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APPLICATION OF INGHAM-BEURLING-TYPE THEOREMS TO COEFFICIENT IDENTIFIABILITY OF VIBRATING SYSTEMS: FINITE TIME IDENTIFIABILITY

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Abstract. The identifiability of spatial variable coefficients for the vibrating string and Euler-Bernoulli beam are considered. It is shown that the coefficients can be determined by means of boundary control and observation in a finite time duration. These results can be considered as the generalization of infinite-time coefficients identifiability through the application of the Ingham-Beurling theorem.

1. Introduction

Identifiability is one of the fundamental problems in parameter identification (see [3] for basic knowledge of identifiability). Roughly speaking, the system parameters are said to be identifiable if they are uniquely determined by observed data. It is well known that parameter identification is an inverse problem and parameter identifiability is equivalent to uniqueness of the solution of the inverse problem. Therefore, the importance of identifiability in parameter identification is because it amounts to the uniqueness of the solution in the theory of inverse problems. It is commonly recognized that the inverse problems are usually ill posed and need to be solved by the

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method called regularization (see, e.g., [14, Chapter 2]), which is initiated by the fact that uniqueness guarantees stability under some natural conditions (see, e.g., Lemma 7.1 of [3, page 135], [14, pages 23-24]). In summary, identifiability tells us that the observed data is sufficient to determine the unknown parameters, and that under certain conditions that are satisfied by almost all parameter identification problems, a stable numerical solution can be obtained by the regularization method. For more details, we refer the reader to Chapter 2 of [14] or Chapter IV of [3].

For coefficient identifiability problems of distributed parameter systems, only a few methods are available at present. One of them is to reduce the identifiability problems to some inverse spectral problems (see, e.g., [4, 11]). This method can be applied to both one-dimensional parabolic and one-dimensional hyperbolic systems. For one-dimensional parabolic systems, by virtue of the theory of Dirichlet series, the coefficient identifiability with finite-time observation can be directly obtained (see, e.g., [13, 16]). However, for one-dimensional hyperbolic systems, the coefficient identifiability can be specified only for those with infinite-time observation (see, e.g., [18]). Recently, this method was improved by the authors in [4] and [5] to solve the identifiability of coefficients for one-dimensional vibrating systems including string and beam equations, and some new identifiability results with infinite-time observation were obtained. One of the objectives of this paper is to generalize these results from infinite-time observation to finite time observation.

It should be indicated that although some special identifiability results of coefficients with finite-time observation for vibrating systems can be obtained directly by the theory of nonlinear integral equations (see, e.g., [14, Section 8.1]), the common feasible way of establishing identifiability is first to investigate the identifiability with infinite-time observation, and then to improve results by extending finite-time data onto infinite-time intervals ([11]). This is natural because, in general, identifiability problems with infinite-time observation are easier to solve.

The main contribution of this paper is to propose a new simpler approach to extend the observation data in a finite-time interval to an infinite time interval. This is realized by the help of an Ingham-Beurling-type theorem. In addition, by this approach, the finite-time coefficient identifiability for one-dimensional vibrating systems in some cases can be directly established as easily as for one-dimensional parabolic systems. This is because spectral data that are required to solve the associated inverse spectral problem can be uniquely determined from some observations in certain finite intervals without extending these observations to an infinite-time interval. Therefore, this paper is not only an extension of our previous results but also some completely new methods are developed such that some results of this paper can be obtained directly without relying on previous ones. We also remark that our method can also be used to obtain the same finite-time identifiability result as mentioned in survey paper [15] for a one-dimensional hyperbolic equation with two boundary observations without any inputs but the initial data required to satisfy a strict condition.

We proceed as follows. In Section 2, we list the main results of the paper. The first one is about a class of functions that are defined on the whole real line but can be determined uniquely by their values in some finite segment. The second result is on the identification of the tension of the vibrating string by finite-time control and observation. The last one is a similar result for mass density and flexural rigidity of the Euler-Bernoulli beam. Section 3 is devoted to the proof of the main results. At the same time, we also illustrate how to obtain the identifiability directly from finite-time observation without relying on the infinite-time observation.

2. MAIN RESULTS

To begin with, we determine, by using an Ingham-Beurling theorem, a special class of functions that are defined on the whole real line but can be determined uniquely by their values in some finite segment. This class of functions plays a key role in our generalization of infinite-time observation identifiability to the finite-time case.

Let $\Omega = \{\omega_n\}_{n\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers, where $\mathbb{Z}$ is an indexed subset of the integers. Define the upper density $D^+(\Omega)$ of the sequence Ω by

$$D^+(\Omega) := \lim_{r\to\infty} \frac{n^+(r,\Omega)}{r},$$

where $n^+(r,\Omega)$ denotes the largest number of terms of the sequence Ω contained in an interval of length r (see, e.g., [1], [12, p.174]). If Ω is a separated set; i.e.,

$$\inf_{m\neq n,\, m,n\in\mathbb{Z}} |\omega_m - \omega_n| > 0,$$

then $\{e^{i\omega_n t}\}_{n\in\mathbb{Z}}$ forms an $\mathcal{L}$-basis in $L^2(I)$, that is, a Riesz basis ([19]) for the closed subspace of $L^2(I)$ spanned by itself, where I is any bounded interval of length $|I| > 2\pi D^+(\Omega)$. Hence, for any nonharmonic Fourier series of the form

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\omega_n t}, \tag{2.1}$$

there exist two constants $D_1, D_2 > 0$ such that

$$D_1 \sum_{n\in\mathbb{Z}} |a_n|^2 \le \int_I |f(t)|^2 dt \le D_2 \sum_{n\in\mathbb{Z}} |a_n|^2. \tag{2.2}$$

Such an f is obviously belonging to $L^2_{loc}(\mathbb{R})$. Consequently, f is uniquely determined by its restriction on I. The above result is usually referred to as the Ingham-Beurling theorem (Theorems 4.3, 9.2 of [12]). If Ω is not a separated set, the first inequality of (2.2) does not hold any more, but if Ω is a relatively separated set; that is, Ω is a union of finitely many separated sequences, the Ingham-Beurling theorem can be generalized in different ways (see Theorem 9.4 of [12] or Proposition 1 of [1]).

As stated in the introduction, the fact that two Dirichlet series which are equal to each other in any finite interval must have the same exponents and coefficients is key for establishing finite-time identifiability of one-dimensional parabolic systems (see, e.g., [13, 16]). Here we give a similar uniqueness result for nonharmonic Fourier series so that the finite-time identifiability of one-dimensional vibrating systems can also be achieved directly. This result, represented by Theorem 2.1 below, is one of the main results of this paper, and its proof depends greatly on the generalized Ingham-Beurling type theorem.

Theorem 2.1. *Let $\Omega_1 = \{\mu_n\}_{n\in\mathbb{Z}}$ and $\Omega_2 = \{\nu_n\}_{n\in\mathbb{Z}}$ be any two strictly increasing sequences of real numbers, satisfying the gap condition*

$$\mu_{n+1} - \mu_n > \gamma,\ \nu_{n+1} - \nu_n > \gamma, \forall\ n \in \mathbb{Z} \tag{2.3}$$

for some positive constant γ. Suppose f is a function given by

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t} - \sum_{n\in\mathbb{Z}} b_n e^{i\nu_n t}, \tag{2.4}$$

where the complex coefficients a_n and b_n are square-summable. If I is a bounded interval of length $|I| > 2\pi(D^+(\Omega_1) + D^+(\Omega_2))$, then:

(i) *f is a function of $L^2_{loc}(\mathbb{R})$ and is uniquely determined by its restriction on I.*

(ii) *$\{a_n\}_{n\in\mathbb{Z}} = \{b_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\in\mathbb{Z}} = \{\nu_n\}_{n\in\mathbb{Z}}$ provided that $a_n \neq 0, b_n \neq 0$ for every $n \in \mathbb{Z}$, and $f = 0$ almost everywhere on I.*

It should be pointed out that Theorem 2.1 plays a crucial role in establishing finite-time identifiability for subsequent vibrating systems. In particular, Theorem 2.1(ii) will be used in the proof of Lemma 3.8 from which the assumption in Theorem 2.1(ii) is clearly demonstrated.

Now we consider the coefficient identification problem for a string equation given by

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0,\ 0 < x < 1,\ t > 0, \\ w(0,t) = 0,\ t \geq 0, \\ a(1)w_x(1,t) = u(t),\ t \geq 0, \\ w(x,0) = w_t(x,0) = 0,\ 0 \leq x \leq 1, \\ y(t) = w_t(1,t), t \geq 0, \end{cases} \tag{2.5}$$

where $a(x)$, an unknown parameter, is the tension of the string, $u(t)$ is the known boundary input, and $y(t)$ is the observation to identify $a(x)$. Assume that $u(\cdot) \in L^2_{loc}(0,\infty)$ is not identical to zero. Furthermore, we assume that $a(\cdot)$ is in a parameter set Q given by

$$Q = \{a(x) \in C^2[0,1] :\ a(x) \geq a_0 > 0, \forall\, x \in [0,1]\}. \tag{2.6}$$

The physical meaning of $a(x) \geq a_0 > 0$ in (2.6) is that the speed $\sqrt{a(x)}$ of wave propagation of the string has a lower bound $\sqrt{a_0}$.

Define the operator $A : D(A)(\subset L^2(0,1)) \mapsto L^2(0,1)$ by

$$\begin{cases} Af = -(a(x)f')', \\ D(A) = \left\{f \in L^2(0,1) \middle| (af')' \in L^2(0,1), f(0) = f'(1) = 0\right\}. \end{cases} \tag{2.7}$$

It is well known that the operator A is positive self-adjoint in $L^2(0,1)$. The state space of the system (2.5) is naturally chosen as the Hilbert space $\mathbb{H} = D(A^{1/2}) \times L^2(0,1) = H^1_L(0,1) \times L^2(0,1)$, $H^1_L(0,1) = \{\varphi \in H^1(0,1) :\ f(0) = 0\}$, with the inner-product-induced norm

$$\|(\varphi,\psi)\|^2_{\mathbb{H}} = \int_0^1 [|\psi(x)|^2 + a(x)|\varphi'(x)|^2]dx,\ \forall\ (\varphi,\psi) \in \mathbb{H}.$$

The system (2.5) is then rewritten as in ([8]):

$$\begin{cases} w_{tt} + Aw = bu \ \text{ in }\ D(A^{1/2})', \\ y(t) = b^* w_t, \end{cases} \tag{2.8}$$

where

$$b = \delta(x-1) \in D(A^{1/2})',\ b^*\varphi = \varphi(1),\ \forall\, \varphi \in D(A^{1/2}) \tag{2.9}$$

with Dirac distribution $\delta(x-1)$.

The eigenvalue problem associated with (2.5) is

$$A\psi_n(x) = \mu_n^2\psi_n(x) \text{ or } \begin{cases} (a(x)\psi_n(x))' = \mu_n^2\psi_n(x), \\ \psi_n(0) = \psi_n'(1) = 0, \end{cases} \tag{2.10}$$

where $\mu_n^2, n = 1, 2, \ldots$, are all eigenvalues of A and ψ_n is the eigenfunction corresponding to μ_n^2. It is easily seen that $\psi_n(1) \neq 0$. Otherwise, $\psi_n(1) = \psi_n'(1) = 0$ implies that $\psi_n \equiv 0$ by the uniqueness of the solution of initial value problems for ordinary differential equations. Thus, we can claim that all eigenvalues μ_n^2 are simple and hence we can assume that all μ_n's are different and positive. In fact, suppose there are two linearly independent eigenfunctions ψ_{n1}, ψ_{n2} corresponding to the same eigenvalue μ_n^2. Then we can choose $\alpha, \beta \neq 0$ such that $\psi_n = \alpha\psi_{n1} + \beta\psi_{n2}$ satisfies $\psi_n(1) = 0$, which contradicts the fact that ψ_n is also an eigenfunction. So all eigenvalues μ_n^2 are simple.

Since A is self-adjoint with compact resolvent in $L^2(0,1)$, $\{\psi_n\}_{n=1}^{\infty}$ forms an orthogonal basis for $L^2(0,1)$, which is normalized so that

$$\psi_n(1) > 0 \text{ and } \int_0^1 \psi_n^2(x)\, dx = 1.$$

Moreover, the following asymptotic expansions hold ([10, pages 270-273])

$$\begin{cases} \mu_n = L^{-1}(n - 1/2)\pi + \mathcal{O}(n^{-1}), \\ \psi_n(1) = c + \mathcal{O}(n^{-1}), \end{cases} \tag{2.11}$$

where c is a positive constant and

$$L = \int_0^1 \frac{1}{\sqrt{a(x)}} dx.$$

By (2.11), it is known that b is an admissible input operator ([6, 8, 9]), and so is b^* as an output operator. Moreover, it is shown in Proposition 2 of [5] that the system (2.8) is well posed in the sense of D.Salamon and regular in the sense of G.Weiss in the state space $\mathbb{H}$ and input (output) space $\mathbb{C}$ ([6]). Actually, as it was indicated in Remark 2 of [5], the system considered in [8] must be regular if it is well posed.

For the identification problem considered here, the coefficient $a(\cdot)$ is called identifiable by $\{(u(t), w_t(1,t)),\ 0 \leq t \leq T\}$ with respect to Q if, for any $a(\cdot), \widetilde{a}(\cdot) \in Q$, $w_t(1,t;a) = w_t(1,t;\widetilde{a})$ for almost all $t \in [0,T]$ implies that $a(x) = \widetilde{a}(x)$ for any $x \in [0,1]$. (see e.g., [3, page 105]).

We have the following finite identifiability for the string equation.

Theorem 2.2. *Suppose there is a positive constant* τ *such that the input* $u(\cdot)$ *in* (2.5) *vanishes in* $[\tau, \infty)$. *Then the coefficient* $a(\cdot)$ *in* (2.5) *can be identified by* $\{(u(t), w_t(1,t)), t \in [0,T]\}$, *where* $T \geq \tau + 4a_0^{-1/2}$.

Corollary 2.1 below tells us that, for the purpose of identification, the function of displacement and velocity makes no big difference theoretically, but the former is easier to measure in practice. This is in sharp contrast to stabilization.

Corollary 2.1. *Suppose there is a positive constant* τ *such that the input* $u(\cdot)$ *in* (2.5) *vanishes in* $[\tau, \infty)$. *Then the coefficient* $a(\cdot)$ *in* (2.5) *can be identified by* $\{(u(t), w(1,t)), t \in [0,T]\}$, *where* $T \geq \tau + 4a_0^{-1/2}$.

Remark 2.1. It should be pointed out that the finite-time coefficient identifiability for the string equation with other boundary conditions can also be obtained by our approach due to how well developed the associated inverse spectral theory is (see, e.g., [13]).

Next we turn to considering coefficients identification for an Euler-Bernoulli beam equation described by

$$\begin{cases} \rho(x)w_{tt}(x,t) + (r(x)w_{xx}(x,t))_{xx} = 0, \ 0 < x < 1, \ t > 0, \\ w(0,t) = w_x(0,t) = 0, \ t \geq 0, \\ r(x)w_{xx}(x,t)|_{x=1} = 0, \ t \geq 0, \\ (r(x)w_{xx}(x,t))_x|_{x=1} = u(t), \ t \geq 0, \\ w(x,0) = 0, w_t(x,0) = 0, \ 0 \leq x \leq 1, \end{cases} \tag{2.12}$$

where $\rho(x)$ and $r(x)$, unknown parameters to be identified, are the mass density and the flexural rigidity of the beam, respectively, and $u(t)$ is the known boundary input. For system (2.12), we assume that $u(\cdot) \in L^2_{loc}(0,\infty)$ is not identical to zero and $(\rho(\cdot), r(\cdot))$ belongs to the parameter set

$$Q = \{(\rho(\cdot), r(\cdot)) \in C^4[0,1] \times C^4[0,1] : \ \rho(x) > 0, r(x) > 0, \forall\, x \in [0,1]\}. \tag{2.13}$$

The formulation of identifiability of $(\rho(\cdot), r(\cdot)) \in Q$ is similar to that for system (2.5).

The eigenvalue problem associated with (2.12) is

$$\begin{cases} (r(x)\phi_n''(x))'' = \omega_n^2 \rho(x)\phi_n(x), 0 < x < 1, \\ \phi_n(0) = \phi_n'(0) = 0, \\ \phi_n''(1) = (r(x)\phi_n''(x))'|_{x=1} = 0, \end{cases} \tag{2.14}$$

where ω_n and $\omega_n^2, n = 1, 2, \ldots$, are eigenfrequencies and eigenvalues, respectively, ϕ_n is the eigenfunction corresponding to the eigenvalue ω_n^2, which is normalized so that

$$\int_0^1 \rho(x)\phi_n^2(x)\,dx = 1.$$

All ω_n's could be assumed to be different and positive according to [2] (see also [7], page 387). Let $L^2_\rho(0,1)$ denote the space of square integrable functions over $[0,1]$ with weight ρ. It is well known that $\{\phi_n\}_{n=1}^\infty$ forms an orthonormal basis for $L^2_\rho(0,1)$. We may assume without loss of generality that ([5])

$$\phi_n(1) > 0,\ \phi_n'(1) > 0. \tag{2.15}$$

It is also known that the following asymptotic properties hold ([5]):

$$\begin{cases} \omega_n = M^{-1}(n-1/2)^2\pi^2 + \mathcal{O}(n^{-1}), \\ \phi_n(1) = \tilde{c} + \mathcal{O}(n^{-1}),\ \tilde{c} > 0, \\ \phi_n'(1) = \mathcal{O}(n-1/2), \end{cases} \tag{2.16}$$

where

$$M = \int_0^1 \left(\frac{\rho(x)}{r(x)}\right)^{\frac{1}{4}} dx.$$

Let $H = L^2_\rho(0,1)$. Define the operator $\mathcal{A} : D(\mathcal{A})(\subset H) \mapsto H$ by

$$\begin{cases} \mathcal{A}f = \dfrac{1}{\rho(x)}(r(x)f'')'',\ \forall\, f \in D(\mathcal{A}), \\ D(\mathcal{A}) = \Big\{ f \in H : (r(x)f'')'' \in L^2(0,1), \\ \qquad\qquad f(0) = f'(0) = f''(1) = (r(x)f''(x))'|_{x=1} = 0 \Big\}. \end{cases} \tag{2.17}$$

The operator $\mathcal{A}$ is positive self-adjoint with compact resolvent in H. It is well known that, in the state space $\mathbf{H} = D(\mathcal{A}^{1/2}) \times H = H_L^2(0,1) \times L^2(0,1)$, $H_L^2(0,1) = \{f \in H^2(0,1) : f(0) = f'(0) = 0\}$, the system (2.12) can be formulated as an SISO second-order collocated system ([8]):

$$\begin{cases} w_{tt} + \mathcal{A}w + bu = 0 \text{ in } D(\mathcal{A}^{1/2})', \\ w_t(1,t) = b^* w_t, \end{cases} \tag{2.18}$$

where

$$b = \frac{1}{\rho(x)}\delta(x-1) \in D(\mathcal{A}^{1/2})',\ b^*g = g(1),\ \forall\, g \in D(\mathcal{A}^{1/2}).$$

From this formulation, it follows that ([5])

(i) there exists a unique solution to (2.12) such that $(w, w_t) \in C([0,\infty), \mathbf{H})$;

(ii) the system (2.18) is well posed in the sense of D.Salamon and regular in the sense of G.Weiss in the sate space $\mathbf{H}$ and input (output) space $\mathbb{C}$ with zero feed-through operator.

Theorem 2.3. *Suppose there is a constant $\tau > 0$ such that the input u in* (2.12) *satisfies*

$$\begin{cases} u(t) \neq 0 & \textit{for almost all } t \in (0,\tau); \\ u(t) = 0 & \textit{for } t \geq \tau. \end{cases}$$

Then, for each $T > \tau$, $(\rho(\cdot), r(\cdot)) \in Q$ can be identified by

$$\{(u(t), w_t(1,t), w_x(1,t)), t \in [0,T]\}.$$

Remark 2.2. Comparing Theorem 2.3 with Theorem 2.2, we see that T in Theorem 2.2 has a positive lower bound, while in Theorem 2.3, T can be taken as an arbitrary small number. This phenomenon is also observed in controllability and stabilization of wave and beam equations. It is caused essentially by the fact that the speed of wave propagation is finite, while that of a beam is infinite.

Corollary 2.2. *Suppose there is a constant $\tau > 0$ such that the input u in* (2.12) *satisfies $u(t) \neq 0$ for almost all $t \in (0,\tau)$. Then, for each $T > \tau$, $(\rho(\cdot), r(\cdot)) \in Q$ can be identified by $\{(u(t), w(1,t), w_x(1,t)), t \in [0,T]\}$.*

We point out that although Corollary 2.2 is regarded here as a corollary of Theorem 2.3, it can be proved independently without using Theorem 2.3. Such a proof can be found in Section 3 in establishing directly the finite-time identifiability. A similar remark is also true for Corollary 2.1.

3. PROOF OF MAIN RESULTS

Proof of Theorem 2.1. Set $\Omega = \{\lambda_n\}_{n\in\mathbb{Z}} = \{\mu_n\}_{n\in\mathbb{Z}} \bigcup \{\nu_n\}_{n\in\mathbb{Z}}$ such that $\{\lambda_n\}$ is a strictly increasing sequence of real numbers. For any given integer $N > 0$, there exist two integers $M \geq N$ and $K \geq N$ such that $\{\lambda_n\}_{n=-K}^{M} = \{\mu_n\}_{n=-N}^{N} \bigcup \{\nu_n\}_{n=-N}^{N}$ and

$$\sum_{n=-K}^{M} c_n e^{i\lambda_n t} = \sum_{n=-N}^{N} a_n e^{i\mu_n t} - \sum_{n=-N}^{N} b_n e^{i\nu_n t}.$$

By Ingham's theorem (see, e.g., Theorem 4.3 of [12]), the terms on the right-hand side of the above equality represent functions in $L^2(I)$, where I is any bounded interval of length $|I| > 2\pi/\gamma$ as $N \to \infty$. This shows that the sum on the left-hand side of the above equality is meaningful as $N \to \infty$ and

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}, \tag{3.1}$$

where, for $n \in \mathbb{Z}$,

$$c_n = \begin{cases} a_k, & \text{if } \lambda_n = \mu_k \neq \nu_m \text{ for some } k \in \mathbb{Z} \text{ and any } m \in \mathbb{Z}, \\ -b_k, & \text{if } \lambda_n = \nu_k \neq \mu_m \text{ for some } k \in \mathbb{Z} \text{ and any } m \in \mathbb{Z}, \\ a_k - b_m, & \text{if } \lambda_n = \mu_k = \nu_m \text{ for some } k, m \in \mathbb{Z}. \end{cases} \tag{3.2}$$

It is now clear that f is a function of $L^2_{loc}(\mathbb{R})$.

By (2.3),

$$\lambda_{n+2} - \lambda_n > \gamma, \forall\, n \in \mathbb{Z}. \tag{3.3}$$

For any $n \in \mathbb{Z}$, denote by $D_{\lambda_n}(\gamma)$ the disk centered at λ_n with radius γ. Due to (3.3), we have, in the direction of increasing n, only two cases:

Case 1. $D_{\lambda_n}(\gamma)$ contains only λ_n. In this case, we denote $e_n(t) = e^{\lambda_n t}$.

Case 2. $D_{\lambda_n}(\gamma)$ contains $\{\lambda_n, \lambda_{n+1}\}$. In this case, we denote $e_n(t) = e^{\lambda_n t}$ and $e_{n+1}(t) = \dfrac{e^{\lambda_n t} - e^{\lambda_{n+1} t}}{\lambda_n - \lambda_{n+1}}$. Then

$$c_n e^{\lambda_n t} + c_{n+1} e^{\lambda_{n+1} t} = (c_n + c_{n+1}) e_n(t) - c_{n+1}(\lambda_n - \lambda_{n+1}) e_{n+1}(t). \tag{3.4}$$

Hence we can further write (3.1) as

$$f(t) = \sum_{n\in\mathbb{Z}} d_n e_n(t), \tag{3.5}$$

where

$$\begin{cases} d_n = c_n & \text{if } D_{\lambda_n}(\gamma) \cap \{\lambda_n\}_{n\in\mathbb{Z}} = \{\lambda_n\}, \\ d_n = c_n + c_{n+1}, d_{n+1} = -c_{n+1}(\lambda_n - \lambda_{n+1}) & \\ & \text{if } D_{\lambda_n}(\gamma) \cap \{\lambda_n\}_{n\in\mathbb{Z}} = \{\lambda_n, \lambda_{n+1}\}. \end{cases} \tag{3.6}$$

By Proposition 9.3 and Theorem 9.4 of [12], there exist constants $C_1, C_2 > 0$ such that the Ingham-type inequality

$$C_1 \sum_{n\in\mathbb{Z}} |d_n|^2 \leq \int_I |f(t)|^2 dt \leq C_2 \sum_{n\in\mathbb{Z}} |d_n|^2 \tag{3.7}$$

holds for any time interval I with length $|I| > 2\pi D^+(\Omega)$.

If $f = 0$ almost everywhere on I, by (3.7), $d_n = 0$ for all $n \in \mathbb{Z}$ and hence $f \equiv 0$ by (3.5). This is (i) due to the trivial fact that $D^+(\Omega) \leq D^+(\Omega_1) + D^+(\Omega_2)$. Moreover, it follows from (3.6) that $c_n = 0$ for all $n \in \mathbb{Z}$.

Since $a_n \neq 0, b_n \neq 0$ for every $n \in \mathbb{Z}$, we have only the third case in (3.2), which claims that $\{a_n\}_{n\in\mathbb{Z}} = \{b_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\in\mathbb{Z}} = \{\nu_n\}_{n\in\mathbb{Z}}$. $\square$

In order to prove Theorem 2.2, we need several lemmas below.

Lemma 3.1. *Suppose $\{\mu_n\}_{n=1}^{\infty}$ is given by the eigenvalue problem* (2.7). *Let $\Lambda = \{\mu_n, -\mu_n\}_{n=1}^{\infty}$. Then $D^+(\Lambda) \leq (\sqrt{a_0}\,\pi)^{-1}$.*

Proof. Since $a(x) \geq a_0$ for any $x \in [0,1]$, it has $L^{-1} \geq \sqrt{a_0}$. By (2.11), it follows that

$$\mu_n - \mu_{n-1} = L^{-1}\pi + \mathcal{O}(n^{-1}) \geq \sqrt{a_0}\,\pi + \mathcal{O}(n^{-1}) \text{ as } n \to +\infty. \tag{3.8}$$

For any $\varepsilon > 0$ with $\varepsilon < \sqrt{a_0}\,\pi$, there exists a positive integer N such that

$$\mu_n - \mu_{n-1} \geq \sqrt{a_0}\,\pi - \varepsilon \quad \text{for } n > N.$$

Let I be an interval that contains $\{\mu_n, -\mu_n\}_{n=1}^{N}$. Then

$$D^+(\Lambda) = \lim_{r\to\infty} \frac{n^+(r,\Lambda)}{r} \leq \lim_{r\to\infty} \frac{n^+(|I|,\Lambda) + 1 + r/(\sqrt{a_0}\pi - \varepsilon)}{r} = \frac{1}{\sqrt{a_0}\,\pi - \varepsilon},$$

showing that $D^+(\Lambda) \leq (\sqrt{a_0}\,\pi)^{-1}$. $\square$

The following infinite-time observation identifiability has been proven in Theorem 3 of [5].

Lemma 3.2. *The coefficient $a(\cdot)$ in* (2.5) *can be identified by*

$$\{(u(t), w_t(1,t)), t \geq 0\}.$$

In view of Theorem 2.1(i) and Lemmas 3.1 and 3.2, we can now prove the finite-time identifiability for the coefficient $a(\cdot)$ in (2.5).

Proof of Theorem 2.2. Since the control u possibly does not vanish in the time interval $[0,\tau)$, from the time moment τ on, the system (2.5) will become a free system. From (2.8), we have

$$\frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = \mathbb{A}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \text{ for } t \geq \tau, \tag{3.9}$$

where

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \ \ D(\mathbb{A}) = \{(\varphi,\psi) \in \mathbb{H} : \mathbb{A}(\varphi,\psi) \in \mathbb{H}\}.$$

It is easily seen that $\mathbb{A}$ is a skew-adjoint operator resolvent in $\mathbb{H}$, and hence generates a C_0-group by Stone's theorem. Moreover, $\mathbb{A}$ has eigenpairs $\{\pm i\mu_n, \Psi_{\pm n}\}_{n=1}^{\infty}$:

$$\mathbb{A}\Psi_{\pm n} = \pm i\mu_n\Psi_{\pm n},$$

where

$$\Psi_n = \begin{pmatrix} -i\mu_n^{-1}\psi_n \\ \psi_n \end{pmatrix}, \quad \Psi_{-n} = \begin{pmatrix} i\mu_n^{-1}\psi_n \\ \psi_n \end{pmatrix}.$$

It is well known that $\{\Psi_{\pm n}\}_{n=1}^{\infty}$ forms an orthonormal basis for $\mathbb{H}$ ($\mathbb{A}$ is of compact resolvent in $\mathbb{H}$). Hence, the state of the system (2.5) at time τ can be represented as

$$\begin{pmatrix} w(\cdot,\tau) \\ w_t(\cdot,\tau) \end{pmatrix} = \sum_{n=1}^{\infty} a_n \Psi_n + \sum_{n=1}^{\infty} c_n \Psi_{-n},$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are square-summable sequences. The solution of the system (2.5) for $t \geq \tau$ can be represented as

$$\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = \sum_{n=1}^{\infty} a_n e^{i\mu_n(t-\tau)} \Psi_n + \sum_{n=1}^{\infty} c_n e^{-i\mu_n(t-\tau)} \Psi_{-n},$$

which yields

$$w_t(1,t;a) = \sum_{n=1}^{\infty} a_n \psi_n(1) e^{i\mu_n(t-\tau)} + \sum_{n=1}^{\infty} c_n \psi_n(1) e^{-i\mu_n(t-\tau)}$$

for $t \geq \tau$. By (3.8) and Ingham's theorem, the above expression makes sense because, by (2.11), both $\{a_n\psi_n(1)\}_{n=1}^{\infty}$ and $\{c_n\psi_n(1)\}_{n=1}^{\infty}$ are square-summable.

In what follows, we write the solution of (2.5) as $w(\cdot,\cdot;a)$ instead of w to show the dependence of w on $a(\cdot)$.

By Lemma 3.2, the proof will be accomplished if we can show that for any $a(\cdot), \widetilde{a}(\cdot) \in Q$, if $w_t(1,t;a) = w_t(1,t;\widetilde{a})$ for almost every $t \in [0, \tau + 4a_0^{-1/2}]$ then $w_t(1,t;a) = w_t(1,t;\widetilde{a})$ for almost all $t \in [0,\infty)$. To do this, let $\widetilde{\mu}_n^2$ and $\widetilde{\psi}_n$ be the eigenpairs corresponding to $\widetilde{a}(\cdot)$. Then we have

$$w_t(1,t;\widetilde{a}) = \sum_{n=1}^{\infty} \widetilde{a}_n \widetilde{\psi}_n(1) e^{i\widetilde{\mu}_n(t-\tau)} + \sum_{n=1}^{\infty} \widetilde{c}_n \widetilde{\psi}_n(1) e^{-i\widetilde{\mu}_n(t-\tau)}$$

for $t \geq \tau$, where $\{\widetilde{a}_n\widetilde{\psi}_n(1)\}_{n=1}^{\infty}$ and $\{\widetilde{c}_n\widetilde{\psi}_n(1)\}_{n=1}^{\infty}$ are square summable. Now set

$$f(t) = w_t(1,t;a) - w_t(1,t;\widetilde{a}).$$

By (3.8), $\Omega_1 = \{\mu_n, -\mu_n\}_{n\in\mathbb{Z}}$ and $\Omega_2 = \{\widetilde{\nu}_n, -\widetilde{\nu}_n\}_{n\in\mathbb{Z}}$ satisfy (2.3). Then it follows from Theorem 2.1(i) that $\{f(t), t \in [\tau,\infty)\}$ is uniquely determined by $\{f(t), t \in [\tau, \tau + 4a_0^{-1/2}]\}$, since by Lemma 3.1 we have that $D^+(\Omega_1) \leq (\sqrt{a_0}\pi)^{-1}$ and $D^+(\Omega_2) \leq (\sqrt{a_0}\pi)^{-1}$. In particular, if $f(t) = 0$ for almost every $t \in [\tau, \tau + 4a_0^{-1/2}]$, then $f(t) = 0$ for almost every $t \in [\tau, \infty)$. This completes the proof. $\square$

Proof of Corollary 2.1. It was found in [5] that the solution of (2.5) can be represented as

$$w(x,t) = \sum_{n=1}^{\infty} \psi_n(1)\psi_n(x) \int_0^t \frac{\sin \mu_n(t-\tau)}{\mu_n} u(\tau)\, d\tau,$$

and

$$w_t(x,t) = \sum_{n=1}^{\infty} \psi_n(1)\psi_n(x) \int_0^t \cos \mu_n(t-\tau) u(\tau)\, d\tau,$$

from which we have

$$w(1,t) = \sum_{n=1}^{\infty} \psi_n^2(1) \int_0^t \frac{\sin \mu_n(t-\tau)}{\mu_n} u(\tau)\, d\tau, \tag{3.10}$$

$$w_t(1,t) = \sum_{n=1}^{\infty} \psi_n^2(1) \int_0^t \cos \mu_n(t-\tau) u(\tau)\, d\tau. \tag{3.11}$$

As was indicated before, the system (2.5) is well posed in the sense of D.Salamon, so $w_t(1,t) \in L^2_{loc}(0,\infty)$ for any $u \in L^2_{loc}(0,\infty)$. From (3.10) and (3.11), it is easily checked that, for any $h > 0$,

$$\int_0^h w_t(1,t)\phi dt = -\int_0^h w(1,t)\phi_t dt, \ \forall\, \phi \in C_0^\infty(0,h). \tag{3.12}$$

This shows that $w_t(1,t)$ is the weak derivative of $w(1,t)$ and is uniquely determined by $w(1,t)$. By Theorem 2.2, we then conclude that $a(\cdot)$ can be uniquely determined by $\{(u(t), w(1,t)), 0 \le t \le \tau + 4a_0^{-1/2}\}$. $\square$

Now we turn to the proof of Theorem 2.3. To this purpose, we need several preliminary lemmas. The following inverse spectral result is due to Barcilon (see [4] for more details).

Lemma 3.3. $(\rho(\cdot), r(\cdot)) \in Q$ *can be uniquely determined by*

$$\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^{\infty}.$$

If follows from [5] that the solution of (2.12) can be represented as

$$w(x,t) = -\sum_{n=1}^{\infty} \phi_n(1)\phi_n(x) \int_0^t \frac{\sin \omega_n(t-\tau)}{\omega_n} u(\tau) d\tau, \tag{3.13}$$

and

$$w_t(x,t) = -\sum_{n=1}^{\infty} \phi_n(1)\phi_n(x) \int_0^t \cos\omega_n(t-\tau)u(\tau)\,d\tau. \tag{3.14}$$

The following Lemma 3.4 comes from Lemma 3 of [5].

Lemma 3.4. *$\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ are uniquely determined by*

$$\{(u(t), w_t(1,t)), t \geq 0\}.$$

Lemma 3.5. *Suppose there is a positive constant τ such that the input $u(\cdot)$ in (2.12) vanishes in $[\tau,\infty)$. Then, for each $T>\tau$, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ are uniquely determined by $\{(u(t), w_t(1,t)), t\in[0,T]\}$.*

Proof. Analogous to the proof of Theorem 2.2, we can obtain

$$w_t(1,t;\rho,r) = \sum_{n=1}^{\infty} a_n\varphi_n(1)e^{i\omega_n(t-\tau)} + \sum_{n=1}^{\infty} c_n\varphi_n(1)e^{-i\omega_n(t-\tau)}$$

for $t\in[\tau,\infty)$, where $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are square summable, and so $\{a_n\psi_n(1)\}_{n=1}^{\infty}$ and $\{c_n\psi_n(1)\}_{n=1}^{\infty}$ are square summable by (2.16). By (2.16), one can easily show that $D^+(\Omega)=0$ for $\Omega=\{\omega_n,-\omega_n\}_{n=1}^{\infty}$. This together with Theorem 2.1(i) shows that for each $T>\tau$, $\{w_t(1,t), t\in[\tau,\infty)\}$ is uniquely determined by $\{w_t(1,t), t\in[\tau,T]\}$; that is, $\{w_t(1,t), t\in[0,\infty)\}$ is uniquely determined by $\{w_t(1,t), t\in[0,T]\}$. Therefore, it follows from Lemma 3.4 that $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ can be uniquely determined by $\{(u(t), w_t(1,t)), 0\leq t\leq T\}$. $\square$

Analogous with Corollary 2.1, we can get Lemma 3.6 below from Lemma 3.5.

Lemma 3.6. *Suppose there is a positive constant τ such that the input $u(\cdot)$ in (2.12) vanishes in $[\tau,\infty)$. Then for each $T>\tau$, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ are uniquely determined by $\{(u(t), w(1,t)), t\in[0,T]\}$.*

In order to obtain another spectral sequence $\{\phi_n'(1)\}_{n=1}^{\infty}$ from finite-time observation, we need the following Lemma 3.7.

Lemma 3.7. [17, Theorem 151] *Assume that $P, g\in L^1(0,T)$, and for a positive constant $\tau<T$, $g(t)\neq 0$ for almost all $t\in(0,\tau)$. If*

$$\int_0^t P(t-s)g(s)ds = 0 \text{ for almost all } t\in[0,T],$$

then $P(t)=0$ for almost all $t\in[0,T-\tau]$.

Corollary 3.1. *Assume that $g \in L^1(0,T)$, and for a positive constant $\tau < T$, $g(t) \neq 0$ for almost all $t \in (0,\tau)$. If the integral equation*

$$\int_0^t P(t-s)g(s)ds = \varphi(t)$$

admits a solution $P \in L^1(0, T-\tau)$ for almost all $t \in [0,T]$, then P is unique.

The proof of the following Lemma 3.8, unlike Lemma 3.5, tells us how to directly obtain spectral data from some observation in a finite-time interval without relying on the infinite-time observation.

Lemma 3.8. *Suppose there is a constant $\tau > 0$ such that the input u in (2.12) satisfies $u(t) \neq 0$ for almost all $t \in (0,\tau)$. Then, for every $T > \tau$, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ can be uniquely determined by*

$$\{(u(t), w_x(1,t)), 0 \leq t \leq T\}.$$

Proof. Since $D^+(\Omega) = 0$ for $\Omega = \{\omega_n, -\omega_n\}_{n=1}^{\infty}$, by the Ingham-Beurling theorem, for any $h > 0$, $\{e^{i\omega_n t}, e^{-i\omega_n t}\}_{n=1}^{\infty}$ forms an $\mathcal{L}$-basis for $L^2(0,h)$. Since

$$\sin \omega_n t = \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}, \tag{3.15}$$

the sequence

$$\left\{ \int_0^t \sin \omega_n (t-s) u(s) ds \right\}_{n=1}^{\infty}$$

is square summable for each $t \geq 0$. This together with (2.16) implies that the series

$$\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(x)}{\omega_n} \int_0^t \sin \omega_n (t-s) u(s)\, ds$$

is uniformly convergent in x, which guarantees, from (3.13), that

$$w_x(x,t) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(x)}{\omega_n} \int_0^t \sin \omega_n (t-s) u(s)\, ds.$$

Hence

$$w_x(1,t) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \int_0^t \sin \omega_n (t-s) u(s)\, ds.$$

It is easily shown that the above series is uniformly convergent on any finite time interval and $w_x(1,t)$ is a continuous function.

Next, set

$$P(t) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \sin \omega_n(t) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}. \tag{3.16}$$

By the basis property of $\{e^{i\omega_n t}, e^{-i\omega_n t}\}_{n=1}^{\infty}$ and (3.15), $P(\cdot)$ is well defined, and so $P(\cdot) \in L^2_{loc}(0,\infty)$. We claim that

$$w_x(1,t) = \int_0^t P(t-s)u(s)ds. \tag{3.17}$$

In fact, set

$$P_n(t) = -\sum_{k=1}^{n} \frac{\phi_k(1)\phi_k'(1)}{\omega_n} \sin \omega_n(t).$$

Then, for any $t > 0$, one has

$$\begin{aligned}
&\left| \int_0^t P_n(t-s)u(s)\,ds - \int_0^t P(t-s)u(s)\,ds \right| \\
&\le \int_0^t |P_n(t-s) - P(t-s)|\,|u(s)|ds \\
&\le \left(\int_0^t |u(s)|^2\,ds \right)^{1/2} \left(\int_0^t |P_n(t-s) - P(t-s)|^2\,ds \right)^{1/2} \\
&\le \left(\int_0^t |u(s)|^2\,ds \right)^{1/2} \left(\int_0^t |P_n(s) - P(s)|^2\,ds \right)^{1/2}.
\end{aligned}$$

This yields

$$\left| \int_0^t P_n(t-s)u(s)ds - \int_0^t P(t-s)u(s)ds \right| \to 0 \text{ as } n \to \infty,$$

since $P_n(\cdot)$ converges to $P(\cdot)$ in $L^2(0,t)$. We thus have proved (3.17).

Finally, by Corollary 3.1, (3.17) implies that $\{P(t), t \in (0, T-\tau)\}$ is uniquely determined by $\{(u(t), w_x(1,t)), t \in [0,T]\}$. Since $D^+(\Omega) = 0$ for $\Omega = \{\omega_n, -\omega_n\}_{n=1}^{\infty}$, it follows from (3.16) and Theorem 2.1(ii) that

$$\{\omega_n\}_{n=1}^{\infty} \quad \text{and} \quad \left\{ \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \right\}_{n=1}^{\infty}$$

which are nonzero by (2.15), are uniquely determined by $\{P(t), t \in (0, T-\tau)\}$, which implies that $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ is uniquely determined by $\{P(t), t \in$

$(0, T-\tau)\}$. Therefore, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ are uniquely determined by $\{(u(t), w_x(1,t)),\, t \in [0,T]\}$. The proof is complete. $\square$

Proof of Theorem 2.3. By Lemma 3.5, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ are uniquely determined by

$$\{(u(t), w_t(1,t)), t \in [0,T]\}.$$

By Lemma 3.8, $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ can be uniquely determined by

$$\{(u(t), w_x(1,t)), 0 \le t \le T\}.$$

Thus, $\{(\omega_n, \phi_n(1), \phi_n'(1))\}_{n=1}^{\infty}$ can be uniquely determined by

$$\{(u(t), w_t(1,t), w_x(1,t)), 0 \le t \le T\}.$$

By virtue of Lemma 3.3, this shows that $(\rho(\cdot), r(\cdot)) \in Q$ can be identified by $\{(u(t), w_t(1,t), w_x(1,t)),\, t \in [0,T]\}$. $\square$

Replace Lemma 3.5 by Lemma 3.6 in the proof of Theorem 2.3 to deduce immediately Corollary 2.2 with an additional condition that $u(t) = 0$ for $t \ge \tau$. But we would rather give a direct proof that does not depend on any infinite-time identifiability results in [4] or [5]. To do this, we need the following Lemma 3.9 which can be proved similarly as Lemma 3.8.

Lemma 3.9. *Suppose there is a constant $\tau > 0$ such that the input u in (2.12) satisfies $u(t) \neq 0$ for almost all $t \in (0,\tau)$. Then for every $T > \tau$, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ can be uniquely determined by*

$$\{(u(t), w(1,t)), 0 \le t \le T\}.$$

Proof of Corollary 2.2. This is a consequence of Lemmas 3.3, 3.8, and 3.9. $\square$

To end this paper, we indicate that the proof for Corollary 2.2 does not rely on any results on infinite-time observation (like that in [4] or [5]).

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