Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnline
CONFERENCE PAPER ARCHIVE

IFAC-PapersOnLine 49-8 (2016) 278–283

Active Disturbance Rejection Control: from ODEs to PDEs ⋆

Bao-Zhu Guo *

* *Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, P.R. China, and School of Computer Science and Applied Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa (e-mail: bzguo@iss.ac.cn).*

Abstract: This paper introduces a new emerging control technology, known as active disturbance rejection control to this day. We start its main idea and two main parts, namely, extended state observer and extended state observer based feedback for lumped parameter systems, and then discuss its application to both state and output feedback stabilization for distributed parameter systems.

Keywords: Active disturbance rejection control, feedback stabilization, PDEs

1. INTRODUCTION

Disturbance rejection is a different paradigm in control theory since the inception of the modern control theory in the later years of 1950's, seeded in Tsien (1954) where it is stated that the control operation "must not be influenced by internal and external disturbances" (Tsien, 1954, p.228). The tradeoff between mathematical rigor by model-based control theory and practicability by model-free engineering applications has been a constantly disputed issue in control community. On the one hand, we have mountains of papers, books, monographes published every year, and on the other hand, the control engineers are nowhere to find, given the difficulty of building (accurate) dynamic model for the system to be controlled, a simple, model free, easy tuning, better performance control technology more than proportional-integral-derivative (PID) control (Silva et al. (2002), see also Bialkowski et al. (2015)). This awkward coexistence of huge modern control theories on the one hand and a primitive control technology that has been dominating engineering applications for one century on the other pushed Jingqing Han, a control scientist at the Chinese Academy of Sciences to propose active disturbance rejection control (ADRC), as an alternative of PID. This is because PID has the advantage of model free nature whereas most parts of modern control theory are based on mathematical models. By model-based control theory, it is hard to cross the boundaries such as time variance, nonlinearity, and uncertainty created mainly by the limitations of mathematics. However, there are some basic limitations for PID in practice to accommodate the liability in the digital processors according to Han (2009).

To address this problem, Han would seek solution from the seed idea of disturbance rejection imbedded in Tsien (1954). Consider stabilization for the following second order Newton system:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = f(x_1(t), x_2(t), d(t), t) + u(t), \\ y(t) = x_1(t), \end{cases} \tag{1.1}$$

where $u(t)$ is the control input, $y(t)$ is the measured output, $d(t)$ is the external disturbance, and $f(\cdot)$ is an unknown function which contains unmodelled dynamics of the system or most possibly, the internal and external disturbance discussed in Tsien (1954).

The total disturbance can certainly be nonlinear, time variant and many other forms. Han considered it just as a signal of time, which is reflected in the measured output and hence can possibly be estimated. Let $a(t) = f(x_1(t), x_2(t), d(t), t)$. Then system (1.1) becomes

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = a(t) + u(t), \\ y(t) = x_1(t). \end{cases} \tag{1.2}$$

A flash of insight arises (Han (1989)): system (1.2) is exactly observable because it is trivially seen that $(y(t), u(t)) \equiv 0, t \in [0, T] \Rightarrow a(t) = 0, t \in [0, T]; (x_1(0), x_2(0)) = 0$ (see, e.g., (Cheng et al., 2015, p.5, Definition 1.2)). This means that $y(t)$ contains all information of $a(t)$! *Why not use $y(t)$ to estimate $a(t)$?*, was perhaps the question in Han's mind. If we can, for instance, $y(t) \Rightarrow \hat{a}(t) \approx a(t)$, then we can cancel $a(t)$ by designing $u(t) = -\hat{a}(t) + u_0(t)$ and system (1.2) amounts to, approximately of course,

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u_0(t), \\ y(t) = x_1(t). \end{cases} \tag{1.3}$$

The nature of the problem is therefore changed now. System (1.3) is just a linear time invariant system for which we have many ways to deal with it. This is likewise feedforward control yet to use output to "transform" the system first. In a different point of view, this part is called

⋆ This work was carried out with the support of the National Natural Science Foundation of China and the National Research Foundation of South Africa.

10.1016/j.ifacol.2016.07.454

the "rejector" of disturbance (Gao (2015)). It seems that a further smarter way would be hardly to find anymore because the control $u(t) = -\hat{a}(t) + u_0(t)$ adopts a strategy of estimation/cancellation, much alike our experience in dealing with uncertainty in daily life. One can imagine and it actually is, one of the most energy saving control strategies as confirmed in Zheng and Gao (2012).

This paradigm-shift is revolutionary for which Han wrote in Han (1989) that "to improve accuracy, it is sometimes necessary to estimate $a(t)$ but it is not necessary to know the nonlinear relationship between $a(t)$ and the states variables". The idea breaks down the garden gates from time varying dynamics (e.g., $f(x_1, x_2, d, t) = g_1(t)x_1 + g_2(t)x_2$), nonlinearity (e.g., $f(x_1, x_2, d, t) = x_1^2 + x_2^3$), and "internal and external disturbance" (e.g., $f(x_1, x_2, d, t) = x_1^2 + x_2^2 + \Delta f(x_1, x_2) + d$). The problem now becomes: how can we realize $y(t) \Rightarrow \hat{a}(t) \approx a(t)$?

Han told us in Han (1995) that it is not only possible but also realizable systematically. This is made possible by what is called extended state observer (ESO). Firstly, Han considered $a(t)$ to be an extended state variable and changed system (1.2) to

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = a(t) + u(t), \\ \dot{a}(t) = a'(t), \\ y(t) = x_1(t). \end{cases} \tag{1.4}$$

A linear observer for system (1.4), or equivalently linear ESO for system (1.2) can be designed as

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + a_1(\hat{x}_1(t) - y(t)), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) + u(t) + a_2(\hat{x}_1(t) - y(t)), \\ \dot{\hat{x}}_3(t) = a_3(\hat{x}_1(t) - y(t)), \end{cases} \tag{1.5}$$

where we can choose high gains

$$a_i = \frac{\alpha_i}{\varepsilon^i}, i = 1, 2, 3, \tag{1.6}$$

so that

$$\begin{aligned} &\hat{x}_1(t) \to x_1(t), \hat{x}_2(t) \to x_2(t), \\ &\hat{x}_3(t) \to a(t) \text{ as } t \to \infty, \varepsilon \to 0. \end{aligned} \tag{1.7}$$

The constants α_i in (1.6) are required to make

$$E = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 1 \\ \alpha_3 & 0 & 0 \end{pmatrix} \tag{1.8}$$

be Hurwitz (Zheng et al. (2007); Guo and Zhao (2011)) and $a'(t)$ is required to be bounded. It is seen that we have obtained estimation $\hat{x}_3(t) \approx a(t)$ from $y(t)$!

Definition 1.1. The ESO (1.5) is said to be convergent, if for any given $\delta > 0$, there exist $T_\delta > 0, \varepsilon_\delta$ such that

$$\begin{aligned} &|\tilde{x}_i(t)| = |\hat{x}_i(t) - x_i(t)| \le \delta, \\ &|\tilde{a}(t)| = |\hat{x}_3(t) - a(t)| \le \delta, \ \forall\, t > T_\delta, \varepsilon > \varepsilon_\delta, i = 1, 2. \end{aligned}$$

Finally, to stabilize system (1.2), we simply cancel the disturbance by using the ESO-based feedback:

$$u(t) = -\hat{x}_3(t) + \beta_1 \hat{x}_1(t) + \beta_2 \hat{x}_2(t), \tag{1.9}$$

where the first term is used to cancel (compensate) the disturbance and the last terms are stabilizing state feedback chosen by separation principle, i.e.

$$F = \begin{pmatrix} 0 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \tag{1.10}$$

is Hurwitz. The closed-loop of (1.2) under the feedback (1.9) becomes

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = a(t) - \hat{x}_3(t) + \beta_1 \hat{x}_1(t) + \beta_2 \hat{x}_2(t), \\ \dot{\hat{x}}_1(t) = \hat{x}_2(t) + a_1(\hat{x}_1(t) - y(t)), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) - \hat{a}(t) + \beta_1 \hat{x}_1(t) + \beta_2 \hat{x}_2(t) \\ \qquad\qquad + a_2(\hat{x}_1(t) - y(t)), \\ \dot{\hat{x}}_3(t) = a_3(\hat{x}_1(t) - y(t)), \end{cases} \tag{1.11}$$

which is equivalent, by setting $\tilde{x}_1(t) = \hat{x}_1(t) - x_1(t)$, $\tilde{x}_2(t) = \hat{x}_2(t) - x_2(t)$ and $\tilde{a}(t) = \hat{x}_3(t) - a(t)$, to

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \beta_1 x_1(t) + \beta_2 x_2(t) + \beta_1 \tilde{x}_1(t) + \beta_2 \tilde{x}_2(t) - \tilde{a}(t), \\ \dot{\tilde{x}}_1(t) = \tilde{x}_2(t) + a_1 \tilde{x}_1(t), \\ \dot{\tilde{x}}_2(t) = \tilde{a}(t) + a_2 \tilde{x}_1(t), \\ \dot{\tilde{a}}(t) = a_3 \tilde{x}_1(t) - a'(t). \end{cases} \tag{1.12}$$

Since $(\tilde{x}_i(t), \tilde{a}(t)) \to 0$ $i = 1, 2$ as $\varepsilon \to 0$ and $t \to \infty$, proved in convergence of ESO, we have immediately that

$$x_i(t) \to 0, i = 1, 2, \text{ as } t \to \infty, \varepsilon \to 0,$$

or equivalently

$$\begin{aligned} &x_i(t) \to 0, \hat{x}_i(t) \to 0, i = 1, 2, \\ &\hat{x}_3(t) - a(t) \to 0 \text{ as } t \to \infty, \varepsilon \to 0. \end{aligned} \tag{1.13}$$

This is the well known separation principle in linear system theory. So, the whole idea not only works and but also works in an extremely wise way of estimating and cancelling the disturbance in real time.

Remark 1.1. System (1.1) is equivalent to second order system:

$$\ddot{x}(t) = f(x(t), \dot{x}(t), d(t), t) + u(t).$$

So the total disturbance and control are matched naturally. If they are not matched, for instance, system like

$$\begin{cases} \dot{x}_1(t) = x_2(t) + d(t), \\ \dot{x}_2(t) = u(t), \\ y(t) = x_1(t), \end{cases} \tag{1.14}$$

we can still apply ADRC to deal with stabilization. Actually, let

$$\bar{x}_2(t) = x_2(t).$$

Then (1.14) becomes

$$\begin{cases} \dot{x}_1(t) = \bar{x}_2(t), \\ \dot{\bar{x}}_2(t) = \dot{d}(t) + u(t), \\ y(t) = x_1(t), \end{cases} \tag{1.15}$$

For stabilization, we can achieve

$$x_1(t) \to 0, \bar{x}_2(t) = x_2(t) + d(t) \to 0 \text{ as } t \to \infty.$$

Certainly, as any other methods, some limitations likely exist in an otherwise perfect setting of ESO in the sense:

- The high gain is resorted in ESO to suppress the effect of the derivative $a'(t)$ of the total disturbance in (1.12);
- the derivative $a'(t)$ of disturbance as shown in (1.12) is supposed to be bounded as well as from (1.4) where $a(t)$ is regarded as an extended state variable.

The second condition can be relaxed to allow some finite order derivative of $a(t)$ to be bounded by increasing the order of ESO.

The first problem is possibly resolved by designing a different type of ESO because in the final analysis when we scrutinize the whole process, ESO (1.5) is nothing more than one of such devices, developed by Han himself only aiming at estimating disturbance from observable measured output which is the ultimate goal of ADRC. It is not, and should not be, a unique way for this purpose. To understand this point, we may think of internal model principle, a similar idea as economic as ADRC on the basis of estimation/cancellation strategy yet no high gain is utilized.

If we let the matter drop here, the ADRC seems not very new idea in control theory. But when we go further to have a comparison, we find from (1.1) that ADRC regards both internal and external disturbance $a(t) = f(x_1(t), x_2(t), d(t), t)$ together as a signal of time which can be estimated by the output. This spans significantly the concept of the disturbance where in adaptive control they are some internal unknown parameters and in internal model principle they are some external disturbance produced from a dynamical exosystem. The ADRC's major component ESO provides a systematical feasible way to estimate total disturbance from measured output. It opens another gate so that we can get rid of mathematical brunt like $a(t) = f(x_1(t), x_2(t), d(t), t)$ to be state dependent or state free, time invariant or time variant, linear or nonlinear and whatever. This is an almost model free control method, carrying PID control forward.

Before we end this opening story, we indicate that the possible improvement of ADRC lies in ESO as what we see in adaptive control and internal model principle where the inherent estimation/cancellation is kept yet no high gain is used.

In the remaining part of this paper, we apply ADRC to stabilization for PDEs. Section 2 is on stabilization for uncertain PDEs via state feedback. The output feedback stabilization is introduced in Section 3.

2. STATE FEEDBACK STABILIZATION FOR UNCERTAIN PDES

The material of this section comes from Guo and Liu (2014) where it applies ADRC to anti-stable Schrödinger equation:

$$\begin{cases} u_t(x,t) = -ju_{xx}(x,t),\ x \in (0,1), t > 0, \\ u_x(0,t) = -jqu(0,t),\ q > 0, t \geqslant 0, \\ u_x(1,t) = U(t) + d(t),\ t \geqslant 0, \end{cases} \tag{2.1}$$

where $u(t)$ is the complex-valued state, j is the imaginary unit, $U(t)$ is the control input. The unknown disturbance $d(t)$ is supposed to be uniformly bounded measurable, that is, $|d(t)| \leqslant M_0$ for some $M_0 > 0$ and all $t \geqslant 0$. The system represents an anti-stable distributed parameter system: all eigenvalues of the free system (with no control and disturbance) are located on the right-half complex plane.

We suppose as usual that $|\dot{d}(t)|$ is also uniformly bounded. Introduce a transformation:

$$w(x,t) = u(x,t) + j(c_0+q)\int_0^x e^{jq(x-y)}u(y,t)dy, c_0 > 0. \tag{2.2}$$

Its inverse transformation is found to be

$$u(x,t) = w(x,t) - j(c_0+q)\int_0^x e^{-jc_0(x-y)}w(y,t)dy. \tag{2.3}$$

The transformation (2.2) transforms system (2.1) into the following system:

$$\begin{cases} w_t(x,t) = -jw_{xx}(x,t), x \in (0,1), t > 0, \\ w_x(0,t) = jc_0 w(0,t), t \geqslant 0, \\ w_x(1,t) = U(t) + d(t) + j(c_0+q)w(1,t) \\ +c_0(c_0+q)\displaystyle\int_0^1 e^{-jc_0(1-x)}w(x,t)dx, t \geqslant 0. \end{cases} \tag{2.4}$$

It is seen that the anti-stable factor $-jqu(0,t)$ in (2.1) becomes the dissipative term $jc_0 w(0,t)$ in (2.4) under the transformation (2.2), both at the end $x = 0$. In what follows, we consider the stabilization of system (2.4) until the final step to go back the system (2.1) under the inverse transformation (2.3). Introduce a new controller $U_0(t)$ so that

$$\begin{aligned} U(t) &= U_0(t) - j(c_0+q)w(1,t) \\ &\quad -c_0(c_0+q)\int_0^1 e^{-jc_0(1-x)}w(x,t)dx. \end{aligned} \tag{2.5}$$

Then (2.4) becomes

$$\begin{cases} w_t(x,t) = -jw_{xx}(x,t), x \in (0,1), t > 0, \\ w_x(0,t) = jc_0 w(0,t), t \geqslant 0, \\ w_x(1,t) = U_0(t) + d(t). \end{cases} \tag{2.6}$$

We write (2.6) into the operator form. Define the operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}f(x) = -jf''(x), \\ D(\mathcal{A}) = \{f \in H^2(0,1) |\ f'(0) = jc_0 f(0), f'(1) = 0\}. \end{cases} \tag{2.7}$$

Then we can write (2.6) in $\mathcal{H}$ as

$$\frac{d}{dt}w(\cdot,t) = \mathcal{A}w(\cdot,t) + \mathcal{B}(U_0(t) + d(t)),\ \mathcal{B} = -j\delta(x-1). \tag{2.8}$$

The following Lemma 2.1 is straightforward.

Lemma 2.1. Let $\mathcal{A}$ be defined by (2.7). Then each eigenvalue of $\mathcal{A}$ is algebraically simple, and there exists a sequence of eigenfunctions of $\mathcal{A}$, which form a Riesz basis for $\mathcal{H}$. Therefore, $\mathcal{A}$ generates an exponential stable C_0-semigroup on $\mathcal{H}$. In addition, $\mathcal{B}$ is admissible to the semigroup $e^{\mathcal{A}t}$ (Weiss (1989)).

Let

$$\begin{cases} y_1(t) = \displaystyle\int_0^1 (2x^3 - 3x^2)w(x,t)dx, \\ y_2(t) = \displaystyle\int_0^1 (12x-6)w(x,t)dx. \end{cases} \tag{2.9}$$

Since $\mathcal{B}$ is admissible to the C_0-semigroup $e^{\mathcal{A}t}$, the solution of (2.6) is understood in the sense of

$$\begin{aligned} \frac{d}{dt}\langle w(\cdot,t), f\rangle &= \langle w(\cdot,t), \mathcal{A}^* f\rangle \\ &\quad -jf(1)(U_0(t) + d(t)),\ \forall\, f \in D(\mathcal{A}^*). \end{aligned} \tag{2.10}$$

Let $f(x)=2x^3-3x^2\in D(\mathcal{A}^*)$ in (2.10) to get

$$\dot{y}_1(t)=jU_0(t)+jd(t)-jy_2(t). \tag{2.11}$$

That is to say, for any initial value $w(\cdot,0)\in\mathcal{H}$, the (weak) solution of (2.6) must satisfy (2.11).

Remark 2.1. From (2.10), $y_1(t)$ and $y_2(t)$ can be chosen as $y_1(t)=\int_0^1 f(x)w(x,t)dx$, $y_2(t)=\int_0^1(\mathcal{A}^*f)(x)w(x,t)dx$ where $f\in D(\mathcal{A}^*)$, $f(1)\neq 0$. Our choice is only a special case by this general principle.

Design the high gain estimators for $y_1(t)$ and $d(t)$ as follows:

$$\begin{cases}\dot{\hat{y}}(t)=j(U_0(t)+\hat{d}(t))-jy_2(t)-\dfrac{1}{\varepsilon}(\hat{y}(t)-y_1(t)),\\ \dot{\hat{d}}(t)=\dfrac{j}{\varepsilon^2}(\hat{y}(t)-y_1(t)),\end{cases} \tag{2.12}$$

where $\varepsilon>0$ is the design small parameter and $\hat{d}(t)$ is regarded as an approximation of $d(t)$.

The state feedback controller to (2.6) is designed as follows:

$$U_0(t)=-\hat{d}(t). \tag{2.13}$$

It is clearly seen from (2.13) that this controller is just used to cancel (compensate) the disturbance d because $\mathcal{A}$ generates an exponential stable C_0-semigroup. This estimation/cancelation strategy (2.13) is just from ADRC. Under the feedback (2.13), the closed-loop system of (2.6) becomes

$$\begin{cases}w_t(x,t)=-jw_{xx}(x,t),\quad x\in(0,1),\quad t>0,\\ w_x(0,t)=jc_0w(0,t),\quad t\geqslant 0,\\ w_x(1,t)=-\hat{d}(t)+d(t),\quad t\geqslant 0.\\ \dot{\hat{y}}(t)=-jy_2(t)-\dfrac{1}{\varepsilon}(\hat{y}(t)-y_1(t)),\\ \dot{\hat{d}}(t)=\dfrac{j}{\varepsilon^2}(\hat{y}(t)-y_1(t)).\end{cases} \tag{2.14}$$

Returning back to system (2.1) by the inverse transformation (2.3), feedback control (2.5) and (2.13), and new variable (2.9), we have the main result of this section.

Theorem 2.1. Suppose that $|d(t)|\leqslant M_0$ and $\dot{d}(t)$ is also uniformly bounded measurable. Then for any initial value $u(\cdot,0)\in\mathcal{H}$, the closed-loop system of (2.1) following:

$$\begin{cases}u_t(x,t)=-ju_{xx}(x,t),\ x\in(0,1),t>0,\\ u_x(0,t)=-jqu(0,t),\ t\geqslant 0,\\ u_x(1,t)=-\hat{d}(t)-j(c_0+q)u(1,t)\\ +q(c_0+q)\displaystyle\int_0^1 e^{jq(1-x)}u(x,t)dx+d(t),\ t\geqslant 0,\end{cases} \tag{2.15}$$

admits a unique solution $(u,u_t)^\top\in C(0,\infty;\mathcal{H})$, and the solution of system (2.15) tends to any arbitrary given vicinity of zero as $t\to\infty,\varepsilon\to 0$, where the feedback control is:

$$\begin{aligned}U(t)=&-\hat{d}(t)-j(c_0+q)u(1,t)\\ &+q(c_0+q)\int_0^1 e^{jq(1-x)}u(x,t)dx,t\geqslant 0\end{aligned} \tag{2.16}$$

and $\hat{d}(t)$ satisfies

$$\begin{cases}\dot{\hat{y}}(t)=-jy_2(t)-\dfrac{1}{\varepsilon}(\hat{y}(t)-y_1(t)),\\ \dot{\hat{d}}(t)=\dfrac{j}{\varepsilon^2}(\hat{y}(t)-y_1(t)),\end{cases} \tag{2.17}$$

$$\begin{cases}y_1(t)=\displaystyle\int_0^1(2x^3-3x^2)\big[u(x,t)\\ +j(c_0+q)\displaystyle\int_0^x e^{jq(x-y)}u(y,t)dy\big]dx,\\ y_2(t)=\displaystyle\int_0^1(12x-6)\big[u(x,t)\\ +j(c_0+q)\displaystyle\int_0^x e^{jq(x-y)}u(y,t)dy\big]dx.\end{cases} \tag{2.18}$$

3. OUTPUT FEEDBACK STABILIZATION FOR UNCERTAIN PDES

The material of this section comes from Feng and Guo (2016) where it considers output feedback stabilization for the following anti-stable one-dimensional wave equation with general disturbance:

$$\begin{cases}w_{tt}(x,t)=w_{xx}(x,t),\ 0<x<1,\ t>0,\\ w_x(0,t)=-qw_t(0,t),\ t\geq 0,\\ w_x(1,t)=d(t)+u(t),\ t\geq 0,\\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x),\ 0\leq x\leq 1,\\ y(t)=\{w(0,t),w_t(0,t),w(1,t)\},\ t\geq 0,\end{cases} \tag{3.1}$$

where $y(t)$ is the output (measurement), $u(t)$ the input (control), $(w_0(x),w_1(x))$ the initial value, and $d\in L^\infty(0,\infty)$ or $d\in L^2(0,\infty)$ which represents an unknown external disturbance.

3.1 Stabilization without disturbance

To stabilize the system with external disturbance, we must first know how to stabilize system without disturbance. In this subsection, we look at system (3.1) without disturbance, which is re-written as

$$\begin{cases}w_{tt}(x,t)=w_{xx}(x,t),\\ w_x(0,t)=-qw_t(0,t),\ w_x(1,t)=u(t),\\ y_o(t)=\{w(0,t),w_t(0,t)\}.\end{cases} \tag{3.2}$$

Here it is noted that the output $y_o(t)$ is fewer than the original output $y(t)$. For stabilization, $y_o(t)$ is almost minimal: the signal $w(0,t)$ only cannot make system (3.2) exactly observable while $w_t(0,t)$ only cannot identify the zero eigenfunction. In this case, we can stabilize system (3.2) by introducing the following transformation:

$$\tilde{w}(x,t)=w(x,t)+W(x,t), \tag{3.3}$$

where $W(x,t)$ is governed by

$$\begin{cases}W_t(x,t)+W_x(x,t)=0,\\ W(0,t)=-c_2w(0,t),\\ W(x,0)=W_0(x),\end{cases} \tag{3.4}$$

with c_2 being a positive turning parameter and $W_0(x)$ the arbitrary initial value. Combining (3.2) and (3.4), $\tilde{w}(x,t)$ satisfies

$$\begin{cases} \tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t), \\ \tilde{w}_x(0,t) = \dfrac{c_2 - q}{1 - c_2}\tilde{w}_t(0,t), \\ \tilde{w}_x(1,t) = u(t) + W_x(1,t). \end{cases} \tag{3.5}$$

It is seen that there is a "passive damper" at the left end $x = 0$, provided we choose $\frac{c_2-q}{1-c_2} > 0$. The right end $x = 1$ can be changed by the control:

$$\begin{aligned} u(t) &= -W_x(1,t) - c_3\tilde{w}(1,t) \\ &= W_t(1,t) - c_3 w(1,t) - c_3 W(1,t), \end{aligned} \tag{3.6}$$

where c_3 is a positive turning parameter. With control (3.6), we have

$$\begin{cases} \tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t), \\ \tilde{w}_x(0,t) = \dfrac{c_2 - q}{1 - c_2}\tilde{w}_t(0,t), \\ \tilde{w}_x(1,t) = -c_3\tilde{w}(1,t). \end{cases} \tag{3.7}$$

The presence of $c_3 > 0$ in (3.6) or (3.7) takes away zero eigenvalue from the corresponding control free system. Under (3.6), we have the following closed-loop of system (3.2):

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = -q w_t(0,t), \\ w_x(1,t) = W_t(1,t) - c_3 w(1,t) - c_3 W(1,t), \\ W_t(x,t) + W_x(x,t) = 0, \\ W(0,t) = -c_2 w(0,t). \end{cases} \tag{3.8}$$

Theorem 3.1. Suppose that $\frac{c_2-q}{1-c_2} > 0$. Then, for any initial value $(w(\cdot,0), w_t(\cdot,0), W(\cdot,0)) \in \mathcal{H} \times H^1(0,1)$, system (3.8) admits a unique solution $(w, w_t, W) \in C(0,\infty; \mathcal{H} \times H^1(0,1))$ which satisfies, for any $t \geq 0$,

$$\|(w(\cdot,t), w_t(\cdot,t), W(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)} \leq L e^{-\omega t} \tag{3.9}$$

for some positive constants L and ω.

3.2 Disturbance estimator

In this subsection, we come back to design a disturbance estimator for system (3.1). To this purpose, we first propose the following auxiliary system to bring the disturbance into an exponentially stable system:

$$\begin{cases} \hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -q w_t(0,t) - c_0[w(0,t) - \hat{w}(0,t)] \\ \qquad\qquad -c_1[w_t(0,t) - \hat{w}_t(0,t)], \\ \hat{w}_x(1,t) = u(t), \end{cases} \tag{3.10}$$

where c_0 and c_1 are two positive turning parameters. It is seen that system (3.10) is completely determined by the input $u(t)$ and the partial output of the original system (3.1). Let

$$\varepsilon(x,t) = w(x,t) - \hat{w}(x,t). \tag{3.11}$$

By (3.1) and (3.10), the error $\varepsilon(x,t)$ is governed by

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = c_0\varepsilon(0,t) + c_1\varepsilon_t(0,t),\ \varepsilon_x(1,t) = d(t), \end{cases} \tag{3.12}$$

which brings disturbance into a stable system. An infinite-dimensional disturbance estimator can be designed as

$$\begin{cases} \hat{d}_{tt}(x,t) = \hat{d}_{xx}(x,t), \\ \hat{d}_x(0,t) = c_0\hat{d}(0,t) + c_1\hat{d}_t(0,t), \\ \hat{d}(1,t) = w(1,t) - \hat{w}(1,t), \\ \hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -q w_t(0,t) - c_0[w(0,t) - \hat{w}(0,t)] \\ \qquad\qquad -c_1[w_t(0,t) - \hat{w}_t(0,t)], \\ \hat{w}_x(1,t) = u(t), \end{cases} \tag{3.13}$$

where $\hat{d}_x(1,t)$ can be considered as an approximation of $d(t)$. This time, system (3.13) is completely determined by the input $u(t)$ and the output $y(t)$ of the original system (3.1).

Theorem 3.2. Suppose that $d \in L^\infty(0,\infty)$ or $d \in L^2(0,\infty)$. Then, for any initial value $(w_0, w_1, \hat{d}(\cdot,0), \hat{d}_t(\cdot,0), \hat{w}(\cdot,0), \hat{w}_t(\cdot,0)) \in \mathcal{H}^3$ with the compatible condition $w_0(1) - \hat{w}(1,0) = \hat{d}(1,0)$, there exists a unique solution $(\hat{d}, \hat{d}_t, \hat{w}, \hat{w}_t) \in C(0,\infty; \mathcal{H}^2)$ to disturbance estimator (3.13) such that

$$\hat{d}_x(1,\cdot) - d(\cdot) \in L^2(0,\infty). \tag{3.14}$$

If we assume further that $(\hat{d}(\cdot,0) - w_0(\cdot) + \hat{w}(\cdot,0), \hat{d}_t(\cdot,0) - w_1(\cdot) + \hat{w}_t(\cdot,0)) \in D(\mathscr{A})$, then there exist two positive constants L_2 and ω_2 such that

$$|\hat{d}_x(1,t) - d(t)| \leq L_2 e^{-\omega_2 t},\ \forall\, t \geq 0, \tag{3.15}$$

where the operator $\mathscr{A} : D(\mathscr{A})(\subset \mathcal{H}_0) \to \mathcal{H}_0$ is defined by

$$\begin{cases} \mathscr{A}(f,g) = (g, f''),\ \forall\,(f,g) \in D(\mathscr{A}), \\ D(\mathscr{A}) = \Big\{(f,g) \in H^2(0,1) \times H^1(0,1) \\ \quad \mid f'(0) = c_0 f(0) + c_1 g(0), f(1) = g(1) = 0\Big\}. \end{cases} \tag{3.16}$$

3.3 Disturbance estimator based output feedback

In view of the target system (3.8), we design naturally an output feedback of control plant (3.1) by compensating the disturbance:

$$u(t) = -\hat{d}_x(1,t) + W_t(1,t) - c_3 w(1,t) - c_3 W(1,t), \tag{3.17}$$

where $\hat{d}_x(1,t)$ is given by the disturbance estimator (3.13) and $W(x,t)$ is given by (3.4). Compared with (3.6), it is seen that the first term of (3.17) on the right is used to cancel (compensate) the disturbance, and the remaining terms are just the stabilizing output feedback (3.6). This is keeping with the spirit of ADRC. Under (3.17), we have the following closed-loop of system (3.1):

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = -qw_t(0,t), \\ w_x(1,t) = d(t) - \hat{d}_x(1,t) + W_t(1,t) - c_3 w(1,t) \\ \qquad -c_3 W(1,t), \\ \hat{d}_{tt}(x,t) = \hat{d}_{xx}(x,t), \\ \hat{d}_x(0,t) = c_0\hat{d}(0,t) + c_1\hat{d}_t(0,t), \\ \hat{d}(1,t) = w(1,t) - \hat{w}(1,t), \\ \hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t), \\ \hat{w}_x(0,t) = -qw_t(0,t) - c_0[w(0,t) - \hat{w}(0,t)] \\ -c_1[w_t(0,t) - \hat{w}_t(0,t)], \\ \hat{w}_x(1,t) = -\hat{d}_x(1,t) + W_t(1,t) - c_3 w(1,t) \\ \qquad -c_3 W(1,t), \\ W_t(x,t) + W_x(x,t) = 0, \\ W(0,t) = -c_2 w(0,t). \end{cases} \tag{3.18}$$

Theorem 3.3. Suppose that $d \in L^\infty(0,\infty)$ or $d \in L^2(0,\infty)$. Then, for any initial value $(w_0, w_1, \hat{d}(\cdot,0), \hat{d}_t(\cdot,0), \hat{w}(\cdot,0), \hat{w}_t(\cdot,0), W_0) \in \mathcal{H}^3 \times H^1(0,1)$ with the compatible condition $w_0(1) - \hat{w}(1,0) = \hat{d}(1,0)$, Then, system (3.18) admits a unique solution $(w, w_t, \hat{d}, \hat{d}_t, \hat{w}, \hat{w}_t, W) \in C(0,\infty; \mathcal{H}^3 \times H^1(0,1))$ such that

$$\begin{cases} \displaystyle\int_0^\infty |\hat{d}_x(1,t) - d(t)|^2 dt \\ \displaystyle + \sup_{t\in[0,\infty)} \|(\hat{d}(\cdot,t), \hat{d}_t(\cdot,t), \hat{w}(\cdot,t), \hat{w}_t(\cdot,t))\|_{\mathcal{H}^2} < \infty, \\ \displaystyle\lim_{t\to\infty} \|(w(\cdot,t), w_t(\cdot,t), W(\cdot,t))\|_{\mathcal{H}\times H^1(0,1)} = 0. \end{cases} \tag{3.19}$$

If we assume further that $(\hat{d}(\cdot,0) - w_0(\cdot) + \hat{w}(\cdot,0), \hat{d}_t(\cdot,0) - w_1(\cdot) + \hat{w}_t(\cdot,0)) \in D(\mathscr{A})$, then

$$|\hat{d}_x(1,t) - d(t)| \le L_3 e^{-\omega_3 t}, \quad t \ge 0 \tag{3.20}$$

for some constants L_3 and ω_3.

Notes: Section 1 comes from Guo and Zhou (2016); Section 2 comes from Guo and Liu (2014), and Section 3 comes from Feng and Guo (2016). For other ADRC application to PDEs, we refer to Guo and Jin (2015, 2013); Guo and Zhou (2015, 2016)

REFERENCES

Bialkowski, W.L. (1996). Control of the pulp and paper making process, in: "*The Control handbook*" (W.S. Levine, Ed.), New York: IEEE Press, 1219-1242.

Cheng, D., Hu, X., Shen, T. (2010). *Analysis and Design of Nonlinear Control Systems*, Science Press Beijing and Springer-Verlag Berlin.

Feng H.Y., Guo B.Z. (2016). Active disturbance rejection control approach to output feedback stabilization for a one-dimensional anti-stable wave equation with disturbance, *submitted*, 2016.

Gao, Z. (2015). Active disturbance rejection control from an enduring idea to an emerging technology, *The 10th International Workshop on Robot Motion and Control*, IEEE, Poznan University of Technology, Poznan, Poland, July 6-8, 269-282.

Guo, B.Z., Liu,J.J. (2015) Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrodinger equation subject to boundary control matched disturbance, *Internat. J. Robust Nonlinear Control*, 24, 2194-2212.

Guo B.Z., Zhao, Z.L. (2013). On convergence of the nonlinear active disturbance rejection control for MIMO systems, *SIAM J. Control Optim.*, 51, 1727-1757.

Guo B.Z., Zhao, Z.L. (2011). On the convergence of an extended state observer for nonlinear systems with uncertainty, *Systems Control Lett.*, 60, 420-430.

Guo B.Z., Zhao, Z.L. (2015). Active disturbance rejection control: theoretical perspectives, *Communications in Information and Systems*, 15, 361-421.

Guo, B.Z., Jin, F.F. (2015). Output feedback stabilization for one-dimensional wave equation subject to boundary disturbance, *IEEE Trans. Automat. Control*, 60, 824-830.

Guo, B.Z., Jin, F.F. (2013). Sliding mode and active disturbance rejection control to stabilization of one-dimensional anti-stable wave equations subject to disturbance in boundary input, *IEEE Trans. Automat. Control*, 58, 1269-1274.

Guo, B.Z., Zhou, H.C. (2015). The active disturbance rejection control to stabilization for multi-dimensional wave equation with boundary control matched disturbance, *IEEE Trans. Automat. Control*, 60, 143-157.

Guo, B.Z., Zhou, H.C. (2016). Output feedback stabilization for multi-dimensional Kirchhoff plate with general corrupted boundary observation, *Eur. J. Control*, 28, 38-48.

Han, J.Q. (1989). Control theory: the doctrince of model or the doctrine of control?, *J. Systems Sci. Math. Sci.*, 9(4), 328-335 (in Chinese).

Han, J.Q. (1995). A class of extended state observers for uncertain systems, *Control & Decision*, 10(1)(1995), 85-88 (in Chinese).

Han, J.Q. (2009). From PID to active disturbance rejection control, *IEEE Trans. Ind. Electron.*, 56, 900-906.

Silva, G.J., Datta,A., Bhattacharyya, S.P. (2002). New Results on the synthesis of PID controllers, *IEEE Trans. Automat. Control*, 47, 241-252.

Tsien, H.S.(1954). *Engineering Cybernetics*, McGraw-Hill, New York.

Weiss, G. (1989). Admissibility of unbounded control operators, *SIAM J. Control Optim.*, 27, 527-545.

Zhao, Z.L., Guo, B.Z. (2016). On convergence of nonlinear active disturbance rejection control for a class of nonlinear systems, *J. Dyn. Control Syst.*, 22, 385-412.

Zheng, Q., Gao, L., Gao, Z. (2007). On stability analysis of active disturbance rejection control for nonlinear time-varying plants with unknow dynamics, *The 47th IEEE Conference on Decision and Control*, IEEE, New Orleans, Louisiana, 3501-3506.

Zheng, Q. Gao, Z. (2012). An energy saving, factory-validated disturbance decoupling control design for extrusion processes, *The 10th World Congress on Intelligent Control and Automation*, IEEE, Piscataway, NJ, 2891-2896.