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Review

Active disturbance rejection control: Old and new results ☆

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ABSTRACT

The active disturbance rejection control (ADRC), first proposed by Jingqing Han in the 1980s is an unconventional design strategy. It has been acknowledged to be an effective control strategy in the absence of proper models and in the presence of model uncertainty. Its power was originally demonstrated by numerical simulations, and later by many engineering practices. For the theoretical problems, namely, the convergence of the tracking differentiator which extracts the derivative of reference signal; the extended state observer used to estimate not only the state but also the "total disturbance", by the output; and the extended state observer based feedback, progresses have also been made in the last few years from nonlinear lumped parameter systems to distributed parameter systems. The aim of this paper is to review the origin, idea and development of this new control technology from a theoretical perspective. Emphasis will be focused on output feedback stabilization for uncertain systems described by partial differential equations.

Contents

1. Introduction

The capability of dealing with uncertainty is one of the major concerns in modern control theory. There are many well developed control design approaches to cope with uncertainty in control systems. These include the adaptive control for vary or initially uncertain parameters; the internal model principle for regulator problems, the sliding mode control and high gain control for uncertain systems, and robust control which is a paradigm shift in control theory for internal variation and external disturbance. Most of these approaches, however, focus on the worst case scenario which makes the controller rather conservative. The two exceptions are the adaptive control and internal model principle in which the idea of real time estimation/cancelation leads to significant saving of control energy. Let us start with these two approaches to see how and why they are working.

The adaptive control approach was emerged in the 1950s and resurged in the 1970s due to study of uncertain system control in large scale after 1970s (Whitaker, Yamron, and Kezer, 1958. For PDEs, we refer to Krstic, 2010). In the adaptive control approach, the bound of uncertainty is not used and the control varies with

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the uncertainty. Consider feedback stabilization for the following system:

$$\dot{x}(t) = \theta f(x(t)) + u(t), \tag{1.1}$$

where θ is an unknown parameter and $u(t)$ is the control. If we can find an estimator $\hat{\theta}(t)$ for the parameter:

$$\hat{\theta}(t) \to \theta \quad \text{as} \quad t \to \infty, \tag{1.2}$$

then a stabilizing feedback control can be designed as follows:

$$u(t) = -x(t) - \hat{\theta}(t) f(x(t)), \tag{1.3}$$

where the second term in the controller (1.3) is used to cancel the corresponding uncertainty term in (1.1). Substituting (1.3) into (1.1), we can obtain the closed-loop system:

$$\begin{cases} \dot{x}(t) = \tilde{\theta}(t) f(x(t)) - x(t), \\ \tilde{\theta}(t) = \theta - \hat{\theta}(t). \end{cases} \tag{1.4}$$

A Lyapunov function for system (1.4) can be chosen as

$$V(t) = \frac{1}{2}x^2(t) + \frac{1}{2}\tilde{\theta}^2(t).$$

The derivative of $V(t)$ along the solution of (1.4) is found to be

$$\frac{dV(x(t), \tilde{\theta}(t))}{dt} = -x^2(t) + \tilde{\theta}(t)[\dot{\tilde{\theta}}(t) + x(t) f(x(t))] = -x^2(t), \tag{1.5}$$

provided $\dot{\tilde{\theta}}(t) = -x(t) f(x(t))$, and the closed-loop system becomes

$$\begin{cases} \dot{x}(t) = \tilde{\theta}(t) f(x(t)) - x(t), \\ \dot{\tilde{\theta}}(t) = -x(t) f(x(t)). \end{cases} \tag{1.6}$$

Notice that the order of the system is increased by one due to the introduction of the variable $\tilde{\theta}(t)$. By Lasalle's invariance principle and (1.5), it follows that the solution of the system (1.6) satisfies

$$x(t) \to 0 \text{ as } t \to \infty. \tag{1.7}$$

The remaining question is: Is $\hat{\theta}(t) \to \theta (t \to \infty)$? or equivalently $\tilde{\theta}(t) \to 0 (t \to \infty)$? Its answer is not necessarily. Actually, by Lasalle's invariance principle, when $\dot{V}(t) = 0$, we can only conclude that $x(t) = 0$. So $\tilde{\theta} = \tilde{\theta}_0$ may be a nonzero constant satisfying $\tilde{\theta}_0 f(0) = 0$. We therefore have two cases: a) $f(0) \neq 0$ and $\tilde{\theta}_0 = 0$; and b) $f(0) = 0$ and $(x(t), \tilde{\theta}(t)) = (0, \tilde{\theta}_0)$ is a solution of (1.6). The latter case implies that $\tilde{\theta}(t) \to 0 (t \to \infty)$ is not necessarily valid. The former case is just the "persistent exciting" (PE) condition which is $f(0) \neq 0$ for this problem. Nevertheless, in either case, we always have

$$\tilde{\theta}(t) f(x(t)) \to 0 \text{ as } t \to \infty, \tag{1.8}$$

regardless of whether the parameter update law $\dot{\tilde{\theta}}(t) = x(t) f(x(t))$ is convergent or not. In other words, the uncertain term $\theta f(x(t))$ of the system (1.1) is always canceled asymptotically by the feedback control (1.3).

Now we look at the process of internal model principle (IMP) in dealing with external disturbance, which was first introduced in Francis and Wonham (1976) (for PDEs, we refer to Rebarber & Weiss, 2003). Consider once again stabilization for the system:

$$\dot{x}(t) = a(t) + u(t), \tag{1.9}$$

where $u(t)$ is the control and $a(t) = \theta \sin \omega t$ is an external disturbance in which the frequency ω is supposed to be known while the constant amplitude θ is unknown. Since $\ddot{a}(t) = -\omega^2 a(t)$, we can increase the order of system (1.9) as

$$\begin{cases} \dot{x}(t) = a(t) + u(t), \\ \ddot{a}(t) = -\omega^2 a(t), \\ y(t) = x(t), \end{cases} \tag{1.10}$$

where the output of system (1.10) is the state of original system (1.9). Write (1.10) in matrix form:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(t), \\ y(t) = CX(t), \end{cases} \tag{1.11}$$

where

$$X(t) = (x(t), a(t), \dot{a}(t))^\top, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = (1, 0, 0).$$

A simple calculation shows that

$$\text{rank} \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = 3.$$

So system (1.10) or (1.11) is observable. Design the Luenberger observer as

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(t) + L(C\hat{X}(t) - x(t)),$$

where $\hat{X}(t) = (\hat{x}(t), \hat{a}(t), z(t))^\top$, and $L = (\ell_1, \ell_2, \ell_3)^\top$ is selected such that $A + LC$ is Hurwitz. Then we have

$$\begin{cases} \dot{\hat{x}}(t) = \hat{a}(t) + u(t) + \ell_1(\hat{x}(t) - x(t)), \\ \dot{\hat{a}}(t) = z(t) + \ell_2(\hat{x}(t) - x(t)), \\ \dot{z}(t) = -\omega^2 \hat{a}(t) + \ell_3(\hat{x}(t) - x(t)). \end{cases} \tag{1.12}$$

In system (1.10), both $a(t)$ and $\dot{a}(t)$ are regarded as extra state variables. The stabilizing feedback control can thus be designed as

$$u(t) = -\hat{a}(t) - x(t), \tag{1.13}$$

where the first term is used to cancel the external disturbance. In other words, as in the case of adaptive control, we also have used the strategy of estimation and cancelation in the IMP approach.

The active disturbance rejection control (ADRC) further systematically developed the estimation and cancelation approach and greatly enhance its power in dealing with uncertainty in systems. We would like to explain this point by considering feedback stabilization of (1.9) again yet in this case,

$$a(t) = f(x(t), d(t), t), \tag{1.14}$$

which can be used to models (combination of) unknown time-varying, state-dependent internal uncertainty, and external disturbance. The term $a(t)$ is referred to as "total disturbance" in ADRC. The key idea is that regardless of the composition nature of the matter what $a(t)$ is, it is considered as a signal of time and is reflected in the measured output of system. We write system (1.9) as

$$\begin{cases} \dot{x}(t) = a(t) + u(t), \\ \dot{a}(t) = \dot{a}(t), \\ y(t) = x(t), \end{cases} \tag{1.15}$$

where $y(t)$ is the output of extended system (1.15). The exact observerbility of system (1.15) is a trivial problem because if $(y(t), u(t)) \equiv 0$, $t \in [0, T]$, then $a(t) = 0, t \in [0, T]$ and $x(0) = 0$ (Cheng, Hu, and Shen, 2010, p.5, Definition 1.2) for any $T > 0$. This means that $y(t)$ contains all information of $a(t)$. Then a natural idea is: if we can estimate $a(t)$ from $y(t)$ to obtain $\hat{a}(t) \approx a(t)$, then we can also cancel the $a(t)$ in the feedback-loop $u(t) = -\hat{a}(t) + u_0(t)$ where $u_0(t)$ is a new control. Consequently, system (1.15) can be approximated as

$$\dot{x}(t) = u_0(t), \tag{1.16}$$

which is a linear time-invariant system and we have therefore many methods to deal with it. Now, the problem is how the estimation of the total disturbance can be achieved: "$\hat{a}(t) \approx a(t)$".

Here came Han with his systematic method of total disturbance estimation (Han, 2009 and references therein for his original papers in Chinese). The answer is his extended state observer (ESO). For system (1.15), a linear ESO can be designed as

$$\begin{cases}\dot{\hat{x}}(t)=\hat{a}(t)+u(t)+c_1(\hat{x}(t)-x(t)),\\ \dot{\hat{a}}(t)=c_2(\hat{x}(t)-x(t)),\end{cases} \tag{1.17}$$

where c_1 and c_2 are tuning parameters. Let $\tilde{x}(t)=\hat{x}(t)-x(t)$ and $\tilde{a}(t)=\hat{a}(t)-a(t)$ be the errors. Then

$$\begin{cases}\dot{\tilde{x}}(t)=\tilde{a}(t)+c_1\tilde{x}(t),\\ \dot{\tilde{a}}(t)=c_2\tilde{x}(t)-\dot{a}(t).\end{cases} \tag{1.18}$$

The solution of system (1.18) is found to be

$$(\tilde{x}(t),\tilde{a}(t))=e^{A_0t}(\tilde{x}(0),\tilde{a}(0))+\int_0^t e^{A_0(t-s)}B_0\dot{a}(s)ds, \tag{1.19}$$

where

$$A_0=\begin{pmatrix}c_1 & 1\\ c_2 & 0\end{pmatrix},\quad B_0=\begin{pmatrix}0\\ -1\end{pmatrix}. \tag{1.20}$$

Hence, the derivative $\dot{a}(t)$ of the total disturbance needs to be uniformly bounded:

$$|\dot{a}(t)|\le M, \forall\, t\ge 0. \tag{1.21}$$

Otherwise, the second term in the error solution (1.19) may not be convergent to zero as $t\to\infty$.

The condition (1.21) can be relaxed to be uniform boundedness of $a(t)$'s derivative of some finite order if an ESO of higher order is used. The second is to make e^{A_0t} exponentially stable with large decay rate. Following the high gain observer design, we may take

$$c_1=-\frac{2}{\varepsilon},\quad c_2=-\frac{2}{\varepsilon^2}. \tag{1.22}$$

In this case, it is easily shown that

$$\|e^{A_0t}\|\le\frac{L}{\varepsilon}e^{-\frac{1}{\varepsilon}t},\quad \|e^{A_0t}B_0\|\le Le^{-\frac{1}{\varepsilon}t} \tag{1.23}$$

for some constant $L>0$ independent of t and ε. By (1.21) and (1.23), it follows from (1.19) that the solution of (1.19) can be estimated as

$$\|(\tilde{x}(t),\tilde{a}(t))\|\le\frac{L}{\varepsilon}e^{-\frac{1}{\varepsilon}t}\|(\tilde{x}(0),\tilde{a}(0))\|+LM\varepsilon. \tag{1.24}$$

The first term on the right-hand side of the above inequality tends to zero as $t\to\infty$, and the second term tends to zero as $\varepsilon\to 0$. In any case, we have

$$\hat{x}(t)\to x(t),\hat{a}(t)\to a(t),\ \text{as } t\to\infty,\varepsilon\to 0. \tag{1.25}$$

Now that we have the estimate of $a(t)$ at hand, we can design an ESO-based stabilizing feedback control for system (1.9) as

$$u(t)=-\hat{a}(t)-x(t), \tag{1.26}$$

where the first term $-\hat{a}(t)$ is used to eliminate the effect of the total disturbance and the second tern $-x(t)$ aims to provide stabilizing feedback control for linear system (1.16). Under the feedback (1.13), the closed-loop system of (1.9) becomes

$$\begin{cases}\dot{x}(t)=-x(t)-\hat{a}(t)+a(t),\\ \dot{\hat{x}}(t)=-x(t)+c_1(\hat{x}(t)-x(t)),\\ \dot{\hat{a}}(t)=c_2(\hat{x}(t)-x(t)).\end{cases} \tag{1.27}$$

Let $\tilde{x}(t)=\hat{x}(t)-x(t)$, $\tilde{a}(t)=\hat{a}(t)-a(t)$. We have the following equivalent description of (1.27):

$$\begin{cases}\dot{x}(t)=-x(t)-\tilde{a}(t),\\ \dot{\tilde{x}}(t)=\tilde{a}(t)+c_1\tilde{x}(t),\\ \dot{\tilde{a}}(t)=c_2\tilde{x}(t)-\dot{a}(t).\end{cases} \tag{1.28}$$

Since $(\tilde{x}(t),\tilde{a}(t))\to 0$ as $t\to\infty$ and $\varepsilon\to 0$, it immediately follows that

$$x(t)\to 0 \text{ as } t\to\infty,\varepsilon\to 0$$

or equivalently

$$x(t)\to 0,\hat{x}(t)\to 0,\hat{a}(t)-a(t)\to 0,\quad \text{as}\quad t\to\infty,\varepsilon\to 0. \tag{1.29}$$

This is the separation principle of estimation and control, which can be carried out separately.

Notice that the ESO (1.17) provides an asymptotic estimation of the total disturbance; and this estimation can be directly used in the control (1.26) to cancel the disturbance. This part of the control is called the "rejector" of disturbance in Gao (2015), albeit from different point of view. This part is also similar to the feed-forward control in that they both modify the system dynamics before further feedback control is applied. Moreover, the fact that the total disturbance can be completely cancelled separately allows the design of feedback control without being over conservative (as in the case of robust control) and of efficient in energy saving, as confirmed in Zheng and Gao (2012). It seems any other control would hardly give better result in dealing with the total disturbance than the control $u(t)=-\hat{a}(t)+u_0(t)$ which adopts the strategy of estimation and cancellation, much alike our experience in dealing with uncertainty in daily life.

The ADRC has been widely applied in many engineering systems since it was proposed. Examples can be found in MEMS electrostatic actuator (Dong & Edwards, 2010), DC-DC power converter (Sun & Gao, 2005), flight vehicles control (Xia & Fu, 2013), gasoline engines (Xue et al., 2015), hydraulic systems control (Yao, Jiao, & Ma, 2014), to name just a few. In our recent monograph (Guo & Zhao, 2016), an overview introduction of many engineering applications is included. Our previous survey paper (Guo & Zhao, 2015) introduces the ADRC from a theoretical perspective for nonlinear lumped parameter systems. In this paper, our focus is mainly on ADRC to distributed parameter control systems. However, to showcase a panoramic view of ADRC, we present some basic materials of ADRC for ODEs in next section. In Section 3, we give a new concept of the observability for systems with disturbance. It is about whether or not the output contains sufficient information on both system state and disturbance. In Section 4, we showcase the whole process of designing a type of ESO for a Euler–Bernoulli beam equation. In Section 5, a multi-dimensional wave equation is briefly discussed, follows by Section 6 with some concluding remarks.

2. ADRC for ODEs

Roughly speaking, the ADRC is composed of three parts. The first part is the tracking differentiator (TD) which extracts the derivative of reference signal, note that in the PID controller used in most of industrial control systems, the derivative action "D" is seldom used because of it's sensitive to high frequency noise. In ADRC, TD severs not only as the derivative extractor, but also as a transient profile that the output of plant can reasonably follow to avoid setpoint jump in PID. The second part, the most important part, is the extended state observer (ESO). As a generalization of the classical state observer in control theory, the ESO provides estimates of both state and total disturbance in terms of output. The last part of ADRC is the TD and ESO based output feedback control which achieve output tracking, which specializes to system stabilization when the reference signal is zero.

Let us begin with the introduction of the TD. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz continuous function, $f(0)=0$. Suppose that the zero equilibrium state of the following reference-free system is globally asymptotically stable:

$$\begin{cases}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=x_3(t),\\ \quad\vdots\\ \dot{x}_n(t)=f(x_1(t),x_2(t),\ldots,x_n(t)),\end{cases}\tag{2.1}$$

for any given initial value. If the reference signal $v(t)$ is differentiable and satisfies $\sup_{t\in[0,\infty)}|v^{(n+1)}(t)|<\infty$, then the solution of the following tracking differentiator:

$$\text{TD}:\begin{cases}\dot{z}_{1R}(t)=z_{2R}(t),\\ \dot{z}_{2R}(t)=z_{3R}(t),\\ \quad\vdots\\ \dot{z}_{nR}(t)=R^n f\left(z_{1R}(t)-v(t),\dfrac{z_{2R}(t)}{R},\ldots,\dfrac{z_{nR}(t)}{R^{(n-1)}}\right),\end{cases}\tag{2.2}$$

is convergent in the sense that for every $a>0$,

$$\lim_{R\to\infty}|z_{1R}-v(t)|=0 \text{ uniformly on } [a,\infty),\tag{2.3}$$

and for any given initial value. This result was first proved in Guo and Zhao (2011a). In the linear case, this tracking differentiator is shown to be the high gain tracking differentiator (Guo & Zhao, 2013b). In control practice, the setpoint $v(t)$ is often given as a step function, which is not appropriate for most dynamics systems because it amounts to asking the output and, therefore, the control signal, to make a sudden jump (Han, 2009). This can be avoided by letting the output track $z_{1R}(t)$ instead of $v(t)$. Other variables produced from TD (2.2) are considered as the derivatives of $v(t)$: $z_{iR}(t)\approx v^{(i-1)}(t)$ in the sense of generalized derivative (Guo & Zhao, 2013b).

For an n-dimensional SISO nonlinear system

$$\begin{cases}x^{(n)}(t)=f(t,x(t),\dot{x}(t),\ldots,x^{(n-1)}(t),w(t))+bu(t),\\ y(t)=x(t),\end{cases}$$

which can be written as

$$\begin{cases}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=x_3(t),\\ \quad\vdots\\ \dot{x}_n(t)=f(t,x_1(t),\ldots,x_n(t),w(t))+bu(t),\\ y(t)=x_1(t),\end{cases}\tag{2.4}$$

where $y(t)$ is the output (observation), $u(t)$ is the input (control), $w\in C^1([0,\infty),\mathbb{R})$ is the external disturbance, $f\in C^1(\mathbb{R}^{n+2},\mathbb{R})$ represents the nonlinear dynamic function of the plant which is possibly unknown, and $b>0$ is a constant control coefficient which is not exactly known, but we have the nominal value b_0 that is sufficiently closed to b. The objective of control design is to make the output $y(t)$ track a given reference signal $v(t)$, and at the same time $x_i(t)$ track $z_{iR}(t)\approx v^{(i-1)}(t)$ for every $i=2,3,\ldots,n$. It is obvious that this general formulation covers not only the special output regulation problem, but also the output feedback stabilization by setting $v(t)\equiv 0$. In configuration of ADRC, we also want to control the convergence rate, that is, $x_i(t)-z_{iR}(t)\approx x_i^*(t)$ where $x_i^*(t)$ satisfies the target asymptotically stable system of the following:

$$\begin{cases}\dot{x}_1^*(t)=x_2^*(t),\\ \dot{x}_2^*(t)=x_3^*(t),\\ \quad\vdots\\ \dot{x}_n^*(t)=\varphi(x_1^*(t),\ldots,x_n^*(t)),\ \varphi(0,0,\ldots,0)=0.\end{cases}\tag{2.5}$$

Now we design an ESO to estimate both the state and the "total disturbance" given by

$$x_{n+1}(t)=f(t,x_1(t),x_2(t),\cdots,x_n(t),w(t))+(b-b_0)u(t),\tag{2.6}$$

which is considered as an extra variable in ADRC. In Guo and Zhao (2011b), a high gain ESO for system (2.4) was designed as

$$\text{ESO}:\begin{cases}\dot{\hat{x}}_1(t)=\hat{x}_2(t)+\varepsilon^{n-1}g_1\left(\dfrac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right),\\ \dot{\hat{x}}_2(t)=\hat{x}_3(t)+\varepsilon^{n-2}g_2\left(\dfrac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right),\\ \quad\vdots\\ \dot{\hat{x}}_n(t)=\hat{x}_{n+1}(t)+g_n\left(\dfrac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right)+u(t),\\ \dot{\hat{x}}_{n+1}(t)=\dfrac{1}{\varepsilon}g_{n+1}\left(\dfrac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right),\end{cases}\tag{2.7}$$

where $g_i(\cdot)(i=1,2,\cdots,n)$ are nonlinear functions to be chosen and ε is the high gain constant. When $g_i(\cdot)$ are linear functions, (2.7) is reduced to linear ESO. The convergence of ESO (2.7) was first proved in Guo and Zhao (2011b):

$$\hat{x}_i(t)-x_i(t)\to 0 \text{ as } t\to\infty,\varepsilon\to 0, i=1,2,\cdots,n+1.\tag{2.8}$$

In particular, $\hat{x}_{n+1}(t)$ gives an estimate of the total disturbance $x_{n+1}(t)$ as $t\to\infty$.

The third and the last link of ADRC is to design an extended state observer-based output feedback control:

$$\text{ADRC}:\ u(t)=\frac{1}{b_0}[\varphi(\hat{x}(t)-z_R(t))+z_{(n+1)R}(t)-\hat{x}_{n+1}(t)],\tag{2.9}$$

where $\hat{x}(t)=(\hat{x}_1(t),\hat{x}_2(t),\ldots,\hat{x}_n(t),\hat{x}_{n+1}(t))$ is the solution of (2.7) and $z_R(t)=(z_{1R}(t),z_{2R}(t),\ldots,z_{nR}(t),z_{(n+1)R}(t))$ is the solution of (2.2). It is seen that the term $\hat{x}_{n+1}(t)$ in controller (2.9) is used to cancel the total disturbance. The other terms are just the linear design to make $x_i(t)-z_{iR}(t)=x_i^*(t)$. The closed-loop system thus becomes

$$\begin{cases}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=x_3(t),\\ \quad\vdots\\ \dot{x}_n(t)=f(t,x(t),w(t))+(b-b_0)u(t)+b_0u(t),\\ \dot{\hat{x}}_1(t)=\hat{x}_2(t)+\varepsilon^{n-1}g_1\left(\dfrac{y(t)-\hat{x}_1(t)}{\varepsilon^n}\right),\\ \quad\vdots\\ \dot{\hat{x}}_{n+1}(t)=\dfrac{1}{\varepsilon}g_{n+1}\left(\dfrac{y-\hat{x}_1(t)}{\varepsilon^n}\right),\\ u(t)=\dfrac{1}{b_0}[\varphi(\hat{x}(t)-z_R(t))+z_{(n+1)R}(t)-\hat{x}_{n+1}(t)].\end{cases}\tag{2.10}$$

The convergence of the closed-loop system (2.10) has been proven in Zhao and Guo (2016b). For MIMO systems, convergence for ESO was presented in Guo and Zhao (2012) and convergence for closed-loop was given in Guo and Zhao (2013a). Other generalization to low triangle systems can be found in Zhao and Guo (2016a). The method of attenuating the peaking value through a time-varying gain was introduced in Zhao and Guo (2015). Generalization to stochastic systems can be found in Guo, Wu, and Zhou (2016). The control unmatched problem was considered recently in Guo and Wu (2017).

Starting from the next section, we shall focus on output feedback stabilization for uncertain PDEs by ADRC. The generalization of ADRC to PDEs is rather complicated. The research started from full state feedback stabilization of PDEs with external disturbances, first for 1-d PDEs in Guo and Jin (2013a, 2013b) and Guo, Liu, and Robust (2014a) and then for multi-dimensional PDEs in

Guo and Zhou (2014, 2015). The main idea is to convert the PDE's problem into associated ODE's by test functions with understanding that a PDE solution is understood as weak solution. The output feedback stabilization for multi-dimensional PDEs with corrupted output was discussed respectively in Guo and Zhou (2016) or Feng and Guo (2016b). In particular, Feng and Guo (2016b) proposed a distributed tracking differentiator which significantly simplifies the related result of Guo and Zhou (2016). A first result on output feedback stabilization for 1-d wave equations was presented in Guo and Jin (2015) where a variable structured unknown input state observer was designed; generalization to other PDEs turns out to be difficult. Very recently, we proposed in Feng and Guo (2017) a different approach on output feedback stabilization for uncertain PDEs by ADRC, which can be easily used to deal with multi-dimensional PDEs. In particular, the approach proposed in Feng and Guo (2017) removes two limitations for ADRC of lumped parameter systems in the context of infinite dimensional systems. More precisely, first, in sharp contrast to (1.21), the boundedness of the derivative of total disturbance or any finite order derivative of total disturbance is not required; and second, the high gain like (1.22) or (2.7) is not explicitly used. The following sections are concentrated on the approach developed in Feng and Guo (2017). As indicated in the beginning for ODEs, for the purpose of the total disturbance estimation, the measured output should be observable in that it contains adequate information of the total disturbance. However, the observability for uncertain PDEs is rather complicated. Our study is based on two basic facts: (a) when there is no disturbance, the system is exactly observable in the classical sense; and (b) the disturbance should be present in the measured output so that it can be recognized to "certain extent". The level of this recognition is dependent on this extent.

3. Observability for uncertain infinite-dimensional systems

Consider the following infinite-dimensional system with external disturbance:

$$\begin{cases} \dot{x}(t) = Ax(t) + B[f(x(t)) + d(t)], \quad t > 0, \\ y(t) = Cx(t), \ t \geq 0, \end{cases} \tag{3.1}$$

where A is the system operator, B the control operator, C the output operator, and $f(x(t)) + d(t)$ the total disturbance that consists of the internal dynamic uncertainty $f(x(t))$ and the external disturbance $d(t)$. We consider system (3.1) in the state space X with the control space U and the output space Y. We use X_{-1} to denote the dual Hilbert space of $D(A)$ with the pivot space X (Weiss, 1989).

As mentioned at the end of last section, observability for system (3.1) should reflects two facts: (a) the uncertainty is identifiable; and (b) the disturbance-free system is exactly observable. The former implies that the output contains sufficient information about the uncertainty. The latter guarantees that when there is no uncertainty, the state can be uniquely continuously recovered from the output. The definition of exactly observability for system (3.1) is given below.

Definition 1. System (3.1) is said to be exactly observable in X if

(i) when $[f(x(t)) + d(t)] = 0$ and $u(t) = 0$, there exist an interval $[0, T]$, $T > 0$ and a constant $c_T > 0$ such that

$$\int_0^T |y(t)|^2 dt \geq c_T \|x(0)\|_X^2, \forall\, x(0) \in X; \tag{3.2}$$

(ii) the total disturbance is asymptotically identifiable in the sense of

$$y(t) = 0, \forall\, t \in [0, \infty) \Rightarrow [f(x(t)) + d(t)] \in L^p(0, \infty; U), \tag{3.3}$$

where $2 \leq p < \infty$.

When $f(x(t)) + d(t) \equiv 0$, the exact observability in the sense of Definition 1 is the same as the traditional exact observability given by Tucsnak and Weiss (2009, Definition 6.1.1, p.173). From condition (ii) of Definition 1, it is seen that we do not "recognize" the total disturbance $f(x(t)) + d(t)$ exactly, but with a possible error in $L^p(0, \infty; U)$. This is based on the asymptotical stability result of Lemma 3.1 which claims that an exponentially stable system remains asymptotically stable even if it is perturbed by an inhomogeneous term of $L^p(0, \infty; U)$.

Lemma 3.1. *Let $\tilde{f} \in L^p(0, \infty; U)$ with $2 \leq p < \infty$. Suppose that the operator $\mathbb{A}$ generates an exponentially stable C_0-semigroup $e^{\mathbb{A}t}$ on X and the control operator $B \in \mathcal{L}(U, X_{-1})$ is admissible for $e^{\mathbb{A}t}$. Then, for any $x(0) \in X$, system*

$$\dot{x}(t) = \mathbb{A}x(t) + B\tilde{f}(t), \quad t \geq 0 \tag{3.4}$$

admits a unique mild solution $x \in C(0, \infty; X) \cap H^1_{loc}(0, \infty; X_{-1})$ such that

$$\lim_{t \to \infty} \|x(t)\|_X = 0. \tag{3.5}$$

Proof. The well-posedness of the solution can be found in Tucsnak and Weiss (2009, Proposition 4.2.5, p.118). For the convergence, we refer the reader to Wolfgang, Charles, Matthias, and Frank (2001, Proposition 1.3.5(b)) and Oostveen and Curtain (1998). □

As an application of Definition 1, let us consider the following uncertain Euler–Bernoulli beam equation:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ w_{xxx}(0,t) = w_{xx}(0,t) = w(1,t) = 0, \quad t \geq 0, \\ w_{xx}(1,t) = f(w(\cdot,t), w_x(\cdot,t)) + d(t) + u(t), \quad t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \quad x \in [0,1], \\ y(t) = (w_t(0,t), w_x(1,t), w_{xt}(1,t)), t \geq 0, \end{cases} \tag{3.6}$$

where $(w_0(x), w_1(x))$ is the initial state, $u(t)$ is the input (control), $y(t)$ is the output (measurement) and $f(w(\cdot,t), w_x(\cdot,t)) + d(t)$ is the total disturbance that consists of the boundary interior uncertainty $f(\cdot)$ and the external disturbance $d(t)$. System (3.6) models a vibrating flexible beam that is free at the end $x = 0$ and is controlled on the other end $x = 1$ and $w(x, t)$ represents the displacement of the beam at $x \in (0, 1)$ and $t \geq 0$. In the rest of the paper, we drop off obvious spatial and time domains without confusion.

Theorem 3.1. *System (3.6) is exactly observable in the sense of Definition 1.*

Proof. When $f(\cdot) + d(t) = 0$ and $u(t) = 0$, by a simple multiplier technique, it can be easily shown that there exists $\tau > 0$ such that

$$\int_0^\tau |w_t(0,t)|^2 dt \geq c_\tau \|(w_0, w_1)\|^2_{\mathcal{H}}, \tag{3.7}$$

where $c_\tau > 0$ is a constant, which means that condition (i) of Definition 1 is satisfied. We only need to show condition (ii). When $y(t) = 0$ for all $t \in [0, \infty)$, $w(x, t)$ is governed by

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \\ w_{xxx}(0,t) = w_{xx}(0,t) = 0, \\ w(1,t) = w_x(1,t) = 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \end{cases} \tag{3.8}$$

and

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \\ w_{xxx}(0,t) = -w_t(0,t), \ w_{xx}(0,t) = 0, \\ w_x(1,t) = w(1,t) = 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x). \end{cases} \tag{3.9}$$

System (3.8) is a conservative system that $\|(w(\cdot,t), w_t(\cdot,t))\| = \|(w_0, w_1)\|$ for all $t \geq 0$. And system (3.9) is a well known exponentially stable system: $\|(w(\cdot, t), w_t(\cdot, t))\| \to 0$ as $t \to \infty$. Therefore, for any $t \geq 0$, $\|(w(\cdot, t), w_t(\cdot, t))\| \equiv 0$ which implies that

$f(w(\cdot,t),w_x(\cdot,t))+d(t)=w_{xx}(1,t)=0$. This completes the proof of the theorem. □

Theorem 3.1 implies that the output $y(t)$ is always observable regardless of the function $f(\cdot)$. As a consequence, the output $y(t)$ contains sufficient information of the total disturbance $f(w(\cdot,t),w_x(\cdot,t))+d(t)$ therefore we can use $y(t)$ to estimate the uncertainty. System (3.6) serves as a benchmark example to show our design idea for uncertain PDEs in next section.

4. Output feedback stabilization

In this section, we design, in terms of the output $y(t)$, a disturbance estimator which can also serve as a new unknown input type state observer for system (3.6). The disturbance estimator design relies on the hidden regularity of the beam equation. This is very different to the ADRC for lumped parameter systems discussed in Section 2 where the high-gain is used in ESO.

4.1. Unknown input observer

We first design a disturbance estimator for system (3.6). To this purpose, we first introduce the following auxiliary system to bring the total disturbance $f(\cdot)+d(t)$ into an exponentially stable system:

$$\begin{cases} z_{tt}(x,t)+z_{xxxx}(x,t)=0,\\ z_{xxx}(0,t)=c_1[w_t(0,t)-z_t(0,t)],\\ z_{xx}(0,t)=z(1,t)=0,\\ z_{xx}(1,t)=c_0[w_x(1,t)-z_x(1,t)]+u(t), \end{cases} \tag{4.1}$$

where c_1 is a positive tuning parameter. System (4.1) depends only on the input and output of the original plant (3.6) and therefore is completely known. Although system (4.1) and the original uncertain system (3.6) are different, the error system is an “ideal environment” for disturbance estimation. Actually, if we set

$$\hat{z}(x,t)=w(x,t)-z(x,t), \tag{4.2}$$

then the error $\hat{z}(x,t)$ is governed by

$$\begin{cases} \hat{z}_{tt}(x,t)+\hat{z}_{xxxx}(x,t)=0,\\ \hat{z}_{xxx}(0,t)=-c_1\hat{z}_t(0,t),\\ \hat{z}_{xx}(0,t)=\hat{z}(1,t)=0,\\ \hat{z}_{xx}(1,t)=-c_0\hat{z}_x(1,t)+f(w(\cdot,t),w_x(\cdot,t))+d(t). \end{cases} \tag{4.3}$$

System (4.3) can be written abstractly as

$$\frac{d}{dt}(\hat{z}(\cdot,t),\hat{z}_t(\cdot,t)) = \mathcal{A}_0(\hat{z}(\cdot,t),\hat{z}_t(\cdot,t)) + \mathcal{B}[f(w(\cdot,t),w_x(\cdot,t))+d(t)], \tag{4.4}$$

where the operators $\mathcal{A}_0$ and $\mathcal{B}$ are defined, respectively, by

$$\begin{cases} \mathcal{A}_0(f,g)=(g,-f^{(4)}),\ \forall (f,g)\in D(\mathcal{A}_0),\\ D(\mathcal{A}_0)=\{(f,g)\in H^4(0,1)\times H^2(0,1)\mid\\ g(1)=f(1)=f''(0)=0,\ f'''(0)=-c_1g(0),\ f''(1)=-c_0f'(1)\}. \end{cases} \tag{4.5}$$

and $\mathcal{B}=(0,-\delta'(x-1))$ with $\delta(\cdot)$ being the Dirac distribution. It is well known that the operator $\mathcal{A}_0$ generates an exponentially stable C_0-semigroup $e^{\mathcal{A}_0 t}$ and the operator $\mathcal{B}$ is admissible to the C_0-semigroup $e^{\mathcal{A}_0 t}$(Guo, Zhou, AL-Fhaid, Younas, & Asiri, 2014b). Since the linear part of system (4.3) is exponentially stable and is independent of the control, the disturbance estimator design for system (4.3) is much easier than for the original system (3.6). Moreover, the exponential stability of linear part of (4.3) guarantees that all the subsystems involved in estimation of total disturbance are uniformly bounded, provided that the total disturbance is uniformly bounded with respect to time t. In this sense, we can say that system (4.1) separates the total disturbance from the original system (3.6) and introduces the total disturbance into a “relatively good” system (4.3) which is our starting point to estimate the total disturbance.

Now, we design an observer for system (4.3):

$$\begin{cases} \hat{d}_{tt}(x,t)+\hat{d}_{xxxx}(x,t)=0,\\ \hat{d}_{xxx}(0,t)=-c_1\hat{d}_t(0,t),\\ \hat{d}_{xx}(0,t)=\hat{d}(1,t)=0,\\ \hat{d}_x(1,t)=\hat{z}_x(1,t)=[w_x(1,t)-z_x(1,t)], \end{cases} \tag{4.6}$$

which turns out to be a total disturbance estimator. Notice that system (4.6) depends only on the input and output of the original system (3.6), as it is in terms of only the signals $w_x(1, t)$ and $z_x(1, t)$. We now show that the observer (4.6) can provide an estimation of the total disturbance in system (4.3). As matter of fact, if we let

$$\tilde{d}(x,t)=\hat{z}(x,t)-\hat{d}(x,t), \tag{4.7}$$

then the error $\tilde{d}(x,t)$ is governed by

$$\begin{cases} \tilde{d}_{tt}(x,t)+\tilde{d}_{xxxx}(x,t)=0,\\ \tilde{d}_{xxx}(0,t)=-c_1\tilde{d}_t(0,t),\quad \tilde{d}_{xx}(0,t)=0,\\ \tilde{d}(1,t)=\tilde{d}_x(1,t)=0, \end{cases} \tag{4.8}$$

which is exponentially stable (Guo & Jin, 2013a). We consider system (4.8) in the state space

$$\mathcal{H}_1=\{(f,g)\in H^2(0,1)\times L^2(0,1)\mid f(1)=f'(1)=0\}, \tag{4.9}$$

whose inner product is given by

$$\langle (f_1,g_1),(f_2,g_2)\rangle_{\mathcal{H}_1}=\int_0^1[f_1''(x)\overline{f_2''(x)}+g_1(x)\overline{g_2}(x)]dx,\quad \forall\ (f_i,g_i)\in\mathcal{H}_1,\ i=1,2. \tag{4.10}$$

System (4.8) can be written as the following evolutionary equation:

$$\frac{d}{dt}(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t)))=\mathcal{A}_1(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t)), \tag{4.11}$$

where the operator $\mathcal{A}_1$ is given by

$$\begin{cases} \mathcal{A}_1(f,g)=(g,-f^{(4)}),\ \forall (f,g)\in D(\mathcal{A}_1),\\ D(\mathcal{A}_1)=\{(f,g)\in H^4(0,1)\times H^2(0,1)|f''(0)=f(1)=f'(1)=0,\\ g(1)=g'(1)=0,\ f'''(0)=-c_1g(0)\}. \end{cases} \tag{4.12}$$

It is well known that the operator $\mathcal{A}_1$ generates a C_0-semigroup of contractions on $\mathcal{H}_1$ (Chen, Delfour, Krall, & Payre, 1987; Guo & Jin, 2013a). As a result, there exist constants $L_{\mathcal{A}_1},\omega_{\mathcal{A}_1}>0$ such that

$$\|e^{\mathcal{A}_1 t}\|\le L_{\mathcal{A}_1}e^{-\omega_{\mathcal{A}_1}t},\ \ t\ge 0. \tag{4.13}$$

Moreover, we have the following results on hidden regularity.

Lemma 4.1. *For any initial state $(\tilde{d}(\cdot,0),\tilde{d}_t(\cdot,0))\in\mathcal{H}_1$, system (4.8) admits a unique solution $(\tilde{d},\tilde{d}_t)\in C(0,\infty;\mathcal{H}_1)$ satisfying the hidden regularity*

$$\tilde{d}_{xx}(1,t)\in L^2(0,\infty). \tag{4.14}$$

If the initial value $(\tilde{d}(\cdot,0),\tilde{d}_t(\cdot,0))\in D(\mathcal{A}_1)$, the solution is classical and satisfies:

$$\tilde{d}_{xx}(1,t)\to 0\ \text{ as }\ t\to\infty. \tag{4.15}$$

Proof. By semigroup theory, system (4.8) admits a unique solution $(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t))\in C(0,\infty;\mathcal{H}_1)$ such that

$$\|(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t))\|_{\mathcal{H}_1}\le L_{\mathcal{A}_1}e^{-\omega_{\mathcal{A}_1}t}\|(\tilde{d}(\cdot,0),\tilde{d}_t(\cdot,0))\|_{\mathcal{H}_1},\quad t\ge 0. \tag{4.16}$$

To prove the hidden regularity (4.14), we define

$$\phi(t)=\int_0^1 x\tilde{d}_x(x,t)\tilde{d}_t(x,t)dx. \tag{4.17}$$

Then

$$|\phi(t)| \le \frac{1}{2}\|(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t))\|^2_{\mathcal{H}_1}, \forall\, t \ge 0. \tag{4.18}$$

Finding the derivative of $\phi(t)$ along the solution of system (4.8), we can obtain

$$\begin{aligned}\dot{\phi}(t) &= \int_0^1 x\tilde{d}_{tx}(x,t)\tilde{d}_t(x,t)dx - \int_0^1 x\tilde{d}_x(x,t)\tilde{d}_{xxxx}(x,t)dx\\ &= \frac{1}{2}\tilde{d}_t^2(1,t) - \frac{1}{2}\int_0^1 \tilde{d}_t^2(x,t)dx\\ &\quad + \int_0^1 (\tilde{d}_x(x,t) + x\tilde{d}_{xx}(x,t))\tilde{d}_{xxx}(x,t)dx\\ &= \frac{1}{2}\tilde{d}_t^2(1,t) + \frac{1}{2}\tilde{d}_{xx}^2(1,t) - \frac{1}{2}\int_0^1 [\tilde{d}_t^2(x,t) + 3\tilde{d}_{xx}^2(x,t)]dx.\end{aligned} \tag{4.19}$$

Hence, for any $\tau > 0$,

$$\begin{aligned}\int_0^\tau \tilde{d}_{xx}^2(1,t)dt &\le 2\phi(\tau) - 2\phi(0) + 3\int_0^t \|(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t))\|^2_{\mathcal{H}_1}dt\\ &\le \|(\tilde{d}(\cdot,0),\tilde{d}_t(\cdot,0))\|^2_{\mathcal{H}_1}\left(2 + 3L^2_{\mathcal{A}_1}\int_0^\tau e^{-2\omega_{\mathcal{A}_1}t}dt\right)\\ &\le \|(\tilde{d}(\cdot,0),\tilde{d}_t(\cdot,0))\|^2_{\mathcal{H}_1}\left(2 + \frac{3L^2_{\mathcal{A}_1}}{2\omega_{\mathcal{A}_1}}\right).\end{aligned} \tag{4.20}$$

This easily leads to (4.14) due to the arbitrariness of τ. When $(\tilde{d}(\cdot,0),\tilde{d}_t(\cdot,0)) \in D(\mathcal{A}_1)$, the solution is classical and satisfies:

$$\|(\tilde{d}_t(\cdot,t),\tilde{d}_{tt}(\cdot,t))\|_{\mathcal{H}_1} \le L_{\mathcal{A}_1}e^{-\omega_{\mathcal{A}_1}t}\|(\tilde{d}_t(\cdot,0),-\tilde{d}_{xxxx}(\cdot,0))\|_{\mathcal{H}_1},\quad \forall\, t \ge 0. \tag{4.21}$$

On the other hand, it follows from (4.17) that

$$\begin{aligned}\dot{\phi}(t) &= \int_0^1 x\tilde{d}_x(x,t)\tilde{d}_{tt}(x,t)dx + \int_0^1 x\tilde{d}_{tx}(x,t)\tilde{d}_t(x,t)dx\\ &\le C[\|(\tilde{d}_t(\cdot,t),\tilde{d}_{tt}(\cdot,t))\|_{\mathcal{H}_1} + \|(\tilde{d}(\cdot,t),\tilde{d}_t(\cdot,t))\|_{\mathcal{H}_1}],\end{aligned} \tag{4.22}$$

where C is a positive constant. In view of (4.16) and (4.21), we can easily obtain (4.15) from (4.22). □

On the other hand, a formal computation from (4.7) and (4.3) shows that

$$\begin{aligned}\tilde{d}_{xx}(1,t) = &-c_0[w_x(1,t) - z_x(1,t)]\\ &+ [f(w(\cdot,t),w_x(\cdot,t)) + d(t)] - \hat{d}_{xx}(1,t),\end{aligned} \tag{4.23}$$

Owing to (4.14), $c_0[w_x(1,t) - z_x(1,t)] + \hat{d}_{xx}(1,t)$ can be regarded as an approximate of $f(w(\cdot,t), w_x(\cdot,t)) + d(t)$ with a possible error in $L^2(0,\infty)$. That is,

$$\{[f(w(\cdot,t),w_x(\cdot,t)) + d(t)] - c_0[w_x(1,t) - z_x(1,t)] - \hat{d}_{xx}(1,t)\} \in L^2(0,\infty). \tag{4.24}$$

The major advantage of this design is that $c_0[w_x(1,t) - z_x(1,t)] + \hat{d}_{xx}(1,t)$ gives an adequate estimation of the total disturbance $f(w(\cdot,t),w_x(\cdot,t)) + d(t)$ in the sense that the total disturbance does not appear in error system (4.8).

Putting systems (4.1) and (4.6) together, we have obtained an unknown input state observer for system (3.6):

$$\begin{cases} z_{tt}(x,t) + z_{xxxx}(x,t) = 0,\\ z_{xxx}(0,t) = c_1[w_t(0,t) - z_t(0,t)],\\ z_{xx}(0,t) = z(1,t) = 0,\\ z_{xx}(1,t) = c_0[w_x(1,t) - z_x(1,t)] + u(t),\\ \hat{d}_{tt}(x,t) + \hat{d}_{xxxx}(x,t) = 0,\\ \hat{d}_{xxx}(0,t) = -c_1\hat{d}_t(0,t),\\ \hat{d}_{xx}(0,t) = \hat{d}(1,t) = 0,\\ \hat{d}_x(1,t) = w_x(1,t) - z_x(1,t), \end{cases} \tag{4.25}$$

which is determined completely by the input and output of system (3.6). Notice that that no high gain is used in (4.25) and boundedness is not required for derivatives of the total disturbance. By (4.2) and (4.7), we have

$$\tilde{d}(x,t) = w(x,t) - z(x,t) - \hat{d}(x,t). \tag{4.26}$$

Hence, the exponential stability of (4.16) implies that the unknown input state observer (4.25) converges exponentially:

$$\|\tilde{d}(\cdot,t)\|_{\mathcal{H}_1} = \|[z(\cdot,t) + \hat{d}(\cdot,t)] - w(\cdot,t)\|_{\mathcal{H}_1} \to 0 \ \text{ as } \ t \to \infty. \tag{4.27}$$

In this way, we obtain an disturbance estimator and a state observer simultaneously. In other words, (4.25) can be considered as an ESO for system (3.6). Introduce

$$\mathcal{H}_0 = \left\{(f,g) \in H^2(0,1) \times L^2(0,1) \mid f(1) = 0\right\} \tag{4.28}$$

whose inner product is given by

$$\langle (f_1,g_1),(f_2,g_2)\rangle_{\mathcal{H}_0} = \int_0^1 f_1''\overline{f_2''}dx + g_1\overline{g_2}dx + c_0 f_1'(1)\overline{f_2'(1)},\quad \forall\ (f_i,g_i) \in \mathcal{H}_0,\ i = 1,2. \tag{4.29}$$

We consider system (4.25) in the space $\mathcal{H}_0^2$.

We first discuss the properties of well-posedness and convergence for the state observer (4.25).

Theorem 4.1. *Suppose that the solution $(w,w_t) \in C(0,\infty;\mathcal{H}_0)$ of system (3.6) with $u \in L^2_{loc}(0,\infty)$ is well-posed and $[f(w(\cdot,t),w_x(\cdot,t)) + d(t)] \in L^2_{loc}(0,\infty)$. Then, for any initial value $(\hat{d}(\cdot,0),\hat{d}_t(\cdot,0),\ z(\cdot,0),z_t(\cdot,0)) \in \mathcal{H}_0^2$, there exists a unique solution $(\hat{d},\hat{d}_t,z,z_t) \in C(0,\infty;\mathcal{H}_0^2)$ to ESO (4.25) such that*

$$\{[f(w(\cdot,t),w_x(\cdot,t)) + d(t)] - c_0[w_x(1,t) - z_x(1,t)] - \hat{d}_{xx}(1,t)\} \in L^2(0,\infty), \tag{4.30}$$

and

$$\begin{aligned}&\|([z(\cdot,t) + \hat{d}(\cdot,t)] - w(\cdot,t),[z_t(\cdot,t) + \hat{d}_t(\cdot,t)] - w_t(\cdot,t))\|_{\mathcal{H}_1}\\ &\le L_{\mathcal{A}_1}e^{-\omega_{\mathcal{A}_1}t}\|([z(\cdot,0) + \hat{d}(\cdot,0)]\\ &\quad - w(\cdot,0),[z_t(\cdot,0) + \hat{d}_t(\cdot,0)] - w_t(\cdot,0))\|_{\mathcal{H}_1},\quad t \ge 0.\end{aligned} \tag{4.31}$$

Proof. By the invertible transformation

$$\begin{pmatrix} w\\ z\\ \hat{d}\end{pmatrix} = \begin{pmatrix} I & 0 & 0\\ I & 0 & -I\\ 0 & -I & I\end{pmatrix}\begin{pmatrix} w\\ \tilde{d}\\ \hat{z}\end{pmatrix}, \tag{4.32}$$

the state observer (4.25) is equivalent to the following system:

$$\begin{cases} \hat{z}_{tt}(x,t) + \hat{z}_{xxxx}(x,t) = 0,\\ \hat{z}_{xxx}(0,t) = -c_1\hat{z}_t(0,t),\\ \hat{z}_{xx}(0,t) = \hat{z}(1,t) = 0,\\ \hat{z}_{xx}(1,t) = -c_0\hat{z}_x(1,t) + f(w(\cdot,t),w_x(\cdot,t)) + d(t),\\ \tilde{d}_{tt}(x,t) + \tilde{d}_{xxxx}(x,t) = 0,\\ \tilde{d}_{xxx}(0,t) = -c_1\tilde{d}_t(0,t),\\ \tilde{d}_{xx}(0,t) = \tilde{d}(1,t) = \tilde{d}_x(1,t) = 0. \end{cases} \tag{4.33}$$

Since $[f(w(\cdot,t),w_x(\cdot,t)) + d(t)] \in L^2_{loc}(0,\infty)$ and the "$\tilde{d}$-part" of (4.33) is independent of the other part, the admissibility of $\mathcal{B}$ and

Lemma 4.1 imply that the solution of system (4.33) is well defined. Owing to (4.26), (4.31) follows from (4.16). Finally, (4.30) can be obtained from (4.14) and (4.23) directly. □

Remark 4.1. It is seen that in state observer (4.25), we use neither high gain nor discontinuous output injection (Guo & Jin, 2015). This is remarkably different from conventional ones for both ODEs and PDEs. More importantly, we did not assume the boundedness for any order derivative of the total disturbance. It should be pointed out that Han's conventional ESO (Han, 2009) as given in (2.7) is just one design in ADRC. There should been other ESO design methods in which the use of high gain is not necessary, as in adaptive control and control based on internal model principle. Our ESO for infinite-dimensional systems developed in this section offers such a new design.

Remark 4.2. The ESO (4.25) is not always convenient for control design because we know from (4.27) that $[z(x,t)+\hat{d}(x,t)]$ is only an approximate of $w(x, t)$ and that the control appears in z-system (4.1) only. To avoid this problem, we may design an approximate $\hat{w}(x,t)$ of $w(x, t)$ directly by compensating the total disturbance with its approximate claimed by (4.30). Specifically, we can design

$$\begin{cases} \hat{w}_{tt}(x,t)+\hat{w}_{xxxx}(x,t)=0,\\ \hat{w}_{xxx}(0,t)=k[w_t(0,t)-\hat{w}_t(0,t)],\ \ k>0,\\ \hat{w}_{xx}(0,t)=\hat{w}(1,t)=0,\\ \hat{w}_{xx}(1,t)=-c_0[w_x(1,t)-z_x(1,t)]+\hat{d}_{xx}(1,t)+u(t), \end{cases} \tag{4.34}$$

which produces another ESO for system (3.6) as follows:

$$\begin{cases} \hat{w}_{tt}(x,t)+\hat{w}_{xxxx}(x,t)=0,\\ \hat{w}_{xxx}(0,t)=k[w_t(0,t)-\hat{w}_t(0,t)],\\ \hat{w}_{xx}(0,t)=\hat{w}(1,t)=0,\\ \hat{w}_{xx}(1,t)=-c_0[w_x(1,t)-z_x(1,t)]+\hat{d}_{xx}(1,t)+u(t),\\ z_{tt}(x,t)+z_{xxxx}(x,t)=0,\\ z_{xxx}(0,t)=c_1[w_t(0,t)-z_t(0,t)],\\ z_{xx}(0,t)=z(1,t)=0,\\ z_{xx}(1,t)=c_0[w_x(1,t)-z_x(1,t)]+u(t),\\ \hat{d}_{tt}(x,t)+\hat{d}_{xxxx}(x,t)=0,\\ \hat{d}_{xxx}(0,t)=-c_1\hat{d}_t(0,t),\\ \hat{d}_{xx}(0,t)=\hat{d}(1,t)=0,\\ \hat{d}_x(1,t)=w_x(1,t)-z_x(1,t). \end{cases} \tag{4.35}$$

In the new ESO (4.35), we consider $\hat{w}(x,t)$ as an approximate of $w(x,\ t)$ and $-c_0[w_x(1,t)-z_x(1,t)]-\hat{d}_{xx}(1,t)$ the approximate of the total disturbance $f(\cdot,t)+d(t)$ in the same of (4.30). In many situations, such as the unstable wave equation in Feng and Guo (2014), the ESO type of (4.35) is more suitable than (4.25) since $\hat{w}(x,t)\approx w(x,t)$. Certainly, (4.35) serves as a different unknown input observer for system (3.6). The issue of unknown input observer is seldom touched for PDEs. By our approach, we have designed two different unknown input observers (4.25) and (4.35) without using discontinuous injection of the output error. The key step in achieving this design is the uncertainty estimation and compensation.

4.2. Stabilizing output feedback control

With the state observer (4.25) at hand, we can naturally design an observer based stabilizing output feedback as follows:

$$\begin{aligned} u(t)=&[-\hat{d}_{xx}(1,t)-c_0w_x(1,t)+c_0z_x(1,t)]\\ &-c_2w_{xt}(1,t)-c_3w_x(1,t),\quad c_2,c_3>0, \end{aligned} \tag{4.36}$$

where the first three terms in bracket are used to cancel the total disturbance and the other terms are stabilizing output feedback control for system (3.6) in the absence of total disturbance. This implies we have adopt the estimation and cancelation strategy in feedback control (4.36) in the spirit of active disturbance rejection control. Under the controller (4.36), we incorporate systems (4.1) and (4.6) to obtain the closed-loop of system (3.6) as follows:

$$\begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=0,\\ w_{xxx}(0,t)=w_{xx}(0,t)=w(1,t)=0,\\ w_{xx}(1,t)=f(w(\cdot,t),w_x(\cdot,t))+d(t)\\ \qquad -[\hat{d}_{xx}(1,t)+c_0w_x(1,t)-c_0z_x(1,t)]\\ \qquad -c_2w_{xt}(1,t)-c_3w_x(1,t),\\ z_{tt}(x,t)+z_{xxxx}(x,t)=0,\\ z_{xxx}(0,t)=c_1[w_t(0,t)-z_t(0,t)],\\ z_{xx}(0,t)=z(1,t)=0,\\ z_{xx}(1,t)=-\hat{d}_{xx}(1,t)-c_2w_{xt}(1,t)-c_3w_x(1,t),\\ \hat{d}_{tt}(x,t)+\hat{d}_{xxxx}(x,t),\\ \hat{d}_{xxx}(0,t)=c_1\hat{d}_t(0,t),\\ \hat{d}_{xx}(0,t)=\hat{d}(1,t)=0,\\ \hat{d}_x(1,t)=w_x(1,t)-z_x(1,t). \end{cases} \tag{4.37}$$

Theorem 4.2. *Suppose that $f\in C(\mathbb{R}^2;\mathbb{R})$ and $d\in L^\infty(0,\infty)$. Then, for any initial state $(w(\cdot,0),w_t(\cdot,0),z(\cdot,0),z_t(\cdot,0),\hat{d}(\cdot,0),\hat{d}_t(\cdot,0))\in\mathcal{H}_0^3$, the closed-loop system (4.37) admits a unique solution $(w,w_t,z,z_t,\hat{d},\hat{d}_t)\in C(0,\infty;\mathcal{H}_0^3)$ such that:*

$$\sup_{t\in[0,\infty)}\|(w(\cdot,t),w_t(\cdot,t),z(\cdot,t),z_t(\cdot,t),\hat{d}(\cdot,t),\hat{d}_t(\cdot,t))\|_{\mathcal{H}_0^3}<+\infty. \tag{4.38}$$

Moreover, there exist $L,\ \omega>0$ such that

$$\|(w(\cdot,t),w_t(\cdot,t))\|_{\mathcal{H}_0}\le Le^{-\omega t},\quad t\ge 0. \tag{4.39}$$

If we assume further that $f(0)=0$ and $d(t)\equiv 0$, then,

$$\|(w(\cdot,t),w_t(\cdot,t),z(\cdot,t),z_t(\cdot,t),\hat{d}(\cdot,t),\hat{d}_t(\cdot,t))\|_{\mathcal{H}_0^3}\to 0\,\text{as}\,t\to\infty. \tag{4.40}$$

In other words, when the external disturbance is disconnected to the system, the closed-loop system is internally asymptotically stable.

Proof. Since system (4.37) is a nonlinear system, it is not easy to dealt with it directly. Fortunately, by the following invertible transformation:

$$\begin{pmatrix} w\\ w_t\\ z\\ z_t\\ \hat{d}\\ \hat{d}_t \end{pmatrix}=\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ I&0&0&0&-I&0\\ 0&I&0&0&0&-I\\ 0&0&-I&0&I&0\\ 0&0&0&-I&0&I \end{pmatrix}\begin{pmatrix} w\\ w_t\\ \tilde{d}\\ \tilde{d}_t\\ \hat{z}\\ \hat{z}_t \end{pmatrix}. \tag{4.41}$$

System (4.37) is transformed into the equivalent system described by the "$(w,\tilde{d})$-part"

$$\begin{cases} w_{tt}(x,t)+w_{xxxx}(x,t)=0,\\ w_{xxx}(0,t)=w_{xx}(0,t)=w(1,t)=0,\\ w_{xx}(1,t)=\tilde{d}_{xx}(1,t)-c_2w_{xt}(1,t)-c_3w_x(1,t),\\ \tilde{d}_{tt}(x,t)+\tilde{d}_{xxxx}(x,t)=0,\\ \tilde{d}_{xxx}(0,t)=c_1\tilde{d}_t(0,t),\\ \tilde{d}_{xx}(0,t)=\tilde{d}(1,t)=\tilde{d}_x(1,t)=0, \end{cases} \tag{4.42}$$

and the "$\hat{z}$-part"

$$\begin{cases} \hat{z}_{tt}(x,t)+\hat{z}_{xxxx}(x,t)=0,\\ \hat{z}_{xxx}(0,t)=c_1\hat{z}_t(0,t),\\ \hat{z}_{xx}(0,t)=\hat{z}(1,t)=0,\\ \hat{z}_{xx}(1,t)=-c_0\hat{z}_x(1,t)+f(w(\cdot,t),w_x(\cdot,t))+d(t), \end{cases} \tag{4.43}$$

Though still nonlinear, the equivalent system can be treated by linear approach, since the "$(w,\tilde{d})$-part" is a linear system; and more importantly, it is decoupled from the "$\hat{z}$-part".

The "$(w, \tilde{d})$-part" can be written, in the Hilbert space $\mathcal{H}_0 \times \mathcal{H}_1$, as the following abstract from:

$$\frac{d}{dt}(w(\cdot,t), w_t(\cdot,t), \tilde{d}(\cdot,t), \tilde{d}_t(\cdot,t)) = \mathscr{A}(w(\cdot,t), w_t(\cdot,t), \tilde{d}(\cdot,t), \tilde{d}_t(\cdot,t)), \tag{4.44}$$

where the operator $\mathscr{A}$ is given by

$$\begin{cases} \mathscr{A}(f,g,\phi,\psi) = (g, -f^{(4)}, \psi, -\phi^{(4)}), \ \forall (f,g,\phi,\psi) \in D(\mathscr{A}), \\ D(\mathscr{A}) = \{(f,g,\phi,\psi) \in H^4(0,1) \times H^2(0,1) \mid \\ \qquad f'''(0) = f''(0) = f(1) = g(1) = 0, \\ f''(1) = \phi''(1) - c_2 g'(1) - c_3 f'(1), \\ \phi(1) = \psi(1) = \phi'(1) = \psi'(1) = \phi''(0) = 0, \phi'''(0) = -c_1\psi(0)\}. \end{cases} \tag{4.45}$$

Now we prove that the operator $\mathscr{A}$ defined by (4.45) generates an exponentially stable C_0-semigroup on $\mathcal{H}_0 \times \mathcal{H}_1$. Indeed, for any $(\hat{f}, \hat{g}, \hat{\phi}, \hat{\psi}) \in \mathcal{H}_0 \times \mathcal{H}_1$, we solve

$$\mathscr{A}(f,g,\phi,\psi) = (g, -f^{(4)}, \psi, -\phi^{(4)}) = (\hat{f}, \hat{g}, \hat{\phi}, \hat{\psi}) \tag{4.46}$$

to obtain $g = \hat{f}$, $\psi = \hat{\phi}$, and

$$\begin{cases} -f^{(4)}(x) = \hat{g}(x), \\ f'''(0) = f''(0) = f(1) = 0, \\ f''(1) = \phi''(1) - c_2\hat{f}'(1) - c_3 f'(1), \end{cases} \tag{4.47}$$

and

$$\begin{cases} -\phi^{(4)}(x) = \hat{\psi}(x), \\ \phi(1) = \phi'(1) = \phi''(0) = 0, \\ \phi'''(0) = -c_1\hat{\phi}(0). \end{cases} \tag{4.48}$$

We solve (4.47) and (4.47) to obtain

$$\begin{cases} f(x) = (1-x)f'(1) - \displaystyle\int_x^1 \int_\tau^1 \int_0^\beta \int_0^\alpha \hat{g}(s)\,ds\,d\alpha\,d\beta\,d\tau, \\ \phi(x) = c_1\hat{\phi}(0)\left(\dfrac{x}{2} - \dfrac{x^3}{6} - \dfrac{1}{3}\right) - \displaystyle\int_x^1 \int_\tau^1 \int_0^\beta \int_0^\alpha \hat{\psi}(s)\,ds\,d\alpha\,d\beta\,d\tau, \\ \phi''(1) = -c_1\hat{\phi}(0) - \displaystyle\int_0^1 \int_0^\alpha \hat{\psi}(s)\,ds\,d\alpha, \\ f'(1) = \dfrac{1}{c_3}\left[\phi''(1) - c_2\hat{f}'(1) + \displaystyle\int_0^1 \int_0^\alpha \hat{g}(s)\,ds\,d\alpha\right]. \end{cases} \tag{4.49}$$

Therefore, $\mathscr{A}^{-1}$ is compact on $\mathcal{H}_0 \times \mathcal{H}_1$. Therefore $\sigma(\mathscr{A})$ consists of isolated eigenvalues only.

By Guo and Yu (2001), the operator $\mathcal{A}_1$, defined by (4.12), generates an exponentially stable C_0-semigroup on $\mathcal{H}_1$; and there is a sequence of generalized eigenfunctions $\{\Phi_n(x)\}$ of $\mathcal{A}_1$ which forms a Riesz basis for $\mathcal{H}_1$, with all associated eigenvalues of sufficiently large module being algebraically simple. Moreover, it is a trivial exercise to show that the same stability and spectral results are true for operator $\mathcal{A}_0$ defined by (4.5). Hence, the spectral growth condition is true for $\mathcal{A}_i$:

$$\omega(\mathcal{A}_i) = s(\mathcal{A}_i) < 0, \quad i = 0, 1, \tag{4.50}$$

where $\omega(\mathcal{A}_i)$ is the growth rate of semigroup $e^{\mathcal{A}_i t}$ and $s(\mathcal{A}_i)$ is the spectral bound of $\mathcal{A}_i, i = 0, 1$. Let the generalized eigenfunctions of $\mathcal{A}_1$ and $\mathcal{A}_0$ be $\{\Phi_n(x)\}$ and $\{\Psi_n(x)\}$, respectively. Define a sequence

$$\{(\Phi_n(x), 0)\} \cup \{(0, \Psi_m(x))\}. \tag{4.51}$$

Then, it is a sequence of generalized eigenfunctions of $\mathscr{A}$, and forms a Riesz basis for $\mathcal{H}_1 \times \mathcal{H}_0$. Therefore $\mathscr{A}$ generates a C_0-semigroup on $\mathcal{X}$ and the spectrum-determined growth condition is true for $\mathscr{A}$:

$$\omega(\mathscr{A}) = s(\mathscr{A}). \tag{4.52}$$

Since the "$(w, \tilde{d})$-part" of system (4.42) is a cascade of the "w-system" and "$\tilde{d}$-system", and the former is independent of the latter. We claim that

$$\sigma(\mathscr{A}) = \sigma(\mathcal{A}_0) \cup \sigma(\mathcal{A}_1). \tag{4.53}$$

Indeed, for any $\lambda \in \sigma(\mathscr{A})$, suppose that $\mathscr{A}(f,g,\phi,\psi) = \lambda(f,g,\phi,\psi)$ with $0 \neq (f,g,\phi,\psi) \in D(\mathscr{A})$. Combining (4.12) and (4.45), we find that $(\phi,\psi) \in D(\mathcal{A}_1)$ and $\mathcal{A}_1(\phi,\psi) = \lambda(\phi,\psi)$. So when $(\phi, \psi) \neq 0$, we have $\lambda \in \sigma(\mathcal{A}_1)$. When $(\phi,\psi) = 0$, then $(f, g) \neq 0$ and it follows from (4.45) and (4.5) that $(f,g) \in D(\mathcal{A}_0)$ and thus

$$\mathscr{A}(f,g,0,0) = (g, -f^{(4)}, 0, 0) = (\mathcal{A}_0(f,g), 0, 0) = \lambda(f,g,0,0). \tag{4.54}$$

This leads to $\mathcal{A}_0(f,g) = \lambda(f,g)$ or $\lambda \in \sigma(\mathcal{A}_0)$. This shows that $\sigma(\mathscr{A}) \subset \sigma(\mathcal{A}_0) \cup \sigma(\mathcal{A}_1)$. The inclusion $\sigma(\mathcal{A}_0) \cup \sigma(\mathcal{A}_1) \subset \sigma(\mathscr{A})$ is trivial. Therefore, (4.53) holds true.

Finally, we combine (4.52) and (4.53) to obtain

$$\omega(\mathscr{A}) = s(\mathscr{A}) = \max\{s(\mathcal{A}_0), s(\mathcal{A}_1)\} < 0. \tag{4.55}$$

Therefore, $e^{\mathscr{A}t}$ is exponentially stable on $\mathcal{H}_0 \times \mathcal{H}_1$. Moreover, the solution of "$(w, \tilde{d})$-part" is well defined and satisfies

$$\|(w(\cdot,t), w_t(\cdot,t), \tilde{d}(\cdot,t), \tilde{d}_t(\cdot,t))\|_{\mathcal{H}_0\times\mathcal{H}_1} \le L_1 e^{-\omega_1 t}, \quad t \ge 0, \tag{4.56}$$

where L_1 and ω_1 are positive constants. From the Sobolev trace-embedding theorem, it follows that

$$[|w_x(1,t)| + |w(1,t)|] \to 0 \ \text{ as } \ t \to \infty, \tag{4.57}$$

which, together the continuity of $f(\cdot)$, leads to that

$$\sup_{t\in[0,\infty)} |f(w(1,t), w_x(1,t)) + d(t)| < +\infty. \tag{4.58}$$

Since the solution of "$(w, \tilde{d})$-part" is well defined, the "$\hat{z}$-part" now becomes an exponentially stable linear system with an inhomogeneous term $f(w(\cdot,t), w_x(\cdot,t)) + d(t)$ that completely comes from the "$(w, \tilde{d})$-part". In this way, we deal with a nonlinear system via a linear approach. Since the "$\hat{z}$-part" is actually a linear system corrupted by an inhomogeneous term, it follows that (Feng & Guo, 2017)

$$\sup_{t\in[0,\infty)} \|(\hat{z}(\cdot,t), \hat{z}_t(\cdot,t))\|_{\mathcal{H}_0} < +\infty, \tag{4.59}$$

which, together with (4.56) and (4.41), readily leads to (4.38).

Finally, when $f(0) = 0$ and $d(t) \equiv 0$, by (4.57) and the continuity of $f(\cdot)$, we have

$$|f(w(1,t), w_x(1,t))| \to 0 \ \text{ as } \ t \to \infty, \tag{4.60}$$

and thus

$$\|(\hat{z}(\cdot,t), \hat{z}_t(\cdot,t))\|_{\mathcal{H}_0} \to 0 \ \text{ as } \ t \to \infty. \tag{4.61}$$

Finally, (4.40) can be obtained by (4.56), (4.61), and (4.41). $\square$

5. Multi-dimensional wave equations

The proposed approach for observer design in previous section is systematic and can be used to deal with many other PDEs such as anti-stable wave equations (Feng & Guo, 2016b), heat equations (Feng & Guo, 2016a), and even the multi-dimensional PDEs. In this section, we consider a multi-dimensional wave equation only.

Let $\Omega \subset \mathbb{R}^n (n \ge 2)$ be an open bounded domain with a smooth C^2-boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where the boundary relative open subsets $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. Let ν be the outward unit normal vector of the boundary Γ. We consider the following multi-dimensional wave equation:

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ w(x,t) = 0 & \text{on} \quad \Gamma_0 \times [0,\infty), \\ \dfrac{\partial w(x,t)}{\partial \nu} = f(w(x,t)) & \text{on} \quad \Gamma_1 \times [0,\infty), \\ \qquad + d(x,t) + u(x,t) & \\ y_o(x,t) = (y_1(x,t), y_2(x,t)) & t \geq 0, \\ \quad = \left(w(x,t)|_{\Gamma_1}, \dfrac{\partial w(x,t)}{\partial \nu}\Big|_{\Gamma_0} \right), & \end{cases} \tag{5.1}$$

where $u(x, t)$ is the control (input) and $y_o(x, t)$ is the measurement (output). The $f(w(x,t)) + d(x,t)$ is regarded as the "total disturbance" which consists of the boundary unknown uncertainty $f(w(x, t))$ and external unknown disturbance $d(x, t)$. The main difficulty for stabilization of system (5.1) is the total disturbance. The problem wound be trivial after we cancel the disturbance by its estimate. Hence, we only give the observer design. Introduce

$$\begin{cases} v_{tt}(x,t) - \Delta v(x,t) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v(x,t)}{\partial \nu} = \dfrac{\partial w(x,t)}{\partial \nu} & \text{on} \quad \Gamma_0 \times [0,\infty), \\ \qquad - c_1 v(x,t) - c_2 v_t(x,t) & \\ \dfrac{\partial v(x,t)}{\partial \nu} = u(x,t) & \text{on} \quad \Gamma_1 \times [0,\infty), \\ \hat{d}_{tt}(x,t) - \Delta \hat{d}(x,t) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial \hat{d}(x,t)}{\partial \nu} = -c_1 \hat{d}(x,t) - c_2 \hat{d}_t(x,t) & \text{on} \quad \Gamma_0 \times [0,\infty), \\ \hat{d}(x,t) = w(x,t) - v(x,t) & \text{on} \quad \Gamma_1 \times [0,\infty), \end{cases} \tag{5.2}$$

where $c_1, c_2 >$ are tuning parameters, $\hat{d}(x,t)$ is used to estimate the total disturbance, and $v(x, t)$ is an auxiliary variable. System (5.2) only depends on the input and output of system (5.1). Let

$$\tilde{v}(x,t) = w(x,t) - v(x,t). \tag{5.3}$$

Then $\tilde{v}(x,t)$ is governed by

$$\begin{cases} \tilde{v}_{tt}(x,t) - \Delta \tilde{v}(x,t) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial \tilde{v}(x,t)}{\partial \nu} = -c_1 \tilde{v}(x,t) - c_2 \tilde{v}_t(x,t) & \text{on} \quad \Gamma_0 \times [0,\infty), \\ \dfrac{\partial \tilde{v}(x,t)}{\partial \nu} = f(w(x,t)) + d(x,t) & \text{on} \quad \Gamma_1 \times [0,\infty), \end{cases} \tag{5.4}$$

which is a type of stable system with the total disturbance $f(w(x,t)) + d(x,t)$ as its inhomogeneous term. We see that "v-system" separates total disturbance from original system (5.1) and control into a stable system. The "$\hat{d}$-system" is actually an observer for system (5.4). If we let

$$\tilde{d}(x,t) = \tilde{v}(x,t) - \hat{d}(x,t) = w(x,t) - v(x,t) - \hat{d}(x,t), \tag{5.5}$$

then $\tilde{d}(x,t)$ is governed by

$$\begin{cases} \tilde{d}_{tt}(x,t) - \Delta \tilde{d}(x,t) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial \tilde{d}(x,t)}{\partial \nu} = -c_1 \tilde{d}(x,t) - c_2 \tilde{d}_t(x,t) & \text{on} \quad \Gamma_0 \times [0,\infty), \\ \tilde{d}(x,t) = 0 & \text{on} \quad \Gamma_1 \times [0,\infty), \end{cases} \tag{5.6}$$

which is a type of stable system and can be served as a target system for the design of observer. More specially, we have

$$\|(\tilde{d}_t(\cdot,t), \tilde{d}(\cdot,t)\|_{H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty, \tag{5.7}$$

where

$$H^1_{\Gamma_1}(\Omega) = \{ f \in H^1(\Omega) \mid f(x) = 0, x \in \Gamma_1 \}. \tag{5.8}$$

Similarly with (4.14) and (4.23), we have

$$\frac{\partial \tilde{d}(\cdot,t)}{\partial \nu} \in L^2(0,\infty; L^2(\Gamma_1)) \tag{5.9}$$

and

$$\frac{\partial \tilde{d}(x,t)}{\partial \nu} = f(w(x,t)) + d(x,t) - \frac{\partial \hat{d}(x,t)}{\partial \nu}. \tag{5.10}$$

Therefore, $\frac{\partial \hat{d}(x,t)}{\partial \nu}$ can be considered as an estimate of the total disturbance $f(w(\cdot,t)) + d(t)$. Moreover, it follows from (5.5) and (5.7)

$$\begin{aligned} &\|(w(\cdot,t) - [v(\cdot,t) + \hat{d}(\cdot,t)], w_t(\cdot,t) \\ &\quad - [v_t(\cdot,t) + \hat{d}_t(\cdot,t)])\|_{H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)} \\ &= \|(\tilde{d}(\cdot,t), \tilde{d}_t(\cdot,t))\|_{H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty, \end{aligned} \tag{5.11}$$

which implies that $(v(\cdot,t) + \hat{d}(\cdot,t), v_t(\cdot,t) + \hat{d}_t(\cdot,t))$ is an estimate of the state $(w(\cdot, t), w_t(\cdot, t))$.

6. Concluding remarks

We have reviewed the development of active disturbance rejection control (ADRC) from its early beginning till this day. The key step toward ADRC is the extended state observer (ESO) which estimates not only the system state but also the total disturbance thus allows the use of the strategy of estimation and cancelation. We first explained how this strategy can be adopted in the ADRC via an ODE example. as in adaptive control and control based internal model principle. The ADRC is capable of dealing with many uncertain systems with various disturbance, thus greatly expand the application area of the strategy in control practice. Some recent progresses toward the theoretical foundation of the ADRC for nonlinear lumped parameter systems are reviewed. Furthermore, we considered the ADRC design for PDEs, especially the design of the ESO. This observer design is a new topic for PDEs and turns out to be more complicated. We showcased the output feedback design process through a 1-d uncertain beam equation, and proposed two types of new ESO given in (4.25) and (4.35). Unlike in the case for lumped parameter systems, the new designs do not use high gain and do not suppose the boundedness of derivative (of any order) of the disturbance. In addition, different from most of unknown input observer designs, the new designs do not use non-smooth method. As a result, the actual nonlinear closed-loop system can be treated via linear method after cancelation. Additional application to multi-dimensional PDEs is briefly illustrated.

As mentioned at the end Section 2, the ADRC is also applicable to PDEs with corrupted measured output in Feng and Guo (2016b). Other applications can be found in Feng and Guo (2014), Guo and Guo (2013) and Guo, Chen, and Feng (2017).

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