n$ such that

$$(FG\sinh G + Q\cosh G)h = 0 \text{ and } Gh = \lambda h. \tag{1.7}$$

Define $v(t) = e^{\lambda t}h$ and

$$w(x,t) = [F\cosh(xG) + xQ\mathcal{G}(xG)]\, v(t) \tag{1.8}$$

where

$$\mathcal{G}(z) = \begin{cases} \dfrac{\sinh z}{z}, & z \neq 0, z \in \mathbb{C} \\ 1, & z = 0. \end{cases} \tag{1.9}$$

By a simple computation, it is easy to check that such a defined (w, w_t, v) is a nonzero solution of system (1.5). However, the measured regulation error $y_{\rm e}(t) = y_{\rm ref}(t) - w(0,t) \equiv 0$. In other words, assumption (1.6) is not stronger than approximately observable for system (1.5).

The problem (1.1) is significantly special from two aspects. First, the measurement for controller design is only the error between performance output and reference. Since the signals of the disturbance and the state are mixed up in the measurement, we need to separate them properly before making an estimation rather estimating them directly. This configuration brings a big obstacle for state observer or disturbance estimator design. Second, the control and performance output are noncollocated. This implies that the control action must propagate through the entire spatial domain to reach the performance output to perform the function, which requires a deep understanding about the control plant.

We proceed as follows. In Section II, we first design an observer for a transformed system of (1.1), which gives an estimate of $v(t)$ by the first time trajectory planning. To transfer the disturbance from noncollocated position to the control end, we use the second time trajectory planning in Section III. As a result, an error-based control is designed. The stability analysis for the closed-loop system is performed in Section IV. In Section V, we consider another noncollocated problem where the regulated output and disturbance are noncollocated. Finally, we apply the result to harmonic disturbance, which covers particularly the result of [15] in Section VI. The proofs of the main results are presented in Section VII. In Section VIII, we present some numerical simulations for illustration, followed by conclusions in Section IX.

Some notations: the spectrum of the operator A is denoted by $\sigma(A)$, the inner product of Hilbert space $L^2(0,1)$ is denoted by $\langle\cdot,\cdot\rangle_{L^2(0,1)}$, and the corresponding norm is denoted by $\|\cdot\|_{L^2(0,1)}$.

II. Trajectory Planning for Observer Design

Before designing an observer for (1.1), we need to separate the disturbance properly from the measured error. In this section, we introduce a new idea by constructing a proper trajectory so that the disturbance appears in both the equation and the output. The motivation will be seen clearly once the observer is designed. Suppose that this "proper trajectory" takes the following form:

$$\begin{cases}\phi_{1tt}(x,t)=\phi_{1xx}(x,t)\\ \phi_1(0,t)=P_1v(t),\quad \phi_{1x}(0,t)=P_2v(t)\end{cases}\tag{2.1}$$

where P_1 and P_2 are n-dimensional row vectors, which will be determined later. Inspired by [8, Ch. 4] and [23, Ch. 12], we try to find a special solution of system (2.1) in the following form:

$$\phi_1(x,t)=\sum_{n=0}^{\infty}\alpha_n(t)\frac{x^n}{n!}.\tag{2.2}$$

Substituting (2.2) into system (2.1), it can be found that the coefficients $\alpha_n(t)$ satisfies

$$\begin{cases}\ddot{\alpha}_n(t)=\alpha_{n+2}(t),\quad n=0,1,2,\ldots\\ \alpha_0(t)=P_1v(t),\quad \alpha_1(t)=P_2v(t)\end{cases}\tag{2.3}$$

which produces in turn

$$\begin{aligned}\phi_1(x,t)&=\sum_{n=0}^{\infty}\left[P_1v^{(2n)}(t)\frac{x^{2n}}{(2n)!}+P_2v^{(2n)}(t)\frac{x^{2n+1}}{(2n+1)!}\right]\\ &=\left[P_1\left(\sum_{n=0}^{\infty}\frac{G^{2n}x^{2n}}{(2n)!}\right)+P_2\left(\sum_{n=0}^{\infty}\frac{G^{2n}x^{2n+1}}{(2n+1)!}\right)\right]v(t).\end{aligned}\tag{2.4}$$

That is

$$\phi_1(x,t)=[P_1\cosh(xG)+xP_2\mathcal{G}(xG)]\,v(t)\tag{2.5}$$

where $\mathcal{G}$ is defined by (1.9). Since $\mathcal{G}$ is continuous on $\mathbb{C}$, the matrix $\mathcal{G}(xG)$ is well defined for any $x\in[0,1]$. In addition

$$\phi_{1x}(1,t)=(P_1G\sinh G+P_2\cosh G)v(t).\tag{2.6}$$

Choose P_1 and P_2 specially so that

$$P_1G\sinh G+P_2\cosh G=0\tag{2.7}$$

which results in $\phi_{1x}(1,t)=0$. Define the transform

$$w_1(x,t)=w(x,t)-\phi_1(x,t).\tag{2.8}$$

System (1.1) is then transformed into

$$\begin{cases}w_{1tt}(x,t)=w_{1xx}(x,t)\\ w_{1x}(0,t)=Q_1v(t),\ w_{1x}(1,t)=u(t)\\ y_{\rm e}(t)=F_1v(t)-w_1(0,t)\end{cases}\tag{2.9}$$

where

$$F_1=F-P_1,\quad Q_1=Q-P_2.\tag{2.10}$$

Superficially, (2.9) and (1.1) take the same form but in (1.1), the Q in $d(t)=Qv(t)$ is fixed whereas, in (2.9), Q_1 can be regulated, which will be seen in the sequel. We then design an observer for system (2.9) in incorporating $\dot{v}(t)=Gv(t)$ as follows:

$$\begin{cases}\hat{w}_{1tt}(x,t)=\hat{w}_{1xx}(x,t)\\ \hat{w}_{1x}(0,t)=c_1[\hat{w}_1(0,t)+y_{\rm e}(t)]\\ \qquad\qquad +c_2[\hat{w}_{1t}(0,t)+\dot{y}_{\rm e}(t)]\\ \hat{w}_{1x}(1,t)=u(t)\\ \dot{\hat{v}}(t)=G\hat{v}(t)+[y_{\rm e}(t)-F_1\hat{v}(t)+\hat{w}_1(0,t)]K\end{cases}\tag{2.11}$$

where c_1,c_2 are positive constants and K is a column vector such that the matrix $G-KF_1$ is Hurwitz. The motivation of this observer design seems a little bit intricate. However, it becomes clear after examination of the error dynamics. In fact, let

$$\tilde{w}_1(x,t)=w_1(x,t)-\hat{w}_1(x,t),\ \ \tilde{v}(t)=v(t)-\hat{v}(t).\tag{2.12}$$

If we choose P_1 and P_2 specially such that

$$\begin{aligned}&Q_1-c_1F_1-c_2F_1G\\ &\quad=Q-P_2-c_1(F-P_1)-c_2(F-P_1)G=0\end{aligned}\tag{2.13}$$

then, the error $(\tilde{w}_1,\tilde{v})$ is governed by "target" stable wave equation

$$\begin{cases}\tilde{w}_{1tt}(x,t)=\tilde{w}_{1xx}(x,t)\\ \tilde{w}_{1x}(0,t)=c_1\tilde{w}_1(0,t)+c_2\tilde{w}_{1t}(0,t)\\ \tilde{w}_{1x}(1,t)=0\\ \dot{\tilde{v}}(t)=(G-KF_1)\tilde{v}(t)+\tilde{w}_1(0,t)K.\end{cases}\tag{2.14}$$

System (2.14) is a cascade of two exponentially stable systems, and hence, is expected to be exponentially stable. Now it is clear: We add the "damping" term $c_1\tilde{w}_1(0,t)+c_2\tilde{w}_{1t}(0,t)$ at the left end of the string, and at the same time, make the matrix $G-KF_1$ be Hurwitz by proper construction of K. The idea of such construction is inspired by [10] where various of coupled systems were proposed by decoupling the coupled system as the controlled plants with their dynamic feedbacks. We consider system (2.14) in the state space

$$\mathcal{H}=H^1(0,1)\times L^2(0,1)\times\mathbb{C}^n\tag{2.15}$$

in which the inner product is equipped with

$$\begin{aligned}&\langle(f_1,g_1,h_1),(f_2,g_2,h_2)\rangle_{\mathcal{H}}=\langle f_1',f_2'\rangle_{L^2(0,1)}\\ &\quad+\langle g_1,g_2\rangle_{L^2(0,1)}+c_1f_1(0)\overline{f_2(0)}+\langle h_1,h_2\rangle_{\mathbb{C}^n}\\ &\quad\forall\,(f_i,g_i,h_i)\in\mathcal{H},\ i=1,2.\end{aligned}\tag{2.16}$$

System (2.14) can be written abstractly as

$$\begin{aligned}&\frac{d}{dt}(\tilde{w}_1(\cdot,t),\tilde{w}_{1t}(\cdot,t),\tilde{v}(t))\\ &\quad=\mathcal{A}_1(\tilde{w}_1(\cdot,t),\tilde{w}_{1t}(\cdot,t),\tilde{v}(t))\end{aligned}\tag{2.17}$$

where the operator $\mathcal{A}_1 : D(\mathcal{A}_1) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\begin{cases} \mathcal{A}_1(f,g,h) = (g, f'', (G-KF_1)h + f(0)K) \\ \forall\, (f,g,h) \in D(\mathcal{A}_1) = \big\{(f,g,h) \mid f \in H^2(0,1) \\ \quad g \in H^1(0,1), h \in \mathbb{C}^n, f'(1) = 0 \\ \quad f'(0) = c_1 f(0) + c_2 g(0)\big\}. \end{cases} \tag{2.18}$$

The following Lemma 2.1 shows that (2.11) is indeed an observer for (2.9).

Lemma 2.1: Suppose that $c_1, c_2 > 0$, $K \in \mathbb{C}^{n\times 1}$, $F_1 \in \mathbb{C}^{1\times n}$, and the matrix $G - KF_1$ is Hurwitz. Then, the operator $\mathcal{A}_1$ defined by (2.18) generates an exponentially stable C_0-semigroup $e^{\mathcal{A}_1 t}$ on $\mathcal{H}$, i.e., there exist two positive constants $L_{\mathcal{A}_1}$ and $\omega_{\mathcal{A}_1}$ such that

$$\left\|e^{\mathcal{A}_1 t}\right\| \le L_{\mathcal{A}_1} e^{-\omega_{\mathcal{A}_1} t} \quad \forall\, t \ge 0. \tag{2.19}$$

The well-posedness of the open-loop system (2.9) and observer (2.11) is presented in Theorem 2.1.

Theorem 2.1: Suppose that $c_1, c_2 > 0, Q_1, F_1 \in \mathbb{C}^{1\times n}, K \in \mathbb{C}^{n\times 1}$, the matrix $G - KF_1$ is Hurwitz, and $Q_1 - c_1F_1 - c_2F_1G = 0$. Then, for any initial state $(w_1(\cdot,0), w_{1t}(\cdot,0), v(0), \hat{w}_1(\cdot,0), \hat{w}_{1t}(\cdot,0), \hat{v}(0)) \in \mathcal{H}^2$ and $u \in L^2_{\rm loc}(0,\infty)$, systems (1.4), (2.9), and (2.11) admit a unique solution $(w_1, w_{1t}, v, \hat{w}_1, \hat{w}_{1t}, \hat{v}) \in C([0,\infty); \mathcal{H}^2)$ such that

$$\begin{aligned} &\|(w_1(\cdot,t) - \hat{w}_1(\cdot,t), w_{1t}(\cdot,t) - \hat{w}_{1t}(\cdot,t) \\ &\quad v(t) - \hat{v}(t))\|_{\mathcal{H}} \le L_1 e^{-\omega_1 t},\ \ t \ge 0 \end{aligned} \tag{2.20}$$

where L_1 and ω_1 are two positive constants.

To end this section, we point out that the introduction of the trajectory planning is mainly for the observer design. Our observer (2.11) is for system (2.9) where Q_1 is regulatable not for original system (1.1) where Q is fixed. The error-based observer design turns out to be the most difficult problem in the output tracking for PDEs.

III. Trajectory Planning for Controller Design

By virtue of (2.9), we can estimate successfully the disturbance $Qv(t)$ by $Q\hat{v}(t)$. However, (2.9) is not suitable for control design because the control and disturbance are still "noncollocated" there. To overcome this difficulty, we need a different trajectory planning to move the disturbance from the left end to the control end so that the control can cancel the disturbance in the feedback loop. This is the objective of this section.

Similar to (2.1), we suppose that the desired trajectory satisfies the following equation:

$$\begin{cases} \phi_{2tt}(x,t) = \phi_{2xx}(x,t) \\ \phi_2(0,t) = Fv(t), \quad \phi_{2x}(0,t) = Qv(t) \end{cases} \tag{3.1}$$

which, similar to (2.1), admits a special solution

$$\phi_2(x,t) = [F\cosh(xG) + xQ\mathcal{G}(xG)]v(t). \tag{3.2}$$

If we define the transform

$$w_2(x,t) = w(x,t) - \phi_2(x,t) \tag{3.3}$$

the controlled plant (1.1) can be converted into

$$\begin{cases} w_{2tt}(x,t) = w_{2xx}(x,t) \\ w_{2x}(0,t) = 0, \quad w_{2x}(1,t) = Q_2 v(t) + u(t) \\ y_{\rm e}(t) = -w_2(0,t) \end{cases} \tag{3.4}$$

where

$$Q_2 = -(FG\sinh G + Q\cosh G). \tag{3.5}$$

Now we can say a few words on the motivation. Compared with the original system (1.1) with control and disturbance not being collocated, the disturbance has been moved to the control end in system (3.4), and more importantly, the measured output tracking error in original system (1.1) has become a usual boundary measurement in system (3.4). Since

$$\begin{aligned} w(0,t) &= w_2(0,t) + \phi_2(0,t) \\ &= w_2(0,t) + Fv(t) = w_2(0,t) + y_{\rm ref}(t) \end{aligned} \tag{3.6}$$

if we can stabilize system (3.4), then, $w_2(0,t) \to 0$ and hence $w(0,t) \to y_{\rm ref}(t)$. In other words, the output tracking problem is transformed into stabilization problem for system (3.4), which is much easy to be dealt with.

By (2.8) and (3.3)

$$\begin{aligned} w_2(x,t) &= w_1(x,t) + \phi_1(x,t) - \phi_2(x,t) \\ &= w_1(x,t) + \Phi(x)v(t) \end{aligned} \tag{3.7}$$

where $\Phi : [0,1] \to \mathbb{C}^n$ is a vector valued function given by

$$\Phi(x) = (P_1 - F)\cosh(xG) + x(P_2 - Q)\mathcal{G}(xG). \tag{3.8}$$

The following Proposition 3.1 gives an estimate of $w_2(x,t)$ in terms of the measured output tracking error.

Proposition 3.1: Under the assumptions in Theorem 2.1, for any initial state

$$(w_2(\cdot,0), w_{2t}(\cdot,0), v(0), \hat{w}_1(\cdot,0), \hat{w}_{1t}(\cdot,0), \hat{v}(0)) \in \mathcal{H}^2$$

and $u \in L^2_{\rm loc}(0,\infty)$, systems (1.4), (2.11), and (3.4) admit a unique solution

$$(w_2, w_{2t}, v, \hat{w}_1, \hat{w}_{1t}, \hat{v}) \in C([0,\infty); \mathcal{H}^2) \tag{3.9}$$

such that

$$\|(\varepsilon_0(\cdot,t), \varepsilon_1(\cdot,t), v(t) - \hat{v}(t))\|_{\mathcal{H}} \le L_2 e^{-\omega_2 t},\ t \ge 0 \tag{3.10}$$

where L_2 and ω_2 are two positive constants, and

$$\begin{cases} \varepsilon_0(x,t) = w_2(x,t) - [\hat{w}_1(x,t) + \Phi(x)\hat{v}(t)] \\ \varepsilon_1(x,t) = w_{2t}(x,t) - [\hat{w}_{1t}(x,t) + \Phi(x)G\hat{v}(t)]. \end{cases} \tag{3.11}$$

This shows that system (2.11) can be regarded as a state observer of system (3.4) as well.

Now we are in a position to design an observer-based stabilizing output feedback control for system (3.4). Thanks to Proposition 3.1, we have an estimate $\hat{w}_1(\cdot,t) + \Phi(\cdot)\hat{v}(t)$ for $w_2(\cdot,t)$, and $\hat{w}_{1t}(\cdot,t) + \Phi(\cdot)G\hat{v}(t)$ for $w_{2t}(\cdot,t)$. Hence, the control design for (3.4) is the same as the disturbance free counterpart after canceling the disturbance $Q_2v(t)$ by $Q_2\hat{v}(t)$, which now reads

$$\begin{aligned} u(t) = & -Q_2\hat{v}(t) - c_3[\hat{w}_{1t}(1,t) + \Phi(1)G\hat{v}(t)] \\ & - c_4[\hat{w}_1(1,t) + \Phi(1)\hat{v}(t)] \\ = & -\Psi_1\hat{v}(t) - c_4\hat{w}_1(1,t) - c_3\hat{w}_{1t}(1,t) \end{aligned} \tag{3.12}$$

where c_3, c_4 are positive tuning parameters and the vector $\Psi_1 \in \mathbb{C}^{1\times n}$ is defined by

$$\Psi_1 = Q_2 + c_3\Phi(1)G + c_4\Phi(1). \tag{3.13}$$

Solving equations (2.7) and (2.13), we obtain that

$$\begin{cases} P_1 = -[Q - F(c_1 I + c_2 G)]\cosh G[f_1(G)]^{-1} \\ P_2 = [Q - F(c_1 I + c_2 G)]G\sinh G[f_1(G)]^{-1} \\ f_1(G) = G\sinh G + (c_1 I + c_2 G)\cosh G \end{cases} \tag{3.14}$$

provided that the matrix $f_1(G)$ is invertible. It then follows from (2.10) and (3.14) that

$$F_1 = (FG\sinh G + Q\cosh G)[f_1(G)]^{-1}. \tag{3.15}$$

Combining (1.1), (2.11), and (3.12), and taking the parameter matrices (3.15), (3.5), and (3.13) into account, we derive the closed-loop system as follows:

$$\begin{cases} \dot{v}(t) = Gv(t) \\ w_{tt}(x,t) = w_{xx}(x,t) \\ w_x(0,t) = Qv(t) \\ w_x(1,t) = -\Psi_1\hat{v}(t) - c_4\hat{w}_1(1,t) - c_3\hat{w}_{1t}(1,t) \\ \hat{w}_{1tt}(x,t) = \hat{w}_{1xx}(x,t) \\ \hat{w}_{1x}(0,t) = c_1[\hat{w}_1(0,t) + y_{\mathrm{e}}(t)] \\ \qquad\qquad +c_2[\hat{w}_{1t}(0,t) + \dot{y}_{\mathrm{e}}(t)] \\ \hat{w}_{1x}(1,t) = -\Psi_1\hat{v}(t) - c_4\hat{w}_1(1,t) - c_3\hat{w}_{1t}(1,t) \\ \dot{\hat{v}}(t) = G\hat{v}(t) + [y_{\mathrm{e}}(t) - F_1\hat{v}(t) + \hat{w}_1(0,t)]K \\ y_{\mathrm{e}}(t) = Fv(t) - w(0,t) \end{cases} \tag{3.16}$$

where K is a column vector such that the matrix $G - KF_1$ is Hurwitz. For the sake of easy reading, the corresponding tuning vectors are clustered as follows:

$$\begin{cases} \Psi_1 = c_3\Phi(1)G + c_4\Phi(1) \\ \qquad -(FG\sinh G + Q\cosh G) \\ F_1 = (FG\sinh G + Q\cosh G)[f_1(G)]^{-1} \\ f_1(G) = G\sinh G + (c_1 I + c_2 G)\cosh G \\ P_1 = -[Q - F(c_1 I + c_2 G)]\cosh G[f_1(G)]^{-1} \\ P_2 = [Q - F(c_1 I + c_2 G)]G\sinh G[f_1(G)]^{-1} \\ \Phi(1) = (P_1 - F)\cosh G + (P_2 - Q)\mathcal{G}(G) \\ = -(FG\sinh G + Q\cosh G)f_0(G)[f_1(G)]^{-1} \\ f_0(G) = \cosh G + (c_1 I + c_2 G)\mathcal{G}(G). \end{cases} \tag{3.17}$$

Remark 3.1: We point out that the trajectory planning can be realized also by a well-established solution using the regulator equations. Actually, put (1.1) and (1.4) together to obtain

$$\begin{cases} \dot{v}(t) = Gv(t) \\ w_{tt}(x,t) = w_{xx}(x,t) \\ w_x(0,t) = Qv(t),\ w_x(1,t) = u(t) \\ y_{\mathrm{e}}(t) = Fv(t) - w(0,t). \end{cases} \tag{3.18}$$

Introduce the tracking error

$$e(x,t) = w(x,t) - \Sigma(x)v(t) \text{ with } \Sigma(x) \in \mathbb{C}^{1\times n} \tag{3.19}$$

in which $\Sigma(x)v(t)$ is the particular steady state to achieve output regulation in the presence of modeled disturbance. A simple calculation gives

$$\begin{cases} \dot{v}(t) = Gv(t) \\ e_{tt}(x,t) = e_{xx}(x,t) \\ e_x(0,t) = 0,\ e_x(1,t) = -\Sigma'(1)v(t) + u(t) \\ y_{\mathrm{e}}(t) = -e(0,t) \end{cases} \tag{3.20}$$

provided that $\Sigma(x)$ is the solution of the following regulator equation:

$$\begin{cases} \Sigma''(x) = \Sigma(x)G^2 \\ \Sigma'(0) = Q,\ \Sigma(0) = F \end{cases} \tag{3.21}$$

which has a solution

$$\Sigma(x) = F\cosh(xG) + xQ\mathcal{G}(xG). \tag{3.22}$$

It is seen that the e-subsystem in (3.20) is almost the same as (3.4). It is therefore sufficient to stabilize system (3.20) to achieve output tracking.

From Sections II and III, we see that double trajectory plannings are performed in the whole process of the controller (3.12) design. Both of them are carried out in the spirit of estimation and cancelation strategy. The first trajectory planning is used to estimate simultaneously the disturbance and the state. After the first planning, observer (2.11) is designed successfully to estimate both the disturbance $d = Qv$ and the state $(w_1(\cdot,t), w_{1t}(\cdot,t))$. In order to compensate the disturbance by its estimation, we need the second trajectory planning to convert the disturbance into the control channel, which can be seen from (2.9) and (3.4) where the disturbance and control are collocated. The second trajectory planning of designing state feedback can be replaced by a well-established output regulation method, but through trajectory planning, we can see clearly why the feedback control should be what it likes by transforming a noncollocated problem into a collocated one.

The initial idea of this trajectory planning method comes from the monograph [23, Ch. 12] where the trajectory planning was used to generate a state reference trajectory for PDEs. This approach has been applied to tracking control design for wave equation in [26]. Theoretically, output regulation problem (1.1) can also be solved by the internal model principal but the existence of solutions of some Sylvester equations needs to be justified, which is not so easy for a PDE (see, e.g., [29, Sec. 9]).

IV. Exponential Stability of the Closed-Loop System

In this section, we consider exponential stability for the closed-loop system (3.16). Let

$$\begin{cases} w_2(x,t) = w(x,t) - \Phi_2(x)v(t) \\ w_{2t}(x,t) = w_t(x,t) - \Phi_2(x)Gv(t) \\ \tilde{w}_1(x,t) = w(x,t) - \Phi_1(x)v(t) - \hat{w}_1(x,t) \\ \tilde{w}_{1t}(x,t) = w_t(x,t) - \Phi_1(x)Gv(t) - \hat{w}_{1t}(x,t) \\ \tilde{v}(t) = v(t) - \hat{v}(t) \end{cases} \tag{4.1}$$

where

$$\begin{cases} \Phi_1(x) = P_1\cosh(xG) + xP_2\mathcal{G}(xG) \\ \Phi_2(x) = F\cosh(xG) + xQ\mathcal{G}(xG). \end{cases} \tag{4.2}$$

In terms of transformation (4.1), the closed-loop system (3.16) can be converted into

$$\begin{cases} w_{2tt}(x,t) = w_{2xx}(x,t) \\ w_{2x}(0,t) = 0 \\ w_{2x}(1,t) = \Psi_1\tilde{v}(t) + c_3\tilde{w}_{1t}(1,t) - c_3w_{2t}(1,t) \\ \qquad\qquad +c_4\tilde{w}_1(1,t) - c_4w_2(1,t) \\ \tilde{w}_{1tt}(x,t) = \tilde{w}_{1xx}(x,t) \\ \tilde{w}_{1x}(0,t) = c_1\tilde{w}_1(0,t) + c_2\tilde{w}_{1t}(0,t) \\ \tilde{w}_{1x}(1,t) = 0 \\ \dot{\tilde{v}}(t) = (G-KF_1)\tilde{v}(t) + \tilde{w}_1(0,t)K \end{cases} \tag{4.3}$$

where F_1, Ψ_1 are given by (3.17), and K is a column vector such that the matrix $G - KF_1$ is Hurwitz. We consider system (4.3) in the state space $\mathcal{X} = H^1(0,1)\times L^2(0,1)\times\mathcal{H}$ equipped with the inner product

$$\begin{aligned} &\langle (f_1,g_1,\phi_1,\psi_1,h_1),(f_2,g_2,\phi_2,\psi_2,h_2)\rangle_{\mathcal{X}} \\ &\quad = \langle f_1', f_2'\rangle_{L^2(0,1)} + \langle g_1,g_2\rangle_{L^2(0,1)} \\ &\qquad + c_4f_1(1)\overline{f_2(1)} + \langle(\phi_1,\psi_1,h_1),(\phi_2,\psi_2,h_2)\rangle_{\mathcal{H}} \\ &\qquad \forall\,(f_i,g_i,\phi_i,\psi_i,h_i)\in\mathcal{X},\ i=1,2. \end{aligned} \tag{4.4}$$

Let

$$X_1(t) = (w_2(\cdot,t), w_{2t}(\cdot,t), \tilde{w}_1(\cdot,t), \tilde{w}_{1t}(\cdot,t), \tilde{v}(t)). \tag{4.5}$$

System (4.3) can be written abstractly as

$$\frac{d}{dt}X_1(t) = \mathscr{A}_1X_1(t) \tag{4.6}$$

where the operator $\mathscr{A}_1 : D(\mathscr{A}_1)(\subset\mathcal{X})\to\mathcal{X}$ is defined by

$$\begin{cases} \mathscr{A}_1(f,g,\phi,\psi,h) = (g, f'', \mathcal{A}_1(\phi,\psi,h)) \\ \forall (f,g,\phi,\psi,h)\in D(\mathscr{A}_1) = \Big\{(f,g,\phi,\psi,h)\in\mathcal{X}\Big| \\ \quad f\in H^2(0,1), g\in H^1(0,1), f'(0)=0 \\ \quad f'(1) = -c_4f(1) + c_4\phi(1) - c_3g(1) + c_3\psi(1) \\ \qquad\qquad +\Psi_1h,\ (\phi,\psi,h)\in D(\mathcal{A}_1)\Big\} \end{cases} \tag{4.7}$$

with $\mathcal{A}_1$ being given by (2.18).

Lemma 4.1: Suppose that $c_i > 0, i = 1,2,3,4$, $K\in\mathbb{C}^{n\times 1}$, $\Psi_1, F_1\in\mathbb{C}^{1\times n}$, and the matrix $G - KF_1$ is Hurwitz. Then, the operator $\mathscr{A}_1$ defined by (4.7) generates an exponentially stable C_0-semigroup $e^{\mathscr{A}_1 t}$ on $\mathcal{X}$, i.e., there exist two positive constants $L_{\mathscr{A}_1}$ and $\omega_{\mathscr{A}_1}$ such that

$$\left\|e^{\mathscr{A}_1 t}\right\| \le L_{\mathscr{A}_1}e^{-\omega_{\mathscr{A}_1}t} \quad \forall\, t\ge 0. \tag{4.8}$$

The main result of this article is Theorem 4.1.

Theorem 4.1: Suppose that (1.6), $c_i > 0, i = 1,2,3,4$, and the matrix G satisfies

$$\sigma(G)\subset\{\lambda \mid \mathrm{Re}\lambda\ge 0\}. \tag{4.9}$$

Then, for any initial state

$$(w(\cdot,0), w_t(\cdot,0), v(0), \hat{w}_1(\cdot,0), \hat{w}_{1t}(\cdot,0), \hat{v}(0))\in\mathcal{H}^2$$

the closed-loop system (3.16) with setting (3.17) admits a unique solution

$$(w, w_t, v, \hat{w}_1, \hat{w}_{1t}, \hat{v})\in C([0,\infty);\mathcal{H}^2) \tag{4.10}$$

satisfying

$$|Qv(t) - Q\hat{v}(t)| + |y_{\mathrm{e}}(t)| \le L_3e^{-\omega_3 t} \quad \forall\, t\ge 0 \tag{4.11}$$

for some constants $L_3 > 0$ and $\omega_3 > 0$. Moreover,

1) If $\sup_{t\in[0,\infty)}\|v(t)\|_{\mathbb{C}^n} < +\infty$, then, the state of closed-loop (3.16) is uniformly bounded, i.e.,

$$\begin{aligned} \sup_{t\in[0,\infty)} \|(w(\cdot,t), w_t(\cdot,t), v(t) \\ \hat{w}_1(\cdot,t), \hat{w}_{1t}(\cdot,t), \hat{v}(t))\|_{\mathcal{H}^2} < +\infty; \end{aligned} \tag{4.12}$$

2) When $v(t)\equiv 0$, then, there exist two positive constants L_4 and ω_4 such that

$$\begin{aligned} &\|(w(\cdot,t), w_t(\cdot,t), v(t), \hat{w}_1(\cdot,t) \\ &\hat{w}_{1t}(\cdot,t), \hat{v}(t))\|_{\mathcal{H}^2} \le L_4e^{-\omega_4 t},\ t\ge 0. \end{aligned} \tag{4.13}$$

V. Different Noncollocated Configuration

In this section, we continue to consider system (1.1) but the performance output is at the control end

$$y_{\mathrm{e}}(t) = y_{\mathrm{ref}}(t) - w(1,t)\to 0 \text{ as } t\to\infty \tag{5.1}$$

where $w(1,t)$ is the performance output and $y_{\mathrm{ref}}(t)$ is still the reference signal generated by exosystem (1.4).

It turns out that the state feedback controller design in this case is slightly easier than the previous one. The reason behind this lies in that the control can take action on the performance output directly, for it is relatively "close" to the performance output. However, the disturbance estimation in this case is more difficult than the previous one. This is because the measurement is noncollocated to the disturbance. In this case, we need the trajectory planning to convert the noncollocated configuration into a collocated one. Suppose now that the desired auxiliary trajectory satisfies the following system:

$$\begin{cases} \phi_{3tt}(x,t) = \phi_{3xx}(x,t), \quad x\in(0,1), \quad t>0 \\ \phi_3(0,t) = P_3v(t), \quad \phi_{3x}(0,t) = Qv(t), \quad t\ge 0 \end{cases} \tag{5.2}$$

where $P_3 \in \mathbb{C}^{1\times n}$ will be determined later. Similar to (2.4), system (5.2) admits a special solution

$$\phi_3(x,t) = \Phi_3(x)v(t) \tag{5.3}$$

where

$$\Phi_3(x) = P_3\cosh(xG) + xQ\mathcal{G}(xG) \tag{5.4}$$

and $\mathcal{G}$ is defined by (1.9). If we let

$$w_3(x,t) = w(x,t) - \phi_3(x,t) \tag{5.5}$$

then, the error $w_3(x,t)$ is governed by

$$\begin{cases} w_{3tt}(x,t) = w_{3xx}(x,t) \\ w_{3x}(0,t) = 0, \;\; w_{3x}(1,t) = Q_0 v(t) + u(t) \\ y_{\mathrm{e}}(t) = F_0 v(t) - w_3(1,t) \end{cases} \tag{5.6}$$

where

$$\begin{cases} F_0 = F - P_3\cosh G - Q\mathcal{G}(G) \\ Q_0 = -P_3 G\sinh G - Q\cosh G. \end{cases} \tag{5.7}$$

Once again, the measurement/disturbance noncollocated configuration in original system (1.1) has been transferred into a collocated one in system (5.6).

Similar to Section II, the observer for system (5.6) and (1.4) can be designed as

$$\begin{cases} \hat{w}_{3tt}(x,t) = \hat{w}_{3xx}(x,t) \\ \hat{w}_{3x}(0,t) = 0 \\ \hat{w}_{3x}(1,t) = -c_1[\hat{w}_3(1,t) + y_{\mathrm{e}}(t)] \\ \qquad\qquad -c_2[\hat{w}_{3t}(1,t) + \dot{y}_{\mathrm{e}}(t)] + u(t) \\ \dot{\hat{v}}(t) = G\hat{v}(t) + [y_{\mathrm{e}}(t) + \hat{w}_3(1,t) - F_0\hat{v}(t)]K_0 \end{cases} \tag{5.8}$$

where c_1, c_2 are positive tuning parameters, and K_0 is a column vector such that the matrix $G - K_0F_0$ is Hurwitz. If we choose specially $P_3 \in \mathbb{C}^{1\times n}$ in (5.7) such that

$$Q_0 + c_1F_0 + c_2F_0G = 0 \tag{5.9}$$

and let

$$\tilde{w}_3(x,t) = w_3(x,t) - \hat{w}_3(x,t), \tilde{v}(t) = v(t) - \hat{v}(t) \tag{5.10}$$

then, the error $(\tilde{w}_3, \tilde{v})$ is governed by

$$\begin{cases} \tilde{w}_{3tt}(x,t) = \tilde{w}_{3xx}(x,t) \\ \tilde{w}_{3x}(0,t) = 0 \\ \tilde{w}_{3x}(1,t) = -c_1\tilde{w}_3(1,t) - c_2\tilde{w}_{3t}(1,t) \\ \dot{\tilde{v}}(t) = (G - K_0F_0)\tilde{v}(t) + \tilde{w}_3(1,t)K_0 \end{cases} \tag{5.11}$$

which is a cascade of two exponentially stable systems. System (5.11) can be written abstractly as

$$\frac{d}{dt}(\tilde{w}_3(\cdot,t), \tilde{w}_{3t}(\cdot,t), \tilde{v}(t)) = \mathcal{A}_2(\tilde{w}_3(\cdot,t), \tilde{w}_{3t}(\cdot,t), \tilde{v}(t))$$

where the operator $\mathcal{A}_2 : D(\mathcal{A}_2) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\begin{cases} \mathcal{A}_2(f,g,h) = (g, f'', (G - K_0F_0)h + f(1)K_0) \\ \forall (f,g,h) \in D(\mathcal{A}_2) = \Big\{(f,g,h) \mid f \in H^2(0,1) \\ \qquad g \in H^1(0,1), h \in \mathbb{C}^n, f'(0) = 0 \\ \qquad f'(1) = -c_1 f(1) - c_2 g(1)\Big\}. \end{cases} \tag{5.12}$$

Lemma 5.1: Suppose that $c_1, c_2 > 0$, $K_0 \in \mathbb{C}^{n\times 1}$, $F_0 \in \mathbb{C}^{1\times n}$, and the matrix $G - K_0F_0$ is Hurwitz. Then, the operator $\mathcal{A}_2$ defined by (5.12) generates an exponentially stable C_0-semigroup $e^{\mathcal{A}_2 t}$ on $\mathcal{H}$, i.e., there exist two positive constants $L_{\mathcal{A}_2}$ and $\omega_{\mathcal{A}_2}$ such that

$$\left\|e^{\mathcal{A}_2 t}\right\| \le L_{\mathcal{A}_2} e^{-\omega_{\mathcal{A}_2} t} \quad \forall\, t \ge 0. \tag{5.13}$$

Proof: System (5.11) is a cascade of two exponentially stable systems and hence is exponentially stable. The proof is very similar to Lemma 2.1 and we omit the details. ■

Theorem 5.1 gives well-posedness of the open-loop system (5.6) and observer (5.8).

Theorem 5.1: Suppose that $c_1, c_2 > 0$, $K_0 \in \mathbb{C}^{n\times 1}$, Q_0, $F_0 \in \mathbb{C}^{1\times n}$, the matrix $G - K_0F_0$ is Hurwitz, and (5.9) holds. Then, for any initial state

$$(w_3(\cdot,0), w_{3t}(\cdot,0), v(0), \hat{w}_3(\cdot,0), \hat{w}_{3t}(\cdot,0), \hat{v}(0)) \in \mathcal{H}^2$$

and $u \in L^2_{loc}(0,\infty)$, systems (1.4), (5.6), and (5.8) admit a unique solution $(w_3, w_{3t}, v, \hat{w}_3, \hat{w}_{3t}, \hat{v}) \in C([0,\infty); \mathcal{H}^2)$ such that

$$\begin{aligned} &\|(w_3(\cdot,t) - \hat{w}_3(\cdot,t), w_{3t}(\cdot,t) - \hat{w}_{3t}(\cdot,t) \\ &\quad v(t) - \hat{v}(t))\|_{\mathcal{H}} \le L_5 e^{-\omega_5 t}, \; t \ge 0 \end{aligned} \tag{5.14}$$

where L_5 and ω_5 are two positive constants.

In order to transfer the output regulation problem into a stabilization problem, we need the following trajectory planning:

$$\begin{cases} \phi_{4tt}(x,t) = \phi_{4xx}(x,t), \; x \in (0,1), \; t > 0 \\ \phi_4(1,t) = Fv(t), \quad \phi_{4x}(0,t) = Qv(t), \quad t \ge 0. \end{cases} \tag{5.15}$$

System (5.15) has a special solution

$$\begin{cases} \phi_4(x,t) = \Phi_4(x)v(t) \\ \Phi_4(x) = P_4\cosh(xG) + xQ\mathcal{G}(xG) \end{cases} \tag{5.16}$$

where P_4 is a row vector such that $\Phi_4(1) = F$. If we define the transform

$$w_4(x,t) = w(x,t) - \phi_4(x,t) \tag{5.17}$$

the controlled plant can then be converted into

$$\begin{cases} w_{4tt}(x,t) = w_{4xx}(x,t) \\ w_{4x}(0,t) = 0, \; w_{4x}(1,t) = -\Phi_4'(1)v(t) + u(t) \\ y_{\mathrm{e}}(t) = -w_4(1,t). \end{cases} \tag{5.18}$$

Now, the output regulation problem of the original system (1.1) has been converted into the stabilization problem of system (5.18). It therefore suffice to stabilize system (5.18). In terms of the observer (5.8), the error-based feedback control is naturally designed as

$$\begin{aligned} u(t) &= \Phi_4'(1)\hat{v}(t) - c_3 w_{4t}(1,t) - c_4 w_4(1,t) \\ &= \Phi_4'(1)\hat{v}(t) + c_3\dot{y}_{\rm e}(t) + c_4 y_{\rm e}(t) \end{aligned} \tag{5.19}$$

where c_3 and c_4 are positive tuning parameters. Combining (5.19), (5.8), and (1.1), we obtain the following closed-loop system:

$$\begin{cases} \dot{v}(t) = Gv(t) \\ w_{tt}(x,t) = w_{xx}(x,t) \\ w_x(0,t) = Qv(t) \\ w_x(1,t) = \Phi_4'(1)\hat{v} + c_3\dot{y}_{\rm e}(t) + c_4 y_{\rm e}(t) \\ \hat{w}_{3tt}(x,t) = \hat{w}_{3xx}(x,t) \\ \hat{w}_{3x}(0,t) = 0 \\ \hat{w}_{3x}(1,t) = -c_1[\hat{w}_3(1,t) + y_{\rm e}(t)] + c_4 y_{\rm e}(t) \\ \quad -c_2[\hat{w}_{3t}(1,t) + \dot{y}_{\rm e}(t)] + \Phi_4'(1)\hat{v} + c_3\dot{y}_{\rm e}(t) \\ \dot{\hat{v}}(t) = G\hat{v}(t) + [y_{\rm e}(t) + \hat{w}_3(1,t) - F_0\hat{v}(t)]K_0 \\ y_{\rm e}(t) = Fv(t) - w(1,t) \end{cases} \tag{5.20}$$

where K_0 is a column vector such that the matrix $G - K_0F_0$ is Hurwitz. For the sake of easy reading, the corresponding tuning vectors are clustered as follows:

$$\begin{cases} F_0 = (FG\sinh G + Q)[f_1(G)]^{-1} \\ f_1(G) = G\sinh G + (c_1 I + c_2 G)\cosh G \\ \Phi_4'(1) = [FG\sinh G + Q][\cosh G]^{-1}. \end{cases} \tag{5.21}$$

In order to prove the stability of the closed-loop system (5.20), we first consider the following transformed system:

$$\begin{cases} w_{4tt}(x,t) = w_{4xx}(x,t) \\ w_{4x}(0,t) = 0 \\ w_{4x}(1,t) = -\Phi_4'(1)\tilde{v}(t) - c_4 w_4(1,t) \\ \qquad\qquad -c_3 w_{4t}(1,t) \\ \tilde{w}_{3tt}(x,t) = \tilde{w}_{3xx}(x,t) \\ \tilde{w}_{3x}(0,t) = 0 \\ \tilde{w}_{3x}(1,t) = -c_1\tilde{w}_3(1,t) - c_2\tilde{w}_{3t}(1,t) \\ \dot{\tilde{v}}(t) = (G - K_0F_0)\tilde{v}(t) + \tilde{w}_3(1,t)K_0. \end{cases} \tag{5.22}$$

System (5.22) can be written abstractly in $\mathcal{X}$ as

$$\frac{d}{dt}X_2(t) = \mathscr{A}_2 X_2(t) \tag{5.23}$$

where

$$X_2(t) = (w_4(\cdot,t), w_{4t}(\cdot,t), \tilde{w}_3(\cdot,t), \tilde{w}_{3t}(\cdot,t), \tilde{v}(t)) \tag{5.24}$$

and the operator $\mathscr{A}_2 : D(\mathscr{A}_2) \subset \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{cases} \mathscr{A}_2(f,g,\phi,\psi,h) = (g, f'', \mathcal{A}_2(\phi,\psi,h)) \\ \forall (f,g,\phi,\psi,h) \in D(\mathscr{A}_2) \\ D(\mathscr{A}_2) = \Big\{(f,g,\phi,\psi,h) \in \mathcal{X} \mid \\ f \in H^2(0,1), g \in H^1(0,1), f'(0) = 0 \\ f'(1) = -c_4 f(1) - c_3 g(1) - \Phi_4'(1)h \\ (\phi,\psi,h) \in D(\mathcal{A}_2)\Big\} \end{cases} \tag{5.25}$$

with $\mathcal{A}_2$ being defined by (5.12).

Lemma 5.2: Suppose that $c_i > 0, i = 1,2,3,4$, and the matrix $G - K_0F_0$ is Hurwitz. Then, the operator $\mathscr{A}_2$ defined by (5.25) generates an exponentially stable C_0-semigroup $e^{\mathscr{A}_2 t}$ on $\mathcal{X}$, i.e., there exist two positive constants $L_{\mathscr{A}_2}$ and $\omega_{\mathscr{A}_2}$ such that

$$\left\|e^{\mathscr{A}_2 t}\right\| \le L_{\mathscr{A}_2} e^{-\omega_{\mathscr{A}_2} t} \quad \forall\, t \ge 0. \tag{5.26}$$

Proof: The proof is very similar to Lemma 4.1 and we omit the details. ■

The main result of this section is in Theorem 5.2.

Theorem 5.2: Suppose that $(G, FG\sinh G + Q)$ is observable, $c_i > 0, i = 1,2,3,4$, $\cosh G$ is invertible, and (4.9) holds. Then, for any initial state

$$(w(\cdot,0), w_t(\cdot,0), v(0), \hat{w}_3(\cdot,0), \hat{w}_{3t}(\cdot,0), \hat{v}(0)) \in \mathcal{H}^2$$

the closed-loop system (5.20) with setting (5.21) admits a unique solution

$$(w, w_t, v, \hat{w}_3, \hat{w}_{3t}, \hat{v}) \in C([0,\infty); \mathcal{H}^2) \tag{5.27}$$

satisfying

$$|Qv(t) - Q\hat{v}(t)| + |y_{\rm e}(t)| \le L_6 e^{-\omega_6 t} \quad \forall\, t \ge 0 \tag{5.28}$$

for some constants $L_6 > 0$ and $\omega_6 > 0$. Moreover,

1) If $\sup_{t\in[0,\infty)} \|v(t)\|_{\mathbb{C}^n} < +\infty$, then, the state of the closed-loop (5.20) is uniformly bounded, i.e.,

$$\begin{aligned} \sup_{t\in[0,\infty)} &\|(w(\cdot,t), w_t(\cdot,t), v(t) \\ &\hat{w}_3(\cdot,t), \hat{w}_{3t}(\cdot,t), \hat{v}(t))\|_{\mathcal{H}^2} < +\infty \end{aligned} \tag{5.29}$$

2) When $v(t) \equiv 0$, there exist two positive constants L_7 and ω_7 such that

$$\begin{aligned} &\|(w(\cdot,t), w_t(\cdot,t), v(t), \hat{w}_3(\cdot,t) \\ &\quad \hat{w}_{3t}(\cdot,t), \hat{v}(t))\|_{\mathcal{H}^2} \le L_7 e^{-\omega_7 t},\ t \ge 0. \end{aligned} \tag{5.30}$$

VI. Application to Harmonic Signals

In this section, we apply the obtained results to a general harmonic disturbance and harmonic reference signal. We only consider the noncollocated performance output tracking (1.3). Another case (5.1) can be treated analogously.

Suppose that all the frequencies of the reference signal and disturbance are

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_m, \quad m \in \mathbb{N}. \tag{6.1}$$

Define the system matrix G by

$$G = \text{diag}\left(G_0, G_1, G_2, \dots, G_m\right) \tag{6.2}$$

where $G_0 = 0$

$$G_j = \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix},\ j = 1, 2, \dots, m. \tag{6.3}$$

Let $F, Q \in \mathbb{C}^{1\times(2m+1)}$. With this setting, the outputs $Fv(t)$ and $Qv(t)$ of exosystem (1.4) can represent the general harmonic signals by proper choosing the initial state.

In [15] and [16], all the amplitudes of the harmonic disturbance were estimated, which increases the order of the system rapidly, and since each component of estimates of the amplitudes is asymptotically convergent only, the convergence there is very slow. By the approach developed in this article, we estimate $d(t)$ as a whole. By Theorem 4.1, the proposed output feedback law works well provided (1.6) and (4.9) hold.

Proposition 6.1: Suppose that the matrix G is defined by (6.2) and $F, Q \in \mathbb{C}^{1\times(2m+1)}$. Then, the $(G, FG\sinh G + Q\cosh G)$ is observable provided that

$$\langle e_1, FG\sinh G + Q\cosh G\rangle^2_{\mathbb{C}^{2m+1}} \neq 0 \tag{6.4}$$

and

$$\begin{aligned}&\langle e_{2j}, FG\sinh G + Q\cosh G\rangle^2_{\mathbb{C}^{2m+1}} \\ &\quad + \langle e_{2j+1}, FG\sinh G + Q\cosh G\rangle^2_{\mathbb{C}^{2m+1}} \neq 0\end{aligned} \tag{6.5}$$

for any $j = 1, 2, \dots, m$, where

$$e_j = (0, \dots, 0, 1, 0 \dots, 0)^\top,\ \ j = 1, 2, \dots, 2m+1. \tag{6.6}$$

VII. Proofs of the Main Results

Proof of Lemma 2.1: For convenience, let $\tilde{G} = G - KF_1$. For any $(\hat{f}, \hat{g}, \hat{h}) \in \mathcal{H}$, we solve the equation $\mathcal{A}_1(f, g, h) = (\hat{f}, \hat{g}, \hat{h})$ to get

$$\begin{cases} f(x) = f(0) + \displaystyle\int_0^x \int_1^\tau \hat{g}(s)dsd\tau \\ g(x) = \hat{f}(x), \quad h = \tilde{G}^{-1}\left(\hat{h} - Kf(0)\right) \\ f(0) = -\dfrac{1}{c_1}\left[c_2\hat{f}(0) + \displaystyle\int_0^1 \hat{g}(s)ds\right]. \end{cases} \tag{7.1}$$

This implies that $\mathcal{A}_1^{-1}$ is compact on $\mathcal{H}$. Hence, $\sigma(\mathcal{A}_1)$, the spectrum of $\mathcal{A}_1$, consists of the isolated eigenvalues of finite algebraic multiplicity only. For any $(f, g, h) \in D(\mathcal{A}_1)$

$$\begin{aligned} &\text{Re}\,\langle \mathcal{A}_1(f, g, h), (f, g, h)\rangle_{\mathcal{H}} \\ &\quad = \text{Re}\langle (g, f'', \tilde{G}h + f(0)K), (f, g, h)\rangle_{\mathcal{H}} \\ &\quad \leq -c_2|g(0)|^2 + \text{Re}\langle \tilde{G}h, h\rangle_{\mathbb{C}^n} + \text{Re}\langle f(0)K, h\rangle_{\mathbb{C}^n} \\ &\quad \leq M_1\|(f, g, h)\|^2_{\mathcal{H}} \end{aligned} \tag{7.2}$$

where M_1 is a positive constant that is independent of (f, g, h). The inequality (7.2) implies that $\mathcal{A}_1 - M_1$ is dissipative in $\mathcal{H}$. This together with $0 \in \rho(\mathcal{A}_1)$ implies that $D(\mathcal{A}_1)$ is densely defined in $\mathcal{H}$ ([32, Proposition 3.1.6, p. 71]). Define the operator A_1 by

$$\begin{cases} A_1(f, g) = (g, f''), \quad \forall (f, g) \in D(A_1) \\ D(A_1) = \{(f, g) \in H^2(0, 1) \times H^1(0, 1) \\ \qquad\qquad \mid f'(1) = 0, f'(0) = c_1 f(0) + c_2 g(0)\}. \end{cases} \tag{7.3}$$

By [14], the operator A_1 generates an exponentially stable C_0-semigroup $e^{A_1 t}$ on $H^1(0, 1) \times L^2(0, 1)$. There is a sequence of generalized eigenfunctions $\{(f_k, g_k)\}_{k=1}^\infty$ of A_1, which forms a Riesz basis for $H^1(0, 1) \times L^2(0, 1)$. Suppose that $\{h_i\}_{i=1}^n$ is a sequence of generalized eigenvectors of $\tilde{G}$, which forms a basis for $\mathbb{C}^n$. A simple computation shows that $\{(0, 0, h_i)\}_{i=1}^n$ is a set of the generalized eigenvectors of $\mathcal{A}_1$. Owing to the finite dimension of $\tilde{G}$, there exits an $M > 0$ such that $\lambda_k \notin \sigma(\tilde{G})$ for all $k \geq M$. Since (f_k, g_k) is a generalized eigenfunction of A_1 associated with λ_k, for each $k \geq M$, there exists an $n_{\lambda_k} \in \mathbb{N}$ such that

$$(\lambda_k - A_1)^{n_{\lambda_k}}(f_k, g_k) = 0. \tag{7.4}$$

We can define a sequence $\{(f_k^j, g_k^j, h_k^j)\}_{j=0}^{n_{\lambda_k}}$ by the following equations:

$$(\lambda_k - \tilde{G})h_k^{j-1} - f_k^{j-1}(0)K = h_k^j, j = 1, 2, \dots, n_{\lambda_k}$$

where $h_k^{n_{\lambda_k}} = 0$, $(f_k^0, g_k^0) = (f_k, g_k)$ and

$$(f_k^j, g_k^j) = (\lambda_k - A_1)(f_k^{j-1}, g_k^{j-1}), j = 1, 2, \dots, n_{\lambda_k}. \tag{7.5}$$

By a simple computation, the sequence

$$\{(f_k^0, g_k^0, h_k^0)\}_{k=M}^\infty \cup \{(0, 0, h_i)\}_{i=1}^n \tag{7.6}$$

is a sequence of generalized eigenfunctions of $\mathcal{A}_1$. Note that the sequence

$$\{(f_k, g_k, 0)\}_{k=1}^M \cup \{(f_k^0, g_k^0, h_k^0)\}_{k=M}^\infty \cup \{(0, 0, h_i)\}_{i=1}^n \tag{7.7}$$

forms a Riesz basis for $\mathcal{H}$. It follows from [13, Th. 6.3] that there are a constant $N > M$ and generalized eigenvectors $\{(\tilde{f}_k, \tilde{g}_k, \tilde{h}_k)\}_{k=1}^N$ of $\mathcal{A}_1$ such that $\{(\tilde{f}_k, \tilde{g}_k, \tilde{h}_k)\}_{k=1}^N \cup \{(f_k^0, g_k^0, h_k^0)\}_{k=N+1}^\infty$ forms a Riesz basis for $\mathcal{H}$. As a result, the spectrum-determined growth condition holds true for $\mathcal{A}_1$.

On the other hand, we have

$$\sigma(\mathcal{A}_1) \subset \sigma(A_1) \cup \sigma(\tilde{G}). \tag{7.8}$$

Indeed, suppose that $\lambda \in \sigma(\mathcal{A}_1)$ and $\mathcal{A}_1(f, g, h) = \lambda(f, g, h)$ with $0 \neq (f, g, h) \in D(\mathcal{A}_1)$. It follows from (2.18) and (7.3) that $(f, g) \in D(A_1)$ and $\lambda \in \sigma(A_1)$, provided $(f, g) \neq 0$. When $(f, g) = 0$, then $h \neq 0$ and it follows from (2.18) that $h \in D(\tilde{G})$ and $\lambda \in \sigma(\tilde{G})$. So (7.8) is true. Owing to (7.8) and the exponential stabilities of $e^{A_1 t}$ and $e^{\tilde{G}t}$, the operator $\mathcal{A}_1$ generates an exponentially stable C_0-semigroup. This completes the proof of the Lemma. ∎

Remark 7.1: The exponential stability of system (2.14) can be proved directly by the Lyapunov analysis like [31]. However, for this 1-D problem, our approach here is much simpler and profound in the sense that the system (2.14) is a Riesz spectral system with spectrum-determined growth condition. Certainly, the Lyapunov approach has merits to deal with possibly the high-dimensional PDEs.

Proof of Theorem 2.1: Since systems (1.4) and (2.9) consist of a cascade system, they admit a unique solution $(w_1, w_{1t}, v) \in C([0,\infty); \mathcal{H})$ for any $(w_1(\cdot,0), w_{1t}(\cdot,0), v(0)) \in \mathcal{H}$ and $u \in L^2_{\rm loc}(0,\infty)$. By Lemma 2.1, system (2.14) with initial state $(\tilde{w}_1(\cdot,0), \tilde{w}_{1t}(\cdot,0), \tilde{v}(0)) = (w_1(\cdot,0) - \hat{w}_1(\cdot,0), w_{1t}(\cdot,0) - \hat{w}_{1t}(\cdot,0), v(0) - \hat{v}(0))$ admits a unique solution $(\tilde{w}_1, \tilde{w}_{1t}, \tilde{v}) \in C([0,\infty); \mathcal{H})$ such that

$$\begin{aligned}&\|(\tilde{w}_1(\cdot,t), \tilde{w}_{1t}(\cdot,t), \tilde{v}(t))\|_{\mathcal{H}} \\ &\quad \le L_{\mathcal{A}_1} e^{-\omega_{\mathcal{A}_1} t} \|(\tilde{w}_1(\cdot,0), \tilde{w}_{1t}(\cdot,0), \tilde{v}(0))\|_{\mathcal{H}}, t \ge 0. \end{aligned} \tag{7.9}$$

Let

$$\begin{cases} \hat{w}_1(x,t) = w_1(x,t) - \tilde{w}_1(x,t) \\ \hat{w}_{1t}(x,t) = w_{1t}(x,t) - \tilde{w}_{1t}(x,t) \\ \hat{v}(t) = v(t) - \tilde{v}(t). \end{cases} \tag{7.10}$$

Noting that $Q_1 - c_1 F_1 - c_2 F_1 G = 0$, it is easy to verify that such a defined $(w_1, w_{1t}, v, \hat{w}_1, \hat{w}_{1t}, \hat{v}) \in C([0,\infty); \mathcal{H}^2)$ solves the systems (1.4), (2.9), and (2.11). Moreover, (2.20) can be obtained directly by combining (7.9) and (7.10). The proof is complete. ■

Proof of Proposition 3.1: For any initial state $(w_2(\cdot,0), w_{2t}(\cdot,0), v(0)) \in \mathcal{H}$ and $u \in L^2_{\rm loc}(0,\infty)$, it is well known that the control plant (3.4) admits a unique solution $(w_2, w_{2t}, v) \in C([0,\infty); \mathcal{H})$. By Theorem 2.1, the solution of systems (1.4), (2.9), and (2.11) is well posed. It follows from (3.7) and (2.12) that

$$\begin{aligned} \varepsilon_0(x,t) &= w_2(x,t) - \hat{w}_1(x,t) - \Phi(x)\hat{v}(t) \\ &= w_1(x,t) + \Phi(x) v(t) - \hat{w}_1(x,t) - \Phi(x)\hat{v}(t) \\ &= \tilde{w}_1(x,t) + \Phi(x)\tilde{v}(t) \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} \varepsilon_1(x,t) &= w_{2t}(x,t) - \hat{w}_{1t}(x,t) - \Phi(x)G\hat{v}(t) \\ &= w_{1t}(x,t) + \Phi(x)Gv(t) - \hat{w}_{1t}(x,t) - \Phi(x)G\hat{v}(t) \\ &= \tilde{w}_{1t}(x,t) + \Phi(x)G\tilde{v}(t) \end{aligned} \tag{7.12}$$

where $\tilde{w}_1(x,t)$ and $\tilde{v}(x,t)$ are defined by (2.12). Combining (2.20), (2.12), (7.12), and (7.11), we can arrive at (3.10). This completes the proof of the proposition. ■

Proof of Lemma 4.1: For any $(\hat{f}, \hat{g}, \hat{\phi}, \hat{\psi}, \hat{h}) \in \mathcal{X}$, we solve $\mathscr{A}_1(f, g, \phi, \psi, h) = (\hat{f}, \hat{g}, \hat{\phi}, \hat{\psi}, \hat{h})$ to obtain

$$\begin{cases} g(x) = \hat{f}(x), f(x) = \displaystyle\int_1^x \int_0^\tau \hat{g}(s) ds d\tau + f(1) \\ f(1) = -\dfrac{1}{c_4}\Big[\displaystyle\int_0^1 \hat{g}(s)ds + c_3 \hat{f}(1) \\ \qquad - (\Psi_1 h + c_3 \psi(1) + c_4 \phi(1))\Big] \\ (\phi, \psi, h) = \mathcal{A}_1^{-1}(\hat{\phi}, \hat{\psi}, \hat{h}). \end{cases} \tag{7.13}$$

Hence, $\mathscr{A}_1^{-1} \in \mathcal{L}(\mathcal{X})$ is compact on $\mathcal{X}$ and $\sigma(\mathscr{A}_1)$, consisting of isolated eigenvalues of finite algebraic multiplicity only. Similar to Lemma 2.1, there exists a positive constant M_2 such that $\mathscr{A}_1 - M_2$ is dissipative in $\mathcal{X}$. Hence, $D(\mathscr{A}_1)$ is densely defined in $\mathcal{H}$ due to [32, Proposition 3.1.6, p. 71]. Define the operator A_2 by

$$\begin{cases} A_2(f,g) = (g, f'') \quad \forall (f,g) \in D(A_2) \\ D(A_2) = \{(f,g) \in H^2(0,1) \times H^1(0,1) \\ \qquad | \; f'(1) = -c_4 f(1) - c_3 g(1), f'(0) = 0\}. \end{cases} \tag{7.14}$$

By [14], A_2 generates an exponentially stable C_0-semigroup $e^{A_2 t}$ on $H^1(0,1) \times L^2(0,1)$ and there is a sequence of generalized eigenfunctions $\{(\gamma_i, \varphi_i)\}_{i=1}^\infty$ of A_2, which forms a Riesz basis for $H^1(0,1) \times L^2(0,1)$. Owing to Lemma 2.1, there exists a sequence of the generalized eigenfunctions $\{(\phi_k^0, \psi_k^0, h_k^0)\}_{k=1}^\infty$ of $\mathcal{A}_1$, which forms a Riesz basis for $\mathcal{H}$.

For any $(\phi_k^0, \psi_k^0, h_k^0)$, suppose that $(f_k^0, g_k^0, \phi_k^0, \psi_k^0, h_k^0)$ is a generalized eigenfunction of $\mathscr{A}_1$ associated with λ_k. Then, there exists a positive integer n_{λ_k} such that

$$(\lambda_k - \mathscr{A}_1)^{n_{\lambda_k}} (f_k^0, g_k^0, \phi_k^0, \psi_k^0, h_k^0) = 0 \tag{7.15}$$

and

$$(\lambda_k - \mathscr{A}_1)^{n_{\lambda_k} - 1} (f_k^0, g_k^0, \phi_k^0, \psi_k^0, h_k^0) \ne 0. \tag{7.16}$$

Owing to the cascade structure of $\mathscr{A}_1$, $(\phi_k^0, \psi_k^0, h_k^0)$ is also a generalized eigenfunction of $\mathcal{A}_1$ and hence

$$(\lambda_k - \mathcal{A}_1)^{n_{\lambda_k}} (\phi_k^0, \psi_k^0, h_k^0) = 0. \tag{7.17}$$

Using (7.17), generalized eigenfunction $(f_k^0, g_k^0, \phi_k^0, \psi_k^0, h_k^0)$ can be represented by a sequence $\{(f_k^j, g_k^j, \phi_k^j, \psi_k^j, h_k^j)\}_{j=0}^{n_{\lambda_k}}$ that is given by the following equations:

$$\begin{cases} \lambda_k f_k^{j-1} - g_k^{j-1} = f_k^j \\ \lambda_k^2 f_k^{j-1} - \lambda_k f_k^j - f_{kxx}^{j-1} = g_k^j \\ f_{kx}^{j-1}(0) = 0 \\ f_{kx}^{j-1}(1) = -c_3[\lambda_k f_k^{j-1}(1) - f_k^j(1)] \\ \qquad -c_4 f_k^{j-1}(1) + \Psi_1 h_k^{j-1} + c_3 \psi_k^{j-1}(1) \\ \qquad +c_4 \phi_k^{j-1}(1) \end{cases} \tag{7.18}$$

where $(f_k^{n_{\lambda_k}}, g_k^{n_{\lambda_k}}) = 0$, and

$$(\phi_k^j, \psi_k^j, h_k^j) = (\lambda_k - \mathcal{A}_1)(\phi_k^{j-1}, \psi_k^{j-1}, h_k^{j-1}) \tag{7.19}$$

$j = 1, 2, \ldots, n_{\lambda_k}$. By the general ordinary differential equation theory, the function (f_k^0, g_k^0) is always well defined. Hence

$$\{(\gamma_i, \varphi_i, 0, 0, 0)\}_{i=1}^\infty \cup \{(f_k^0, g_k^0, \phi_k^0, \psi_k^0, h_k^0)\}_{k=1}^\infty \tag{7.20}$$

is a sequence of the generalized eigenfunctions of $\mathscr{A}_1$ that forms a Riesz basis for $\mathcal{X}$. As a result, the spectrum-determined growth condition holds true for $\mathscr{A}_1$.

On the other hand, similar to (7.8), we also have

$$\sigma(\mathscr{A}_1) \subset \sigma(A_2) \cup \sigma(\mathcal{A}_1). \tag{7.21}$$

In fact, suppose that $\mu \in \sigma(\mathscr{A}_1)$ and $\mathscr{A}_1(f, g, \phi, \psi, h) = \mu(f, g, \phi, \psi, h)$ with $0 \ne (f, g, \phi, \psi, h) \in D(\mathscr{A}_1)$. It follows from (4.7) and (7.14) that $(\phi, \psi, h) \in D(\mathcal{A}_1)$ and $\mu \in \sigma(\mathcal{A}_1)$, provided $(\phi, \psi, h) \ne 0$. When $(\phi, \psi, h) = 0$, one can obtain that $(f, g) \ne 0$. By (4.7), it follows that $(f, g) \in D(A_2)$ and $\mu \in \sigma(A_2)$. Therefore, (7.21) holds.

Owing to the exponential stabilities of $e^{\mathcal{A}_1 t}$ and $e^{A_2 t}$, (7.21) shows that the operator $\mathscr{A}_1$ generates an exponentially stable C_0-semigroup. The proof is complete. ■

Proof of Theorem 4.1: By assumptions (4.9) and (1.6), it follows from Lemma 9.1 and Proposition 9.1 that the pair (G, F_1) is observable. Hence, there exists a vector $K \in \mathbb{C}^{n\times 1}$ such that the matrix $G - KF_1$ is Hurwitz. Using the initial state of the closed-loop system (3.16), we can define

$$\begin{cases} w_2(x,0) = w(x,0) - \Phi_2(x)v(0) \\ w_{2t}(x,0)) = w_t(x,0) - \Phi_2(x)Gv(0) \\ \tilde{w}_1(x,0) = w(x,0) - \Phi_1(x)v(0) - \hat{w}_1(x,0) \\ \tilde{w}_{1t}(x,0) = w_t(x,0) - \Phi_1(x)Gv(0) - \hat{w}_{1t}(x,0) \\ \tilde{v}(0) = v(0) - \hat{v}(0) \end{cases} \tag{7.22}$$

where Φ_1 and Φ_2 are defined by (4.2). By Lemma 4.1, system (4.3) with the initial state (7.22) admits a unique solution $(w_2, w_{2t}, v, \tilde{w}_1, \tilde{w}_{1t}, \tilde{v}) \in C([0,\infty); \mathcal{X})$ such that (7.9) holds. Define

$$\begin{cases} w(x,t) = w_2(x,t) + \Phi_2(x)v(t) \\ w_t(x,t) = w_{2t}(x,t) + \Phi_2(x)Gv(t) \\ \hat{w}_1(x,t) = w(x,t) - \Phi_1(x)v(t) - \tilde{w}_1(x,t) \\ \hat{w}_{1t}(x,t) = w_t(x,t) - \Phi_1(x)Gv(t) - \tilde{w}_{1t}(x,t) \\ \hat{v}(t) = v(t) - \tilde{v}(t). \end{cases} \tag{7.23}$$

Then, such a defined $(w, w_t, v, \hat{w}_1, \hat{w}_{1t}, \hat{v})$ satisfies (4.10) and solves the closed-loop system (3.16). The convergence (4.11) and the boundedness (4.12) can be obtained by (7.9) and (7.23) easily. When $v(t) \equiv 0$, (4.13) follows directly from (4.8) and transform (7.23). This completes the proof of the theorem. ■

Proof of Theorem 5.1: For any $(w_3(\cdot,0), w_{3t}(\cdot,0), v(0)) \in \mathcal{H}$ and $u \in L^2_{loc}(0,\infty)$, it is well known that the cascade of systems (1.4) and (5.6) admits a unique solution $(w_3, w_{3t}, v) \in C([0,\infty); \mathcal{H})$. By Lemma 5.1, system (5.11) with the initial state

$$\begin{aligned} &(\tilde{w}_3(\cdot,0), \tilde{w}_{3t}(\cdot,0), \tilde{v}(0)) \\ &\quad = (w_3(\cdot,0) - \hat{w}_3(\cdot,0), w_{3t}(\cdot,0) - \hat{w}_{3t}(\cdot,0), v(0) - \hat{v}(0)) \end{aligned}$$

admits a unique solution $(\tilde{w}_3, \tilde{w}_{3t}, \tilde{v}) \in C([0,\infty); \mathcal{H})$ such that

$$\begin{aligned} &\|(\tilde{w}_3(\cdot,t), \tilde{w}_{3t}(\cdot,t), \tilde{v}(t))\|_{\mathcal{H}} \\ &\quad \le L_{\mathcal{A}_2} e^{-\omega_{\mathcal{A}_2} t} \|(\tilde{w}_3(\cdot,0), \tilde{w}_{3t}(\cdot,0), \tilde{v}(0))\|_{\mathcal{H}}, t \ge 0. \end{aligned} \tag{7.24}$$

Define

$$\begin{cases} \hat{w}_3(x,t) = w_3(x,t) - \tilde{w}_3(x,t) \\ \hat{w}_{3t}(x,t) = w_{3t}(x,t) - \tilde{w}_{3t}(x,t) \\ \hat{v}(t) = v(t) - \tilde{v}(t). \end{cases} \tag{7.25}$$

In view of (5.9), it is easy to verify that such a defined $(w_3, w_{3t}, v, \hat{w}_3, \hat{w}_{3t}, \hat{v}) \in C([0,\infty); \mathcal{H}^2)$ solves the systems (1.4), (5.6), and (5.8). Moreover, (5.14) can be obtained directly by combining (7.24) and (7.25). ■

Proof of Theorem 5.2: Owing to the assumption in Theorem 5.2, Proposition 9.1 and Lemma 9.1 in Appendix, the pair (G, F_0) is observable. Hence, there exists a vector $K_0 \in \mathbb{C}^{n\times 1}$ such that the matrix $G - K_0F_0$ is Hurwitz. Using the initial state of the closed-loop system (5.20), we can define

$$\begin{cases} w_4(x,0) = w(x,0) - \Phi_4(x)v(0) \\ w_{4t}(x,0)) = w_t(x,0) - \Phi_4(x)Gv(0) \\ \tilde{w}_3(x,0) = w(x,0) - \Phi_3(x)v(0) - \hat{w}_3(x,0) \\ \tilde{w}_{3t}(x,0) = w_t(x,0) - \Phi_3(x)Gv(0) - \hat{w}_{3t}(x,0) \\ \tilde{v}(0) = v(0) - \hat{v}(0) \end{cases} \tag{7.26}$$

where Φ_1 and Φ_2 are defined, respectively, by (5.4) and (5.16). By Lemma 5.2, system (5.22) with the initial state (7.26) admits a unique solution $(w_4, w_{4t}, v, \tilde{w}_3, \tilde{w}_{3t}, \tilde{v}) \in C([0,\infty); \mathcal{X})$ such that (7.24) holds. Define

$$\begin{cases} w(x,t) = w_4(x,t) + \Phi_4(x)v(t) \\ w_t(x,t) = w_{4t}(x,t) + \Phi_4(x)Gv(t) \\ \hat{w}_3(x,t) = w(x,t) - \Phi_3(x)v(t) - \tilde{w}_3(x,t) \\ \hat{w}_{3t}(x,t) = w_t(x,t) - \Phi_3(x)Gv(t) - \tilde{w}_{3t}(x,t) \\ \hat{v}(t) = v(t) - \tilde{v}(t). \end{cases} \tag{7.27}$$

Then, such a defined $(w, w_t, v, \hat{w}_3, \hat{w}_{3t}, \hat{v})$ satisfies (5.27) and solves the closed-loop system (5.20). The convergence (5.28) and the boundedness (5.29) can be obtained by (5.17) and (7.27) easily. When $v(t) \equiv 0$, (5.30) follows directly from (5.26) and transform (7.27). This completes the proof. ■

Proof of Proposition 6.1: By a simple computation, $f_1(\lambda) = \lambda \sinh\lambda + (c_1 + c_2\lambda)\cosh\lambda \neq 0$ for any $\lambda \in \sigma(G) \subset i\mathbb{R}$. This implies that matrix $f_1(G)$ is invertible. The first part of this proposition is proved.

Now, we prove the remaining part of the proposition. Let

$$FG\sinh G + Q\cosh G = (J_0, J_1, J_2, \dots, J_m) \tag{7.28}$$

with $J_i \in \mathbb{C}^{1\times 2}$, $i = 1, 2, \dots, m$, $J_0 \in \mathbb{C}$. Then, the conditions (6.4) and (6.5) imply that

$$|J_0| \neq 0, \|J_i\|_{\mathbb{C}^2} \neq 0, \quad i = 1, 2, \dots, m. \tag{7.29}$$

By the Kalman rank condition for observability, (G_i, J_i) is observable for all $i = 0, 1, 2, \dots, m$. Using (6.2) and (7.28), we can conclude that $(G, FG\sinh G + Q\cosh G)$ is observable. ■

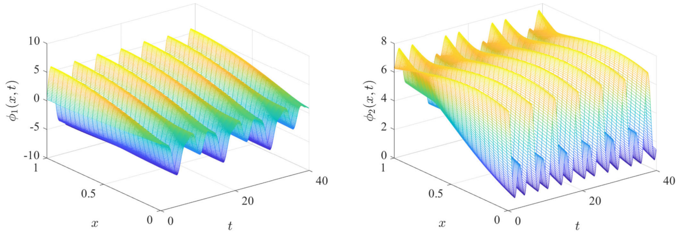
VIII. Numerical Simulations

In order to validate the theoretical results, we make some numerical simulations for closed-loop system (3.16). The disturbance and the reference signal are generated by a system matrix (6.2) with $m = 3$, $\omega_1 = 1$, $\omega_2 = 2$, and $\omega_3 = 0.5$. The corresponding parameters are chosen as

$$\begin{cases} F = (1,1,1,1,1,0,0), Q = (1,0,0,0,0,1,0) \\ c_1 = 1, c_2 = 0.9, c_3 = 0.9, c_4 = 1 \\ K = (6.075, -8.4588, -2.9198, -9.8477 \\ \qquad 1.8211, 12.8729, -1.7207)^\top. \end{cases} \tag{8.1}$$

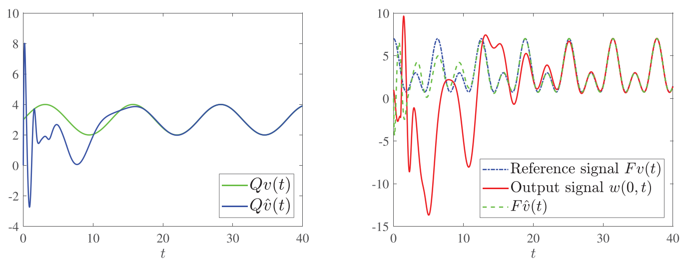
With this setting, the eigenvalues of $G - KF_1$ are

$$\sigma(G - KF_1) = \{-1, -1, -1, -1.5, -1.5, -1.5, -2\}.$$



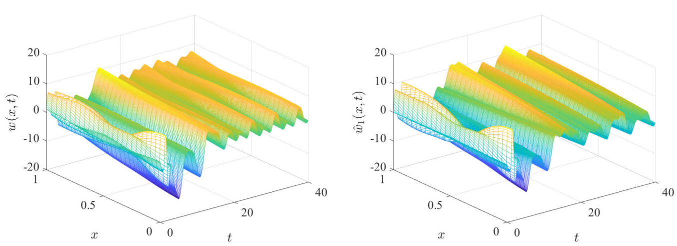
(a) (b)

Fig. 1. Planning trajectories. (a) First trajectory ϕ_1. (b) Second trajectory ϕ_2.



(a) (b)

Fig. 2. Tracking performance. (a) Qv and its estimate $Q\hat{v}$. (b) Output tracking.



(a) (b)

Fig. 3. Displacement of closed-loop system (3.16). (a) $w(x,t)$. (b) $\hat{w}_1(x,t)$.

The initial states are selected as

$$\begin{cases} w(x,0) = \cos 2\pi x, w_t(x,0) = \hat{w}_1(x,0) = \hat{w}_{1t}(x,0) = 0 \\ v = (3,1,1,1,1,0,1), \hat{v} = (0,0,0,0,0,0,0). \end{cases}$$

The numerical results are programmed in MATLAB. The time step and the space step are taken as 0.0005 and 0.001, respectively. The first and the second planning trajectories are plotted in Fig. 1(a) and 1(b), respectively. The ϕ_1 is used for disturbance estimation, and ϕ_2 is used for output regulation. The disturbance estimation is plotted in Fig. 2(a) and the output tracking is plotted in Fig. 2(b). Both of them show that the convergence is effectively and smoothly. The solution of the closed-loop system (3.16) is plotted in Fig. 3, which shows that all states are obviously bounded.

IX. Conclusion

In this article, we demonstrate the panorama of the output tracking for a 1-D wave equation via error-based feedback control. Our main focus is on the difficult one: the control is at the right end yet the performance output is at the left end. To deal with this noncollocated problem, we use double trajectory plannings. The first trajectory planning is to estimate the state and disturbance and the second one is to design an error-based feedback control. Another case where the output is noncollocated to the disturbance is also discussed by the trajectory planning. The exponential output tracking is concluded and all the states of subsystems in the closed loop are shown to be uniformly bounded, and the closed loop is shown to be internally exponentially stable. The results cover all similar researches in literature with harmonic external disturbance as a consequence where the control and performance are always at the same end. The numerical simulations validate the theoretical results. Finally, we point out that all convergence including the tracking error are exponentially, which is much stronger than asymptotic convergence in many existing literatures.

Appendix

Proposition 9.1: Let $G \in \mathbb{C}^{n\times n}$, $F \in \mathbb{C}^{1\times n}$, and $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. Suppose that the matrix $f(G) \in \mathbb{C}^{n\times n}$ is invertible. Then, (G, F) is observable if and only if $(G, Ff(G))$ is observable.

Proof: For any $v \in \mathrm{Ker}(\lambda - G) \cap \mathrm{Ker}(F)$ with $\lambda \in \sigma(G)$, it has

$$Ff(G)v = f(\lambda)Fv = 0. \tag{9.1}$$

Hence

$$\mathrm{Ker}(\lambda - G) \cap \mathrm{Ker}(F) \subset \mathrm{Ker}(\lambda - G) \cap \mathrm{Ker}(Ff(G)). \tag{9.2}$$

On the other hand, for $v \in \mathrm{Ker}(\lambda - G) \cap \mathrm{Ker}(Ff(G))$

$$0 = Ff(G)v = f(\lambda)Fv. \tag{9.3}$$

Since $f(G) \in \mathbb{C}^{n\times n}$ is invertible, $f(\lambda) \neq 0$ for any $\lambda \in \sigma(G)$. It then follows from (9.3) that $Fv = 0$, that is,

$$\mathrm{Ker}(\lambda - G) \cap \mathrm{Ker}(Ff(G)) \subset \mathrm{Ker}(\lambda - G) \cap \mathrm{Ker}(F). \tag{9.4}$$

By Hautus's lemma, (9.2) and (9.4), (G, F) is observable if and only if $(G, Ff(G))$ is observable. ∎

Lemma 9.1: Suppose that the matrix G satisfies (4.9). Then, for any $c_1, c_2 > 0$, the matrix $G \sinh G + (c_1 I + c_2 G) \cosh G$ is always invertible.

Proof: Consider the following wave equation:

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t) \\ v_x(0,t) = c_1 v(0,t) + c_2 v_t(0,t) \\ v_x(1,t) = 0. \end{cases} \tag{9.5}$$

By a simple computation, the characteristic equation of (9.5) is found to be

$$\lambda \sinh \lambda + (c_1 + c_2\lambda) \cosh \lambda = 0. \tag{9.6}$$

Since system (9.5) is exponentially stable, it follows from (4.9) that

$$\lambda_0 \sinh \lambda_0 + (c_1 + c_2\lambda_0) \cosh \lambda_0 \neq 0 \quad \forall\, \lambda_0 \in \sigma(G). \quad (9.7)$$

This implies that all the eigenvalues of $G \sinh G + (c_1 I + c_2 G) \cosh G$ are nonzero and hence $G \sinh G + (c_1 I + c_2 G) \cosh G$ is invertible.

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