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Blow-up and global solutions for quasilinear parabolic equations with Neumann boundary conditions

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Using the upper and lower solution techniques and Hopf's maximum principle, the sufficient conditions for the existence of blow-up positive solution and global positive solution are obtained for a class of quasilinear parabolic equations subject to Neumann boundary conditions. An upper bound for the ‘blow-up time’, an upper estimate of the ‘blow-up rate’, and an upper estimate of the global solution are also specified.

Keywords: quasilinear parabolic equation; blow-up solution; global solution; Neumann boundary conditions; blow-up time

AMS Subject Classifications: 35K55; 35K10; 35K57

1. Introduction

In this article, we are concerned with the following equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (a(u)\nabla u) + f(u) & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) &= u_0(x) > 0 & \text{in } \overline{D},
\end{align*}
\]

where $D$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary, $\nabla$ is the gradient operator, $\frac{\partial}{\partial n}$ denotes the outward normal derivative, $T$ is the maximum existence time of $u(x, t)$, and $\overline{D}$ is the closure of $D$.

Throughout this article, we always assume that $a$, $b$ and $f$ are positive $C^2(\mathbb{R}^+)$ functions, and the function $u_0$ is a positive $C^2(\overline{D})$ function satisfying

\[
\frac{\partial u_0(x)}{\partial n} = 0 \quad \text{on } \partial D.
\]

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It is known from the classical parabolic equation theory (see e.g. [1]) that there exists a unique local classical solution $u$ to problem (1), that is $T > 0$. Moreover, the Hopf’s maximum principle [2] ensures that $u > 0$ over $\mathcal{D} \times [0, T)$.

**Definition 1.1** We say that the classical solution $u(x, t)$ of (1) is said to blow up at finite time $T$ if $u(x, t)$ satisfies (1) for $t \in [0, T)$ and

$$\lim_{t \to T} u(x, t) = +\infty$$

for all $x \in \mathcal{D}$. $u(x, t)$ is said to be a global solution if it satisfies (1) for all $t \in [0, \infty)$.

The blow-up positive solutions of quasilinear parabolic equations have been studied extensively in literature, see for instance [3–8]. Lair and Oxley in [5] gave the necessary and sufficient condition for the existence of global and blow-up solutions for the following problem:

$$u_t = \nabla \cdot (a(u)\nabla u) + f(u) \quad \text{in } D \times (0, T),$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D \times (0, T),$$

$$u(x, 0) = u_0(x) > 0 \quad \text{in } \mathcal{D},$$

where $D$ is a bounded domain of $\mathbb{R}^N$ with smooth boundary. Under some suitable conditions, Imai and Mochizuki [6] studied the following problem:

$$b'(u)_t = \Delta u + f(u) \quad \text{in } D \times (0, T),$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D \times (0, T),$$

$$u(x, 0) = u_0(x) > 0 \quad \text{in } \mathcal{D},$$

where $D$ is a bounded region of $\mathbb{R}^N$, with smooth boundary. The sufficient conditions for the existence of global solution and blow-up solution were given. The objective of this article is to extend the results in [5,6] to the Equation (1) that covers Equations (2) and (3) as special cases with $b(u) \equiv u$ and $a(u) \equiv 1$, respectively.

We proceed as follows. In Section 2, the blow-up and and global solutions for Equation (1) are studied. Several examples in which the functions $a(s)$, $b(s)$ and $f(s)$ are of power and exponential types are presented in Section 3 as the application of main results.

### 2. Blow-up solutions and global solutions

**Theorem 2.1** Let $u(x, t)$ be a $C^3(D \times (0, T)) \cap C^2(\mathcal{D} \times [0, T))$ solution of Equation (1). Assume that

(i) For any $s \in \mathbb{R}^+$,

$$b'(s) > 0, \quad \left(\frac{a(s)}{b'(s)}\right)' \geq 0, \quad \left(\frac{f'(s)}{a(s)}\right)' \geq 0. \quad (4)$$

(ii)

$$\int_{m_0}^{+\infty} \frac{a(s)}{f(s)} ds < +\infty \text{ where } m_0 = \min_{\mathcal{B}} u_0(x). \quad (5)$$
Then $u(x, t)$ must blow up in finite time $T$,

$$T \leq \frac{1}{\alpha} \int_{m_0}^{+\infty} \frac{a(s)}{f(s)} \, ds \quad \text{and} \quad u(x, t) \leq H^{-1}(\bar{a}(T - t)),$$

where

$$\alpha = \frac{a(m_0)}{b'(m_0)}, \quad \bar{\alpha} = \frac{a(M_0)}{b'(M_0)}, \quad M_0 = \max u_0(x), \quad H(z) = \int_z^{+\infty} \frac{a(s)}{f(s)} \, ds, \quad \forall z \geq 0,$$

$H^{-1}$ is the inverse function of $H$.

**Proof** Let $u(x, t)$ be a solution of the following equation:

$$\begin{cases}
(b(u))_t = \nabla \cdot (a(u) \nabla u) + f(u) & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) = m_0 & \text{in } \bar{D}.
\end{cases}$$

(7)

Then $u(x, t)$ is a lower solution of Equation (1). Now we show that $u(x, t)$ blows up, and so does $u(x, t)$.

Consider the function

$$\varphi = -a(u) u_t + \alpha f(u).$$

(8)

We find that

$$\nabla \varphi = -a' \nabla u \nabla u - a \nabla u_t + \alpha f' \nabla u,$$

$$\Delta \varphi = -a' \Delta u |\nabla u|^2 - a \nabla u_t \cdot \nabla u + a f' \Delta u + a f'' |\nabla u|^2 - a \Delta u,$$

(9)

(10)

and

$$\varphi_t = -a'(u)_t^2 - a(u)_t + \alpha f' \Delta u $$

$$= -a'(u)_t^2 - a \left[ \frac{a}{b'} \Delta u + \frac{a'}{b'} |\nabla u|^2 + \frac{1}{b'} f \right] + \alpha f' \Delta u $$

$$= -a'(u)_t^2 + \left[ \frac{aa''}{(b')^2} - \frac{aa'}{b'} \right] u_t |\nabla u|^2 - \frac{2a a'}{b'} \nabla u \cdot \nabla u - \frac{a^2}{b'} \Delta u $$

$$+ \left[ \frac{aa''}{(b')^2} - \frac{aa'}{b'} \right] u_t \Delta u + \left[ \frac{ab''}{(b')^2} - \frac{a}{b'} f' + \alpha f' \right] u_t.$$  

(11)

By (10) and (11), it follows that

$$\frac{a}{b'} \Delta \varphi - \varphi_t = a'(u)_t^2 + \frac{a}{b'} a f' \Delta u + \frac{a}{b'} a f'' |\nabla u|^2 - \frac{aa''}{(b')^2} u_t |\nabla u|^2 $$

$$- \frac{a^2}{(b')^2} u_t \Delta u + \left[ \frac{a}{b'} f' - \frac{ab''}{(b')^2} f - \alpha f' \right] u_t.$$  

(12)

By the first equation of (7), we have

$$a \Delta u = b' u_t - a' |\nabla u|^2 - f(u).$$  

(13)
Combining (12) and (13) yields

$$\frac{a}{b'} \Delta \varphi - \varphi_t = \frac{a' b' - a b''}{b'} (u_t)^2 + \frac{a f'' - a' f'}{b'} \varphi |\nabla u|^2 + \frac{a}{b'} f' u_t - \alpha f' f'. \quad (14)$$

By (8), it has

$$a u_t = \alpha f - \varphi. \quad (15)$$

It then follows from (14) and (15) that

$$\frac{a}{b'} \Delta \varphi + \frac{f'}{b'} \varphi - \varphi_t = b' \left( \frac{a}{b'} \right)' (u_t)^2 + \frac{a^2}{b'} \left( \frac{f'}{a} \right)' \varphi |\nabla u|^2. \quad (16)$$

From (4) and (6), we see that the right-hand side of equality (16) is non-negative, i.e.

$$\frac{a}{b'} \Delta \varphi + \frac{f'}{b'} \varphi - \varphi_t \geq 0. \quad (17)$$

Let $t = \varepsilon > 0$. Then by (8) and continuity, we have

$$\varphi(x, 0) = \lim_{\varepsilon \to 0^+} \varphi(x, \varepsilon)$$

$$= \lim_{\varepsilon \to 0^+} \left\{ -a(u(x, \varepsilon)) \left[ \frac{1}{b'(u(x, \varepsilon))} \nabla \cdot (a(u(x, \varepsilon)) \nabla u(x, \varepsilon)) + \frac{1}{b'(u(x, \varepsilon))} f(u(x, \varepsilon)) \right] + \alpha f(u(x, \varepsilon)) \right\}$$

$$= -a(m_0) \left[ \frac{1}{b'(m_0)} \nabla \cdot (a(m_0) \nabla m_0) + \frac{1}{b'(m_0)} f(m_0) \right] + \alpha f(m_0)$$

$$= -a(m_0) \frac{b'(m_0)}{b'(m_0)} f(m_0) + \alpha f(m_0) = 0. \quad (18)$$

On the other side,

$$\frac{\partial \varphi}{\partial n} = -a' u_t \frac{\partial u_t}{\partial n} - a f' \frac{\partial u}{\partial n} + \alpha f' \frac{\partial u}{\partial n} = -a \left( \frac{\partial u}{\partial n} \right)_t = 0 \quad \text{on } \partial D \times (0, T). \quad (19)$$

Combining (17)–(19) and applying the Hopf’s maximum principles ([2]), we find that the maximum of $\varphi$ in $\overline{D} \times [0, T)$ is zero. Hence

$$\varphi \leq 0 \quad \text{in } \overline{D} \times [0, T),$$

and

$$\alpha \leq \frac{a(u)}{f(u)} u_t. \quad (20)$$

For any fixed $x \in \overline{D}$, integrate (20) over $[0, t]$ to get

$$t \leq \frac{1}{\alpha} \int_{m_0}^{u(x, t)} \frac{a(s)}{f(s)} ds.$$
By assumption (5), this implies that \( u(x, t) \) must blow up in a finite time \( t = T \). Moreover,

\[
T \leq \frac{1}{\alpha} \int_{m_0}^{+\infty} \frac{a(s)}{f(s)} \, ds.
\]

Therefore \( u(x, t) \) also blows up in a finite time \( t = T \) and

\[
T \leq T \leq \frac{1}{\alpha} \int_{m_0}^{+\infty} \frac{a(s)}{f(s)} \, ds.
\]

On the other hand, let \( \tilde{u}(x, t) \) be a solution of the following equation:

\[
\begin{aligned}
(b(u))_t &= \nabla \cdot (a(u)\nabla u) + f(u) \quad \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \times (0, T), \\
u(x, 0) &= M_0 \quad \text{in } \overline{D}.
\end{aligned}
\]

Then \( \tilde{u}(x, t) \) is an upper solution of Equation (1). Define the function

\[
\psi = -a(\tilde{u})\tilde{u}_t + \overline{\alpha} f(\tilde{u}).
\]

Repeating the above-mentioned process, we find that the maximum of \( \psi \) in \( \overline{D} \times [0, T) \) is zero as well. Hence

\[
\psi \leq 0 \quad \text{in } \overline{D} \times [0, T),
\]

i.e.

\[
\overline{\alpha} \leq \frac{a(\tilde{u})}{f(\tilde{u})} \tilde{u}_t. \tag{21}
\]

For any fixed \( x \), integrate the inequality (21) over \([t, s] \), \( 0 < t < s < T \), to get

\[
H(\tilde{u}(x, t)) \geq H(\tilde{u}(x, t)) - H(\tilde{u}(x, s)) = \int_{\tilde{u}(x, t)}^{\tilde{u}(x, s)} \frac{a(s)}{f(s)} \, ds = \int_t^s \frac{a(\tilde{u})}{f(\tilde{u})} \tilde{u}_t \, dt \geq \overline{\alpha}(s - t),
\]

and

\[
\tilde{u}(x, t) \leq H^{-1}(\overline{\alpha}(s - t)).
\]

Letting \( s \to T \), we get

\[
\tilde{u}(x, t) \leq H^{-1}(\overline{\alpha}(T - t)).
\]

By virtue of the comparison principle, we have

\[
u(x, t) \leq \tilde{u}(x, t).
\]

Therefore,

\[
u(x, t) \leq H^{-1}(\overline{\alpha}(T - t)).
\]

The proof is complete. \( \square \)
THEOREM 2.2  Let \( u(x,t) \) be a \( C^3(\tilde{D} \times (0,T)) \cap C^2(\overline{D} \times [0,T]) \) solution of (1). Assume that

(i) For any \( s \in \mathbb{R}^+ \),
\[
\frac{a'(s)}{b'(s)} > 0, \quad \left( \frac{a(s)}{b(s)} \right)' \leq 0, \quad \left( \frac{f(s)}{a(s)} \right)' \leq 0. \tag{22}
\]

(ii)
\[
\int_{M_0}^{+\infty} \frac{a(s)}{f(s)} \, ds = +\infty \text{ where } M_0 = \max_{\tilde{D}} u_0(x). \tag{23}
\]

Then \( u(x,t) \) must be a global solution and
\[
u(x,t) \leq G^{-1}(\bar{\alpha} t), \quad \bar{\alpha} = \frac{a(M_0)}{b'(M_0)},
\]
where
\[
G(z) = \int_{M_0}^{z} \frac{a(s)}{f(s)} \, ds, \quad \forall \ z \geq M_0,
\]
and \( G^{-1} \) is the inverse function of \( G \).

Proof  Repeating the arguments of the proof of Theorem 1, we find that the minimum of \( \psi \) in \( \tilde{D} \times [0,T) \) is also zero. Hence
\[
\psi \geq 0 \quad \text{ in } \tilde{D} \times [0,T),
\]
i.e.
\[
\bar{\alpha} \geq \frac{a(\bar{u})}{f(\bar{u})} \bar{u}_t. \tag{24}
\]
For any fixed \( x \in \tilde{D} \), we get, by integration (24), that
\[
\bar{u}_t \geq \frac{1}{\bar{\alpha}} \int_{M_0}^{\bar{u}(x,t)} \frac{a(s)}{f(s)} \, ds. \tag{25}
\]
It then follows from (23) and (25) that \( \bar{u}(x,t) \) must be a global solution, and so is \( u(x,t) \). Furthermore, by (25), one has
\[
G(\bar{u}(x,t)) = \int_{M_0}^{\bar{u}(x,t)} \frac{a(s)}{f(s)} \, ds \leq \bar{\alpha} t.
\]
Hence
\[
\bar{u}(x,t) \leq G^{-1}(\bar{\alpha} t).
\]
By the comparison principle, we finally get
\[
u(x,t) \leq \bar{u}(x,t) \leq G^{-1}(\bar{\alpha} t).
\]
The proof is complete.
3. Application

Example 3.1 When \( b(u) \equiv u \) (or \( a(u) \equiv 1 \)), Equation (1) reduces to (2) (or (3)). It is easy to check that in this case, the conclusions of Theorems 2.1 and 2.2 are also valid. We thus extend the results in [5] (or [6]) for Equations (2) (or (3)) to (1).

Example 3.2 Let \( u(x, t) \) be a \( C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T]) \) solution of the following equation:

\[
\begin{cases}
(u^m)_t = \nabla \cdot (u^n \nabla u) + u^q & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) = u_0(x) > 0 & \text{in } \overline{D},
\end{cases}
\]

where \( D \) is a bounded domain of \( \mathbb{R}^N \) with smooth boundary. By Theorem 1, if \( 0 < m \leq n+1 < q \), then \( u(x, t) \) must blow up in a finite time \( t = T \),

\[
T \leq \frac{m}{q - n - 1}m_0^{m-q}, \quad m_0 = \min_{\overline{D}} u_0(x),
\]

and

\[
u(x, t) \leq \left[ \frac{1}{m} M_0^{n-m+1}(T-t) \right]^{n-q+1} \cdot M_0 = \max_{\overline{D}} u_0(x).
\]

By Theorem 2.1, if \( 0 \leq q < n+1 \leq m \), then \( u(x, t) \) must be a global solution and

\[
u(x, t) \leq \left[ \frac{(n-q+1)M_0^{n-m+1}}{m} t + M_0^{n-q+1} \right]^{(1/(n-q+1))}.
\]

Example 3.3 Let \( u(x, t) \) be a \( C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T]) \) solution of the following equation:

\[
\begin{cases}
(e^{mu})_t = \nabla \cdot (e^{nu} \nabla u) + e^{qu} & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) = u_0(x) > 0 & \text{in } \overline{D},
\end{cases}
\]

where \( D \) is a bounded domain of \( \mathbb{R}^N \) with smooth boundary. By Theorem 1, if \( 0 < m \leq n < q \), then \( u(x, t) \) must blow up in a finite time \( t = T \),

\[
T \leq \frac{m}{q - n} e^{(m-q)m_0}, \quad m_0 = \min_{\overline{D}} u_0(x),
\]

and

\[
u(x, t) \leq \frac{1}{n-q} \ln \frac{q-n}{m} + \frac{(n-m)M_0}{n-q} + \frac{1}{n-q} \ln(T-t), \quad M_0 = \max_{\overline{D}} u_0(x).
\]

By Theorem 2.1, if \( 0 \leq q < n \leq m \), then \( u(x, t) \) must be a global solution and

\[
u(x, t) \leq \frac{1}{n-q} \ln \left[ \frac{(n-q)e^{(n-m)M_0}}{m} t + e^{(n-q)M_0} \right].
\]
Example 3.4 Let \( u(x, t) \) be a \( C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T]) \) solution of the following equation:

\[
\begin{align*}
(u^2)_t &= \nabla \cdot (ue^u \nabla u) + u^3 e^u & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) &= u_0(x) = 2 & \text{in } \overline{D},
\end{align*}
\]

where \( D = \{ x = (x_1, x_2, x_3) \mid \sum_{i=1}^3 x_i^2 < 1 \} \) is the unit ball of \( \mathbb{R}^3 \). Now

\[
f(u) = u^3 e^u, \quad a(u) = ue^u, \quad b(u) = u^2.
\]

It is easy to check that (4) and (5) hold. Hence, \( u(x, t) \) must blow up in a finite time \( t = T \),

\[
T \leq \frac{1}{\varrho} \int_{m_0}^{+\infty} \frac{a(s)}{f(s)} ds = \frac{2}{e^2} \int_2^{+\infty} \frac{1}{s^2} ds = \frac{1}{e^2},
\]

and

\[
u(x, t) \leq H^{-1}(\varrho(T - t)) = \frac{2}{e^2(T - t)}.
\]

Example 3.5 Let \( u(x, t) \) be a \( C^3(D \times (0, T)) \cap C^2(\overline{D} \times [0, T]) \) solution of the following equation:

\[
\begin{align*}
(e^u)_t &= \nabla \cdot \left( \frac{1}{\sqrt{u}} \right) \nabla u + \sqrt{u} & \text{in } D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial D \times (0, T), \\
u(x, 0) &= u_0(x) = 2 & \text{in } \overline{D},
\end{align*}
\]

where \( D = \{ x = (x_1, x_2, x_3) \mid \sum_{i=1}^3 x_i^2 < 1 \} \). Now we have

\[
f(u) = \sqrt{u}, \quad a(u) = \frac{1}{\sqrt{u}}, \quad b(u) = e^u.
\]

It is easy to check that both (22) and (23) hold true. Therefore, \( u(x, t) \) must be a global solution and

\[
u(x, t) \leq G^{-1}(\varrho t) = 2 \exp \left( \frac{t}{\sqrt{2e^2}} \right).
\]

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