Boundary stabilization for axially moving Kirchhoff string under fractional PI control

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In this paper, we investigate vibration control for an axially moving Kirchhoff string. A fractional PI boundary control is applied transversally to the right boundary and the transverse vibration of the system is shown to be suppressed. By constructing an auxiliary equivalent system (PDE-ODE) and a novel energy like function, the well-posedness and exponential stability of the closed-loop system are established by means of the Faedo-Galerkin method and the Lyapunov method. The effectiveness of the proposed controller is verified by a numerical simulation using finite element method.

1 | INTRODUCTION

The stabilization problem of the axially string systems arise in many engineering applications like elastic robot arms, plastic films, band saws, paper sheets, magnetic tapes, aerial cable, and conveyor belts, which has been attracted much attention, see, for instance, [1–5]. Especially, the Kirchhoff string, as a typical distributed parameter system, has been extensively investigated from perspectives of both engineering and mathematics, see, e.g., [6, 7] and the references therein. However, these systems frequently suffer from transverse vibrations in the process, which has motivated many researchers to explore different vibration control methods. Among many of them, the boundary control is considered as one of the most effective implementable methods for infinite-dimensional dynamic systems to suppress the vibration of these systems via sensors and non-intrusive actuators [8–12]. Therefore, designing a suitable stabilizing boundary feedback is an important topic for the control of the mechanical systems, which possesses many advantages like fewer sensors and actuators.

In paper [13], two linear feedback controls were designed to suppress the transverse and longitudinal vibration for a coupled nonlinear flexible marine riser. Actually, under the quasi-static stretch assumption [14], the change of longitudinal vibration is small and can be ignored. The average value of the distributed strain can be approximated by a function of
1/L ∫₀ᴸ y²ₓ(x,t)dx so that the coupling system considered in [13] can be reduced to the classical Kirchhoff string proposed by [15]. It should be noted that the stability analysis of a three-dimensional string system was presented in [16] by the direct Lyapunov method with three boundary control inputs applied at the boundary.

In this paper, we consider an axially moving Kirchhoff string described by the following nonlinear partial differential equation (PDE):

\[
\begin{align*}
\left\{ \begin{array}{l}
y_{tt}(x,t) + 2 vy_{xt}(x,t) = \left[ \frac{\tau_0}{\hat{\rho} h} - v^2 + \frac{\varepsilon}{2 \hat{\rho} L} \int_0^L y²_x(x,t)dx \right] y_{xx}(x,t), \quad x \in (0, L), t > 0, \\
\tau_0 \frac{\hat{\rho} h}{\hat{\rho}} - v^2 + \frac{\varepsilon}{2 \hat{\rho} L} \int_0^L y²_x(x,t)dx \right] y_x(L, t) - vy_x(L, t) = U(t), \\
y(0, t) = 0, \quad y(x, 0) = f(x), \quad y(t, 0) = g(x),
\end{array} \right.
\end{align*}
\]

where \( y(x, t) \) stands for the transversal displacement of the Kirchhoff string at the position \( x \) and at time \( t \), \( v \) is in proportional to the speed of the string through the eyelets, \( U(t) \) is the control input, \( \hat{\rho}, \varepsilon, h, \tau_0, f, g \) denote the mass density, the Young modulus, the cross-section area, the initial axial tension, the initial displacement and velocity of the string, respectively. When \( v = 0 \), system (1) is reduced to the classical Kirchhoff string, which has been discussed extensively in literature like those [6, 7, 17, 18], to name just a few. For the investigation of the vibration control of some non-moving flexible string (or beam) systems, we refer the interested reader to [19–21].

For an axially moving Kirchhoff string system (i.e., \( v \neq 0 \)), by constructing an energy-like function, the paper [22] proposed a linear boundary velocity feedback control: \( U(t) = -ky_x(L, t) \) with \( k > 0 \), and proved that the resulting closed-loop system is exponentially stable. The absolute exponential stability of an axially moving Kirchhoff system with nonlinear boundary controller which satisfies a sector constraint was studied in [18] by employing an integral type multiplier method. In addition, using the linear boundary feedback, the exponential stability of two kinds of nonlinear axially moving strings were established in [23] and [24]. Recently, the direct Lyapunov’s method was used to address the boundary stabilization of a nonlinear axially moving string with a low-gain adaptive velocity feedback in [25] and an axially moving Kirchhoff system with high-gain adaptive velocity feedback in [26]. The boundary stabilization is also available for two kinds of nonlinear axially moving beam systems in [27, 28]. In addition to vibration control, the dynamics or vibration behavior of the axially moving structures also becomes an important topic and we refer this to [29–34].

In early 20th century, the paper [35] first studied the PI (Proportional-Integral) control for some finite dimensional systems. In the past decades, the stability of one-dimensional infinite dimensional linear systems with PI controller has been well investigated [36, 37]. In [38], the stabilization of a hyperbolic density flow system governed by linear hyperbolic equation with PI boundary control has been addressed by frequency domain approach. Moreover, under PI boundary control, feedback stabilization for a class of density-flow systems was also investigated by the frequency domain methods in [39], where the integral term in PI controllers is the decisive factor to eliminate the steady-state errors. In [40], a necessary and sufficient condition on the stability of a nonlinear transport equation under a PI boundary controller has been developed. When the parameters in controlled plants are uncertain, deterioration may occur in control performance. In order to address this problem, adaptive control is usually adopted. In [41], an adaptive PID controller was presented for the vibration suppression of a flexible beam and the stability of the closed loop PDE systems was discussed by means of infinite-dimensional Lyapunov method.

On the other hand, design of fractional order control such as fractional order PID (Proportion Integration Differentiation) controls has been regarded as a generalization of the integer order PID control, which presents more flexible control law to improve the performance. For the application of fractional PID control, we refer to a recent review paper [42]. It is generally considered that in dealing with the stability of nonlinear systems, the performance of the fractional PID controller is usually better than integer order PID controller, which is attributed to the fact that the anti-interference effect of the fractional PID control is better than the traditional method [43]. In particular, when the variation of gain and high frequency noise appear, fractional PID control has robustness, and can eliminate the steady-state errors by the fractional integral term [44]. In addition, the boundary control is often relaxed to the slope of the boundary deflection or first-order time differentiation if the sensor is a deflection sensor. The noise amplification caused by the high-frequency differential process can also be reduced by the fractional derivatives [45]. Motivated by the works aforementioned, a fractional PI boundary control applied at the right boundary is proposed in this paper to suppress the vibration of the Kirchhoff string system (1).
Compared with the existing results, the contributions of present paper are summarized as follows. The fractional integral term presented in the control which is different from the nonlocal term of [46] involves singular and non-integrable kernels ($t^{\alpha-1}$; $0 < \alpha < 1$). Because all the previously developed methods deal with regular convolution terms only [47], the string with a time delay [48, 49] or integrable kernels [46, 50, 51] are no longer regular, which leads to some substantial mathematics challenges. In addition, the well-posedness of the axially moving Kirchhoff string systems were not addressed in previous studies due to the difficulty of establishing the regularity of the solutions except our works for nonlinear beam systems [27, 28]. Inspired by [52], an augmented system (PDE-ODE) which is equivalent to the original system is constructed to overcome the difficulties caused by the fractional terms. By applying the Galerkin approximation method, the existence and uniqueness of the local solution of the linear closed-loop system are proved. The well-posedness of the closed-loop system driven by a nonlinear PDE with fractional PI boundary damping is then established by the fixed point theorem. Finally, by means of the direct Lyapunov method, the exponential stability of the closed-loop is established by constructing a novel energy like function. The fractional PI boundary controller proposed in this paper extends linear control [22], and the suppression effect of nonlinear system (1) is better than the linear one as shown in the numerical simulations in Section 4.

The remainder of this paper is organized as follows. The generalized fractional order integral, the equivalent system model, its controller and main results are introduced in Section 2. In Section 3, the well-posedness and the global stability of the closed-loop system are established by the Galerkin approximation method and the direct Lyapunov method. In Section 4, some numerical simulations are presented to illustrate the proposed theoretical results, where the finite element method is used to solve the closed-loop system, followed up by concluding remarks in Section 5.

2 | NOTATION AND MAIN RESULTS

2.1 | Fractional PI control

For notational simplicity, let $y_x := \frac{\partial y}{\partial x}, y_{xx} := \frac{\partial^2 y}{\partial x^2}, y_{xt} := \frac{\partial^2 y}{\partial x \partial t}, y_{tt} := \frac{\partial^2 y}{\partial t^2}$ and $(\cdot) := \frac{d(\cdot)}{dt}$. The tension in the string system (1) is defined by

$$M(\|y_x(t)\|^2) = a + b \int_0^L y_{x}^2(x,t) dx$$

where $a = \frac{\tau_0}{\rho h} - v^2 > 0$ and $b = \frac{\varepsilon}{2 \rho L} \geq 0$. Since the left boundary $x = 0$ is fixed, the boundary value condition is $y(0, t) = 0$. The boundary condition $M(\|y_x(t)\|^2) y_x(L, t) - v y_t(L, t) = U(t)$ means that the control input $U(t)$ is applied transversally at the right boundary in order to drive the transversal component of the tension in the string to be balanced. Since the moving state of the string can produce extra damping, we add a velocity term $v y_t(L, t)$ to the boundary condition of (1) so that all the results in this paper can be extended to the case of non-moving strings (i.e., $v = 0$).

We first introduce the following spaces. Let $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and the inner product of $L^2(0, L)$ respectively. Set $V_1 := \{ y \in H^1(0, L) : y(0) = 0 \}$ and $W := L^2(-\infty, +\infty)$ with the norm $\| \cdot \|_W$. It is clear that $V_1$ is a closed subspace of $H^1(0, L)$.

**Definition 2.1.** The generalized fractional integral of order $\alpha > 0$ of the function $h(\cdot)$ defined over $[a, b]$ is given by

$$I^{\alpha, \beta}_{a,b} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{-\beta (t-s)} h(s) ds, \ \alpha > 0, \ \beta \geq 0, \ t > a,$$

where $\Gamma$ is the Gamma function. When $\beta = 0$, it is reduced to the classical fractional order integral [53].

In this work, we consider a fractional PI boundary control given by

$$\begin{cases}
U(t) = -k_p(t) y_t(L, t) - k_I I^{\alpha, \beta}_{0,t} y_t(L, t), & t > 0, \\
k_p(t) = y_t^2(L, t), & k_p(0) = k_0,
\end{cases}$$

(3)

where $0 < \alpha < 1$, $k_p > 0$, $y_t(L, t)$ is the output velocity of the right boundary of the moving string system (1), $k_0 > 0$ and $k_I > 0$. 


Remark 2.1. It is worth noting that, the additional parameters (differential order and integral order) provided by fractional PI controller can effectively improve the performance of the systems, and adjust the dynamic behavior of the control system. The fractional integrator makes the steady-state error of the system be very small, and the oscillation suppression effect of nonlinear system be better, as illustrated in the experimental results of [54].

Remark 2.2. The exponential stability of the Kirchhoff string systems under the linear feedback $U(t) = -ky_t(L, t)$, that is, $k_p(t) \equiv \text{Const.} \, k_j = 0$ in (3), was established in [55] for the static case (i.e. $v = 0$). Under the fractional damping at the boundary, the asymptotic stability of linear wave equation and Euler-Bernoulli beam equation were available in [56, 57] via the semigroup theory of linear operators.

### 2.2 Well-posedness of closed loop systems

Now, we substitute (3) into (1) to obtain a closed-loop system governed by the following nonlinear PDE:

\[
\begin{aligned}
&y_{tt}(x,t) = M(\|y_x(t)\|^2)y_{xx}(x,t) - 2vy_{xt}(x,t), \\
&M(\|y_x(t)\|^2)y_x(L,t) - vy_t(L,t) = -k_p(t)y_t(L,t) - k_I t_0^\alpha y_x(L,t), \\
&\dot{k}_p(t) = y_t^2(L,t), \quad k_p(0) = k_0, \\
&y(0,t) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = g(x),
\end{aligned}
\]

for all $x \in [0,L]$ and $t > 0$, where $k_0, k_I$ are the constants given in (3).

The main objective addressed hereafter is about the well-posedness and exponential stability of the closed-loop system (4) through dealing with an auxiliary equivalent system given by the following proposition.

**Proposition 2.1.** The closed-loop system (4) is equivalent to the following PDEs system: for all $x \in [0,L]$, $\eta \in \mathbb{R}$ and $t > 0$,

\[
\begin{aligned}
&y_{tt}(x,t) = M(\|y_x(t)\|^2)y_{xx}(x,t) - 2vy_{xt}(x,t), \\
&M(\|y_x(t)\|^2)y_x(L,t) - vy_t(L,t) = -k_p(t)y_t(L,t) - \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi(\eta, t)d\eta, \\
&\dot{\phi}(\eta, t) + (\eta^2 + \beta)\phi(\eta, t) - y_t(L,t)\sigma(\eta) = 0, \\
&\dot{k}_p(t) = y_t^2(L,t), \quad k_p(0) = k_0, \\
&y(0,t) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = g(x), \quad \phi(\eta,0) = 0,
\end{aligned}
\]

where $\sigma(\eta) := |\eta|^{\frac{1-2\alpha}{2}}$ with $0 < \alpha < 1$, $\rho = \frac{\sin \pi \alpha}{\pi} k_I$, $\beta > 0$, and $k_0, k_I$ are given in (3).

For notation simplicity, we omit the obvious variables $(x, t)$ in the related functions in what follows. To prove the dissipativity of system (5), set

\[
E(t) := \frac{1}{2} \int_0^L \left[ y_t^2 + ay_x^2 \right] dx + \frac{b}{4} \left[ \int_0^L y_x^2 dx \right]^2 + \frac{\rho}{2} \int_{-\infty}^{+\infty} \phi^2(\eta, t)d\eta
\]

\[
= \frac{1}{2} \|y_t(\cdot, t)\|^2 + \frac{1}{2} M(\|y_x(t)\|^2) + \frac{\rho}{2} \|\phi(\cdot, t)\|_W^2,
\]

to be the modified energy, where $M(s) := \int_0^s M(w)dw$, $\rho$ is the constant given in Proposition 2.1. The boundary conditions in (5) imply that the initial energy

\[
E(0) = \frac{1}{2} \int_0^L (g^2(x) + af_x^2(x))dx + \frac{b}{4} \left( \int_0^L f_x^2(x)dx \right)^2 > 0.
\]
Corresponding to the nonlinear PDEs system (5), we first consider the following linear PDEs system

\[
\begin{align*}
\dot{y}_t(x,t) &= \varphi(t)y_{xx}(x,t) - 2v y_x(x,t), \\
\varphi(t)y_x(L,t) - v y_t(L,t) &= -k_p(t)y(L,t) - \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi(\eta,t)d\eta, \\
\phi_t(\eta,t) + (\eta^2 + \beta)\phi(\eta,t) - y(L,t)\sigma(\eta) &= 0, \\
k_p(t) &= y^2_t(L,t), \quad k_p(0) = k_0, \\
y(0,t) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = g(x), \quad \phi(\eta,0) = 0,
\end{align*}
\]

(7)

for all \( x \in [0,L], \eta \in \mathbb{R} \) and \( t > 0 \), where \( k_0, \sigma, \rho \) are given in Proposition 2.1 and \( \varphi \in W^{1,\infty}(0,\infty) \) with \( \varphi(t) \geq \varphi(0) = a \). The local existence of the solutions of the PDEs system (7) is given by the following Theorem 2.1, which plays a significant role in the proof of the well-posedness of the closed-loop system (5).

**Theorem 2.1.** Suppose the compatibility condition \( af_x(L) - vg(L) = -k_0g(L) \). Then, there exists a unique solution \((y, \phi)\) satisfying \( y \in L^\infty([0,T);V_1 \cap H^2(0,L)) \cap C([0,T);V_1) \), \( y_t \in L^\infty([0,T);V_1) \cap C([0,T);V_2(0,L)) \), and \( \phi \in C([0,T);W) \) with \( T > 0 \) such that \((y, \phi)\) satisfies the PDEs system (7).

With the help of Theorem 2.1, the global existence of the solutions of the closed-loop system (5) is obtained by the fixed point theorem.

**Theorem 2.2.** If \( f, g \in V_1 \cap H^2(0,L) \) and the compatibility condition \( M(\|f_x\|^2)f_x(L) - vg(L) = -k_0g(L) \) holds, then there exists a unique solution \( y : [0,L] \times [0,T_{\text{max}}) \to \mathbb{R}, \phi : \mathbb{R} \times [0,T_{\text{max}}) \to \mathbb{R} \) with \( 0 < T_{\text{max}} \leq \infty \) to the closed loop system (5) such that

\[
\begin{align*}
y &\in L^\infty([0,T_{\text{max}});V_1 \cap H^2) \cap C([0,T);V_1), \\
y_t &\in L^\infty([0,T_{\text{max}});V_1) \cap C([0,T_{\text{max}});L^2(0,L)), \\
\phi &\in C([0,T_{\text{max}});W).
\end{align*}
\]

**Remark 2.3.** From Theorem 2.1, we know that the energy function \( E(t) \) is a non-increasing function for \( t \geq 0 \). Integrating both sides of (66) over \((0,t)\) gives

\[
\begin{align*}
k_0 \int_0^t y^2_t(L,s)ds + \rho \int_0^t \int_{-\infty}^{+\infty} \phi^2(\eta,s)(\eta^2 + \beta)d\eta ds \\
+ \frac{1}{2}\|y(\cdot,t)\|^2 + \frac{1}{2}M(\|y_x(t)\|^2) + \frac{\rho}{2}\|\phi(\cdot,t)\|^2_W \leq E(0),
\end{align*}
\]

(8)

which, together with (3), yields that for any \( t > 0 \),

\[
k_p(t) = k_0 + \int_0^t y^2_t(L,s)ds \leq k_0 + \frac{E(0)}{k_0},
\]

(9)

where \( k_0 \) is the constant given in (3). From (8), one can conclude that \( y_t(L, \cdot) \in L^2(0,\infty) \), and \( \int_{-\infty}^{+\infty} \phi^2(\eta, \cdot)(\eta^2 + \beta)d\eta \in L(0, +\infty) \).

**Remark 2.4.** Based on the auxiliary equivalent system, it is natural to apply LaSalle’s invariant principle or the frequency domain method to the asymptotic stability of the closed loop system (5) for linear case of \( M(s) \equiv a \). However, for the nonlinear distributed parameter system (4), it is difficult to establish the general results by these methods. In the next section, we construct an energy-type Lyapunov function for the auxiliary equivalent model, and derive directly the exponential stability of the closed-loop Kirchhoff system (5).
2.3 | Global stability

In this subsection, we develop the exponential stability of the closed-loop system (5). First, we introduce an energy-type Lyapunov function defined by

\[ V(t) := E(t) + \gamma \int_0^L x y_1 y_3 \, dx + v \gamma \int_0^L x y_1^2 \, dx \] (10)

Lemma 2.1. If \( \gamma < \frac{1}{\max\{1, \frac{L^2 + 2Lv}{a}\}} \), then the function \( V(t) \) defined by (10) satisfies

\[ 0 \leq \omega_1 E(t) \leq V(t) \leq \omega_2 E(t) \] (11)

for all \( t \geq 0 \), where \( \omega_1 = 1 - \gamma \max\{1, \frac{L^2 + 2Lv}{a}\} \) and \( \omega_2 = 1 + \gamma \max\{1, \frac{L^2 + 2Lv}{a}\} \).

Remark 2.5. Because of the choice of number \( \gamma \) in Lemma 2.1 and \( E(0) > 0 \), it is easy to see that \( V(0) > 0 \).

Next, we show that the energy function \( E(t) \) decays exponentially to zero as \( t \to \infty \).

Theorem 2.3. Let the number \( \gamma \) satisfy

\[ \gamma < \min \left\{ \frac{1}{\max\{1, \frac{L^2 + 2Lv}{a}\}}, \frac{2\beta a}{a + 2\beta L \rho C_{\beta}}, \frac{2ak_0}{aL + 2L \max\{v^2, \left(\frac{E(0)}{k_0}\right)^2}\} \right\} \] (12)

where \( \rho = \frac{\sin \pi \alpha}{\pi \sin \pi \alpha} k_t, C_{\beta} = \beta^{-\alpha} \frac{\pi}{\sin \pi \alpha} \). Suppose that all assumptions of Theorem 2.2 are satisfied. Then, the energy function \( E(t) \) along the solution of (5) satisfies:

\[ E(t) \leq \kappa e^{-\delta t} E(0) \]

for all \( t > 0 \), where \( \kappa, \delta \) are two positive constants.

Finally, we analyze the exponential stability of the nonlinear Kirchhoff string under the control (3).

Theorem 2.4. Suppose that all assumptions of Theorem 2.3 are satisfied. Then, the string response \( y(x, t) \) of the closed-loop system (5) decays exponentially, that is,

\[ |y(x, t)| \leq \hat{\kappa} e^{-\frac{\tilde{\delta} t}{2}} \]

for all \( t > 0 \) and \( x \in [0, L] \), where \( \hat{\kappa} = \sqrt{\frac{2L E(0)}{a \omega_1}}, \tilde{\delta}, \omega_1 \) are given in Theorem 2.3 and Lemma 2.1, respectively.

Remark 2.6. It is worth noting that when the string is non-moving (i.e., \( v = 0 \)), the same exponential stability of the Kirchhoff string system can be obtained by the same approach. In addition, if \( M(s) \) given in (2) is an abstract function like paper [7] satisfying the condition \( M \in C^2(0, \infty) \) with \( M(s) \geq a > 0 \) for all \( s \in \mathbb{R} \), Theorems 2.2, 2.3 and 2.4 are still valid for a general model of the closed-loop system (5) by the similar arguments.

3 | PROOF OF MAIN RESULTS

Proof of Proposition 2.1. Note that for all \( \eta \in \mathbb{R} \) and \( t > 0 \),

\[ \phi_t(\eta, t) + (\eta^2 + \beta)\phi(\eta, t) - y_t(L, t)\sigma(\eta) = 0, \] (13)
and $\phi(\eta, 0) = 0$. The solution of (13) is found to be

$$
\phi(\eta, t) = \int_0^t \sigma(\eta)e^{-(\eta^2 + \beta)(t-s)}y_i(L, s)ds.
$$

(14)

Due to the boundary condition of (5) and $\sigma(\eta) = \left|\eta\right|^{\frac{1-2\alpha}{2}}$, it follows from (14) that

$$
M(\|y_x(t)\|^2)y_x(L, t) - vy_i(L, t)
$$

$$
= -k_p(t)y_i(L, t) - \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi(\eta, t)d\eta
$$

$$
= -k_p(t)y_i(L, t) - \frac{k_i \sin \alpha \pi}{\pi} e^{-\beta t} \int_0^t \left[ 2 \int_0^{+\infty} \eta^{1-2\alpha} e^{-\eta^2(t-s)}d\eta \right] e^{\beta s} y_i(L, s)ds
$$

$$
= -k_p(t)y_i(L, t) - \frac{k_i \sin \alpha \pi}{\pi} e^{-\beta t} \int_0^t [(t-s)^{\alpha-1}\Gamma(1-\alpha)] e^{\beta s} y_i(L, s)ds.
$$

(15)

Since $\frac{\sin \alpha \pi}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$, the (15) becomes

$$
M(\|y_x(t)\|^2)y_x(L, t) - vy_i(L, t)
$$

$$
= -k_p(t)y_i(L, t) - \frac{k_i}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}y_i(L, s)ds
$$

$$
= -k_p(t)y_i(L, t) - k_i I_0^\alpha y_i(L, t).
$$

(16)

This completes the proof of the proposition.

\[\square\]

**Proof of Theorem 2.1.** To begin with, the variational structure corresponding to (7) is provided by

$$
\int_0^L y_{tt}w dx + 2v \int_0^L y_{xt}w dx + \varphi(t) \int_0^L y_x w_x dx + [k_p(t)y_i(L, t)
$$

$$+\rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi(\eta, t)d\eta - vy_i(L, t)]w(L) = 0,
$$

(17)

and

$$
\int_{-\infty}^{+\infty} \phi_t(\eta, t)v(\eta)d\eta + \int_{-\infty}^{+\infty} (\eta^2 + \beta)\phi(\eta, t)v(\eta)d\eta - \int_{-\infty}^{+\infty} y_i(L, t)\sigma(\eta)v(\eta)d\eta = 0,
$$

(18)

for any $w \in V_1, v \in W$. We apply the Faedo-Galerkin method to get the existence of the solution to (7). Since $V_1 \cap H^2(0, L)$ is a separable Banach space, there exists an orthonormal basis $\{w_i\}_{i=1}^{\infty}$, for which $f, g \in \Delta_2 := \text{Span}\{w_1, w_2\}$. Let $\{\mu_i\}_{i=1}^{\infty}$ be an orthogonal basis for $W$. Set $\Delta_n = \text{Span}\{w_1, \cdots, w_n\}$ and $V_n = \text{Span}\{\mu_1, \cdots, \mu_n\}$. We give an approximate solution $(y^n, \phi^n)$ of (17)-(18) defined by $y^n(\cdot, t) = \sum_{j=1}^{n} c_{jn}(t)w_j(\cdot) \in \Delta_n$, $\phi^n(\cdot, t) = \sum_{j=1}^{n} b_{jn}(t)\mu_j(\cdot) \in V_n$, satisfying the initial conditions $y^n(0) = f$, $y^n_0(0) = g$, and $\phi^n(\eta, 0) = 0$, for any $n \in N$. The proof will be accomplished by four steps.

**Step 1: For any $T > 0$, there exists a constant $\tilde{C}_T > 0$ independent of $n$ such that**

$$
\sup_{n \in \mathbb{N}} \{\|y^n(t)\|^2 + \varphi(t)\|y^n_0(t)\|^2 + \rho\|\phi^n(t)\|^2_W\} \leq \tilde{C}_T
$$

(19)

**for almost all $t \in [0, T]$.**
Since \( y_n(t,0) = 0 \), it has
\[
2 \int_0^L y_n^2(x) \, dx = \int_0^L |y_n(L,t)|^2 \, dx.
\]
Plugging \( w = y_n \) and \( v = \phi_n \) into (17) and (18), respectively, gives
\[
\dot{E}_1^n(t) := \frac{1}{2} \frac{d}{dt} \left\{ |y_n^{\prime}(\cdot,t)|^2 + \varphi(t) |y_n^{\prime}(\cdot,t)|^2 + \rho \| \phi_n^{\prime}(\cdot,t) \|_W^2 \right\}
\]
\[
= \int_0^L y_n^2 \, dx + \varphi(t) \int_0^L y_n^2 \, dx + \rho \int_{-\infty}^{+\infty} \phi_n^{\prime}(t) \phi_n(t) \, d\eta + \frac{\dot{\varphi}(t)}{2} \| y_n^{\prime}(\cdot,t) \|^2
\]
\[
= -2v \int_0^L y_n^2 \, dx - \left[ k_p^n(t) y_n(L,t) + \rho \int_{-\infty}^{+\infty} \sigma(\eta) \phi_n(t) \, d\eta \right] y_n(L,t)
\]
\[
+ v [y_n(L,t)]^2 + \rho \int_{-\infty}^{+\infty} \phi_n(t) \phi_n(t) \, d\eta + \frac{\dot{\varphi}(t)}{2} \| y_n^2(\cdot,t) \|^2
\]
\[
= -k_p^n(t) [y_n(L,t)]^2 - \rho \int_{-\infty}^{+\infty} \sigma(\eta) \phi_n(t) \, d\eta y_n(L,t) + \frac{\dot{\varphi}(t)}{2} \| y_n^2(\cdot,t) \|^2
\]
\[
+ \rho \int_{-\infty}^{+\infty} \phi_n(t) \phi_n(t) \, d\eta.
\]
(20)

Since \( \phi_n(t,\eta) = - (\eta^2 + \beta) \phi_n(\eta, t) + y_n(L,t) \sigma(\eta) \), it follows from (20) that
\[
\dot{E}_1^n(t) = -k_p^n(t) [y_n(L,t)]^2 - \rho \int_{-\infty}^{+\infty} \sigma(\eta) \phi_n(t) \, d\eta y_n(L,t)
\]
\[
+ \rho \int_{-\infty}^{+\infty} \phi_n(t) [- (\eta^2 + \beta) \phi_n(\eta, t) + y_n(L,t) \sigma(\eta)] \, d\eta + \frac{\dot{\varphi}(t)}{2} \| y_n^2(\cdot,t) \|^2
\]
\[
= -k_p^n(t) [y_n(L,t)]^2 - \rho \int_{-\infty}^{+\infty} [\phi_n(t)]^2 (\eta^2 + \beta) \, d\eta + \frac{\dot{\varphi}(t)}{2} \| y_n^2(\cdot,t) \|^2
\]
\[
\leq -k_p^n(t) [y_n(L,t)]^2 - \rho \int_{-\infty}^{+\infty} [\phi_n(t)]^2 (\eta^2 + \beta) \, d\eta + \frac{\dot{\varphi}(t)}{2} E_1^n(t).
\]
(21)

which, together with \( k_p^n(t) \geq k_0 > 0 \), leads to
\[
E_1^n(t) + k_0 \int_0^t [y_n^2(L,s)]^2 \, ds + \rho \int_0^t \int_{-\infty}^{+\infty} [\phi_n(s)]^2 (\eta^2 + \beta) \, d\eta \, ds
\]
\[
\leq E_1^n(0) + \int_0^t \frac{\dot{\varphi}(s)}{\varphi(s)} E_1^n(s) \, ds.
\]
(22)

The Gronwall’s inequality gives (19).

**Step 2:** For any \( T > 0 \), there exists a constant \( C_T > 0 \) depending on \( T \) such that
\[
\sup_{n \in \mathbb{N}} \{ |y_n^2(\cdot,t)|^2 + \varphi(t) |y_n^2(\cdot,t)|^2 + \rho \| \phi_n^{\prime}(\cdot,t) \|_W^2 \} \leq C_T,
\]
(23)

for almost all \( t \in [0,T] \).

First, we show \( \sup_{n \in \mathbb{N}} \| y_n^2(\cdot,0) \| < \infty \). Let \( t = 0 \) in (17). Then, for any \( w \in V_1 \),
\[
\int_0^L y_n^2(x,0) w(x) \, dx + 2v \int_0^L y_n^2(x,0) w(x) \, dx + \varphi(0) \int_0^L y_n^2(x,0) w(x) \, dx
\]
\[
+ \left[ k_p^n(0) y_n(L,0) - vy_n(L,0) \right] w(L) = 0.
\]
(24)
From the compatibility condition and taking \( w = y^n_{tt}(0) \) in (24), one obtains

\[
\|y^n_{tt}(-,0)\|^2 - \varphi(0) \int_0^L y^n_{xx}(x,0)y^n_{tt}(x,0)dx + 2v \int_0^L y^n_{tt}(x,0)y^n_{xx}(x,0)dx = 0.
\] (25)

The Cauchy-Schwarz inequality implies that

\[
\|y^n_{tt}(-,0)\|^2 = \varphi(0) \int_0^L y^n_{xx}(x,0)y^n_{tt}(x,0)dx - 2v \int_0^L y^n_{tt}(x,0)y^n_{xx}(x,0)dx
\]
\[
\leq \varphi(0)\|y^n_{xx}(-,0)\|\|y^n_{tt}(-,0)\| + 2v\|y^n_{tt}(-,0)\|\|y^n_{xx}(-,0)\|
\]
\[
= a\|f_{xx}\|\|y^n_{tt}(-,0)\| + 2v\|y^n_{tt}(-,0)\|\|g_x\|,
\] (26)

and then

\[
\|y^n_{tt}(-,0)\| \leq a\|f_{xx}\| + 2v\|g_x\|. \tag{27}
\]

Differentiating (17) with respect to \( t \) gives

\[
\int_0^L y^n_{tt}(x,t)wdx + \dot{\varphi}(t) \int_0^L y^n_{xx}w_x dx + \varphi(t) \int_0^L y^n_{tt}w dx
\]
\[
= -2v \int_0^L y^n_{xx}wdx - \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi^n_t(\eta, t)d\eta w(L)
\]
\[
- k^n_p(t)y^n_t(L,t)w(L) - (k^n_p(t) - v)y^n_{tt}(L,t)w(L).
\] (28)

Plugging \( w = y^n_{tt} \) into (28) yields

\[
\int_0^L y^n_{tt}y^n_{tt} dx + \dot{\varphi}(t) \int_0^L y^n_{xx}y^n_{tt} dx + \varphi(t) \int_0^L y^n_{tt}y^n_{tt} dx
\]
\[
= -2v \int_0^L y^n_{xx}y^n_{tt} dx - \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi^n_t(\eta, t)d\eta y^n_{tt}(L,t)
\]
\[
- k^n_p(t)y^n_t(L,t)y^n_{tt}(L,t) - (k^n_p(t) - v)[y^n_{tt}(L,t)]^2.
\] (29)

Performing the integration by parts and noticing \( y^n_{tt}(0,t) = 0 \), we obtain

\[
\dot{\varphi}(t) \int_0^L y^n_{xx}y^n_{tt} dx = \dot{\varphi}(t)y^n_{tt}(L,t)y^n_{tt}(L,t) - \dot{\varphi}(t) \int_0^L y^n_{xx}y^n_{tt} dx
\]
\[
= \dot{\varphi}(t)y^n_{xx}(L,t)y^n_{tt}(L,t) - \dot{\varphi}(t) \int_0^L y^n_{xx}y^n_{tt} dx,
\] (30)

and

\[
\int_0^L y^n_{xx}y^n_{tt} dx = \frac{1}{2}(y^n_{tt}(L,t))^2.
\] (31)

Plugging (30) and (31) into (29) produces

\[
\int_0^L y^n_{tt}y^n_{tt} dx + \varphi(t) \int_0^L y^n_{xx}y^n_{tt} dx + \frac{\dot{\varphi}(t)}{2}\|y^n_{xx}\|^2
\]
\[
= \varphi(t) \int_0^L y^n_{xx}y^n_{tt} dx - \dot{\varphi}(t)y^n_{xx}(L,t)y^n_{tt}(L,t) + \frac{\dot{\varphi}(t)}{2}\|y^n_{xx}\|^2 - k^n_p(t)(y^n_{tt}(L,t))^2
\]
\[
- \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi^n_t(\eta, t)d\eta y^n_{tt}(L,t) - (y^n_t(L,t))^3y^n_{tt}(L,t).
\] (32)
Set
\[
E_2^n(t) = \frac{1}{2} ||y^n_{tt}(\cdot, t)||^2 + \frac{1}{2} \varphi(t) ||y^n_{xx}(\cdot, t)||^2 + \frac{1}{2} \rho \|\phi^n(t, \cdot)\|_W^2 + \frac{1}{4} Q^n(L, t)^4. \tag{33}
\]

From (32), we can infer that
\[
\dot{E}_2^n(t) = \int_0^L y^n_{tt}(t) y^n_{tt} x^n_{tt} d \gamma + \phi(t) \int_0^L y^n_{xx}(t) y^n_{xx} x^n_{xx} d \gamma
\]
\[
+ \varphi(t) \int_0^L \phi^n(t, \cdot) \phi^n(t, \cdot) d \eta + (y^n_{xx}(L, t))^3 y^n_{xx}(L, t)
\]
\[
= \frac{\phi(t)}{2} ||y^n_{tt}(\cdot, t)||^2 + \frac{1}{2} \rho \int_0^L \phi^n(t, \cdot) \phi^n(t, \cdot) d \eta + \varphi(t) \int_0^L y^n_{xx}(t) y^n_{xx} x^n_{xx} d \gamma
\]
\[
- \varphi(t) y^n_{xx}(L, t) y^n_{xx}(L, t) - \rho \int_{-\infty}^{+\infty} (\eta^2 + \beta)(\phi^n(t, \cdot))^2 d \eta - k^n_p(t)(y^n_{xx}(L, t))^2. \tag{34}
\]

Since \(\phi^n_{tt}(\eta, t) = - (\eta^2 + \beta) \phi^n(t, \cdot) + \sigma(\eta) y^n_{tt}(L, t)\) and \(y^n_{tt} = \varphi(t) y^n_{xx} - 2v y^n_{x}\), we find from (34) that
\[
\dot{E}_2^n(t) \leq \mathcal{T} P(t) \left[ \frac{\phi(t)}{2} ||y^n_{tt}(\cdot, t)||^2 + \frac{1}{2} ||y^n_{xx}(\cdot, t)||^2 \right] + P(t) \varphi(t) y^n_{tt}(L, t) y^n_{xx}(L, t)
\]
\[
- \rho \int_{-\infty}^{+\infty} (\eta^2 + \beta)(\phi^n(t, \cdot))^2 d \eta - k^n_p(t)(y^n_{xx}(L, t))^2, \tag{35}
\]

where \(\mathcal{T} = \max\{2(v + 1), \frac{2v}{a}\} + 1\), \(P(t) = \frac{\phi(t)}{\varphi(t)}\) for any \(t \in [0, T]\), Using Young’s inequality \((ab \leq \frac{a^2}{4c} + \frac{\epsilon c}{\epsilon} b^2)\) and the second equation of (7), we have
\[
|\varphi(t) y^n_{xx}(L, t) y^n_{xx}(L, t)|
\]
\[
= |(v y^n_{xx}(L, t) - k^n_p(t)(y^n_{xx}(L, t))) y^n_{xx}(L, t)|
\]
\[
\leq C_1 \left( y^n_{xx}(L, t) \right) + \epsilon C_1 \left( y^n_{xx}(L, t) \right) + \rho \int_{-\infty}^{+\infty} \sigma(\eta) \phi^n(t, \cdot) d \eta y^n_{xx}(L, t) \right] + \epsilon C_1 \left( y^n_{xx}(L, t) \right)^2
\]
\[
\leq C_1 \left( y^n_{xx}(L, t) \right) + 2\epsilon C_1 \left( y^n_{xx}(L, t) \right)^2
\]
\[
+ \rho \int_{-\infty}^{+\infty} \sigma(\eta)(\beta + \eta^2)^{-1} d \eta \int_{-\infty}^{+\infty} (\eta^2 + \beta)(\phi^n(t, \cdot))^2 d \eta
\]
\[
\leq C_1 \left( y^n_{xx}(L, t) \right) + 2\epsilon C_1 \left( y^n_{xx}(L, t) \right)^2
\]
\[
+ \frac{C_1 \rho C_\beta}{4\epsilon} \int_{-\infty}^{+\infty} (\eta^2 + \beta)(\phi^n(t, \cdot))^2 d \eta,
\tag{36}
\]

where the following formula was used:
\[
C_\beta = \int_{-\infty}^{+\infty} \sigma(\eta)(\eta^2 + \beta)^{-1} d \eta = \int_{-\infty}^{+\infty} \frac{|\eta|^{1-2\alpha}}{\beta + \eta^2} d \eta
\]
\[
= \int_{0}^{+\infty} \frac{x^{-\alpha}}{x + \beta} d x (x := \eta^2)
\]
\[
= \beta^{-\alpha} \int_{1}^{+\infty} v^{-1}(v - 1)^{-\alpha} d v (v := \frac{x}{\beta} + 1)
\]
\[
= \beta^{-\alpha} \int_{1}^{1} w^{\alpha-1}(1 - w)^{-\alpha} d w (w := \frac{1}{v})
\]
\[
= \beta^{-\alpha} B(\alpha, 1 - \alpha) = \beta^{-\alpha} \Gamma(\alpha)\Gamma(1 - \alpha) = \beta^{-\alpha} \frac{\pi}{\sin \pi \alpha},
\]
\( C_1 = k_0 + \frac{E(0)}{k_0} + v \), and \( \varepsilon > 0 \) is Young’s constant. Plugging (36) into (35) yields

\[
\dot{E}_2^n(t) \leq TP(t) \left[ \frac{\varphi(t)}{2} \| y_{n1}^n(\cdot,t) \|^2 + \frac{1}{2} \| y_{n2}^n(\cdot,t) \|^2 \right] - k_0(y_{n1}^n(L,t))^2 + P(t) \frac{C_1}{4\varepsilon} (y_{n2}^n(L,t))^2 \\
+ P(t) 2c C_1 (y_{n1}^n(L,t))^2 + P(t) \frac{C_1 \rho C_\beta}{2\varepsilon} \int_{-\infty}^{+\infty} (\eta^2 + \beta) (\phi^n(\eta, t))^2 d\eta \\
- \rho \int_{-\infty}^{+\infty} (\eta^2 + \beta) [\phi^n(\eta, t)]^2 d\eta.
\]  

Let \( M_p = \sup_{t \in [0, T]} P(t) \) and choose sufficiently small \( \varepsilon \) such that \( \varepsilon = \frac{k_0}{2c M_p} \). Then, (37) becomes

\[
\dot{E}_2^n(t) \leq C_2 E_2^n(t) + C_2 (y_{n1}^n(L,t))^2 + C_2 \int_{-\infty}^{+\infty} (\eta^2 + \beta) (\phi^n(\eta, t))^2 d\eta,
\]

where \( C_2 = \max\left\{ TM_p, \frac{C_2^1 M_p^2 \rho C_\beta}{2k_0}, \frac{C_2^2 M_p^2}{2k_0} \right\} \) is a constant. Integrating (38) over \((0, t)\), we obtain

\[
E_2^n(t) \leq E_2^n(0) + \int_0^t G(s) ds + C_2 \int_0^t E_2^n(s) ds,
\]

where

\[
G(t) = C_2 (y_{n1}^n(L,t))^2 + C_2 \int_{-\infty}^{+\infty} (\eta^2 + \beta) (\phi^n(\eta, t))^2 d\eta.
\]

In light of (22), one can show that

\[
\int_0^t G(s) ds \leq C_3 E_1^n(0)
\]

for any \( t \in [0, T] \), where \( C_3 = \frac{C_2 (1 + M_p T e^{M_p T})}{\min\{k_0, \rho, v\}} \). Applying Gronwall’s inequality to (39), together with (26), shows that for almost all \( t > 0 \), there exists a constant \( C_T \) depending on \( T \) such that \( \sup_{n \in N} E_2^n(t) \leq C_T \).

**Step 3: Existence of solution to closed loop system (7).** From (23) and the system equation (7), we can deduce that \( y \in H^2(0, L) \). As a result, we can conclude from Steps 1 and 2 that \( y^n, y^n_1, y^n_{11} \) and \( \phi^n \) belong to bounded sets in \( L^\infty([0, T); V_1 \cap H^2), L^\infty([0, T); V_1), L^\infty([0, T); L^2(0, L)) \) and \( L^\infty([0, T); W) \), respectively. By the weak compactness, there exists subsequences still denoted by \( y^n, y^n_1, y^n_{11}, \) and \( \phi^n \) such that

\[
\begin{align*}
&\text{y}^n \rightharpoonup \text{y} \text{ in } L^\infty([0, T); V_1 \cap H^2) \text{ weak}^*, \\
&\text{y}^n_1 \rightharpoonup \text{y}_1 \text{ in } L^\infty([0, T); V_1) \text{ weak}^*, \\
&\text{y}^n_{11} \rightharpoonup \text{y}_{11} \text{ in } L^\infty([0, T); L^2(0, L)) \text{ weak}^*, \\
&\phi^n \rightharpoonup \phi \text{ in } L^\infty([0, T); W) \text{ weak}^* \text{ as } n \to \infty.
\end{align*}
\]

Using the Lions-Aubin lemma to get the necessary compactness and passing (17)-(18) to the limit yields that \( (y, \phi) \) is a solution of (17)–(18). To obtain existence of the solution, it remains to show that

\[
y \in C([0, T); V_1), \ y_1 \in C([0, T); L^2(0, L)), \ \phi \in C([0, T); W).
\]
Actually, from (19) and (23), we can conclude from Lemma 3.3 of [58], p.74 that $y$ is weakly continuous from $[0, T)$ in $V_1$. Let $V'_1$ represent the reflexive space of $V_1$. Since $y \in L^\infty([0, T); V_1)$, it follows from (2) that $y_x \in L^\infty([0, T); L^2(0, L))$ and $y_{xx} \in L^\infty([0, T); V'_1)$, which implies further that $y_{tt} \in L^\infty([0, T); V'_1)$. Thus, $y_t$ is weakly continuous from $[0, T)$ in $\mathcal{W}$. Since $y$ satisfies the estimate (19), it has

$$ t \rightarrow \|y_t(\cdot, t)\|^2 + \varphi(t)\|y_x(\cdot, t)\|^2 + \rho\|\phi(\cdot, t)\|_W^2 $$

is continuous over $[0, T)$. This, together with the properties of weak continuity, gives

$$ y \in L^\infty([0, T); V_1 \cap H^2) \cap C([0, T); V_1), $$

$$ y_t \in L^\infty([0, T); \mathcal{W}) \cap C([0, T); \mathcal{W}) $$

and $\phi \in C([0, T); \mathcal{W})$.

**Step 4: The uniqueness of the solution to closed loop system (7).** Assume that $(y, \phi)$, $(\hat{y}, \hat{\phi})$ are two solutions of the closed-loop system (7) with the same initial values. Then, $(y, \phi)$ and $(\hat{y}, \hat{\phi})$ satisfy the variational equations (17)–(18), respectively. Subtracting these two equations and letting $\tilde{y} = y - \hat{y}$, $\tilde{\phi} = \phi - \hat{\phi}$, $w = \tilde{y}_t$ and $v = \tilde{\phi}$, we obtain

$$ \int_0^L \tilde{y}_{tt} \tilde{y}_t dx + 2v \int_0^L \tilde{y}_{xt} \tilde{y}_t dx + \varphi(t) \int_0^L \tilde{y}_x \tilde{y}_{xt} dx + [k_p(t)y_t(L, t) - \hat{k}_p(t)\hat{y}_t(L, t)] $$

$$ + \rho \int_+^\infty \sigma(\eta)\tilde{\phi}(\eta, t) d\eta - v\tilde{y}_t(L, t)\tilde{y}_t(L, t) = 0, $$

and

$$ \int_{-\infty}^+ \tilde{\phi}_t(\eta, t)\tilde{\phi}(\eta, t) d\eta + \int_{-\infty}^+ (\eta^2 + \beta)\tilde{\phi}^2(\eta, t) d\eta - \int_{-\infty}^+ \tilde{y}_t(L, t)\sigma(\eta)\tilde{\phi}(\eta, t) d\eta = 0, $$

where $\hat{k}_p(t) = [\hat{y}_t(L, t)]^2$, $\check{k}_p(0) = k_0$. Similar to the estimate (21), we can deduce that

$$ \frac{1}{2} E_3(t) = \frac{\hat{\phi}(t)}{2} \|\tilde{y}_x(\cdot, t)\|^2 - \rho \int_{-\infty}^+ \sigma(\eta)\tilde{\phi}(\eta, t) d\eta\tilde{y}_t(L, t) $$

$$ + \frac{1}{2} (k_p(t) - \hat{k}_p(t))[y_t(L, t)]^2 - [\hat{y}_t(L, t)]^2 $$

$$ - [k_p(t)y_t(L, t) - \hat{k}_p(t)\hat{y}_t(L, t)]\tilde{y}_t(L, t) $$

$$ = \frac{\hat{\phi}(t)}{2} \|\tilde{y}_x(\cdot, t)\|^2 - \rho \int_{-\infty}^+ (\eta^2 + \beta)\tilde{\phi}^2(\eta, t) d\eta $$

$$ - \frac{k_p(t) + \hat{k}_p(t)}{2} [y_t(L, t) - \hat{y}_t(L, t)]^2 $$

where

$$ E_3(t) := \|\tilde{y}_t(\cdot, t)\|^2 + \varphi(t)\|\tilde{y}_x(\cdot, t)\|^2 + \rho\|\tilde{\phi}(\cdot, t)\|_W^2 + \frac{1}{2} [k_p(t) - \hat{k}_p(t)]^2. $$

With the same arguments as in the estimation of $E_1^n(t)$ in Step 1, we can deduce

$$ E_3(t) \leq \frac{|\hat{\phi}(t)|}{\varphi(t)}E_3(t). $$

This, together with $E_3(0) = 0$ and Gronwall’s inequality, shows that $E_3(t) \equiv 0$ for any $t \geq 0$. This completes the proof of the theorem.
**Proof of Theorem 2.2.** Let $T_1 < T_{\text{max}}$ be a real number to be determined. Define a vector set:

$$
\Omega_R : = \begin{cases} 
    y \in L^\infty([0, T_1]; V_1), \quad \dot{y} \in L^\infty([0, T_1]; V_1) \cap C([0, T_1]; L^2(0, L)), \\
    \|y\|_{L^\infty([0, T_1]; V_1)} + \|\dot{y}\|_{L^\infty([0, T_1]; V_1)} \leq R, \\
    y(x, 0) = f(x), \quad y_t(x, 0) = g(x).
\end{cases} \quad (48)
$$

Clearly, the set $\Omega_R$ is a complete metric space equipped with the metric:

$$
d(y_1, y_2) = \|y_1 - y_2\|_{L^\infty([0, T_1]; V_1)} + \|\dot{y}_1 - \dot{y}_2\|_{C([0, T_1]; L^2(0, L))}. \quad (49)
$$

Define a solution mapping $S : \Omega_R \rightarrow Y$ as follows: $z \rightarrow S(z) = y$, where $Y$ is the solution set of following PDEs:

$$
\begin{align*}
    y_{tt}(x, t) &= M(\|z_x(t)\|^2)y_{xx}(x, t) - 2vy_{xt}(x, t), \\
    M(\|z_x\|^2)y_x(L, t) - vy_x(L, t) &= -k_p(t)y_t(L, t) - \rho \int_{-\infty}^{+\infty} \sigma(\eta)\phi(\eta, t) d\eta, \\
    \phi_t(\eta, t) + (\eta^2 + \beta)\phi(\eta, t) - y_t(L, t)\sigma(\eta) &= 0, \\
    \dot{k}_p(t) &= y_t^2(L, t), \quad k_p(0) = k_0, \\
    y(0, t) &= \phi(\eta, 0) = 0, x(0, 0) = f(x), y_t(x, 0) = g(x),
\end{align*} \quad (50)
$$

for $z \in \Omega_R$. Let $\varphi(t) = M(\|z_x(t)\|^2) \in W^{1,\infty}(0, T_1)$. By Theorem 2.1, there exists a unique solution $y = S(z)$ to equation (50) for every $z \in \Omega_R$. Now we show that $S(\Omega_R) \subset \Omega_R$ and the mapping $S$ is strictly contraction.

To this end, let $M_R = \max\left\{\frac{dM(t)}{ds} : 0 \leq s \leq R^2\right\}$. Then, $|\varphi(t)| \leq M_R(1 + 2R^2)$ for any $t \in [0, T_1]$. Assume that there exists a sufficient large number $R$ such that $aR > M_1 + M_2$ where

$$
M_1 = \|g\|^2 + M(\|f_x\|^2)\|f_x\|^2, \quad (51)
$$

$$
M_2 = M(\|f_x\|^2)\|f_{xx}\| + C_3 M(\|f_x\|^2)\|f_x\|^2 + (v + C_3)\|g_x\|^2, \quad (52)
$$

with $C_3 > 0$ being the constant given in (40). In light of (22), we have

$$
a \|y_x(\cdot, t)\|^2 \leq E_1(t) \leq E_1(0)e^{\int_0^t \frac{\|g_0\|}{\|v\|} ds}e^{\frac{M_R(1 + 2R^2)}{a} t}, \quad (53)
$$

where $M_\varphi = \frac{M_R(1 + 2R^2)}{a}$. From the estimate (39), it follows that

$$
E_2(t) \leq [E_2(0) + C_3 E_1(0)]e^{M_\varphi T_1}, \quad (54)
$$

which implies that $a \|\dot{y}_x(\cdot, t)\| \leq M_2 e^{M_\varphi T_1}$ for $t \in [0, T_1]$. Since $R > \frac{(M_1 + M_2)}{a}$, it follows from (53) that there exists $0 < T_1 < 1$ such that

$$
\|y\|_{L^\infty([0, T_1]; V_1)} + \|\dot{y}\|_{L^\infty([0, T_1]; V_1)} \leq R, \quad (55)
$$

which implies that $S(\Omega_R) \subset \Omega_R$. 
We next show the strict contraction of the operator $S$. To this purpose, let $S(z) = y$, $S(\hat{z}) = \hat{y}$ and $w = y - \hat{y}$. By (5), it follows that

$$
\begin{aligned}
\begin{cases}
\dot{w}_t(x,t) = M(\|z_x(t)\|^2)w_{xx}(x,t) + [M(\|z_x(t)\|^2) - M(\|\hat{z}_x(t)\|^2)]\hat{y}_{xx} - 2w_t w_{xt}(x,t), \\
M(\|z_x(t)\|^2)y_x(L,t) - M(\|\hat{z}_x(t)\|^2)\hat{y}_x(L,t) - \nu w_t(L,t)
\end{cases}
\end{aligned}

$$

(56)

Taking the inner product in $L^2(0,L)$ on the first equation of (56) with $w_t$, we find

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left\{ \|w_t(\cdot,t)\|^2 + M(\|z_x(t)\|^2)\|w_x(\cdot,t)\|^2 + \rho \|\phi(\cdot,t)\|^2_W + \frac{1}{2}[k_p(t) - \hat{k}_p(t)]^2 \right\}
\end{aligned}
$$

(57)

With the same arguments as in (22) and (45), (57) becomes

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left\{ \|w_t(\cdot,t)\|^2 + M(\|z_x(t)\|^2)\|w_x(\cdot,t)\|^2 + \rho \|\phi(\cdot,t)\|^2_W + \frac{1}{2}[k_p(t) - \hat{k}_p(t)]^2 \right\}
\end{aligned}
$$

(58)

where $k_0$ and $\rho$ are given in (5). Observe that

$$
M(\|z_x(t)\|^2) - M(\|\hat{z}_x(t)\|^2) = \frac{\partial M(s)}{\partial s}(\|z_x(t)\|^2 - \|\hat{z}_x(t)\|^2)
$$

$$
\leq M_R(\|z_x(t)\| + \|\hat{z}_x(t)\|)d(z, \hat{z})
$$

(59)

From (54), one has

$$
\|\hat{y}_{tt}(\cdot,t)\| \leq M_2 e^{2M_R T_1}, \ \forall t \in [0, T_1],
$$
where $M_2$ is a positive constant and $M_\varphi$ is given in (53). By $\dot{y}_t(t) = M(\|\dot{z}_x(t)\|^2)\dot{y}_{xx} - 2v\dot{y}_{xt}$, it has
\begin{equation}
\|\dot{y}_{xx}(\cdot, t)\| \leq \frac{2M_2e^{2M_\varphi T_1}}{a}
\end{equation}
which holds for all $t \in [0, T_1]$. Since $y_x(L, t) = \frac{1}{L} \int_0^L (xy)_x \, dx$, by the Cauchy-Schwarz inequality, it has
\begin{equation}
|y_x(L, t)| \leq \frac{1}{\sqrt{L}}\|y_x\| + \sqrt{L}\|y_{xx}\|.
\end{equation}
Similarly, from $w_t(0, t) = 0$, there holds $|w_x(L, t)| = \sqrt{L}\|w_{xx}(\cdot, t)\|$. Plugging (61) with (59) and (60) into (58) produces
\begin{equation}
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ \|w_t(\cdot, t)\|^2 + M(\|z_x(t)\|^2)\|w_x(\cdot, t)\|^2 + \rho\|\phi(\cdot, t)\|^2_W + \frac{1}{2}[k_\mu(t) - \hat{k}_\mu(t)]^2 \right\} \\
& \quad \leq \Lambda_1d(z, \hat{z}) + \Lambda_2\|w_x(\cdot, t)\|^2 + \Lambda_3d(z, \hat{z})\|w_t(\cdot, t)\|^2,
\end{aligned}
\end{equation}
where $\Lambda_1, \Lambda_2, \Lambda_3$ are positive constants. Integrate over $[0, t]$ on both sides of above equation (62) to obtain
\begin{equation}
\begin{aligned}
\|w_t(\cdot, t)\|^2 + a\|w_x(\cdot, t)\|^2 & \leq \|w_t(\cdot, t)\|^2 + M(\|z_x(t)\|^2)\|w_x(\cdot, t)\|^2 + \rho\|\phi(\cdot, t)\|^2_W \\
& \leq \Lambda_1d(z, \hat{z})t + \Lambda_2\int_0^t \|w_x(\cdot, s)\|^2 ds + \Lambda_3d(z, \hat{z})\int_0^t \|w_t(\cdot, s)\|^2 ds.
\end{aligned}
\end{equation}
By Gronwall’s inequality,
\begin{equation}
\begin{aligned}
\|w_t(\cdot, t)\|^2 & \leq \Lambda_1d(z, \hat{z})T_1 e^{\Lambda_2 T_1}, \quad \|w_x(\cdot, t)\|^2 \leq a^{-1}\Lambda_1d(z, \hat{z})T_1 e^{\Lambda_2 T_1}, \forall t \in [0, T_1],
\end{aligned}
\end{equation}
where $b_1 = \max\{\frac{\Lambda_2}{a}, 2\Lambda_3R\}$. Let $T_1$ be small enough so that
\begin{equation}
d(S(z), S(\hat{z})) \leq \Lambda d(z, \hat{z}),
\end{equation}
where $0 < \Lambda < 1$. We then obtain the existence of the local solution to the closed-loop system (5).

With the same arguments as Step 1 in the proof of Theorem 2.1, we can deduce that
\begin{equation}
E(t) : = \frac{1}{2} \frac{d}{dt} \left\{ \|y_x(t)\|^2 + M(\|y_x(t)\|^2) + \rho\|\phi(\cdot, t)\|^2_W \right\} \\
\leq -k_0y_t^2(L, t) - \rho \int_{-\infty}^{+\infty} [\phi(\eta, t)]^2(\eta^2 + \beta) \, d\eta.
\end{equation}
Applying the same approach of [7], we can conclude from (66) that the local solution can be extended to be global solution, which completes the proof of the theorem. \[\square\]

**Proof of Lemma 2.1.** By the mean value inequality with (6), we obtain
\begin{equation}
\begin{aligned}
\int_0^L xy_t y_x dx + v \int_0^L xy_x^2 dx \\
& \leq \frac{1}{2} \int_0^L y_t^2 dx + \frac{L^2}{2} \int_0^L y_x^2 dx + vL \int_0^L y_x^2 dx \\
& \leq \max\{1, \frac{L^2 + 2Lv}{a}\} \left( \frac{1}{2} \int_0^L y_t^2 dx + \frac{1}{2}M(\|y_x(t)\|^2) \right) \\
& \leq \max\{1, \frac{L^2 + 2Lv}{a}\} E(t),
\end{aligned}
\end{equation}
for all \( t \geq 0 \). Analogously,

\[
\int_0^L xy_1 y_2 \, dx + v \int_0^L xy_2^2 \, dx \geq - \max \left\{ 1, \frac{L^2 + 2Lv}{a} \right\} E(t). \tag{68}
\]

This, together with (10), gives the estimate (11).

\[ \square \]

**Proof of Theorem 2.3.** In view of (10), the derivatives of the function \( V(t) \) is

\[
V(t) = \dot{E}(t) + \gamma \int_0^L xy_1 y_1 \, dx + \gamma \int_0^L xy_1 y_2 \, dx + \gamma v \int_0^L (xy_2^2) \, dx. \tag{69}
\]

Performing the integration by parts, we obtain from (4) that for all \( t > 0 \),

\[
\int_0^L xy_1 y_2 \, dx = \int_0^L x \left[ M(\|y_2(x)\|_2) y_{xx} - 2vy_{xt} \right] y_1 \, dx
\]

\[
= M(\|y_2(x)\|_2^2) \int_0^L xy_{xx} y_1 \, dx - 2v \int_0^L xy_{xt} y_1 \, dx
\]

\[
= \frac{1}{2} M(\|y_2(x)\|_2^2) \int_0^L (xy_2^2) y_1 \, dx - \frac{1}{2} M(\|y_2(x)\|_2^2) \int_0^L y_2^2 \, dx - v \int_0^L (xy_2^2) y_1 \, dx
\]

\[
= \frac{L}{2} M(\|y_2(x)\|_2^2) y_2^2(L,t) - \frac{1}{2} M(\|y_2(x)\|_2^2) \int_0^L y_2^2 \, dx - v \int_0^L (xy_2^2) y_1 \, dx,
\]

and

\[
\int_0^L xy_{xt} y_1 \, dx = \frac{1}{2} \int_0^L [xy_1^2]_{x} \, dx - \frac{1}{2} \int_0^L y_1^2 \, dx = \frac{L}{2} y_1^2(L,t) - \frac{1}{2} \int_0^L y_1^2 \, dx. \tag{71}
\]

Plugging (70)–(71) into (69) leads to

\[
\dot{V}(t) \leq \dot{E}(t) + \frac{\gamma L}{2} y_1^2(L,t) + \frac{\gamma L}{2} M(\|y_2(x)\|_2^2) y_2^2(L,t)
\]

\[
- \gamma \left[ \frac{1}{2} M(\|y_2(x)\|_2^2) \int_0^L y_2^2 \, dx + \frac{1}{2} \int_0^L y_1^2 \, dx \right]. \tag{72}
\]

From (2), it is easy to see that \( \frac{1}{2} M(\|y_2(x)\|_2^2) \int_0^L y_2^2 \, dx \geq \frac{1}{2} \tilde{M}(\|y_2(x)\|_2^2) \). Then, (72) becomes

\[
\dot{V}(t) \leq \dot{E}(t) + \frac{\gamma L}{2} y_1^2(L,t) + \frac{\gamma L}{2} M(\|y_2(x)\|_2^2) y_2^2(L,t) - \gamma \left[ E(t) - \frac{\rho}{2} \|\phi(\cdot,t)\|_W^2 \right]. \tag{73}
\]

Similar to (36) of Theorem 2.1, we can see that

\[
\frac{\gamma L}{2} M(\|y_2(x)\|_2^2) y_2^2(L,t) \leq \frac{\gamma L}{2a} \left[ M(\|y_2(x)\|_2^2) y_2(L,t) \right]^2 \tag{74}
\]

\[
\leq \frac{\gamma L(v - k_\rho(t))^2}{\alpha} y_1^2(L,t) + \frac{\gamma L\rho^2}{\alpha} C_\beta \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta, t) \, d\eta, \tag{75}
\]

where \( \rho, C_\beta \) are the constants given in (36), and

\[
\frac{\gamma \rho}{2} \int_{-\infty}^{+\infty} \phi^2(\eta,t) \, d\eta = \frac{\gamma \rho}{2} \int_{-\infty}^{+\infty} (\eta^2 + \beta)^{-1} (\eta^2 + \beta) \phi^2(\eta,t) \, d\eta
\]

\[
\leq \frac{\gamma \rho}{2} \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta,t) \, d\eta. \tag{76}
\]
Plugging (66), (74) and (76) into (73) yields

\[
\dot{V}(t) \leq -\gamma E(t) + \frac{\gamma L^2}{2a} \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta, t) d\eta + \frac{\gamma aL}{2a} \frac{y^2(L, t)}{2a} + \gamma L \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta, t) d\eta \\
+ \gamma \rho C_\beta \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta, t) d\eta
\]

\[
= -\gamma E(t) + \gamma \rho C_\beta \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta, t) d\eta
\]

\[
= -\gamma E(t) + \frac{2\gamma L^2}{a} \frac{v - k_p(t)}{a} + \gamma \rho \int_{-\infty}^{+\infty} (\eta^2 + \beta) \phi^2(\eta, t) d\eta.
\]

(77)

By (9) in Remark 2.3, \(k_p(t) \leq k_0 + \frac{E(0)}{k_0}\). By (12), \(\gamma \frac{2L^2}{a} \frac{V(t)}{a} \leq k_0\) for any \(t > 0\), and \(\gamma \frac{1}{2\beta} + \frac{L \rho C_\beta}{a} < 1\). Therefore, (77) becomes

\[
\dot{V}(t) \leq -\gamma E(t).
\]

(78)

It follows from (11) in Lemma 2.1 that

\[
\dot{V}(t) \leq -\frac{\gamma}{\omega_2} V(t).
\]

(79)

In light of Gronwall’s inequality, we arrive from (79) at

\[
V(t) \leq V(0) e^{-\frac{\gamma}{\omega_2} t}.
\]

(80)

This, together with Lemma 2.1, yields that for any \(t \geq 0\),

\[
E(t) \leq \frac{V(0)}{\omega_1} e^{-\frac{\gamma}{\omega_1} t},
\]

(81)

where \(\omega_1\) and \(\omega_2\) are constants stated in Lemma 2.1. This completes the proof of the theorem. \(\square\)

**Proof of Theorem 2.4.** By the boundary condition \(y(0, t) = 0\), we can derive from the Cauchy-Schwarz inequality that

\[
|y(x, t)| = \left| \int_0^x y_x(s, t) ds \right| \leq \int_0^L |y_x(x, t)| dx \leq \left( L \int_0^L |y_x(x, t)|^2 dx \right)^\frac{1}{2}
\]

\[
\leq \left( \frac{L}{a} M(\|y_x(t)\|^2) \right)^\frac{1}{2} \leq \sqrt{\frac{2L}{a}} E(t),
\]

(82)

for all \(t \geq 0, x \in [0, L]\). The result follows immediately from Theorem 2.3 to (82). \(\square\)

## 4 | NUMERICAL SIMULATION

A simulation example is carried out in this section for the closed-loop system (4). To show the performance of the proposed control law (3), the simulation is performed by using the finite element method, where the Lagrange “hat” basis of the finite element equidistant meshes is used.
FIGURE 1 Transverse vibration of the open-loop moving string in simulation

(a) Vibration response of closed-loop system under the fractional PI (3).

(b) Vibration response of closed-loop system under the linear control law $L$.

(c) Vibration response of closed-loop system under the fractional integral control law $F$. 
To present the numerical results, the non-dimensional parameters are assigned as follows: \( a = 2, \ b = 3, \ v = 0.1, \ L = 1, \ k_0 = 1, \ \alpha = 0.1, \ \beta = 1, \ k_I = 0.5 \). The initial conditions are \( f(x) = 0.3 \sin(5x) \) and \( g(x) = 0.3 \cos(3x) \). When the control law \( U \equiv 0 \) in (1), the vibration response of the open-loop system case is illustrated by Figure 1, where the left boundary \( x = 0 \) is fixed, and the right boundary \( x = L \) is free. To show the effect of the fractional PI (3), we provide other two different controllers: the fractional integral control \( F(w) = I_{0+}^{0.1} w \) and the linear control \( L(w) = 3w \) which are special cases of the proposed control law (3) by Remark 2.2.
FIGURE 6

The string transverse response $y(x, t)$ of the closed-loop system (4) with the fractional PI control law (3), linear control law $L$, and the fractional integral control law $F$, are depicted in Figure 2a–c. Under the action of these three different controllers, the transversal displacement at the boundary point $x = 1$ and the corresponding control input are also shown in Figures 3-5.

It is seen from Figures 2–5 that, in the case of the fractional PI (3), the decay of the transverse vibration is relatively faster compared to the other two different controls. Moreover, under the different control laws, the norm $\|y(\cdot, t)\|_2$ of the closed-loop system are illustrated in Figure 6, which shows that the transverse vibration has been exponentially suppressed. Comparatively speaking, under the proposed controller (3), the effect of vibration suppression is relatively better.

5 CONCLUDING REMARKS

The present paper provides a fractional PI boundary control for the axially moving Kirchhoff string system in terms of the velocity feedback. By constructing an auxiliary equivalent system, the existence of the global solution to the closed-loop auxiliary Kirchhoff system is established by applying the Galerkin approximation method and the fixed point theorem. The exponential stability of the closed-loop auxiliary Kirchhoff system are established by the direct Lyapunov method. If the tension function of the string is not specific, the stabilization of this system can also be solved under the assumption stated in Remark 2.6. The results give some new insight into the boundary control of the distributed parameter systems. An interesting problem about exponential stability in the presence of bounded disturbance on the boundary like the paper [59] would be next future work.

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REFERENCES


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