Well-posedness and regularity of Naghdi’s shell equation under boundary control and observation

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A system of Naghdi’s shell equation with Dirichlet boundary control and collocated observation is considered. Results on the associated nonhomogeneous boundary value problem are presented. Based on these results, it is shown that the system is well-posed in the sense of D. Salamon and regular in the sense of G. Weiss. The expression of the corresponding feedthrough operator is explicitly found by means of Riemannian geometric method and partial Fourier transform. These properties make this partial differential control system parallel in many ways finite-dimensional ones in the general framework of well-posed and regular infinite-dimensional systems. © 2010 Elsevier Inc. All rights reserved.

1. Introduction and main results

The well-posedness and regularity are major issues in the study of linear infinite-dimensional systems over the last two decades. Well-posed and regular systems represent a broad class of infinite-dimensional systems, which covers many control systems described by partial differential equations with actuators and sensors supported on sub-domain or on a part of the boundary of the spatial region. The most advantage of well-posed and regular systems is that this class of systems parallel in
many ways finite-dimensional ones. Due to this fact, verification of well-posedness and regularity for partial differential equation systems become an active research topic in recent years. We refer to [1–4,11–16,20] for the study of well-posedness and regularity of multi-dimensional partial differential control systems.

In this paper, we are concerned with the well-posedness and regularity of a thin shell equation derived from classical Naghdi’s model [23] in [28] where the middle surface was viewed as a Riemannian manifold with the induced metric of $\mathbb{R}^3$. It is shown that the system is well-posed in the sense of D. Salamon and regular in the sense of G. Weiss. The expression of the corresponding feedthrough operator is explicitly found by means of Riemannian geometric method and partial Fourier transform. The geometric method was introduced in [29] for the controllability of wave equation with variable coefficients and was extended late for other problems in [7,8,28,30–32]. Our results make this complicated partial differential control system parallels in many ways finite-dimensional systems in the general framework of well-posed and regular infinite-dimensional systems.

We mentioned here that some other properties of Naghdi’s shell equation have been developed already in literature. In [28], some observability inequalities for boundary control of Naghdi’s shell system were established. The boundary feedback stabilizations of the same system were studied in [5] and [6]. However, as a consequence, our well-posedness result will establish the equivalence between the exact controllability (observability) and exponential stability for this system.

Before stating the main results in Section 1.3, we need some preliminary knowledge from Riemannian geometry which is presented in Section 1.1. A simple introduction of modeling of Naghdi’s shell equation is introduced in Section 1.2.

1.1. Some notation in Riemannian geometry

Let us first introduce some notation from Riemannian geometry. It is noted that all these definitions and notation are standard and classical in literature.

Denote by $\langle \cdot, \cdot \rangle$ the usual dot inner product of $\mathbb{R}^3$. Let $M$ be a surface of $\mathbb{R}^3$, which is assumed to be smooth for simplicity. In the natural way, $M$ produces a Riemannian manifold of dimension two with the induced metric of $\mathbb{R}^3$ that is denoted by $g$ or simply by $\langle \cdot, \cdot \rangle$. For each $x \in M$, $M_x$ represents the tangential space of $M$ at $x$. It is assumed that $M$ is orientable in terms of the unit normal field $N$ on $M$. Denote the set of all vector fields on $M$ by $X(M)$. The set of all $k$-order tensor fields and the set of all $k$-forms on $M$ are denoted by $\mathcal{T}_k(M)$ and $\Lambda^k(M)$, respectively, where $k$ is a nonnegative integer. Then

$$\Lambda^k(M) \subset \mathcal{T}_k(M). \quad (1.1)$$

In particular, $\Lambda^0(M) = T^0(M) = \mathcal{C}^\infty(M)$ is the set of all $\mathcal{C}^\infty$ functions on $M$ and

$$T^1(M) = T(M) = \Lambda(M) = \mathcal{X}(M), \quad (1.2)$$

where $\Lambda(M) = \mathcal{X}(M)$ is defined to be the following isomorphism: For any given $X \in \mathcal{X}(M)$, the equation

$$U(Y) = \langle Y, X \rangle, \quad \forall Y \in \mathcal{X}(M), \quad (1.3)$$

determines a unique solution $U \in \Lambda(M)$.

It is well known that for each $x \in M$, the $k$-order tensor space $T^k_x$ on $M_x$ is an inner product space defined as follows. Let $e_1, e_2$ be an orthonormal basis of $M_x$. For any $\alpha, \beta \in T^k_x$, the inner product of $T^k_x$ is given by

$$\langle \alpha, \beta \rangle_{T^k_x} = \sum_{i_1,\ldots,i_k=1}^2 \alpha(e_{i_1}, \ldots, e_{i_k}) \beta(e_{i_1}, \ldots, e_{i_k}). \quad (1.4)$$
In particular, for $k = 1$ definition (1.4) becomes

$$g(\alpha, \beta) = \langle \alpha, \beta \rangle_{T_x} = \langle \alpha, \beta \rangle, \quad \forall \alpha, \beta \in M_x,$$

which is the induced inner product of $M_x$ in $\mathbb{R}^3$.

Let $\Omega$ be a bounded region of $M$ with a regular boundary $\Gamma$ or without boundary (when $\Gamma$ is empty). From (1.4), $T^k(\Omega)$ is an inner product space in the following sense:

$$(T_1, T_2)_{T^k(\Omega)} = \int_{\Omega} \langle T_1, T_2 \rangle_{T^k_x} \, dx, \quad \forall T_1, T_2 \in T^k(\Omega),$$

where $dx$ is the volume element of surface $M$ in its Riemannian metric $g$.

The completion of $T^k(\Omega)$ with the inner product (1.6) is denoted by $L^2(\Omega, T^k)$. In particular, $L^2(\Omega, \Lambda) = L^2(\Omega, T)$. $L^2(\Omega)$ is the completion of $C^\infty(\Omega)$ under the inner product:

$$(f, h)_{L^2(\Omega)} = \int_{\Omega} f(x)h(x) \, dx, \quad \forall f, h \in C^\infty(\Omega).$$

Let $D$ be the Levi-Civita connection on $M$ with Riemannian metric $g$. For $U \in \mathcal{X}(M)$, $DU$ is the covariant differential of $U$ which is a 2-order covariant tensor field in the following sense:

$$DU(X, Y) = D_Y U(X) = \langle D_Y U, X \rangle, \quad \forall X, Y \in M_x, \ x \in M.$$

We also define $D^* U \in T^2(M)$ by

$$D^* U(X, Y) = DU(Y, X), \quad \forall X, Y \in M_x, \ x \in M,$$

that is, $D^* U \in T^2(M)$ is the transpose of $DU$. For any $T \in T^2(M)$, the trace of $T$ at $x \in M$ is defined by

$$\text{tr} \ T = \sum_{i=1}^{2} T(e_i, e_i),$$

where $e_1, e_2$ is an orthonormal basis of $M_x$. It is obvious that $\text{tr} \ T \in C^\infty(M)$ whenever $T \in T^2(M)$.

For $T \in T^k(M)$ and $X \in \mathcal{X}(M)$, we define $l_X T \in T^{k-1}(M)$ by

$$l_X T(X_1, \ldots, X_{k-1}) = T(X, X_1, \ldots, X_{k-1}), \quad \forall X_1, \ldots, X_{k-1} \in \mathcal{X}(M).$$

The Sobolev space $H^k(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to the norm:

$$\| f \|_{H^k(\Omega)}^2 = \sum_{i=1}^{k} \| D^i f \|_{L^2(\Omega, T)}^2 + \| f \|_{L^2(\Omega)}^2, \quad \forall f \in C^\infty(\Omega),$$

where $D^i f$ is the $i$-th covariant differential of $f$ in terms of the Riemannian metric $g$ of $M$, which is an $i$-order tensor field on $\Omega$, and $\| \cdot \|_{L^2(\Omega, T)}$ and $\| \cdot \|_{L^2(\Omega)}$ are the induced norms by the inner products (1.6) and (1.7), respectively. For details on Sobolev spaces on Riemannian manifold, we refer to [17] or [25].
Another important Sobolev space that will be used later is $H^k(\Omega, \Lambda)$:

$$H^k(\Omega, \Lambda) = \left\{ U \mid U \in L^2(\Omega, \Lambda), \ D^i U \in L^2(\Omega, T^{i+1}), \ 1 \leq i \leq k \right\} \quad (1.13)$$

with the inner product

$$(U, V)_{H^k(\Omega, \Lambda)} = \sum_{i=0}^{k} (D^i U, D^i V)_{L^2(\Omega, T^{i+1})}, \quad \forall U, V \in H^k(\Omega, \Lambda) \quad (1.14)$$

(see for instance [27]). In particular, $H^0(\Omega, \Lambda) = L^2(\Omega, \Lambda)$.

For $\hat{\Gamma} \subset \Gamma$, set

$$H^1_{\hat{\Gamma}}(\Omega, \Lambda) = \left\{ W \mid W \in H^1(\Omega, \Lambda), \ W|_{\hat{\Gamma}} = 0 \right\}; \quad (1.15)$$

$$H^2_{\hat{\Gamma}}(\Omega) = \left\{ w \mid w \in H^2(\Omega), \ w|_{\hat{\Gamma}} = \frac{\partial w}{\partial n} \right\} \quad (1.16)$$

In particular, $H^1_0(\Omega, \Lambda) = H^1_{\Gamma}(\Omega, \Lambda)$ and $H^2_0(\Omega) = H^2_{\Gamma}(\Omega)$.

### 1.2. Naghdi’s shell equation

Let us turn to geometrical displacement equation of Naghdi’s shell model developed in [28]. Suppose that the middle surface of the shell occupies a bounded region $\Omega$ of surface $M$ in $\mathbb{R}^3$. The shell, now a body in $\mathbb{R}^3$, is defined by

$$S = \left\{ p \mid p = x + zN(x), \ x \in \Omega, \ -h/2 < z < h/2 \right\}, \quad (1.17)$$

where the small positive number $h$ is the thickness of the shell.

In Naghdi’s model, the displacement vector $\xi(p)$ at point $p \in S$ can be approximated as:

$$\xi(p) = \xi_1(x) + z \cap(x) \quad (1.18)$$

(see [23, (7.67)]), where $\xi_1(x) \in \mathbb{R}^3$ is the displacement vector of the middle surface and $\cap(x) \in \mathbb{R}^3$ is the director displacement vector, both at $x \in \Omega$. We decompose vectors $\xi_1$ and $\cap$ into sums

$$\xi_1(x) = W_1(x) + w_1N(x) \quad \text{and} \quad \cap(x) = U(x) + w_2N(x), \quad (1.19)$$

where $W_1, U \in X(\Omega)$. In [23, (7.59) and (7.55)], the following tensor fields on the middle surface was directly defined:

$$\Upsilon_0(\xi) = \frac{1}{2} (DW_1 + D^*W_1) + w_1\Pi, \quad (1.20)$$

$$\chi_0(\xi) = \frac{1}{2} [DU + D^*U + \Pi(\cdot, D.W_1) + \Pi(D.W_1, \cdot)] + w_2\Pi + w_1c, \quad (1.21)$$

$$\varphi_0(\xi) = \frac{1}{2} (Dw_1 + U - lW_1\Pi), \quad (1.22)$$

where $\Pi$, and $c$ are the second and third fundamental forms on the surface $M$, respectively.
The shell strain energy associated to the displacement $\zeta$ can be written as:

$$\alpha h \int_{\Omega} P_0(\zeta, \zeta) \, dx,$$

(1.23)

where

$$P_0(\zeta, \zeta) = \left| \Upsilon_0(\zeta) \right|^2 + 2\left| \phi_0(\zeta) \right|^2 + \beta \left( \text{tr} \Upsilon_0(\zeta) + w_2 \right)^2$$

$$+ w_2^2 + \gamma \left[ \left| \chi_0(\zeta) \right|^2 + \frac{1}{2} |Dw_2|^2 + \beta (\text{tr} \chi_0(\zeta))^2 \right]$$

(1.24)

with $\alpha = E/(1+\mu)$, $\beta = \mu/(1-2\mu)$, $\gamma = h^2/12$ at $x \in \Omega$; $E$ and $\mu$ denote respectively Young’s modulus and Poisson’s coefficient of the material.

Make a change of variable:

$$W_2 = U + I_{W_1} \Pi.$$

(1.25)

**Formula I.** Assume that there are no external loads on the shell and that the shell is clamped along a portion $\Gamma_0$ of $\Gamma$ and free on $\Gamma_1$, where $\Gamma_0 \cup \Gamma_1 = \Gamma$. After changing $(W_1, \sqrt{\gamma} W_2, w_1, \sqrt{\gamma} w_2)$ and $t$ to $(W_1, W_2, w_1, w_2)$ and $t/\lambda$ with $\lambda^2 \alpha = 2$ respectively, it was shown in [28] that the variable $\eta = (W_1, W_2, w_1, w_2)$ satisfies the following boundary value problem:

$$\begin{cases}
W_1'' - \Delta \mu W_1 + \mathcal{F}_1(\eta) = 0 & \text{in } Q_\infty, \\
W_2'' - \Delta \mu W_2 + \mathcal{F}_2(\eta) = 0 & \text{in } Q_\infty, \\
w_1'' - \Delta w_1 + f_1(\eta) = 0 & \text{in } Q_\infty, \\
w_2'' - \Delta w_2 + f_2(\eta) = 0 & \text{in } Q_\infty, \\
\eta(0) = \eta^0, \quad \eta'(0) = \eta^1 & \text{in } \Omega, \\
\eta = 0 & \text{on } \Sigma_{0\infty}, \\
B_1(\eta) = B_2(\eta) = 0 & \text{on } \Sigma_{1\infty}, \\
b_1(\eta) = b_2(\eta) = 0 & \text{on } \Sigma_{1\infty},
\end{cases}$$

(1.26)

where $\Delta \mu = -[\delta d + 2(1 + \beta) d \delta]$ is an operator of Hodge–Laplace type, $d$ the exterior differential, $\delta$ the formal adjoint of $d$, $\Delta$ the Laplacian on Riemannian manifold $M$, $\mathcal{F}_i(\eta)$, $f_i(\eta)$ the low terms in which the order of derivative with respect to $x$ is equal to or less than one for $i = 1, 2$, and

$$\begin{cases}
B_1(\eta) = 2\ln Y(\eta) + 2\beta \left( \text{tr} Y(\eta) + \frac{1}{\sqrt{\gamma}} w_2 \right) n, \\
B_2(\eta) = 2\ln X(\eta) + 2\beta \text{tr} X(\eta) n, \\
b_1(\eta) = 2[\phi(\eta), n], \\
b_2(\eta) = \frac{\partial w_2}{\partial n},
\end{cases}$$

(1.27)

with
\[
\begin{align*}
\Upsilon(\eta) &= \frac{1}{2}(DW_1 + D^*W_1) + w_1 \Pi, \\
X(\eta) &= \frac{1}{2}(DW_2 + D^*W_2) + w_2 \Pi - \sqrt{\gamma}(lW_1D\Pi - w_1c), \\
\phi(\eta) &= \frac{1}{2} Dw_1 - lW_1\Pi + \frac{1}{2\sqrt{\gamma}} W_2,
\end{align*}
\]

(1.28)

and \( n \) the normal vector along the curve \( \Gamma \) in metric \( g \).

\[
Q_\infty = \Omega \times (0, \infty), \quad \Sigma_{0\infty} = \Gamma_0 \times (0, \infty), \quad \Sigma_{1\infty} = \Gamma_1 \times (0, \infty).
\]

(1.29)

Set

\[
P(\eta, \zeta) = \left( \frac{2}{\gamma} \langle \Upsilon(\eta), \Upsilon(\zeta) \rangle_T^2 + 2 \langle X(\eta), X(\zeta) \rangle_T^2 + 4 \langle \phi(\eta), \phi(\zeta) \rangle + 2\beta \text{tr} \ X(\eta) \text{tr} \ X(\zeta)
\right.
\]

\[
+ \frac{2}{\gamma} \left[ \text{tr} \ X(\eta) + \frac{1}{\sqrt{\gamma}} W_2 \right] \left[ \text{tr} \ X(\zeta) + \frac{1}{\sqrt{\gamma}} W_2 \right] + \langle Dw_2, Du_2 \rangle,
\]

(1.30)

\[
P(\eta, \zeta) = \int_\Omega P(\eta, \zeta) \, dx
\]

(1.31)

and

\[
\mathcal{A} \eta = - (\Delta_\mu W_1 - \mathcal{F}_1(\eta), \Delta_\mu W_2 - \mathcal{F}_2(\eta), \Delta w_1 - f_1(\eta), \Delta w_2 - f_2(\eta)),
\]

(1.32)

where \( \eta = (W_1, W_2, w_1, w_2), \ \zeta = (U_1, U_2, u_1, u_2) \).

The following Green's formula for geometrical equation of Naghdi's shell model was proved in [28].

**Formula II.** Let bilinear \( \mathcal{P}(\cdot, \cdot) \) be given by (1.31). Then for any \( \eta = (W_1, W_2, w_1, w_2), \ \zeta = (U_1, U_2, u_1, u_2) \in (H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2 \), it has

\[
\mathcal{P}(\eta, \zeta) = (\mathcal{A} \eta, \zeta)_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2} + \int_\Gamma \partial(\mathcal{A} \eta, \zeta) \, d\Gamma,
\]

(1.33)

where

\[
\partial(\mathcal{A} \eta, \zeta) = [B_1(\eta), U_1] + [B_2(\eta), U_2] + b_1(\eta)u_1 + b_2(\eta)u_2.
\]

(1.34)

### 1.3. Control model and main results

Collecting the discussions in Sections 1.1 and 1.2, we can now formulate the Dirichlet boundary control problem for the geometrical equation of Naghdi's shell model as following:

\[
\begin{align*}
\eta'' + \mathcal{A} \eta &= 0 & \text{in } Q, \\
\eta &= 0 & \text{on } \Sigma_1, \\
\eta &= \zeta & \text{on } \Sigma_0, \\
\eta(0) &= \eta^0, \quad \eta'(0) &= \eta^1 & \text{in } \Omega, \\
\mathcal{O} &= -(B_1(A^{-1} \eta'), B_2(A^{-1} \eta'), b_1(A^{-1} \eta'), b_2(A^{-1} \eta')) & \text{on } \Sigma_0,
\end{align*}
\]

(1.35)

where \( \eta = (W_1, W_2, w_1, w_2) \), \( \zeta = (U_1, U_2, u_1, u_2) \).
where
\[ A\eta = \mathcal{A}\eta, \quad D(A) = \left(H^2(\Omega, \Lambda) \cap H^1_0(\Omega, \Lambda)\right)^2 \times \left(H^2(\Omega) \cap H^1_0(\Omega)\right)^2, \]
\[ \varsigma = (\mathcal{K}_1, \mathcal{K}_2, \kappa_1, \kappa_2) \text{ is the input (or control) and } \mathcal{O} \text{ is the output (or observation),} \]
\[ Q = \Omega \times (0, T), \quad \Sigma = \Gamma \times (0, T), \quad \Sigma_i = \Gamma_i \times (0, T), \quad i = 0, 1. \]

We consider the system (1.35) in the state Hilbert space \( \mathcal{H} = [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2] \), control and observation space \( \mathcal{U} = (L^2(\Gamma_0, \Lambda))^2 \times (L^2(\Gamma_0))^2 \).

**Theorem 1.1.** Let \( T > 0, (\eta^0, \eta^1) \in \mathcal{H} \) and \( \varsigma \in \mathcal{U} \). Then there exists a unique solution \((\eta, \eta') \in C([0, T]; \mathcal{H})\) to the system (1.35), which satisfies \( \eta(0) = \eta^0 \) and \( \eta'(0) = \eta^1 \). Moreover, there exists a constant \( C_T > 0 \), independent of \((\eta^0, \eta^1, \varsigma)\), such that
\[ \left\| (\eta(T), \eta'(T)) \right\|_{\mathcal{H}}^2 + \left\| \mathcal{O} \right\|_{L^2(0,T;\mathcal{U})}^2 \leq C_T \left[ \left\| (\eta^0, \eta^1) \right\|_{\mathcal{H}}^2 + \left\| \varsigma \right\|_{L^2(0,T;\mathcal{U})}^2 \right]. \]

Theorem 1.1 implies that the open-loop system (1.35) is well-posed in the sense of D. Salamon with the state space \( \mathcal{H} \), input and output space \( \mathcal{U} \). Since system (1.35) is a collocated system by formulation (2.6) in the next section, by virtue of our Theorem 1.1 and Theorem 2.2 of [2] (see also Theorem 3 of [10]), we know that the system (1.35) is exactly controllable in some time interval \([0, T]\) if and only if its closed-loop system under the proportional output feedback \( \varsigma = -k\mathcal{O}, \ k > 0, \) is exponentially stable. Since the exact controllability of system (1.35) was proved in [28], we can state the exponential stability for system (1.35) as follows.

Let \( \varpi(\cdot, \cdot) \) be a bilinear form on \( T^2(\Omega) \), defined by
\[ \varpi(T_1, T_2) = (T_1, T_2) + \beta \text{tr} T_1 \text{tr} T_2, \quad \forall T_1, T_2 \in T^2(\Omega). \]

For any \( W \in H^1(\Omega, \Lambda) \), set
\[ S(W) = \frac{1}{2}(DW + D^*W). \]

**Assumption (H1).** Suppose that there is a vector field \( V \in \mathcal{X}(M) \) such that
\[ DV(X, X) = b(x)|X|^2, \quad X \in M_x, \ x \in \overline{\mathcal{O}}, \]
where \( b \) is a function on \( \Omega \). Set
\[ a(x) = \frac{1}{2} \langle DV, \mathcal{E} \rangle_{T^2_\mathcal{O}}, \quad x \in \overline{\mathcal{O}}, \]
where \( \mathcal{E} \) is the volume element of \( M \). It is assumed that the above functions \( b \) and \( a \) meet the following inequality
\[ 2 \min_{x \in \overline{\Omega}} b(x) > \lambda_0(1 + 2\beta) \max_{x \in \overline{\Omega}} |a(x)|, \]
where \( \lambda_0 \) satisfies
\[ \lambda_0 \int_{\Omega} \left[ \varpi(S(W), S(W)) + |W|^2 \right] dx \geq \|DW\|^2, \quad \forall W \in H^1(\Omega, \Lambda). \]
Assumption (H2). $\Gamma_0$ and $\Gamma_1$ satisfy

$$\Gamma_1 = \{ x \mid x \in \Gamma, \langle V, n \rangle \leq 0 \}; \quad \Gamma_0 = \Gamma / \Gamma_0.$$ 

Corollary 1.1. Suppose that Assumptions (H1) and (H2) hold. Then the closed-loop system (1.35) under the proportional output feedback $\zeta = -kO$, $k > 0$, is exponentially stable, that is, there exist constants $K \geq 1$ and $\omega > 0$ such that

$$E(t) \leq Ke^{-\omega t} E(0), \quad \forall t \geq 0,$$

where

$$E(t) = \frac{1}{2} \| (\eta, \eta') \|_{H} = \frac{1}{2} \left[ \| \eta \|^2_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2} + \| A^{-1/2} \eta' \|^2_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2} \right]$$

is the energy of the system (1.35).

The following Theorem 1.2 shows that the system (1.35) is also regular in the sense of G. Weiss [13]. The analytic expression of feedthrough operator is also presented.

Theorem 1.2. The system (1.35) is regular as well. More precisely, if $\eta(0) = \eta'(0) = 0$ and $\zeta(x, t) = (K_1, K_2, \kappa_1, \kappa_2), x \in \Gamma_0$ is a step input: $\zeta(\cdot, t) \equiv \zeta(0) \in \mathcal{U}$, then the corresponding output $O$ satisfies

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \int_0^\sigma O(t) dt = (N_1(\zeta), N_2(\zeta), \kappa_1, \kappa_2),$$

where

$$N_j(\zeta) = \langle K_j, \tau \rangle \tau + \sqrt{2(1 + \beta)} \langle K_j, n \rangle n, \quad j = 1, 2.$$ 

The remaining part of this paper is organized as follows. In Section 2, we cast the system (1.35) into an abstract setting studied in [2] or [10]. Preliminary results on associated nonhomogeneous boundary value problem are given in Section 3. The proofs of Theorems 1.1 and 1.2 are presented in Sections 4 and 5, respectively.

2. Collocated formulation

Let $H = (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2$ be the dual space of the product Sobolev space $(H^1_0(\Omega, \Lambda))^2 \times (H^1_0(\Omega))^2$ with the usual inner product. Let $A$ be the positive self-adjoint operator in $H$ induced by the bilinear form $P(\cdot, \cdot)$ defined by (1.31) as follows:

$$\langle A\eta, \xi \rangle_{H', H} = P(\eta, \xi) = \int_\Omega P(\eta, \xi) dx, \quad \forall \eta, \xi \in (H^1_0(\Omega, \Lambda))^2 \times (H^1_0(\Omega))^2.$$

By means of the Lax–Milgram theorem, $A$ is a canonical isomorphism from $D(A) = (H^1_0(\Omega, \Lambda))^2 \times (H^1_0(\Omega))^2$ to $H$. It is easy to show that $A\eta = A\eta$ whenever $\eta \in (H^1_0(\Omega, \Lambda))^2 \times (H^1_0(\Omega))^2$. Hence $A$ is an extension of the operator $A$ to the space $(H^1_0(\Omega, \Lambda))^2 \times (H^1_0(\Omega))^2$. 
It can be easily shown that $D(A^{1/2}) = (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2$ and $A^{1/2}$ is an isomorphism from $(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2$ onto $H$ [12]. Define the Dirichlet map $D \in \mathcal{L}((L^2(\Gamma_0, \Lambda))^2 \times (L^2(\Gamma_0))^2$, $(H^{1/2}(\Omega, \Lambda))^2 \times (H^{1/2}(\Omega))^2)$ by $D\xi = \xi$ if and only if

\[
\begin{cases}
    A\xi = 0 & \text{in } \Omega, \\
    \xi = \zeta & \text{on } \Gamma_0, \\
    \zeta = 0 & \text{on } \Gamma_1.
\end{cases}
\]

By virtue of the map $D$, one can write the system (1.35) as

\[
\eta'' + A(\eta - D\zeta) = 0. \tag{2.1}
\]

Since $D(A)$ is dense in $H$, and so is $D(A^{1/2})$. We identify $H$ with its dual $H'$. Then the following relations hold:

$$D(A^{1/2}) \hookrightarrow H = H' \hookrightarrow (D(A^{1/2}))^{'}. $$

An extension $\tilde{A} \in \mathcal{L}(D(A^{1/2}), (D(A^{1/2}))'$ of $A$ is defined by

$$\langle \tilde{A}\eta, \zeta \rangle_{(D(A^{1/2}))', D(A^{1/2})} = \langle A^{1/2}\eta, A^{1/2}\zeta \rangle_H, \quad \forall \eta, \zeta \in D(A^{1/2}). \tag{2.2}$$

So (2.1) can be further written in $(D(A^{1/2}'))'$ as

$$\eta'' + \tilde{A}\eta + \mathbb{B}\zeta = 0,$$

where $\mathbb{B} \in \mathcal{L}(U, (D(A^{1/2})))'$ is given by

$$\mathbb{B}\zeta = -\tilde{A}D\zeta, \quad \forall \zeta \in U. \tag{2.3}$$

Define $\mathbb{B}^* \in \mathcal{L}(D(A^{1/2}), U)$ by

$$\langle \mathbb{B}^*\eta, \zeta \rangle_U = \langle \eta, \mathbb{B}\zeta \rangle_{D(A^{1/2}), (D(A^{1/2}))'}, \quad \forall \eta \in D(A^{1/2}), \ \zeta \in U.$$ 

Then for any $\eta \in D(A^{1/2})$ and $\zeta \in (C_0^\infty(\Gamma_0, \Lambda))^2 \times (C_0^\infty(\Gamma_0))^2$, we have

$$\langle \eta, \mathbb{B}\zeta \rangle_{D(A^{1/2}), (D(A^{1/2}))'} = \langle \eta, \tilde{A}\tilde{A}^{-1}\mathbb{B}\zeta \rangle_{D(A^{1/2}), (D(A^{1/2}))'} = -\langle A^{1/2}\eta, A^{1/2}\mathbb{B}\zeta \rangle_H = -\langle \eta, D\zeta \rangle_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2} = -\langle AA^{-1}\eta, D\zeta \rangle_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2}$$

$$= \int_{\Gamma_0} \partial(AA^{-1}\eta, \zeta) \, d\Gamma. \tag{2.4}$$

Since $(C_0^\infty(\Gamma_0, \Lambda))^2 \times (C_0^\infty(\Gamma_0))^2$ is dense in $U = (L^2(\Gamma_0, \Lambda))^2 \times (L^2(\Gamma_0))^2$, we finally obtain that

$$\mathbb{B}^*\eta = \big( B_1(A^{-1}\eta), B_2(A^{-1}\eta), b_1(A^{-1}\eta), b_2(A^{-1}\eta) \big) |_{\Gamma_0}. \tag{2.5}$$

Now, we have formulated the open-loop system (1.35) into the abstract form of a second-order system in the space $\mathcal{H} = [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]$: 
\[ \begin{aligned}
\eta''(t) + \mathcal{A}\eta(t) + \mathcal{B}\zeta(t) &= 0, \\
\eta(0), \eta'(0) &\in \mathcal{H},
\end{aligned} \tag{2.6} \]

where \( \mathcal{B} \) and \( \mathcal{B}^* \) are defined by (2.3) and (2.5), respectively. The abstract system (2.6) has been studied in detail in [2] and [10], respectively.

### 3. Results on nonhomogeneous boundary value problems

In this section, we give some results on the existence and regularity of solutions to the nonhomogeneous boundary value problems associated with Naghdi’s shell equation.

For brevity in notation, we denote \( Q = \Omega \times (0, T), \) \( \Sigma = \Gamma \times (0, T), \) \( L^2(\Sigma, \Lambda) = H^0(\Sigma, \Lambda) = L^2(0, T; L^2(\Gamma, \Lambda)), \) \( H^1(\Sigma, \Lambda) = L^2(0, T; H^1(\Gamma, \Lambda)) \cap H^1(0, T; L^2(\Gamma, \Lambda)), \) \( H^{-1}(\Sigma, \Lambda) = (H^1_0(\Sigma, \Lambda))^*, \)

\( L^2(\Sigma) = H^0(\Sigma) = L^2(0, T; L^2(\Gamma)), \) \( H^1(\Sigma) = L^2(0, T; H^1(\Gamma)) \cap H^1(0, T; L^2(\Gamma)), \) \( H^{-1}(\Sigma) = (H^1_0(\Sigma))^*. \)

Consider system (1.35) in a more general case:

\[ \begin{aligned}
\eta'' + A\eta &= \Phi \quad \text{in } Q, \\
\eta &= \zeta \quad \text{on } \Sigma, \\
\eta(0) = \eta^0, \quad \eta'(0) = \eta^1 \quad \text{in } \Omega.
\end{aligned} \tag{3.1} \]

**Proposition 3.1.** For any given \( T > 0, \) suppose that

\[ \begin{aligned}
\Phi &\in L^2(0, T; (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2), \\
\eta^0 &\in (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2, \quad \eta^1 \in (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2, \\
\zeta &\in L^2(0, T; (L^2(\Gamma, \Lambda))^2 \times (L^2(\Gamma))^2).
\end{aligned} \]

Then there exists a unique solution \( \eta \) to Eq. (3.1) satisfying

\[ (\eta, \eta') \in C([0, T]; [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]), \tag{3.2} \]

and

\[ (B_1(\eta), B_2(\eta), b_1(\eta), b_2(\eta)) \in (H^{-1}(\Sigma, \Lambda))^2 \times (H^{-1}(\Sigma))^2. \tag{3.3} \]

**Remark 3.1.** With the same reason as the discussion after Remark 2.5 of [18], we only need to show Proposition 3.1 in \( L^\infty([0, T]; [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]) \) instead of \( C([0, T]; [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]). \)

**Remark 3.2.** In Proposition 3.1, one can show that \( \eta, \eta' \) and \( (B_1(\eta), B_2(\eta), b_1(\eta), b_2(\eta)) \) depend continuously on the related given datum. Similar remarks apply to all regularity results in the sequel.

**Remark 3.3.** Let \( \Phi = 0 \) and \( \zeta = 0 \) in Proposition 3.1. Then (3.2) associates with a \( C_0 \)-semigroup on \( (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2, \) that is,

\[ (\eta, \eta') = e^{\mathcal{C}t}(\eta^0, \eta^1), \]

where \( e^{\mathcal{C}t} \) is a \( C_0 \)-semigroup on \( (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2. \)
In order to prove Proposition 3.1, we need several preliminary lemmas. First, we consider the dual system of (3.1) in unknown \( \zeta = (U_1, U_2, u_1, u_2) \).

\[
\begin{cases}
  \zeta'' + A\zeta = \Psi & \text{in } Q, \\
  \zeta = 0 & \text{on } \Sigma, \\
  \zeta(0) = \zeta'(0) = 0 & \text{in } \Omega.
\end{cases}
\]  

(3.4)

Lemma 3.1. Suppose that \( \Psi \in L^1(0, T; (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2) \) for some \( T > 0 \). Then there exists a unique solution \( \zeta \) to (3.4), which satisfies

\[
(\zeta, \zeta') \in L^\infty(0, T; \left[(H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2\right] \times \left[(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2\right]),
\]

(3.5) and

\[
(B_1(\zeta), B_2(\zeta), b_1(\zeta), b_2(\zeta)) \in (L^2(\Sigma, \Lambda))^2 \times (L^2(\Sigma))^2.
\]

(3.6)

Proof. Applying the \( C_0 \)-semigroup theory [24], we know that the system (3.4) admits a unique solution satisfying (3.5). Now we only need to show (3.6). First, since \( \Gamma \) is smooth, we can always take a vector field \( V \) such that \( V = n \) on \( \Gamma \) [14]. Set

\[
m(\zeta) = (DV U_1, DV U_2, V(u_1), V(u_2)).
\]

Multiply the both sides of the first equation in (3.4) by \( m(\zeta) \) and integrate over \( Q \), to get

\[
\int_Q \langle \Psi, m(\zeta) \rangle dQ = \int_Q \langle \zeta'' + A\zeta, m(\zeta) \rangle dQ
\]

\[
= \int_Q \langle \zeta', m(\zeta) \rangle \bigg|_0^T - \int_Q \langle \zeta', m(\zeta) \rangle dQ
\]

\[
+ \int_Q P(\zeta, m(\zeta)) dQ - \int_\Sigma \partial(A\zeta, m(\zeta)) d\Sigma.
\]

(3.7)

By modifying slightly the proof of Proposition 4.2 of [28], we can easily obtain that

\[
\int_\Omega P(\zeta, m(\zeta)) d\zeta = \frac{1}{2} \int_\Gamma P(\zeta, \zeta)(V, n) d\Gamma - \frac{1}{2} \int_\Omega P(\zeta, \zeta) \text{div}(V) d\zeta + \int_\Omega e(\zeta, \zeta) d\zeta + l(\zeta).
\]

(3.8)

where

\[
e(\zeta, \zeta) = 2 \sum_{i=1}^2 \sigma((S(U_i), G(V, U_i)) + 4\varphi(\zeta), \varphi(\zeta)(D.V)) + DV(Du_2, Du_2),
\]

\[
\sigma(T_1, T_2) = \langle T_1, T_2 \rangle + \beta \text{tr} T_1 \text{tr} T_2, \quad \forall T_1, T_2 \in T^2(\Omega),
\]

\[
S(U) = \frac{1}{2}(DU + DU^*), \quad \forall U \in H^1(\Omega, \Lambda),
\]

\[
G(V, U)(X, Y) = \frac{1}{2}[DU(X, D_Y V) + DU(Y, D_X V)], \quad \forall X, Y \in M_x, \ x \in \overline{\Omega}.
\]
in which \( \cdot \) represents the position of variable and \( l(\zeta) \) all the lower terms. From (3.7), (3.8) and b) in Proposition 4.4 of [28], it follows that

\[
\int_{Q} \langle \Psi, m(\zeta) \rangle dQ = \int_{\Omega} \langle \zeta', m(\zeta) \rangle dx \bigg|_{0}^{T} - \frac{1}{2} \int_{Q} V(|\zeta'|^2) dQ - \frac{1}{2} \int_{\Sigma} P(\zeta, \zeta) \langle V, n \rangle d\Sigma \\
- \frac{1}{2} \int_{Q} P(\zeta, \zeta) \text{div}(V) dQ + \int_{Q} e(\zeta, \zeta) dQ + \text{lot}(\zeta),
\]

(3.9)

where \( \text{lot}(\zeta) = \int_{0}^{T} l(\zeta) dt \). Therefore,

\[
\int_{\Sigma} P(\zeta, \zeta) d\Sigma = -2 \int_{Q} \langle \Psi, m(\zeta) \rangle dQ + 2 \int_{\Omega} \langle \zeta', m(\zeta) \rangle dx \bigg|_{0}^{T} + 2 \int_{Q} e(\zeta, \zeta) dQ \\
+ \int_{Q} \left[ |\zeta'|^2 - P(\zeta, \zeta) \right] \text{div}(V) dQ + \text{lot}(\zeta)
\leq C_T \left( \|\zeta\|_{L^\infty(0,T;L^2(\Omega,\Lambda))^2 \times (L^2(\Omega))^2)}^2 + \|\xi\|_{L^\infty(0,T;H^1(\Omega,\Lambda))^2 \times (H^1(\Omega))^2)}^2 + \|\Psi\|_{L^1(0,T;L^2(\Omega,\Lambda))^2 \times (L^2(\Omega))^2)}^2 \right).
\]

(3.10)

On the other hand, since \( \zeta = 0 \) on \( \Gamma \), it is easy to check that

\[
\left| \left( B_1(\zeta), B_2(\zeta), b_1(\zeta), b_2(\zeta) \right) \right|^2 = P(\zeta, \zeta) \quad \text{on} \ \Gamma.
\]

(3.11)

(3.6) then follows from (3.10) and (3.11). \( \Box \)

**Lemma 3.2.** (3.2) in Proposition 3.1 holds true.

**Proof.** We first prove the result by taking \( \Phi = 0 \) and assume that all data are smooth. The proof will be split into several steps.

**Step 1.** Let \( \eta \) be the solution of (3.1) and let \( \zeta \) be the solution of the following system:

\[
\begin{cases}
\zeta'' + A\zeta = \Psi & \text{in} \ Q, \\
\zeta = 0 & \text{on} \ \Sigma, \\
\zeta(T) = \zeta'(T) = 0 & \text{in} \ \Omega,
\end{cases}
\]

(3.12)

where \( \Psi \in L^1(0,T;L^2(\Omega,\Lambda))^2 \times (L^2(\Omega))^2) \). Then

\[
0 = \int_{Q} \langle \Phi, \zeta \rangle dQ + \int_{\Sigma} \partial(A\eta, \zeta) d\Sigma \\
= \int_{Q} \langle \eta'', A\eta, \zeta \rangle dQ + \int_{\Sigma} \partial(A\eta, \zeta) d\Sigma \\
= \int_{\Omega} \left( \langle \eta', \zeta \rangle - \langle \eta, \zeta' \rangle \right) dx \bigg|_{0}^{T} + \int_{Q} \langle \eta, \zeta'' \rangle dQ + \int_{Q} P(\eta, \zeta) dQ
\]
From [19], we know that there is a unique solution proving (3.2) in the case of $\eta$. Furthermore, an appeal to "lifting theorem" in [21] arrives at that is,

$$
\eta = -\int \langle \eta' (0), \zeta (0) \rangle \, dx + \int Q \langle \eta, \zeta'' + A \zeta \rangle \, dx + \int \partial (A \zeta, \eta) \, d \Sigma
$$

$$
= -\int \langle \eta' (0), \zeta (0) \rangle \, dx + \int \langle \eta (0), \zeta' (0) \rangle \, dx + \int \langle \eta, \Psi \rangle \, d Q + \int \partial (A \zeta, \eta) \, d \Sigma. \tag{3.13}
$$

**Step 2.** From (3.13), it follows that

$$
\int Q \langle \eta, \Psi \rangle \, d Q = \int \langle \eta' (0), \zeta (0) \rangle \, dx - \int \langle \eta (0), \zeta' (0) \rangle \, dx - \int \partial (A \zeta, \eta) \, d \Sigma. \tag{3.14}
$$

By Lemma 3.1, the assumptions of Proposition 3.1 and the definition of $\partial (A \zeta, \eta)$, we know that the right side of (3.14) is continuous on $L^1 (0, T; (L^2 (\Omega, \Lambda)) \times (L^2 (\Omega)) ^2)$. So $\eta$ belongs to the dual space of $L^1 (0, T; (L^2 (\Omega, \Lambda)) \times (L^2 (\Omega)) ^2)$, that is,

$$
\eta \in L^\infty (0, T; (L^2 (\Omega, \Lambda)) ^2 \times (L^2 (\Omega)) ^2).
$$

**Step 3.** From the result of **Step 2** and system (3.1), we have

$$
\eta'' = -A \eta \in L^\infty (0, T; (H^{-2} (\Omega, \Lambda)) ^2 \times (H^{-2} (\Omega)) ^2).
$$

By Theorems 2.3 and 12.4 in Chapter 1 of [19], we get

$$
\eta' \in L^2 (0, T; [(L^2 (\Omega, \Lambda)) ^2 \times (L^2 (\Omega)) ^2, (H^{-2} (\Omega, \Lambda)) ^2 \times (H^{-2} (\Omega)) ^2]_{1/2}).
$$

that is,

$$
\eta' \in L^2 (0, T; (H^{-1} (\Omega, \Lambda)) ^2 \times (H^{-1} (\Omega)) ^2).
$$

Furthermore, an appeal to "lifting theorem" in [21] arrives at

$$
\eta' \in C (0, T; (H^{-1} (\Omega, \Lambda)) ^2 \times (H^{-1} (\Omega)) ^2) \subset L^\infty (0, T; (H^{-1} (\Omega, \Lambda)) ^2 \times (H^{-1} (\Omega)) ^2),
$$

proving (3.2) in the case of $\Phi = 0$.

The proof will be accomplished if we can show the same conclusion for $\Phi \neq 0$ and $\Phi \in L^2 (0, T; (H^{-1} (\Omega, \Lambda)) ^2 \times (H^{-1} (\Omega)) ^2)$. Let $\xi$ be the solution of the following system:

$$
\begin{cases}
\xi'' + A \xi = \Phi & \text{in } Q, \\
\xi = 0 & \text{on } \Sigma, \\
\xi (0) = \xi' (0) = 0 & \text{in } \Omega. \tag{3.15}
\end{cases}
$$

From [19], we know that there is a unique solution $\xi$ to system (3.15), which satisfies

$$
\xi \in L^\infty (0, T; (L^2 (\Omega, \Lambda)) ^2 \times (L^2 (\Omega)) ^2), \quad \xi' \in L^\infty (0, T; (H^{-1} (\Omega, \Lambda)) ^2 \times (H^{-1} (\Omega)) ^2).
$$

Let $\eta$ be the solution of (3.1) with $\Phi = 0$. Then $\eta + \xi$ is the solution of (3.1) with $\Phi \neq 0$, and it satisfies (3.2). The proof is complete. \( \square \)
Lemma 3.3. Consider the system (3.1) with $\Phi = 0$ and suppose that

$$\begin{align*}
\zeta, \zeta' &\in L^2(0, T; (L^2(\Gamma, \Lambda))^2 \times (L^2(\Omega))^2), \\
\eta^0 &\in (H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2, \\
\eta^1 &\in (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2,
\end{align*}$$

and

$$\zeta|_{t=0} = \eta^0|_{\Gamma}.$$

Then

$$\eta' \in L^\infty(0, T; (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2), \quad \eta'' \in L^\infty(0, T; (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2).$$

Proof. We may assume all data to be smooth and the proof will be accomplished by continuous extension and density argument. Let $\theta = \eta'$. Then $\theta$ satisfies

$$\begin{align*}
\theta'' + A\theta &= 0 \quad \text{in } Q, \\
\theta &= \zeta' \quad \text{on } \Sigma, \\
\theta(0) = \eta^1, \quad \theta'(0) = -A\eta^0 \quad \text{in } \Omega.
\end{align*}$$

(3.16)

Since $\theta'(0) = -A\eta^0 \in (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2$, apply Lemma 3.2 to system (3.16), to get

$$\begin{align*}
\theta &\in L^\infty(0, T; (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2), \\
\theta' &\in L^\infty(0, T; (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2).
\end{align*}$$

Lemma 3.3 then follows from the fact $\theta = \eta'$. \Box

Lemma 3.4. Consider the system in Lemma 3.3. If in addition

$$\zeta \in L^\infty(0, T; (H^{1/2}(\Gamma, \Lambda))^2 \times (H^{1/2}(\Gamma))^2),$$

then

$$\eta \in L^\infty(0, T; (H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2).$$

(3.17)

Proof. By Lemma 3.3,$n$,$n$

$$A\eta = -\eta'' \in L^\infty(0, T; (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2).$$

Take $t$ as a parameter and consider the Dirichlet problem of the following system:

$$\begin{align*}
A\eta &\in L^\infty(0, T; (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2), \\
\eta|_{\Gamma} &= \zeta \in L^\infty(0, T; (H^{1/2}(\Gamma, \Lambda))^2 \times (H^{1/2}(\Gamma))^2).
\end{align*}$$

Since operator $A$ is elliptic, we get (3.17) from the elliptic theory (see e.g. [5]). \Box
Lemma 3.5. Suppose that

\[
\begin{align*}
\Phi &\in L^1(0, T; (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2), \\
\varsigma &\in (H^1(\Sigma, \Lambda))^2 \times (H^1(\Sigma))^2, \\
\eta^0 &\in (H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2, \quad \eta^1 \in (L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2
\end{align*}
\]  

(3.18)

with the compatibility condition

\[
\varsigma|_{t=0} = \eta^0|_{\Gamma}.
\]

Then the solution \( \eta \) of system (3.1) satisfies

\[
(\eta, \eta') \in C([0, T]; [(H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2] \times [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2]),
\]

and

\[
(B_1(\eta), B_2(\eta), b_1(\eta), b_2(\eta)) \in (L^2(\Sigma, \Lambda))^2 \times (L^2(\Sigma))^2.
\]

(3.20)

**Proof.** We first prove (3.19). From the proof of Lemma 3.2, we need only consider the case of \( \Phi = 0 \). Since \( \varsigma \in (H^1(\Sigma, \Lambda))^2 \times (H^1(\Sigma))^2 \), it follows that

\[
\varsigma \in L^2(0, T; (H^1(\Sigma, \Lambda))^2 \times (H^1(\Sigma))^2), \quad \varsigma' \in L^2(0, T; (L^2(\Sigma, \Lambda))^2 \times (L^2(\Sigma))^2).
\]

By Theorem 3.1 of [19, Chapter 1], we have

\[
\varsigma \in L^\infty(0, T; (H^{1/2}(\Sigma, \Lambda))^2 \times (H^{1/2}(\Sigma))^2).
\]

(3.22)
On the other hand, by the definition of $B_1(\eta)$, it has
\[
|B_1(\eta)| = \left|2\ln \Upsilon(\eta) + 2\beta \left( \text{tr} \Upsilon(\eta) + \frac{1}{\sqrt{\Upsilon}} w_2 \right) \right| \leq C \left[ |\Upsilon(\eta)| + \left| \left( \text{tr} \Upsilon(\eta) + \frac{1}{\sqrt{\Upsilon}} w_2 \right) \right| \right].
\] (3.23)

Combining (3.22) and (3.23) yields $B_2(\eta) \in L^2(\Sigma, \Lambda)$.

Similarly, by $\varsigma \in L^\infty(0,T; (H^{1/2}(\Gamma, \Lambda))^2 \times (H^{1/2}(\Gamma))^2)$, the definitions of $B_2(\eta)$, $b_1(\eta)$, $b_2(\eta)$, and (3.22), we obtain that $B_1(\eta) \in L^2(\Sigma, \Lambda)$, $b_1(\eta) \in L^2(\Sigma)$, and (3.22), we obtain that $b_2(\eta) \in L^2(\Sigma)$. Thus (3.20) holds.

**Proof of Proposition 3.1.** Since (3.2) has been proven in Lemma 3.2, it remains to show (3.3). Let $\zeta$ be the solution of the following system:
\[
\begin{cases}
\zeta'' + A\zeta = 0 & \text{in } Q, \\
\zeta = \vartheta & \text{on } \Sigma, \\
\zeta(T) = \zeta'(T) = 0 & \text{in } \Omega,
\end{cases}
\]
(3.24)

where $\vartheta$ satisfies
\[
\vartheta \in \left( H^1(\Sigma, \Lambda) \right)^2 \times \left( H^1(\Sigma) \right)^2, \quad \vartheta(T) = 0 \quad \text{on } \Gamma.
\]

Assuming all data being smooth, using (3.13) and noting that $\Psi = 0$ and $\zeta = \vartheta$ on $\Gamma$, we have
\[
\int_\Sigma \partial(A\eta, \zeta) \, d\Sigma = -\int_Q \langle \Phi, \zeta \rangle \, dQ - \int_\Omega \langle \eta'(0), \zeta(0) \rangle \, dx + \int_\Omega \langle \eta(0), \zeta'(0) \rangle \, dx + \int_\Sigma \partial(A\zeta, \eta) \, d\Sigma.
\]

Apply Lemma 3.5 to system (3.24), to obtain
\[
\int_\Sigma \partial(A\eta, \zeta) \, d\Sigma \leq C \| \vartheta \|_{(H^1(\Sigma, \Lambda))^2 \times (H^1(\Sigma))^2}
\]
for some positive constant $C$ independent of $\vartheta$. (3.3) then follows by taking $\vartheta \in (H^1_0(\Sigma, \Lambda))^2 \times (H^1_0(\Sigma))^2$. \(\square\)

4. **Proof of Theorem 1.1**

Since $\eta = 0$ on $\Gamma_1$, we may assume without loss of generality that $\Gamma_0 = \Gamma = \partial \Omega$. Consider then the system (1.35) with zero initial data:
\[
\begin{cases}
\eta'' + A\eta = 0 & \text{in } Q, \\
\eta = \varsigma & \text{on } \Sigma, \\
\eta(0) = 0, \quad \eta'(0) = 0 & \text{in } \Omega, \\
O = (B_1(A^{-1}\eta'), B_2(A^{-1}\eta'), b_1(A^{-1}\eta'), b_2(A^{-1}\eta')) & \text{on } \Sigma.
\end{cases}
\]
(4.1)

By Theorem A.1 in Appendix A, Theorem 1.1 is equivalent to saying that the solution of (4.1) satisfies
\[
\|O\|_{L^2(0,T;\mathcal{U})} \leq C_T \|\varsigma\|_{L^2(0,T;\mathcal{U})},
\]
(4.2)
where $\mathcal{U} = (L^2(\Gamma, \Lambda))^2 \times (L^2(\Gamma))^2$. 

Remark 4.1. By Remark 3.2, (4.2) is equivalent to (see also [9])

\[ B_i(A^{-1} \eta') \in L^2(\Sigma, \Lambda), \quad b_i(A^{-1} \eta') \in L^2(\Sigma), \quad i = 1, 2. \quad (4.3) \]

Remark 4.2. For any \( \eta = (W_1, W_2, w_1, w_2) \in D(\mathcal{P}) \), set

\[ \mathcal{P} \eta = (\mathcal{P}W_1, \mathcal{P}W_2, \mathcal{P}w_1, \mathcal{P}w_2) = -(\Delta_\mu W_1, \Delta_\mu W_2, \Delta w_1, \Delta w_2), \quad (4.4) \]

where

\[ D(\mathcal{P}) = (H^2(\Omega, \Lambda) \cap H^1_0(\Omega, \Lambda))^2 \times (H^2(\Lambda) \cap H^1_0(\Omega))^2. \]

We claim that for any \( \xi \in (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2 \),

\[ B_i(A^{-1} \xi) \in L^2(\Sigma, \Lambda) \Leftrightarrow B_i(\mathcal{P}^{-1} \xi) \in L^2(\Sigma, \Lambda), \quad i = 1, 2, \quad (4.5) \]

and

\[ b_i(A^{-1} \xi) \in L^2(\Sigma) \Leftrightarrow b_i(\mathcal{P}^{-1} \xi) \in L^2(\Sigma), \quad i = 1, 2. \quad (4.6) \]

In fact, it is easy to check that

\[ A^{-1} \xi = A^{-1}(A - (A - \mathcal{P})) \mathcal{P}^{-1} \xi = \mathcal{P}^{-1} \xi - A^{-1}(A - \mathcal{P}) \mathcal{P}^{-1} \xi. \quad (4.7) \]

From \( \xi \in (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2 \) and the definitions of \( D(A) \) and \( D(\mathcal{P}) \), it follows that

\[ A^{-1}(A - \mathcal{P}) \mathcal{P}^{-1} \xi \in C([0, T]; (L^2(\Omega, \Lambda))^2 \times (H^1(\Omega))^2). \]

By trace theorem, we have

\[ \begin{cases} B_i(A^{-1}(A - \mathcal{P}) \mathcal{P}^{-1} \xi) \in C([0, T]; H^{3/2}(\Sigma, \Lambda)) \subset L^2(\Sigma, \Lambda), \\ b_i(A^{-1}(A - \mathcal{P}) \mathcal{P}^{-1} \xi) \in C([0, T]; H^{3/2}(\Sigma)) \subset L^2(\Sigma), \quad i = 1, 2. \end{cases} \quad (4.8) \]

Combine (4.7) and (4.8) to obtain (4.5) and (4.6).

Remark 4.3. Let \( \Phi, \eta^0 \) and \( \eta^1 \) be in Proposition 3.1. Similar to Remark 4.3 in [16], we know that for any \( \Phi, \eta^0 \) and \( \eta^1 \), the following system:

\[ \begin{cases} \eta'' + A\eta - A_{\text{lot}}(\eta) = \Phi \quad \text{in } Q, \\ \eta = 0 \quad \text{on } \Sigma, \\ \eta(0) = \eta^0, \quad \eta'(0) = \eta^1 \quad \text{in } \Omega, \end{cases} \quad (4.9) \]

where \( A_{\text{lot}}(\eta) \) is the lower order term from \( A\eta \), admits a unique solution which satisfies

\[ (\eta, \eta') \in C([0, T]; [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]). \quad (4.10) \]
We briefly recall the construction of geodesic normal coordinate on the boundary. For each $q \in \partial \Omega$, let $\gamma_q : [0, \varepsilon) \to \tilde{\Omega}$ denote the unit-speed geodesic starting at $q$ and normal to $\partial \Omega$. If $x_1$ is any local coordinate for $\partial \Omega$ near $q \in \partial \Omega$, we can extend it smoothly to a function on a neighborhood of $q$ in $\tilde{\Omega}$ by letting it be constant along each normal geodesic $\gamma_q$. If we define $x_2$ to be the parameter along $\gamma_q$, it is then easily seen that $\{x_1, x_2\}$ forms a coordinate for $\tilde{\Omega}$ in some neighborhood of $q$, which we call the boundary normal coordinate. In this defined coordinate, locally characterized by $x_2$, we can change it to a function on a neighborhood of $\tilde{\Omega}$.

By using the above geodesic normal coordinate, we can change locally $\Omega$ and $\Gamma$ to $\tilde{\Omega} := \{(x_1, x_2) \in \mathbb{R}^2, \ x_2 > 0, \ x_1 \in \mathbb{R}\}$ and $\tilde{\Gamma} := \{(x_1, x_2) \in \mathbb{R}^2, \ x_2 = 0, \ x_1 \in \mathbb{R}\}$. Under such coordinate,

$$(E_1, E_2) =: \left( \frac{1}{\sqrt{g_1}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$$

is the orthonormal basis of vector fields. Then for any function $u$ and any vector $U = v_1 E_1 + v_2 E_2$, we have

$$\Delta u = -\delta du = E_1 E_1 u + E_2 E_2 u + l(u), \quad (4.11)$$

$$\delta dU = -(E_1 E_1 v_1 + E_1 E_2 v_2) E_1 - (E_2 E_1 v_1 + E_2 E_2 v_2) E_2 + l(U), \quad (4.12)$$

and

$$\delta dU = -(E_2 E_1 v_2 + E_2 E_2 v_1) E_1 - (E_1 E_1 v_2 - E_1 E_2 v_1) E_2 + l(U). \quad (4.13)$$

Thus

$$\Delta U = \left( 2(1 + \beta) \frac{1}{g_1} \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + (1 + 2\beta) \frac{1}{\sqrt{g_1}} \frac{\partial^2 v_2}{\partial x_1 \partial x_2} \right) E_1$$

$$+ \left( \frac{1}{g_1} \frac{\partial^2 v_2}{\partial x_1^2} + 2(1 + \beta) \frac{\partial^2 v_2}{\partial x_2^2} + (1 + 2\beta) \frac{1}{\sqrt{g_1}} \frac{\partial^2 v_1}{\partial x_1 \partial x_2} \right) E_2 + l(U), \quad (4.14)$$

and

$$\Delta u = \frac{1}{g_1} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + l(u). \quad (4.15)$$

Furthermore, a simple computation gives that for $\eta = (W_1, W_2, w_1, w_2)$ and $W_i = w_{i1} E_1 + w_{i2} E_2$,

$$B_i(\eta) = 2 \left( -\frac{\partial w_{i1}}{\partial x_2} + \frac{1}{\sqrt{g_1}} \frac{\partial w_{i2}}{\partial x_1} \right) E_1$$

$$+ 2 \left( (1 + \beta) \frac{\partial w_{i2}}{\partial x_2} + \beta \frac{1}{\sqrt{g_1}} \frac{\partial w_{i1}}{\partial x_1} \right) E_2 + l(\eta), \quad i = 1, 2, \quad (4.16)$$

and

$$b_i(\eta) = -\frac{\partial w_i}{\partial x_2} + l(\eta), \quad i = 1, 2, \quad (4.17)$$
where \( l(\eta) \) is the lower order term which we use the same notation without confusion although they may have different values from different contexts.

**Proof of Theorem 1.1.** By Remarks 4.1, 4.2, we only need to prove

\[
B_i(\mathcal{D}^{-1} \eta') \in L^2(\Sigma, \Lambda) \quad \text{and} \quad b_i(\mathcal{D}^{-1} \eta') \in L^2(\Sigma), \quad i = 1, 2.
\]

The proof will be split into four steps.

**Step 1.** Let \( \zeta \in L^2(0, T; (L^2(\Gamma, \Lambda))^2 \times (L^2(\Gamma))^2) \). Then by Proposition 3.1, the solution to (4.1) satisfies

\[
(\eta, \eta') \in C([0, T]; [(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2], \quad (4.18)
\]

and

\[
(B_1(\eta), B_2(\eta), b_1(\eta), b_2(\eta)) \in (H^{-1}(\Sigma, \Lambda))^2 \times (H^{-1}(\Sigma))^2. \quad (4.19)
\]

**Step 2.** Let \( q \) be a given point of the boundary \( \Gamma \). By using the above geodesic normal coordinate, we can choose a given coordinate neighborhood \( \mathcal{V} \) of \( q \) and change \( \Omega \cap \mathcal{V} \) and \( \Gamma \cap \mathcal{V} \) to \( \tilde{\Omega} := \{(x_1, x_2) \in \mathbb{R}^2, \ x_2 > 0, \ x_1 \in \mathbb{R} \} \) and \( \tilde{\Gamma} := \{(x_1, x_2) \in \mathbb{R}^2, \ x_2 = 0, \ x_1 \in \mathbb{R} \} \). Let \( \phi \in C^\infty_0(M), \ 0 < \phi < 1, \) be a smooth cutoff function in \( M \) with \( \text{supp}(\phi) \subset \mathcal{V} \). Set \( \tilde{\eta} = \phi \eta \) and \( \tilde{\zeta} = \phi \zeta \). Then from (4.1), we have

\[
\begin{cases}
\tilde{\eta}'' + \mathcal{A}\tilde{\eta} + [\phi, \mathcal{A}]\eta = 0 & \text{in } \tilde{\Omega} \times (0, \infty), \\
\tilde{\eta} = \tilde{\zeta} & \text{on } \tilde{\Gamma} \times (0, \infty), \\
\tilde{\eta}(0) = \tilde{\eta}'(0) = 0 & \text{in } \tilde{\Omega},
\end{cases} \quad (4.20)
\]

where \( [\phi, \mathcal{A}]\eta = \phi\mathcal{A}\eta - \mathcal{A}(\phi\eta) \).

Since the solution \( \tilde{\eta} \) of (4.20) has zero initial value, one can extend \( \tilde{\eta} \) to be zero for \( t < 0 \). Let \( \rho \in C^\infty_0(\mathbb{R}), \ |\rho| \leq 1, \) be a smooth cutoff function in \( \mathbb{R} \) with \( \rho(t) = 0 \) for \( t \geq (3/2)T \) and \( \rho(t) = 1 \) while \( t \in [0, T] \), and put

\[
\zeta = (U_1, U_2, u_1, u_2) = \rho \tilde{\eta}.
\]

Then \( \zeta \) satisfies

\[
\begin{cases}
\zeta'' + \mathcal{Q}\zeta + l(\eta) = 0 & \text{in } \tilde{\Omega} \times (0, \infty), \\
\zeta = \rho \tilde{\zeta} & \text{on } \tilde{\Gamma} \times (0, \infty), \\
\zeta(0) = \zeta'(0) = 0 & \text{in } \tilde{\Omega},
\end{cases} \quad (4.21)
\]

where \( \mathcal{Q}\zeta = (\mathcal{Q}U_1, \mathcal{Q}U_2, \mathcal{Q}u_1, \mathcal{Q}u_2) \).

\[
\mathcal{Q}U_i = \left( 2(1 + \beta) \frac{1}{g_1} D_{x_1}^2 u_{11} + D_{x_2}^2 u_{11} + (1 + 2\beta) \frac{1}{\sqrt{g_1}} D_{x_1} D_{x_2} u_{12} \right) E_1
\]

\[
+ \left( \frac{1}{g_1} D_{x_1}^2 u_{12} + 2(1 + \beta) D_{x_2}^2 u_{12} + (1 + 2\beta) \frac{1}{\sqrt{g_1}} D_{x_1} D_{x_2} u_{11} \right) E_2, \quad i = 1, 2,
\]

and
\[ q u_i = \frac{1}{g_i} D_{x_1}^2 u_i + D_{x_2}^2 u_i, \quad i = 1, 2. \]

Here we used
\[ D_{x_1} x_1 = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_1}, \quad D_{x_2} x_2 = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_2}, \quad U_i = u_{i1} E_1 + u_{i2} E_2. \]

Now, decompose \( \zeta = \theta + \vartheta \), where \( \theta = (\Psi_1, \Psi_2, \psi_1, \psi_2) \) and \( \vartheta = (V_1, V_2, v_1, v_2) \) satisfy (4.22) and (4.23) below, respectively.

\[
\begin{cases}
\theta'' + Q \theta = 0 & \text{in } \tilde{\Omega} \times (0, \infty), \\
\theta = \rho \tilde{\varsigma} & \text{on } \tilde{\Gamma} \times (0, \infty), \\
\theta(0) = \theta'(0) = 0 & \text{in } \tilde{\Omega},
\end{cases}
\] (4.22)

and

\[
\begin{cases}
\vartheta'' + Q \vartheta = \mathcal{F} := -\text{lot}(\eta) & \text{in } \tilde{\Omega} \times (0, \infty), \\
\vartheta = 0 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
\vartheta(0) = \vartheta'(0) = 0 & \text{in } \tilde{\Omega},
\end{cases}
\] (4.23)

From Step 1, \( \zeta = \rho \tilde{\eta} \) and \( \mathcal{F} = -\text{lot}(\eta) \), it is easy to obtain that

\[ (\zeta, \zeta') \in C([0, T]; [\left( L^2(\Omega, \Lambda) \right)^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]). \] (4.24)

and

\[ \mathcal{F} = -\text{lot}(\eta) \in C([0, T]; (H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2). \] (4.25)

By Remark 4.3, the solution \( \vartheta \) to (4.23) satisfies

\[ (\vartheta, \vartheta') \in C([0, T]; [\left( L^2(\Omega, \Lambda) \right)^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]). \] (4.26)

Since \( \theta = \zeta - \vartheta \), using (4.24) and (4.26), we obtain that

\[ (\theta, \theta') \in C([0, T]; [\left( L^2(\Omega, \Lambda) \right)^2 \times (L^2(\Omega))^2] \times [(H^{-1}(\Omega, \Lambda))^2 \times (H^{-1}(\Omega))^2]). \] (4.27)

Step 3. In this step, we prove

\[ b_i(\mathcal{P}^{-1} \eta') \in L^2(\Sigma), \quad i = 1, 2. \] (4.28)

From \( \zeta = \theta + \vartheta \) and (4.21)-(4.23), it is easy to check that \( u_1 = \psi_1 + v_1 \) and \( u_1, \psi_1, \nu_1 \) satisfy the following equations:
\[
\begin{align*}
\begin{cases}
    u_1'' + q u_1 + \lambda t(\eta) &= 0 & \text{in } \tilde{Q} \times (0, \infty), \\
u_1 &= \kappa_1 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
u_1(0) &= u_1'(0) = 0 & \text{in } \tilde{Q},
\end{cases}
\end{align*}
\] (4.29)

and

\[
\begin{align*}
\begin{cases}
    v_1'' + q v_1 &= f := -\lambda t(\eta) & \text{in } \tilde{Q} \times (0, \infty), \\
v_1 &= 0 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
v_1(0) &= v_1'(0) = 0 & \text{in } \tilde{Q},
\end{cases}
\end{align*}
\] (4.31)

where \( \kappa_1 = \rho \kappa_1. \)

To prove (4.28), we first show that for the nonhomogeneous problem (4.31), the map

\[
\zeta \mapsto \frac{\partial}{\partial n}(\varphi^{-1} v_1')
\]

is continuous from \((L^2(\mathcal{S}, \Lambda))^2 \times (L^2(\mathcal{S}))^2\) to \(L^2(\tilde{\mathcal{S}}).\) (4.32)

Indeed, since the map \( \zeta \mapsto f := -\lambda t(\eta) \) is continuous from \((L^2(\mathcal{S}, \Lambda))^2 \times (L^2(\mathcal{S}))^2\) to \(L^2(0, T; H^{-1}(\tilde{\mathcal{Q}}))\), it suffices to show that

\[
f \mapsto \frac{\partial}{\partial n}\varphi^{-1}
\]

is continuous from \(L^2(0, T; H^{-1}(\tilde{\mathcal{Q}}))\) to \(L^2(\tilde{\mathcal{S}}).\) (4.33)

Set \( \varphi_0 v = \varphi v, \forall v \in D(\varphi_0) = H^2(\tilde{\mathcal{Q}}) \cap H^1_0(\tilde{\mathcal{Q}}). \)

Applying \( \varphi_0^{-1} \) to (4.31) gives

\[
\begin{align*}
\begin{cases}
    \Phi'' - q \Phi &= \varphi_0^{-1} f & \text{in } \tilde{\mathcal{Q}} \times (0, \infty), \\
\Phi &= 0 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
\Phi(0) &= \Phi'(0) = 0 & \text{in } \tilde{\mathcal{Q}},
\end{cases}
\end{align*}
\] (4.34)

where \( \Phi = \varphi_0^{-1} v_1 \) satisfies, by the definition of \(D(\varphi_0),\) that

\[
\Phi \in H^2(\tilde{\mathcal{Q}}) \cap H^1_0(\tilde{\mathcal{Q}}), \quad \varphi_0^{-1} f \in L^2(0, T; H^1_0(\tilde{\mathcal{Q}})), \quad \varphi_0^{-1} \Phi' \in C([0, T]; H^1_0(\tilde{\mathcal{Q}})).
\]

Apply Theorem 3.11 of [18] to Eq. (4.34) to obtain

\[
\frac{\partial \Phi}{\partial n} \in H^1(\tilde{\mathcal{S}}) = L^2(0, T; H^1(\tilde{\Gamma})) \cap H^1(0, T; L^2(\tilde{\Gamma})),
\]

which implies

\[
\frac{\partial \Phi'}{\partial n} \in L^2(0, T; L^2(\tilde{\Gamma})) = L^2(\tilde{\mathcal{S}}).
\]

Since \( \Phi' = \varphi_0^{-1} v' \in C([0, T]; H^1_0(\tilde{\mathcal{Q}})), \) it follows that \( \lambda t(\Phi') \in C([0, T]; L^2(\tilde{\mathcal{Q}})) \) and so \( \varphi_0^{-1} \lambda t(\Phi') \in C([0, T]; H^2(\tilde{\mathcal{Q}})) \) that implies
Thus
\[
\partial_n (q_0^{-1} \text{lot}(\Phi')) \in C([0, T]; H^{1/2}(\tilde{\Gamma})) \subset L^2(\tilde{\Sigma}).
\] (4.35)

On the other hand, using the result of Step 4 in the proof of Theorem 1.1 of [20], we obtain that
\[
\kappa_1 \to \frac{\partial}{\partial n} \tilde{\psi}_1^i is continuous from \ L^2(\Sigma) to \ L^2(\tilde{\Sigma}).
\] (4.36)

From Step 2, it is easy to check that
\[
\kappa_1 \to \kappa_1 is continuous from \ L^2(\Sigma) to \ L^2(\tilde{\Sigma}).
\] So
\[
\kappa_1 \to \frac{\partial}{\partial n} \tilde{\psi}_1^i is continuous from \ L^2(\Sigma) to \ L^2(\tilde{\Sigma}).
\] (4.37)

Since \( u_1 = \tilde{w}_1 \rho \) and \( \rho \) is independent of space variable, it follows from (4.35) and (4.37) that
\[
\frac{\partial}{\partial n} (\tilde{\psi}_1^i) \in L^2(\tilde{\Sigma}).
\] (4.38)

Thus, by the definitions of \( b_1 \) and \( D(\mathcal{P}) \), we obtain that
\[
b_1 (\mathcal{P}^{-1} \eta') \in L^2(\Sigma).
\] (4.39)

Similarly,
\[
b_2 (\mathcal{P}^{-1} \eta') \in L^2(\Sigma).
\] (4.40)

**Step 4.** This step is aiming at proving
\[
B_i (\mathcal{P}^{-1} \eta') \in L^2(\Sigma, \Lambda), \quad i = 1, 2.
\] (4.41)

From (4.21)–(4.23), it is easy to check that \( U_1 = \Psi_1 + V_1 \) and \( U_1, \Psi_1, V_1 \) satisfy the following equations:

\[
\begin{cases}
U''_1 - \mathcal{Q} U_1 + \text{lot}(\eta) = 0 & \text{in } \tilde{\Omega} \times (0, \infty), \\
U_1 = \kappa_1 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
U_1(0) = U'_1(0) = 0 & \text{in } \tilde{\Omega}, \\
\Psi''_1 - \mathcal{Q} \Psi_1 = 0 & \text{in } \tilde{\Omega} \times (0, \infty), \\
\Psi_1 = \kappa_1 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
\Psi_1(0) = \Psi'_1(0) = 0 & \text{in } \tilde{\Omega},
\end{cases}
\] (4.42)

and
\[
\begin{aligned}
V_1'' - Q V_1 &= F := \log(\eta) & \text{in } \tilde{\Omega} \times (0, \infty), \\
V_1 &= 0 & \text{on } \tilde{\Gamma} \times (0, \infty), \\
V_1(0) &= V'_1(0) = 0 & \text{in } \tilde{\Omega},
\end{aligned}
\]  \quad (4.44)

where \( \kappa_1 = \rho \kappa_1 \). Furthermore, we have the following regularity:

\[
(U_1, U'_1), (\Psi_1, \Psi'_1), (V_1, V'_1) \in C([0, T]; L^2(\Omega, \Lambda) \times H^{-1}(\Omega, \Lambda)).
\]  \quad (4.45)

Set \( Q_0 U = Q U, \forall U \in D(Q_0) = H^2(\tilde{\Omega}, \Lambda) \cap H^0_0(\tilde{\Omega}, \Lambda) \), and

\[
\bar{B}_1(U_1) = \left( \frac{\partial u_{11}}{\partial x_2} - \frac{1}{\sqrt{g_1}} \frac{\partial u_{12}}{\partial x_1}, \frac{\beta}{(1 + \beta)\sqrt{g_1}} \frac{\partial u_{11}}{\partial x_1} + \frac{\partial u_{12}}{\partial x_2} \right).
\]  \quad (4.46)

Here we used

\[
(u_{11}, u_{12}) \triangleq U_1 = u_{11}E_1 + u_{12}E_2.
\]

From Steps 3, 4 in the proof of Theorem 1.1 of [16], we have

\[
\bar{B}_1(Q_0^{-1} U'_1) \in L^2(\Sigma).
\]  \quad (4.47)

From the definitions of \( P, Q_0 \) and (4.47), it follows that

\[
\bar{B}_1(P^{-1} U'_1) = \bar{B}_1(P^{-1} Q_0^{-1} U'_1) = \bar{B}_1(Q_0^{-1} U'_1) - \bar{B}_1(P^{-1} (P - Q_0) Q_0^{-1} U'_1) \in L^2(\Sigma).
\]  \quad (4.48)

Collecting (4.16), (4.46) and (4.48) and the definition of \( \mathcal{P} \), we obtain that

\[
B_1(\mathcal{P}^{-1} \zeta') \in L^2(\Sigma).
\]  \quad (4.49)

Thus we have

\[
B_1(\mathcal{P}^{-1} \eta') \in L^2(\Sigma).
\]  \quad (4.50)

Similarly,

\[
B_2(\mathcal{P}^{-1} \eta') \in L^2(\Sigma).
\]  \quad (4.51)

5. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need some preliminary results. It follows from the Appendix of [10] that the transfer function of the system (2.6) is

\[
H(\lambda) = \lambda \mathcal{B}^* (\lambda^2 + \hat{\mathcal{A}})^{-1} \mathcal{B},
\]  \quad (5.1)

where \( \hat{\mathcal{A}}, \mathcal{B} \) and \( \mathcal{B}^* \) are given by (2.2), (2.3) and (2.5), respectively. Moreover, from the well-posedness claimed by Theorem 1.1, it follows that there are constants \( M, \sigma \) such that [26]

\[
\sup_{\text{Re} \lambda \geq \sigma} \| H(\lambda) \|_{L(\mathcal{H})} = M < \infty.
\]  \quad (5.2)
We then have the following Proposition 5.1.

**Proposition 5.1.** Theorem 1.2 is valid if for any \( \varsigma = (K_1, K_2, \kappa_1, \kappa_2) \in (C_0^\infty (\Gamma, \Lambda))^2 \times (C_0^\infty (\mathcal{G}))^2 \), the solution \( \eta \) to the following equation:

\[
\begin{cases}
\lambda^2 \eta + A \eta = 0 & \text{in } \Omega, \\
\eta = \varsigma & \text{on } \Gamma,
\end{cases}
\]  

(5.3)

satisfies

\[
\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} \int_{\Gamma} \left| \frac{1}{\lambda} B(\eta) - (N_1(\varsigma), N_2(\varsigma), \kappa_1, \kappa_2) \right|^2 d\Gamma = 0,
\]

(5.4)

where

\[
B(\eta) = (B_1(\eta), B_2(\eta), b_1(\eta), b_2(\eta)), \quad N_j(\varsigma) = (K_j, \tau) \tau + \sqrt{2(1 + \beta)} (K_j, n)n, \quad j = 1, 2.
\]

**Proof.** It was shown in [26] that in the frequency domain, (5.4) is equivalent to

\[
\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} H(\lambda) \varsigma = (N_1(\varsigma), N_2(\varsigma), \kappa_1, \kappa_2), \quad \forall \varsigma \in \mathcal{U}
\]

(5.5)

in the strong topology of \( \mathcal{U} \), where \( H(\lambda) \) is given by (5.1). Thanks to (5.2) and the density argument, it suffices to show that (5.5) is satisfied for all \( \varsigma \in (C_0^\infty (\Gamma, \Lambda))^2 \times (C_0^\infty (\mathcal{G}))^2 \).

Now assume that \( \varsigma \in (C_0^\infty (\Gamma, \Lambda))^2 \times (C_0^\infty (\mathcal{G}))^2 \), and put

\[
\eta(x) = -((\lambda^2 + \tilde{\lambda})^{-1} B) \varsigma)(x).
\]

Then \( \eta \) satisfies (5.3) and

\[
(H(\lambda) \varsigma)(x) = -\lambda B(A^{-1} \eta), \quad \forall x \in \Gamma.
\]

(5.6)

Take \( \xi \in (H^2(\Omega, \Lambda))^2 \times (H^2(\mathcal{G}))^2 \) to satisfy

\[
\begin{cases}
A \xi = 0 & \text{in } \Omega, \\
\xi = \varsigma & \text{on } \Gamma.
\end{cases}
\]

(5.7)

Then (5.3) can be written as

\[
\begin{cases}
\lambda^2 \eta + A(\eta - \xi) = 0 & \text{in } \Omega, \\
\eta - \xi = 0 & \text{on } \Gamma
\end{cases}
\]

or

\[
-\lambda^2 (A^{-1} \eta) = \eta - \xi.
\]

So (5.6) becomes
\[
(H(\lambda)\xi)(\eta) = \frac{1}{\lambda} B(\eta) - \frac{1}{\lambda} B(\xi). \tag{5.9}
\]

Since \(B(\xi)\) is independent of \(\lambda\), the required result then follows from (5.5) and (5.9). \(\Box\)

We have the following Proposition 5.2 by Fourier transform which was used in [13].

**Proposition 5.2.** Let \(a > 0, a + 2b > 0\) and \(F \in L^2(\Omega, \Lambda)\). Then for any \(K \in C^\infty_0(\Gamma, \Lambda)\), the solution vector field \(W\) to the following equation

\[
\begin{cases}
\lambda^2 W + (a \delta d + (a+b) d\delta) W = F & \text{in } \Omega, \\
W = K & \text{on } \Gamma,
\end{cases}
\tag{5.10}
\]

satisfies, for any \(\lambda \in (1, \infty)\), that

\[
\int_{\Gamma} \left| \frac{1}{\lambda} \left( \sqrt{a} (D_n W, \tau) + \sqrt{a+b} (D_n W, n)n \right) - \kappa \right|^2 d\Gamma \leq \frac{C}{\lambda^2} \left( \lambda \|W\|^2_{H^{1/2}(\Gamma, \Lambda)} + \|W\|^2_{H^{1}(\Omega, \Lambda)} + \|F\|^2_{L^2(\Omega, \Lambda)} \right) \tag{5.11}
\]

for some constant \(C\) that depends on \(a, b\) only.

**Proof.** The proof is divided into four steps.

**Step 1: Flattening and localization.** Let us use the geodesic normal coordinate introduced in Section 4. For any given \(x_0 \in \Gamma\), let \((h, s)\) be geodesic normal coordinate in a neighborhood \(V_0\) of \(x_0\) and \(\Psi\) be the corresponding diffeomorphism such that for some \(r > 0\) and an open neighborhood \(\Omega_{x_0}(\subset V_{x_0})\) of \(x_0\),

(i) \(\Psi^{-1}(\Omega_{x_0}) = B_r = \{(h, s) \in \mathbb{R}^2, \quad \| (h, s) \| \leq r \}, \)

(ii) \(\Psi^{-1}((\Omega_{x_0} \cap \Omega) = B_r^+ = \{(h, s) \in B_r, \quad s > 0 \}, \)

(iii) \(\Psi^{-1}(\Omega_{x_0} \cap \partial \Omega) = \{(h, s) \in B_r, \quad s = 0 = \{|h| < r\} \times \{0\}, \)

(iv) \(\Psi^{-1}(x_0) = (0, 0),\)

where \(\| \cdot \|\) denotes the Euclidean norm. Under this coordinate, we have

\[
E_1 = g_1^{-1/2}(h, s) \partial_h \triangleq \frac{1}{\sqrt{g_1(h, s)}} \frac{\partial}{\partial h}, \quad E_2 = \partial_s \triangleq \frac{\partial}{\partial s}, \quad \text{and} \quad \frac{\partial}{\partial n} = -\partial_s |_{\Gamma}.
\]

We may assume without loss of generality that \(g_1(0, 0) = 1\). Set

\[
(\phi_1, \phi_2) \triangleq W = \phi_1 E_1 + \phi_2 E_2, \quad (\nu_1, \nu_2) \triangleq K = \nu_1 E_1 + \nu_2 E_2.
\]

By (5.10), (4.12) and (4.13), we obtain that \((\phi_1, \phi_2)\) satisfies

\[
\begin{cases}
a \partial_s^2 \phi_1 + P_1(h, s, \partial_h, \partial_s) W + Q_1 W - \lambda^2 \phi_1 = -F_1, \\
(a+b) \partial_s^2 \phi_2 + P_2(h, s, \partial_h, \partial_s) W + Q_2 W - \lambda^2 \phi_2 = -F_2, \\
\phi_j(h, 0) = \nu_j(h), \quad j = 1, 2,
\end{cases}
\tag{5.12}
\]

where
Applying the above Fourier transform to system (5.16) gives

\[
\begin{align*}
P_1(h, s, \partial_h, \partial_s)W &= (a + b)g_1^{-1/2}_{1} \partial_h^2 \phi_1 + bg_1^{-1/2}_{1} \partial_h \partial_s \phi_2, \\
P_2(h, s, \partial_h, \partial_s)W &= ag_1^{-1/2}_{1} \partial_h^2 \phi_2 + bg_1^{-1/2}_{1} \partial_h \partial_s \phi_1.
\end{align*}
\]

(5.13)

and \( Q_k, k = 1, 2 \), are linear differential operators of order 1 with continuous coefficients in \( B_r \).

Let \( \gamma > 0 \) be fixed and be small enough. Since \( g_1 \) is positive continuous in \( \Psi^{-1} \cap B_r \), one can find a scalar \( r_0 \in (0, r) \) such that for all \((h, s) \in B_{r_0}\),

\[
\left| g_1^{-1/2}_{1}(h, s) - g_1^{-1/2}_{1}(0, 0) \right| \leq \gamma.
\]

(5.14)

Introduce a cutoff function \( \phi_0 = \phi_0(h, s) \in C_0^\infty(\mathbb{R}^2) \) and \( \text{supp}(\phi_0) \subset B_{r_0} \) such that \( 0 \leq \phi_0 \leq 1 \) and \( \phi_0 = 1 \) in \( B_{r_0/2} \). Set

\[
\chi(h, s) = \phi_0(h, s)W(h, s), \quad f(h) = \phi_0(h, 0)\mathcal{K}(h), \quad \forall (h, s) \in \mathbb{R} \times \mathbb{R}^+.
\]

(5.15)

Then one can check that \( \chi \in H^2(\mathbb{R} \times \mathbb{R}^+) \) and \( \chi(h, s) = 0 \) in \( \mathbb{R} \times \{ s \geq r_0 \} \). By (5.12), \( \chi \) satisfies

\[
\begin{align*}
\begin{cases}
\partial_s^2 \chi_1 + P_1(0, 0, \partial_h, \partial_s)\chi - \lambda^2 \chi_1 &= G_1\chi + L_1(W, \mathcal{F}), \\
(a + b)\partial_s^2 \chi_2 + P_2(0, 0, \partial_h, \partial_s)\chi - \lambda^2 \chi_2 &= G_2\chi + L_2(W, \mathcal{F}), \\
\chi(h, 0) &= f(h),
\end{cases}
\end{align*}
\]

(5.16)

where

\[
\begin{align*}
G_k\chi &= P_k(0, 0, \partial_h, \partial_s)\chi - P_k(h, s, \partial_h, \partial_s)\chi, \\
L_k(W, \mathcal{F}) &= -\phi_0(h, s)(\mathcal{F}_k - Q_kW) + [\partial_s^2, \phi_0]\phi_k + [P_k(h, s, \partial_h, \partial_s), \phi_0]W, \\
k &= 1, 2,
\end{align*}
\]

(5.17)

with

\[
\begin{align*}
[\partial_s^2, \phi_0]\phi_k &= \partial_s^2(\phi_0\phi_k) - \phi_0\partial_s^2\phi_k, \\
[P_k(h, s, \partial_h, \partial_s), \phi_0]W &= P_k(h, s, \partial_h, \partial_s)(\phi_0W) - \phi_0P_k(h, s, \partial_h, \partial_s)W.
\end{align*}
\]

Clearly, \( G_k \) and \( L_k \) are two linear differential operators of order 2 and order 1, respectively, and

\[
\begin{align*}
P_1(0, 0, \partial_h, \partial_s)\chi &= (a + b)\partial_h^2 \chi_1 + b\partial_h \partial_s \chi_2, \\
P_2(0, 0, \partial_h, \partial_s)\chi &= a\partial_h^2 \chi_2 + b\partial_h \partial_s \chi_1.
\end{align*}
\]

(5.18)

**Step 2: Partial Fourier transform.** Fix \( s \) for any \( \chi(\cdot, s) \in (L^2(\mathbb{R}))^2 \). From now on, we denote by \( \hat{\chi} \) the partial Fourier transform with respect to \( h \), for instance

\[
\hat{\chi}(\xi, s) = \int_\mathbb{R} e^{-i(h, \xi)} \chi(h, s) dh.
\]

Applying the above Fourier transform to system (5.16) gives

\[
\begin{align*}
\begin{cases}
\partial_s^2 \hat{\chi}_1 - p_1(i\xi, \partial_s)\hat{\chi}_1 - \lambda^2 \hat{\chi}_1 &= \hat{G}_1\chi + \hat{L}_1(W, \mathcal{F}), \\
(a + b)\partial_s^2 \hat{\chi}_2 - p_2(i\xi, \partial_s)\hat{\chi}_2 - \lambda^2 \hat{\chi}_2 &= \hat{G}_2\chi + \hat{L}_2(W, \mathcal{F}), \\
\hat{\chi}(\xi, 0) &= \hat{f}(\xi).
\end{cases}
\end{align*}
\]

(5.19)
where \( p_k(iξ, \partial_s) = P_k(0, 0, iξ, \partial_s), \ k = 1, 2, \) that is,
\[
\begin{align*}
\begin{cases}
 p_1(iξ, \partial_s)\hat{X} = (a + b)ξ^2\hat{X}_1 + ibξ\partial_s\hat{X}_2, \\
p_2(iξ, \partial_s)\hat{X} = aξ^2\hat{X}_2 + ibξ\partial_s\hat{X}_1.
\end{cases}
\end{align*}
\]
(5.20)

Since
\[
\hat{X}(ξ, s) = 0, \quad ∀(ξ, s) ∈ \mathbb{R} \times [r_0, ∞),
\]
(5.21)
in order to analyze the solution of (5.19) satisfying (5.21), we decompose \( \hat{X}(ξ, s) \) as follows:
\[
\hat{X}(ξ, s) = ψ(ξ, s) + v(ξ, s), \quad ∀(ξ, s) ∈ \mathbb{R} \times \mathbb{R}^+.
\]
(5.22)

where \( ψ \) satisfies
\[
\begin{align*}
\begin{cases}
 aξ^2\psi_1 - p_1(iξ, \partial_s)ψ - λ^2ψ_1 = 0, & (ξ, s) ∈ \mathbb{R} \times \mathbb{R}^+, \\
(a + b)ξ^2\psi_2 - p_2(iξ, \partial_s)ψ - λ^2ψ_2 = 0, & (ξ, s) ∈ \mathbb{R} \times \mathbb{R}^+, \\
ψ(ξ, 0) = f(ξ), & ξ ∈ \mathbb{R}, \\
\lim_{s → ∞} ψ(ξ, s) = 0, & ξ ∈ \mathbb{R},
\end{cases}
\end{align*}
\]
(5.23)

and \( v \) satisfies
\[
\begin{align*}
\begin{cases}
 aξ^2v_1 - p_1(iξ, \partial_s)v - λ^2v_1 = \hat{G}_1χ + L_1(W, \mathcal{F}), & (ξ, s) ∈ \mathbb{R} \times \mathbb{R}^+, \\
(a + b)ξ^2v_2 - p_2(iξ, \partial_s)v - λ^2v_2 = \hat{G}_2χ + L_3(W, \mathcal{F}), & (ξ, s) ∈ \mathbb{R} \times \mathbb{R}^+, \\
v(ξ, 0) = 0, & ξ ∈ \mathbb{R}, \\
v(ξ, s) = Φ(ξ, s), & (ξ, s) ∈ \mathbb{R} \times [r_0, ∞)
\end{cases}
\end{align*}
\]
(5.24)

with \( Φ(ξ, s) \) a vector function determined by the system (5.23), which will be given below in detail. The validity of the last equality comes from (5.22) and the explicit expression of the solution of (5.23).

In the following two steps, we give the estimates of \( ψ \) and \( v \). In the sequel, \( C \) will denote some positive constant independent of \( λ \) although it may have different values in different contexts, and
\[
|v|^2 = |v_1|^2 + |v_2|^2
\]
for any vector field:
\[
(v_1, v_2) ≜ v = v_1E_1 + v_2E_2.
\]

**Step 3.** We claim that for all \( λ ∈ (1, ∞) \),
\[
\int_{\mathbb{R}} \left| \frac{1}{λ}(\sqrt{a}\partial_sψ_1, \sqrt{a + b}\partial_sψ_2 + ψ) \right|_{s=0}^2 dξ ≤ \frac{C}{λ^2} \|W\|^2_{H^1(∂Ω, A)}.
\]
(5.25)

Actually, set
\[
Z_1 = ψ_1, \quad Z_2 = \partial_sψ_1, \quad Z_3 = ψ_2, \quad Z_4 = \partial_sψ_2.
\]
Then it follows from the definition of $p_k$, $k = 1, 2$, that (5.23) can be rewritten as the following first order system.

$$
\begin{align*}
&\begin{cases}
z' = Tz,
(z_1, z_3)(0) = (\hat{f}_1(\xi), \hat{f}_2(\xi)) = \hat{f}(\xi),
\end{cases}
\lim_{s \to \infty} (z_1, z_3) = 0,
\end{align*}
$$

(5.26)

where $z = (z_1, z_2, z_3, z_4)$, $z' = \partial_s z$, and

$$
T = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{\lambda^2 + (a + b)\xi^2}{a} & 0 & 0 & \frac{bi\xi}{a} \\
0 & 0 & 0 & 1 \\
\frac{bi\xi}{a+b} & \frac{\lambda^2 + a\xi^2}{a+b} & 0
\end{pmatrix}.
$$

A direct calculation shows that the eigenvalues $\{\omega_i\}_{i=1}^4$ of $T$ are given by

$$
\omega_1 = \sqrt{(a + b)(\lambda^2 + (a + b)\xi^2)} \frac{1}{a + b},
\omega_2 = -\sqrt{(a + b)(\lambda^2 + (a + b)\xi^2)} \frac{1}{a + b},
\omega_3 = \sqrt{a(\lambda^2 + a\xi^2)} \frac{1}{a},
\omega_4 = -\sqrt{a(\lambda^2 + a\xi^2)} \frac{1}{a}.
$$

The eigenvectors $q_2$ and $q_4$ of $T$ corresponding to the negative eigenvalues $\omega_2$ and $\omega_4$ are

$$
q_2 = \begin{pmatrix}
1, -\omega_1, -i\omega_1 \xi, i\omega_1 \xi
\end{pmatrix}^\top,
q_4 = \begin{pmatrix}
i\omega_3 \xi, -i\omega_3 \xi, 1, -\omega_3
\end{pmatrix}^\top.
$$

Let

$$
M(s) = \begin{pmatrix}
e^{-\omega_1 s} & \frac{i\omega_1}{\xi} e^{-\omega_3 s} \\
-\frac{i\omega_1}{\xi} e^{-\omega_1 s} & e^{-\omega_3 s}
\end{pmatrix}.
$$

(5.27)

After a direct computation, we obtain

$$
\beta_0 = \det(M(0)) = \frac{\xi^2 - \omega_1 \omega_3}{\xi^2}
$$

and

$$
M_0^{-1} := (M(0))^{-1} = \frac{1}{\beta_0} \begin{pmatrix}
1, -\frac{i\omega_1}{\xi} \\
\frac{i\omega_1}{\xi}, 1
\end{pmatrix}.
$$

From the theory of ordinary differential equations, we know that the solution of (5.23) is

$$
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = M(s)M_0^{-1} \begin{pmatrix}
\hat{f}_1(\xi) \\
\hat{f}_2(\xi)
\end{pmatrix}
= \frac{1}{\xi^2 - \omega_1 \omega_3} \begin{pmatrix}
\xi^2 e^{-\omega_1 s} - \omega_1 \omega_3 e^{-\omega_3 s} & -i\xi \omega_3 (e^{-\omega_1 s} - e^{-\omega_3 s}) & \hat{f}_1(\xi) \\
-i\xi \omega_1 (e^{-\omega_1 s} - e^{-\omega_3 s}) & -\omega_1 \omega_3 e^{-\omega_1 s} + \xi^2 e^{-\omega_3 s} & \hat{f}_2(\xi)
\end{pmatrix}.
$$

(5.28)
Moreover, we also determine $\Phi(\xi, s)$ in (5.24) which satisfies

$$\Phi(\xi, s) = -M(s)M_0^{-1}\tilde{f}^\top(\xi), \quad s \geq r_0.$$ 

From (5.27) and (5.28), it follows that

$$\left(\frac{1}{\lambda}(\sqrt{a}\partial_3\psi_1, \sqrt{a + b}\partial_3\psi_2) + \psi\right)^\top_{s=0} = \frac{1}{\lambda}\left(-\sqrt{a}\omega_1 - \frac{i\sqrt{a}\omega_2}{\xi}, -\sqrt{a + b}\omega_3\right) \right)M_0^{-1}\tilde{f}^\top(\xi) + \tilde{f}^\top(\xi) = (N_1 + N_2)\tilde{f}^\top(\xi), \quad (5.29)$$

where

$$N_1 = \frac{\omega_1 - \omega_3}{\lambda\beta} \left(\sqrt{a} - \frac{a+b}{\xi}\omega_3\right), \quad N_2 = \left(\frac{\sqrt{a}\omega_1}{\lambda} - 1, 0, \frac{a+b}{\lambda\omega_1} - 1\right).$$

It is easy to check that

$$\left\{\begin{array}{l}
\frac{1}{|\beta|} = \frac{\xi^2}{\omega_1\omega_3 - \xi^2} = \frac{\xi^2(\xi^2 + \omega_1\omega_3)}{\omega_1^2\omega_3^2 - \xi^4} \leq \frac{(2a + b)\xi^2}{\lambda^2}, \\
|\omega_1 - \omega_3| = \frac{a|\xi|^2}{(a + b)(\lambda^2 + a|\xi|^2) + a(\lambda^2 + (a + b)\xi^2)}, \\
\left|\frac{\sqrt{a}\omega_3}{\lambda} - 1\right| = \frac{1}{\lambda}\frac{a|\xi|^2}{\sqrt{a}|\xi|^2 + \lambda^2} \leq \frac{\sqrt{a}|\xi|}{\lambda},
\end{array}\right. \quad (5.30)$$

and

$$\left|\frac{\sqrt{a + b}\omega_1}{\lambda} - 1\right| \leq \frac{\sqrt{a + b}|\xi|}{\lambda}. \quad (5.31)$$

To obtain (5.25), we need only to estimate all entries of matrices $N_1$ and $N_2$. But from (5.29)–(5.31), they are bounded by $C|\xi|/\lambda$. Thus

$$\int_{\mathbb{R}} \left|\left(N_1 + N_2\right)\tilde{f}^\top(\xi)\right|^2 d\xi \leq \frac{C}{\lambda^2} \int_{\mathbb{R}} |\xi|^2 |\tilde{f}(\xi)|^2 d\xi, \quad (5.32)$$

which implies (5.25).

**Step 4: Estimating $\frac{1}{\lambda}(\sqrt{a}\partial_3 v_1(\cdot, 0), \sqrt{a + b}\partial_3 v_2(\cdot, 0) + v(\cdot, 0))$.**

We will estimate $(\sqrt{a}\partial_3 v_1(\cdot, 0), \sqrt{a + b}\partial_3 v_2(\cdot, 0) + v(\cdot, 0))$ by means of a classical trace theorem. This requires the computation of $\partial^2_3 v$ and $\partial_3 v$. To do it, we first estimate $L_k(W, \mathcal{F})$ and $G_k \chi$, $k = 1, 2$.

**(a) Estimating $L_k(W, \mathcal{F})$ and $G_k \chi$.**

Clearly, we have

$$\|L_k(W, \mathcal{F})\|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C\left(\|W\|_{H^1(\Omega, A)} + \|\mathcal{F}\|_{L^2(\Omega, A)}\right). \quad (5.33)$$
By (5.17) and the Plancherel formula, it follows that

\[
\| \hat{G}_k \chi \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} = (2\pi)^{\frac{1}{2}} \| G_k \chi \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \\
\leq \gamma C \left( \sum_{k=1}^{2} \| \hat{\partial}_h \chi_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} + \sum_{k=1}^{2} \| \partial_h \partial_s \chi_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \right) \\
\leq \gamma C \left( \| |\xi|^2 \hat{\chi} \|_{L^2(\mathbb{R} \times \mathbb{R}^+))^2} + \| |\xi| \partial_s \hat{\chi} \|_{L^2(\mathbb{R} \times \mathbb{R}^+))^2} \right) .
\] (5.34)

From (5.22) and (5.34), we find

\[
\| \hat{G}_k \chi \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq \gamma C \left( \| |\xi|^2 \psi \|_{L^2(\mathbb{R} \times \mathbb{R}^+))^2} + \| |\xi|^2 v \|_{L^2(\mathbb{R} \times \mathbb{R}^+))^2} \\
+ \| |\xi| \partial_s \psi \|_{L^2(\mathbb{R} \times \mathbb{R}^+))^2} + \| |\xi| \partial_s v \|_{L^2(\mathbb{R} \times \mathbb{R}^+))^2} \right) .
\] (5.35)

On the other hand, multiplying the \( k \)-th equation of the system (5.24) by \(-\xi^2 \bar{v}_k\), \( k = 1, 2 \), respectively, then integrating by parts over \( \mathbb{R} \times \mathbb{R}^+ \), taking the expression of \( \Phi(\xi, s) \) and the last equality of (5.24) into account and adding the results, we obtain

\[
\int_{\mathbb{R} \times \mathbb{R}^+} (I_1 + I_2 + I_3 + I_4) \, d\xi \, ds = - \sum_{k=1}^{2} \int_{\mathbb{R} \times \mathbb{R}^+} (\hat{G}_k \chi + L_k(\hat{W}, \mathcal{F})) \xi^2 \bar{v}_k \, d\xi \, ds ,
\] (5.36)

where
\[
I_1 = a \xi^2 |\partial_s v_1|^2 + (a + b) \xi^2 |\partial_s v_2|^2 , \quad I_2 = (a + b) \xi^4 |v_1|^2 + a \xi^4 |v_2|^2 , \quad I_3 = 2 \text{Re}\left( ib \xi^3 \bar{v}_1 \partial_s v_2 \right) , \quad I_4 = \lambda \xi^2 |v_1|^2 + \lambda \xi^2 |v_2|^2 .
\]

By the Cauchy inequality, we have

\[
|I_3| \leq |b| \xi^4 |v_1|^2 + |b| \xi^2 |\partial_s v_2|^2
\] (5.37)

and

\[
\left| \int_{\mathbb{R} \times \mathbb{R}^+} (\hat{G}_k \chi + L_k(\hat{W}, \mathcal{F})) \xi^2 \bar{v}_k \, d\xi \, ds \right| \\
\leq \| \hat{G}_k \chi + L_k(\hat{W}, \mathcal{F}) \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} \| \xi^2 \bar{v}_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \\
\leq C \left( \| \hat{G}_k \chi \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} + \| L_k(\hat{W}, \mathcal{F}) \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} \right) \| \xi^2 \bar{v}_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} .
\] (5.38)

From (5.36)–(5.38), it follows that

\[
\theta_0 \| \xi^2 v \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} \leq C \left( \| \hat{G}_k \chi \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} + \| L(\hat{W}, \mathcal{F}) \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} \right) \| \xi^2 v \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} ,
\] (5.39)

and

\[
\theta_0 \| \xi \partial_s v \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} \leq C \left( \| \hat{G}_k \chi \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} + \| L(\hat{W}, \mathcal{F}) \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} \right) \| \xi^2 v \|_{L^2(\mathbb{R} \times \mathbb{R}^+))} ,
\] (5.40)

where \( \theta_0 = \min[a, a + 2b] \), \( G = (G_1, G_2) \) and \( L = (L_1, L_2) \).
From (5.39) and (5.40), we have
\[
\|\xi^2 v\|_{L^2(\mathbb{R}^n)} \leq C(\|\hat{G}X\|_{L^2(\mathbb{R}^n)} + \|L(\hat{W}, \mathcal{F})\|_{L^2(\mathbb{R}^n)}) \tag{5.41}
\]
and
\[
\|\xi \partial_s v\|_{L^2(\mathbb{R}^n)} \leq C(\|\hat{G}X\|_{L^2(\mathbb{R}^n)} + \|L(\hat{W}, \mathcal{F})\|_{L^2(\mathbb{R}^n)}). \tag{5.42}
\]
Substituting (5.41) and (5.42) into (5.35) gives
\[
(1 - \gamma C)\|\hat{G}X\|_{L^2(\mathbb{R}^n)}^2 \leq \gamma C(\|\xi^2 \psi\|_{L^2(\mathbb{R}^n)} + \|\xi \partial_s \psi\|_{L^2(\mathbb{R}^n)} + \|L(\hat{W}, \mathcal{F})\|_{L^2(\mathbb{R}^n)}). \tag{5.43}
\]
On the other hand, it follows from (5.28) that for \(k = 1, 2\),
\[
\psi_k(\xi, s) = \sum_{j=1}^2 (a_{kj}(\xi)e^{-\omega_1 s} + b_{jk}(\xi)e^{-\omega_3 s})\hat{f}_j(\xi), \tag{5.44}
\]
and
\[
\partial_s \psi_k(\xi, s) = -\sum_{j=1}^2 (a_{kj}(\xi)\omega_1 e^{-\omega_1 s} + b_{jk}(\xi)\omega_3 e^{-\omega_3 s})\hat{f}_j(\xi). \tag{5.45}
\]
In what follows, we estimate \(\|\xi^2 \psi\|_{L^2(\mathbb{R}^n)}^2\) and \(\|\xi \partial_s \psi\|_{L^2(\mathbb{R}^n)}^2\). For \(k = 1, 2\),
\[
\|\xi^2 a_{kj}(\xi)e^{-\omega_1 s}\hat{f}_j(\xi)\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \left|\xi^2 a_{kj}(\xi)\hat{f}_j(\xi)\right|^2 \left(\int_0^\infty e^{-2\omega_1 s} ds\right) d\xi\right)^{1/2}
\]
\[
= \left\|\frac{1}{\sqrt{2\omega_1}}\xi^2 a_{kj}(\xi)\hat{f}_j(\xi)\right\|_{L^2(\mathbb{R})}
\]
\[
\leq C\left(1 + |\xi|\right)^{1/2}\hat{f}_j(\xi)\|_{L^2(\mathbb{R})} \leq C\|W\|_{H^{3/2}(\Gamma, \Lambda)}, \tag{5.46}
\]
and
\[
\|\xi a_{kj}(\xi)\omega_1 e^{-\omega_1 s}\hat{f}_j(\xi)\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \left|\omega_1 a_{kj}(\xi)\hat{f}_j(\xi)\right|^2 \left(\int_0^\infty e^{-2\omega_1 s} ds\right) d\xi\right)^{1/2}
\]
\[
= \left\|\frac{\sqrt{\omega_1}}{\sqrt{2}}\xi a_{kj}(\xi)\hat{f}_j(\xi)\right\|_{L^2(\mathbb{R})}
\]
\[
\leq C\left((\sqrt{\lambda} + |\xi|)\xi a_{kj}(\xi)\hat{f}_j(\xi)\|_{L^2(\mathbb{R})}
\]
\[
\leq C\sqrt{\lambda}\|W\|_{H^{3}(\Gamma, \Lambda)} + \|W\|_{H^{3/2}(\Gamma, \Lambda)}^2
\]
\[
\leq C\sqrt{\lambda}\|W\|_{H^{3/2}(\Gamma, \Lambda)}. \tag{5.47}
\]
Similarly,
\[ \| \xi^2 b_{jk}(\xi) e^{-\alpha_2^2 \hat{f}_j(\xi)} \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \| W \|_{H^{7/2}(\Gamma, A)}, \] (5.48)
and
\[ \| \xi b_{jk}(\xi) \alpha_2 e^{-\alpha_2^2 \hat{f}_j(\xi)} \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \sqrt{\lambda} \| W \|_{H^{7/2}(\Gamma, A)}. \] (5.49)
From (5.44)–(5.49), it follows that
\[ \| \xi^2 \psi \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \| W \|_{H^{7/2}(\Gamma, A)}, \] (5.50)
and
\[ \| \xi \partial_s \psi \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \sqrt{\lambda} \| W \|_{H^{7/2}(\Gamma, A)}. \] (5.51)
Finally, it follows from (5.33), (5.43), (5.50) and (5.51) that
\[ \| \overline{G^i \chi} \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 \leq C \left( \sqrt{\lambda} \| W \|_{H^{7/2}(\Gamma, A)} + \| W \|_{H^1(\Omega, A)} + \| \mathcal{F} \|_{L^2(\Omega, A)} \right). \] (5.52)

**b. Estimating \( \partial_s^2 \mathbf{v} \).** Multiply the \( k \)-th equation of the system (5.24) by \( \overline{\partial_s^2 v_k} \), \( k = 1, 2 \), integrate by parts over \( \mathbb{R} \times \mathbb{R}^+ \), and notice the last equality of the system (5.24), to obtain
\[
\begin{align*}
& a \| \partial_s^2 v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 + \lambda^2 \| \partial_s v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 \\
& \leq \int_{\mathbb{R} \times \mathbb{R}^+} |\partial_s (p_k(i, \xi, \partial_s) v) \partial_s v_k| d\xi ds + \int_{\mathbb{R} \times \mathbb{R}^+} \left| (\overline{G_k \chi} + L_k(W, \mathcal{F})) \overline{\partial_s^2 v_k} \right| d\xi ds \\
& \leq \frac{a}{2} \| \partial_s^2 v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 + C \| \overline{G_k \chi} + L_k(W, \mathcal{F}) \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2}^2 + C \| \xi \partial_s v \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2}^2. \tag{5.53}
\end{align*}
\]
Substituting (5.42) into (5.53) yields
\[ \| \partial_s^2 v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \left( \| \overline{G^i \chi} \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2} + \| L(W, \mathcal{F}) \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2} \right). \] (5.54)
This together with (5.33) and (5.52) gives
\[ \| \partial_s^2 v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \left( \sqrt{\lambda} \| W \|_{H^{7/2}(\Gamma, A)} + \| W \|_{H^1(\Omega, A)} + \| \mathcal{F} \|_{L^2(\Omega, A)} \right). \] (5.55)

**c. Estimating \( \partial_s \mathbf{v} \).** By the last equality in (5.24), multiply the \( k \)-th equation of (5.24) by \( -\overline{v_k} \), \( k = 1, 2 \), and integrate by parts over \( \mathbb{R} \times \mathbb{R}^+ \), to get
\[
\begin{align*}
& a \| \partial_s v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 + \lambda^2 \| v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 \\
& \leq \frac{C}{\lambda^2} \left( \| \xi^2 v \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2}^2 + \| \xi \partial_s v \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2}^2 \\
& + \| \overline{G_k \chi} + L_k(W, \mathcal{F}) \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2}^2 + \frac{\lambda^2}{2} \| v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 \right). \tag{5.56}
\end{align*}
\]
This together with (5.41) and (5.42) gives
\[
\| \partial_s v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq \frac{C}{\lambda^2} \left( \| \hat{G} \hat{\chi} \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2} + \| L(W, \mathcal{F}) \|_{(L^2(\mathbb{R} \times \mathbb{R}^+))^2} \right). \tag{5.57}
\]

It then follows from (5.33), (5.52) and (5.57) that
\[
\| \partial_s v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)} \leq C \left( \sqrt{\lambda} \| W \|_{H^{7/2}(\Gamma, \Lambda)} + \| W \|_{H^1(\Omega, \Lambda)} + \| \mathcal{F} \|_{L^2(\Omega, \Lambda)} \right). \tag{5.58}
\]

(d). Estimating \( \partial_s v_k(\cdot, 0) \). We use the following standard inequality:
\[
\int_{\mathbb{R}} \int_0^\infty \Re(\partial_s v_k(\xi, s) \partial_s^2 v_k(\xi, s)) \, ds \, d\xi \leq \| \partial_s v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2 + \| \partial^2_s v_k \|_{L^2(\mathbb{R} \times \mathbb{R}^+)}^2. \tag{5.59}
\]

This together with (5.55) and (5.58) gives the desired estimate for \( v \):
\[
\int_{\mathbb{R}} \left| \left( \frac{1}{\lambda} (\sqrt{a} \partial_s \chi_1, \sqrt{a + b} \partial_s \chi_2) + v \right) \right|^2 \, d\xi \leq \frac{C}{\lambda^2} \left( \lambda \| W \|_{H^{7/2}(\Gamma, \Lambda)}^2 + \| W \|_{H^1(\Omega, \Lambda)}^2 + \| \mathcal{F} \|_{L^2(\Omega, \Lambda)}^2 \right). \tag{5.60}
\]

Here we used the fact \( v(\xi, 0) = 0 \) given by the system (5.24). Combining (5.22), the estimates (5.25) and (5.60) yield
\[
\int_{\mathbb{R}} \left| \left( \frac{1}{\lambda} (\sqrt{a} \partial_s \chi_1, \sqrt{a + b} \partial_s \chi_2) + \chi \right) \right|^2 \, dx \leq \frac{C}{\lambda^2} \left( \lambda \| W \|_{H^{7/2}(\Gamma, \Lambda)}^2 + \| W \|_{H^1(\Omega, \Lambda)}^2 + \| \mathcal{F} \|_{L^2(\Omega, \Lambda)}^2 \right), \tag{5.61}
\]

and hence by the Parseval formula, it has
\[
\int_{|h| < r_0/2} \left| \left( \frac{1}{\lambda} (\sqrt{a} \partial_s \phi_1, \sqrt{a + b} \partial_s \phi_2) + W \right) \right|^2 \, dh \leq \frac{C}{\lambda^2} \left( \lambda \| W \|_{H^{7/2}(\Gamma, \Lambda)}^2 + \| W \|_{H^1(\Omega, \Lambda)}^2 + \| \mathcal{F} \|_{L^2(\Omega, \Lambda)}^2 \right). \tag{5.63}
\]
On the other hand, under the local coordinate \((h, s)\), we have

\[ n = -\partial_z = -E_2, \quad \tau = E_1. \quad \tag{5.64} \]

and

\[ D_n W = -D_{\partial_z} W = -\partial_z \phi_1 E_1 - \partial_z \phi_2 E_2 - (\phi_1 D_{\partial_z} E_1 + \phi_2 D_{\partial_z} E_2). \quad \tag{5.65} \]

From (5.64) and (5.65), it follows that

\[
\left| \frac{1}{\lambda} \sqrt{a} \langle D_n W, \tau \rangle - \langle W, \tau \rangle \right|^2 \\
= \left| -\frac{1}{\lambda} \sqrt{a} [\partial_z \phi_1 E_1 + \partial_z \phi_2 E_2 + (\phi_1 D_{\partial_z} E_1 + \phi_2 D_{\partial_z} E_2), E_1] - (\phi_1 E_1 + \phi_2 E_2, E_1) \right|^2 \\
= \left| \frac{1}{\lambda} \sqrt{a} \partial_z \phi_1 + \phi_1 + \frac{1}{\lambda} \sqrt{a} \langle \phi_1 D_{\partial_z} E_1 + \phi_2 D_{\partial_z} E_2, E_1 \rangle \right|^2 \\
\leq 2 \left| \frac{1}{\lambda} \sqrt{a} \partial_z \phi_1 + \phi_1 \right|^2 + \frac{C}{\lambda^2} \left| (\phi_1 D_{\partial_z} E_1 + \phi_2 D_{\partial_z} E_2, E_1) \right|^2 \\
\leq 2 \left| \frac{1}{\lambda} \sqrt{a} \partial_z \phi_1 + \phi_1 \right|^2 + \frac{C}{\lambda^2} |W|^2, \quad \tag{5.66} \]

and

\[
\left| \frac{1}{\lambda} \sqrt{a + b} \langle D_n W, n \rangle - \langle W, n \rangle \right|^2 \\
= \left| \frac{1}{\lambda} \sqrt{a + b} \partial_z \phi_1 E_1 + \partial_z \phi_2 E_2 + (\phi_1 D_{\partial_z} E_1 + \phi_2 D_{\partial_z} E_2), E_2 \right|^2 \\
= \left| \frac{1}{\lambda} \sqrt{a + b} \partial_z \phi_2 E_1 + \phi_2 + \frac{1}{\lambda} \sqrt{a + b} \langle \phi_1 D_{\partial_z} E_1 + \phi_2 D_{\partial_z} E_2, E_2 \rangle \right|^2 \\
\leq 2 \left| \frac{1}{\lambda^2} \sqrt{a + b} \partial_z \phi_2 E_1 + \phi_2 \right|^2 + \frac{C}{\lambda^2} |W|^2. \quad \tag{5.67} \]

From (5.63), (5.66), (5.67) and the change of coordinate involving \(\Psi\), we obtain that

\[
\int_{\tilde{Q} \cap \partial \Omega} \left| \frac{1}{\lambda} \sqrt{a} \langle D_n W, \tau \rangle - \langle W, \tau \rangle \right|^2 \, dx \\
\leq \frac{C}{\lambda^2} \left( \lambda \| W \|^2_{H^{2/3}(\Gamma, \Lambda)} + \| W \|^2_{H^1(\Omega, \Lambda)}, \| \mathcal{F} \|^2_{L^2(\Omega, \Lambda)} \right), \quad \tag{5.68} \]

and
\[
\int_{\tilde{\Omega}_0 \cap \partial \Omega} \left| \frac{1}{\lambda} \sqrt{a + b \langle D_n W, n \rangle} - \langle W, n \rangle \right|^2 dx \\
\leq \frac{C}{\lambda^2} \left( \lambda \| W \|_{H^{7/2}(\Gamma, \Lambda)}^2 + \| W \|_{H^1(\Omega, \Lambda)}^2 + \| F \|_{L^2(\Omega, \Lambda)}^2 \right),
\]  
(5.69)

where \( \tilde{\Omega}_0 \subset \Omega_0 \) is an open neighborhood of \( x_0 \in \partial \Omega \). Since \( x_0 \) is arbitrarily chosen, one easily deduces from (5.68) and (5.69) that

\[
\left\| \frac{1}{\lambda} (\sqrt{a} \langle D_n W, \tau \rangle + \sqrt{a + b} \langle D_n W, n \rangle n) - W \right\|_{L^2(\Gamma, \Lambda)}^2 \\
\leq \frac{C}{\lambda^2} \left( \lambda \| W \|_{H^{7/2}(\Gamma, \Lambda)}^2 + \| W \|_{H^1(\Omega, \Lambda)}^2 + \| F \|_{L^2(\Omega, \Lambda)}^2 \right),
\]  
(5.70)

This ends the proof. \( \square \)

**Remark 5.1.** The system (5.26) is actually a boundary-value problem. In general, a boundary-value problem may have many solutions. Here, we get the unique solution explicitly by solving the ordinary differential equation and applying the boundary conditions in system (5.26).

Notice that \( \delta dw = -\Delta w \) for \( w \in \mathcal{C}^\infty(\overline{\Omega}) \). By the same process as the proof of Proposition 5.2, we have the following Proposition 5.3.

**Proposition 5.3.** Let \( f \in L^2(\Omega) \). Then for any function \( \kappa \in \mathcal{C}_0^\infty(\Gamma) \), the solution \( w \) to the following equation

\[
\begin{cases}
\lambda^2 w - \Delta w = f & \text{in } \Omega, \\
w = \kappa & \text{on } \Gamma,
\end{cases}
\]  
(5.71)

satisfies, for any \( \lambda \in (1, \infty) \), that

\[
\int_{\Gamma} \left| \frac{1}{\lambda} \frac{\partial w}{\partial n} - \kappa \right|^2 d\Gamma \\
\leq \frac{C}{\lambda^2} \left( \lambda \| w \|_{H^{7/2}(\Gamma)}^2 + \| w \|_{H^1(\Omega)}^2 + \| f \|_{L^2(\Gamma)}^2 \right).
\]  
(5.72)

**Proof of Theorem 1.2.** Since \( \eta = (W_1, W_2, w_1, w_2) \) is the solution of system (5.3) and \( \Delta \mu = -(\delta d + 2(1 + \beta)d\delta) \), we know that \( W_k \) and \( w_k \), \( k = 1, 2 \), satisfy, respectively

\[
\begin{cases}
\lambda^2 W_k + (\delta d + 2(1 + \beta)d\delta) W_k = -\mathcal{F}_k(\eta) & \text{in } \Omega, \\
W_k = \kappa_k & \text{on } \Gamma,
\end{cases}
\]  
(5.73)

and

\[
\begin{cases}
\lambda^2 w_k - \Delta w_k = -\mathcal{F}_k(\eta) & \text{in } \Omega, \\
w_k = \kappa_k & \text{on } \Gamma.
\end{cases}
\]  
(5.74)

Apply Propositions 5.2 and 5.3 with \( W = W_k \) and \( w = w_k \), \( k = 1, 2 \), respectively, to obtain
In what follows, we estimate term by term for the right side of (5.79).

\[
\left\| \frac{1}{\lambda} (D_n W_k, \tau) \tau + \sqrt{2(1 + \beta)} (D_n W_k, n) n - \kappa_k \right\|_{L^2(\Gamma, \Lambda)}^2 \leq \frac{C}{\lambda^2} \left( \lambda \| W_k \|_{H^{3/2}(\Gamma, \Lambda)}^2 + \| W_k \|_{H^1(\Omega, \Lambda)}^2 + \| F_k(\eta) \|_{L^2(\Omega)}^2 \right)
\]

\[
\leq \frac{C}{\lambda^2} \left( \lambda \| W_k \|_{H^{3/2}(\Gamma, \Lambda)}^2 + \| \eta \|_{(H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2)}^2 \right), \tag{5.75}
\]

and

\[
\int_\Gamma \left| \frac{1}{\lambda} \frac{\partial w_k}{\partial n} - \kappa_k \right|^2 d\Gamma \leq \frac{C}{\lambda^2} \left( \lambda \| W_k \|_{H^{3/2}(\Gamma)}^2 + \| W_k \|_{H^1(\Omega)}^2 + \| f_k(\eta) \|_{L^2(\Omega)}^2 \right)
\]

\[
\leq \frac{C}{\lambda^2} \left( \lambda \| W_k \|_{H^{3/2}(\Gamma)}^2 + \| \eta \|_{(H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2)}^2 \right). \tag{5.76}
\]

Now, multiply the both sides of the first equation in (5.3) by \( \overline{\eta} \), apply (1.33) and integrate the result over \( \Omega \) by parts, to get

\[
P(\eta, \overline{\eta}) + \lambda^2 \| \eta \|_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2)}^2 = \int_\Gamma \partial(A\eta, \overline{\eta}) d\Gamma. \tag{5.77}
\]

From the expressions of \( P(\eta, \overline{\eta}) \) and \( \partial(A\eta, \overline{\eta}) \), it follows that

\[
C \| \eta \|_{(H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2)}^2 \leq P(\eta, \overline{\eta}) + \lambda^2 \| \eta \|_{(L^2(\Omega, \Lambda))^2 \times (L^2(\Omega))^2)}^2, \tag{5.78}
\]

and

\[
\partial(A\eta, \overline{\eta}) = \{B_1(\eta), \kappa_1\} + \{B_2(\eta), \kappa_2\} + b_1(\eta)\kappa_1 + b_2(\eta)\kappa_2. \tag{5.79}
\]

In what follows, we estimate term by term for the right side of (5.79).

\[
\|B_1(\eta), \kappa_1\| = \|2 \gamma(\eta)(n, n) + 2\beta(\tau \gamma(\eta) + w_2/\sqrt{\gamma}) \| \kappa_1, n \| + 2 \gamma(\eta)(n, \tau) \kappa_1, \tau\|
\]

\[
= \|2(1 + \beta)DW_1(n, n)\kappa_1, n \| + 2[\kappa_1, n] + \beta(DW_1(n, \tau) + DW_1(n, \tau) + DW_1(n, \tau) + 2w_1 \Pi(n, \tau)]\kappa_1, \tau\|
\]

\[
\leq \lambda \sqrt{2(1 + \beta)} \left( \frac{1}{\lambda} \sqrt{2(1 + \beta)} (D_n W_1, n) - \kappa_1, n \right) \kappa_1, n \right]
\]

\[
+ \lambda \sqrt{2(1 + \beta)} |(\kappa_1, n)|^2 + \left| \left( \frac{1}{\lambda} (D_n W_1, \tau) - \kappa_1, \tau \right) \kappa_1, \tau \right|
\]

\[
+ \lambda \left| \kappa_1, \tau \right|^2 + |\eta|^2 + |D_\tau W_1|^2 + |\kappa_1|^2
\]

\[
\leq \lambda \sqrt{2(1 + \beta)} (D_n W_1, n) - \kappa_1, n \right|^2 + \lambda C |(\kappa_1, n)|^2
\]

\[
+ \lambda C |\kappa_1|^2 + \lambda \varepsilon \left| \left( \frac{1}{\lambda} (D_n W_1, \tau) - \kappa_1, \tau \right) \kappa_1, \tau \right|^2 + \lambda C |(\kappa_1, \tau)|^2
\]
\[ + \lambda |\mathcal{K}_1|^2 + |\eta|^2 + |D_\tau W_1|^2 + |\mathcal{K}_1|^2 \]
\[ \leq \lambda \varepsilon \left| \frac{1}{\lambda} \sqrt{2(1 + \beta)} (D_n W_2, n) n + \frac{1}{\lambda} (D_n W_2, \tau) \tau - \mathcal{K}_2 \right|^2 + \lambda C_\varepsilon |\mathcal{K}_1|^2 + C |\eta|^2, \]

Similarly,
\[ \left\| B_2(\eta), \mathcal{K}_2 \right\| \leq \lambda \varepsilon \left| \frac{1}{\lambda} \sqrt{2(1 + \beta)} (D_n W_2, n) n + \frac{1}{\lambda} (D_n W_2, \tau) \tau - \mathcal{K}_2 \right|^2 + \lambda C_\varepsilon |\mathcal{K}_2|^2 + C |\eta|^2, \]

and
\[ |b_1(\eta)\mathcal{K}_1| \leq \lambda \varepsilon \left| \frac{1}{\lambda} \frac{\partial w_1}{\partial n} - \mathcal{K}_1 \right|^2 + \lambda C_\varepsilon |\mathcal{K}_1|^2 + C |\eta|^2, \]

where \( \varepsilon \) is a positive real number and \( C_\varepsilon \) is a positive real number independent of \( \lambda \).

Substituting (5.79)–(5.83) into (5.78) yields
\[ \frac{1}{\lambda^2} \| \eta \|^2_{(H^1(\Omega, \Lambda))^2 \times (H^1(\Omega))^2} \]
\[ \leq \frac{\varepsilon}{\lambda} \sum_{j=1}^{2} \left| \frac{1}{\lambda} \sqrt{2(1 + \beta)} (D_n W_2, n) n + \frac{1}{\lambda} (D_n W_2, \tau) \tau - \mathcal{K}_2 \right|^2_{L^2(\Gamma)} + \frac{\varepsilon}{\lambda} \sum_{j=1}^{2} \left| \frac{1}{\lambda} \frac{\partial w_1}{\partial n} - \mathcal{K}_j \right|^2_{L^2(\Gamma)} + \frac{C_\varepsilon}{\lambda} \sum_{j=1}^{2} \| D_\tau W_j \|_{L^2(\Gamma)}^2 \]  

Combining (5.75), (5.76) and (5.84), and using the facts \( H^{7/2}(\Gamma, \Lambda) \subset H^1(\Gamma, \Lambda) \subset L^2(\Gamma, \Lambda) \) and \( H^{7/2}(\Gamma) \subset H^4(\Gamma) \subset L^2(\Gamma) \), we obtain that for all \( \lambda \in (1, \infty) \), any solution \( \eta \in (H^4(\Omega, \Lambda))^2 \times (H^4(\Omega))^2 \) of (5.3) satisfies
\[ (1 - C \varepsilon) \sum_{j=1}^{2} \left( \left| \frac{1}{\lambda} \sqrt{2(1 + \beta)} (D_n W_2, n) n + \frac{1}{\lambda} (D_n W_2, \tau) \tau - \mathcal{K}_j \right|^2_{L^2(\Gamma, \Lambda)} + \left| \frac{1}{\lambda} \frac{\partial w_j}{\partial n} - \mathcal{K}_j \right|^2_{L^2(\Gamma)} \) \]
\[ \leq \frac{C_\varepsilon}{\lambda} \| \eta \|^2_{(H^{7/2}(\Gamma, \Lambda))^2 \times (H^{7/2}(\Gamma))^2}, \]

which implies that
\[ \sum_{j=1}^{2} \left( \left| \frac{1}{\lambda} \sqrt{2(1 + \beta)} (D_n W_2, n) n + \frac{1}{\lambda} (D_n W_2, \tau) \tau - \mathcal{K}_j \right|^2_{L^2(\Gamma, \Lambda)} + \left| \frac{1}{\lambda} \frac{\partial w_j}{\partial n} - \mathcal{K}_j \right|^2_{L^2(\Gamma)} \) \]
\[ \leq \frac{C}{\lambda} \| \eta \|^2_{(H^{7/2}(\Gamma, \Lambda))^2 \times (H^{7/2}(\Gamma))^2} \]  

(5.85)
with $\varepsilon = 1/(2\mathcal{C})$. Therefore,

$$\lim_{\lambda \to \infty} \left\| \frac{1}{\lambda} \sqrt{2(1+\beta)} (D_n W_j, n) + \frac{1}{\lambda} (D_n W_j, \tau) - \mathcal{K}_j \right\|^2_{L^2(\Gamma, \Lambda)} = 0, \quad j = 1, 2, \quad (5.86)$$

and

$$\lim_{\lambda \to \infty} \left\| \frac{1}{\lambda} \frac{\partial w_j}{\partial n} - \kappa_j \right\|^2_{L^2(\Gamma)} = 0, \quad j = 1, 2. \quad (5.87)$$

By the definition of $B_1(\eta)$, we have

$$\left| \frac{1}{\lambda} B_1(\eta) - \sqrt{2(1+\beta)} (\mathcal{K}_1, n) n - (\mathcal{K}_1, \tau) \right|^2 \leq \left| \frac{1}{\lambda} (2(1+\beta)(D_n W_1, n) + 2 w_1 \Pi (n, n) + 2 \beta (D_\tau W_1 + w_2 / \sqrt{\gamma})) - \sqrt{2(1+\beta)} (\mathcal{K}_1, n) \right|^2 \right. \left. + \left| \frac{1}{\lambda} (D_n W_1, \tau) + 2 w_1 \Pi (n, \tau) - (\mathcal{K}_1, \tau) \right|^2 \right. \left. + \frac{2}{\lambda} \left| 2 w_1 \Pi (n, n) + 2 \beta (D_\tau W_1 + w_2 / \sqrt{\gamma}) \right|^2 \right. \left. + 2 \left| \frac{1}{\lambda} (D_n W_1, \tau) - (\mathcal{K}_1, \tau) \right|^2 + 2 \left| (D_\tau W_1, n) + 2 w_1 \Pi (n, \tau) \right|^2 \right. \left. \leq 2 \left| \frac{1}{\lambda} \sqrt{2(1+\beta)} (D_n W_1, n) n - (\mathcal{K}_1, \tau) \right|^2 + \frac{1}{\lambda} \left| (\mathcal{K}_1, n) \right|^2 + 2 \left| \frac{1}{\lambda} (D_n W_1, \tau) \right|^2 + \frac{C}{\lambda} \left( |\eta|^2 + |D_\tau W_1|^2 \right). \quad (5.88)$$

Similarly,

$$\left| \frac{1}{\lambda} B_2(\eta) - \sqrt{2(1+\beta)} (\mathcal{K}_2, n) n - (\mathcal{K}_2, \tau) \right|^2 \leq 2 \left| \frac{1}{\lambda} \sqrt{2(1+\beta)} (D_n W_2, n) n + \frac{1}{\lambda} (D_n W_2, \tau) - \mathcal{K}_2 \right|^2 + \frac{C}{\lambda} \left( |\eta|^2 + |D_\tau W_2|^2 \right), \quad (5.89)$$

$$\left| \frac{1}{\lambda} b_1(\eta) - \kappa_1 \right|^2 \leq 2 \left| \frac{1}{\lambda} \frac{\partial w_1}{\partial n} - \kappa_1 \right|^2 + \frac{C}{\lambda} |\eta|^2, \quad (5.90)$$

and

$$\left| \frac{1}{\lambda} b_2(\eta) - \kappa_2 \right|^2 = \left| \frac{1}{\lambda} \frac{\partial w_2}{\partial n} - \kappa_2 \right|^2. \quad (5.91)$$

Collecting (5.86)–(5.91) yields

$$\lim_{\lambda \to \infty} \left\| \frac{1}{\lambda} B_j(\eta) - \sqrt{2(1+\beta)} (\mathcal{K}_j, n) n - (\mathcal{K}_j, \tau) \right\|^2_{L^2(\Gamma, \Lambda)} = 0, \quad j = 1, 2. \quad (5.92)$$
and
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \left\| \frac{d}{d\lambda} b_j(\eta) - \kappa_j \right\|_{L^2(\Gamma)}^2 = 0, \quad j = 1, 2.
\]

This completes the proof of Theorem 1.2. □

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Appendix A. Continuous of input–output map implies well-posedness

In this appendix, we give a much simpler proof of an abstract result for the systems with the same structure of (2.6). This result is similar to Proposition 4.1 of [20] for the first order systems. R. Triggiani first gave a proof in [22].

Consider the following second-order system:
\[
\begin{cases}
\ddot{x}(t) + Ax(t) + Bu(t) = 0, \\
y(t) = B^* \dot{x}(t),
\end{cases}
\]

where the following assumptions are presumed:

(i) \(A\) is a self-adjoint positive operator in the Hilbert space \(H\);

(ii) \(B \in \mathcal{L}(U, (D(A^{1/2}))')\) or equivalently \(A^{-1/2}B \in \mathcal{L}(U, H)\).

By definition, \(B^* \in \mathcal{L}(D(A^{1/2}), U)\) is given by
\[
\langle B^* z, u \rangle_U = \langle z, Bu \rangle_{D(A^{1/2})}, \quad \forall z \in D(A^{1/2}), \ u \in U.
\]

An extension \(\tilde{A} \in \mathcal{L}(D(A^{1/2}), (D(A^{1/2}))')\) of \(A\) is defined by
\[
\langle \tilde{A} x, z \rangle_{D(A^{1/2})'}, D(A^{1/2})) = \langle A^{1/2} x, A^{1/2} z \rangle_H, \quad \forall x, z \in D(A^{1/2}).
\]

Consider the system (A.1) in the state Hilbert space \(X = D(A^{1/2}) \times H\).

Theorem A.1. With reference to problem (A.1) subject to assumptions (i) and (ii). Assume that for a given \(T > 0\), the input–output map with zero initial value is bounded:
\[
\|y\|_{L^2(0, T; U)} \leq C_T \|u\|_{L^2(0, T; U)} \quad \text{with } x(0) = \dot{x}(0) = 0, \ \forall u \in L^2_{\text{loc}}(0, \infty; U),
\]

where \(C_T\) is a constant independent of \(u\). Then system (A.1) is well-posed. Precisely
\[
\left\| (x(T), \dot{x}(T)) \right\|_X \leq \sqrt{2C_T} \|u\|_{L^2(0, T; U)} \quad \text{with } x(0) = \dot{x}(0) = 0, \ \forall u \in L^2_{\text{loc}}(0, \infty; U).
\]

Proof. Suppose \(x(0) = \dot{x}(0) = 0\), \(u \in C^2(0, T; U)\), \(u(0) = \dot{u}(0) = 0\). The variation of parameter formula for (A.1) is
\[
x(t) = -\int_0^t \frac{e^{i\sqrt{A}(t-s)} - e^{-i\sqrt{A}(t-s)}}{2i} A^{-1/2}Bu(s) \, ds
\]

\[
= -A^{-1}Bu(t) + \int_0^t \frac{e^{i\sqrt{A}(t-s)} + e^{-i\sqrt{A}(t-s)}}{2} A^{-1/2}Bu(s) \, ds \in D(A^{1/2}), \quad \forall t \geq 0,
\]

(A.6)

where \( e^{i\sqrt{A}} \) is the \( C_0 \)-group on \( H \) generated by \( iA^{1/2} \), and so is \( e^{-i\sqrt{A}} \) by \(-iA^{1/2} \). Hence

\[
\dot{x}(t) = -\int_0^t \frac{e^{i\sqrt{A}(t-s)} - e^{-i\sqrt{A}(t-s)}}{2i} A^{-1/2}\dot{u}(s) \, ds
\]

\[
= -A^{-1}\dot{u}(t) + \int_0^t \frac{e^{i\sqrt{A}(t-s)} + e^{-i\sqrt{A}(t-s)}}{2} A^{-1/2}\dot{u}(s) \, ds \in D(A^{1/2}), \quad \forall t \geq 0,
\]

(A.7)

and

\[
\ddot{x}(t) = -\int_0^t \frac{e^{i\sqrt{A}(t-s)} - e^{-i\sqrt{A}(t-s)}}{2i} A^{-1/2}\ddot{u}(s) \, ds \in H, \quad \forall t \geq 0.
\]

(A.8)

Now, since \( x(t) \) given by (A.6) is the solution of (A.1) with zero initial value, that is,

\[
\dot{x}(t) + Ax(t) + Bu(t) = 0, \quad x(0) = \dot{x}(0) = 0 \quad \text{in } (D(A^{1/2}))',
\]

(A.9)

take duality product with \( \dot{x}(t) \in D(A^{1/2}) \) between \( D(A^{1/2}) \) and \( D(A^{1/2})' \) with pivot space \( H \) on both sides of (A.9), and take (A.3)–(A.8) into account, to obtain

\[
\langle \ddot{x}(t), \dot{x}(t) \rangle_{D(A^{1/2})', D(A^{1/2})} + \langle A\dot{x}(t), \dot{x}(t) \rangle_{D(A^{1/2})', D(A^{1/2})} + \langle Bu(t), \dot{x}(t) \rangle_{D(A^{1/2})', D(A^{1/2})} + \langle u(t), B^*\dot{x}(t) \rangle_U
\]

\[
= \langle \ddot{x}(t), \dot{x}(t) \rangle_H + \langle A^{1/2}x(t), A^{1/2}\dot{x}(t) \rangle_H + \langle u(t), B^*\dot{x}(t) \rangle_U = 0, \quad \forall t \geq 0.
\]

(A.10)

Given \( T > 0 \). Integrate the second equality of (A.10) over \([0, T]\) with respect to \( t \), to get

\[
\langle \langle x(T), \dot{x}(T) \rangle \rangle_X^2 = \langle \ddot{x}(T) \rangle_H^2 + \langle A^{1/2}x(T) \rangle_H^2 + \langle u(t), B^*\dot{x}(t) \rangle_U = -2 \int_0^T \langle u(t), B^*\dot{x}(t) \rangle_U \, dt.
\]

(A.11)

This together with (A.4) gives

\[
C_T \|u\|_{L^2(0,T;U)} \geq \|\langle y, u \rangle_{L^2(0,T;U)}\| = \|\langle B^*\dot{x}, u \rangle_{L^2(0,T;U)}\| = \frac{1}{2} \|\langle x(T), \dot{x}(T) \rangle \rangle_X^2
\]

for any \( u \in C^2(0, T; U) \) with \( u(0) = \dot{u}(0) = 0 \). This is (A.5). Since the functions of \( C^2(0, \infty; U) \) with \( u(0) = \dot{u}(0) = 0 \) are dense in \( L^2_{loc}(0, \infty; U) \), (A.5) then follows from the density argument. \( \square \)
References