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CHARACTERIZATION AND IDENTIFICATION OF MATRIX FRACTION DESCRIPTIONS FOR LTI SYSTEMS*

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Abstract. This paper shows that the matrix fraction description, given by a pair $\{A(z)\ B(z)\}$ of matrix polynomials of z, for a linear time-invariant system may not be unique even if $A(z)$ is monic, $A(z)$ and $B(z)$ have no common left factor, and the matrix coefficients corresponding to the highest-order terms of $A(z)$ and $B(z)$ are full row rank. The orders of all possible matrix fraction descriptions (MFDs) of a given system are completely characterized. Testing criteria for determining whether a matrix pair is an MFD of the system are derived, which involve rank tests of certain Toeplitz matrices derived from either the impulse response or output correlation functions of the system. A decision procedure is devised that generates sequentially all MFDs for a given system. Identification algorithms are introduced that estimate all MFDs of a given system from its input-output data or output data only. The results are then extended to cover ARMAX systems.

Key words. matrix fraction description, impulse response, correlation function, order characterization, Toeplitz matrix

AMS subject classifications. 93E12, 93E03, 93E10

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1. Introduction.

Consider the following multi-input-multi-output (MIMO) linear time-invariant (LTI) autoregressive moving average (ARMA) system:

$$A(z)y_k = B(z)u_k, \quad \text{where} \tag{1.1}$$

$$A(z) = I + A_1 z + \cdots + A_p z^p, \tag{1.2}$$

$$B(z) = B_0 + B_1 z + \cdots + B_q z^q \tag{1.3}$$

are matrix polynomials of the backward-shift operator $z : zy_k = y_{k-1}$ with $A_i \in R^{n\times n}$, $B_j \in R^{n\times m}$. The orders of the polynomials are denoted by (p, q). $A(z)$ is assumed to be asymptotically stable, namely, $\det A(z) \neq 0\ \forall |z| \leq 1$.

Typical system identification consists of several essential steps, including model structure selection, model order determination, input design, data acquisition, parameter estimation, model validation, etc. [16]. This paper will use the model structure (1.1) in which the pair $\{A(z)\ B(z)\}$ of order (p, q) is a matrix fraction description (MFD) of the system's transfer matrix. This paper presents new results that completely characterize the orders of these MFDs for a given system, and algorithms for identifying the matrix coefficients after order selection.

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Model order determination is a critical task in a modeling process, and has drawn extensive research effort. Most well-established information criteria for order selection are the AIC [1], the BIC [19, 20], and the ΦIC [14]. The method proposed by [15] extends the minimum description length criterion introduced in [19]. This method is later extended to order estimation of ARMA models under noisy outputs [2], and subsequently to estimation of non-Gaussian processes that are corrupted by colored additive Gaussian noises using higher order cumulants [3]. A recursive algorithm of order selection is proposed in [11] for MIMO ARMA and ARMAX models. Parameter estimation is a classical topic, and we refer the reader to [16] for a comprehensive coverage of estimation algorithms, and to [9] for detailed convergence analysis of some recursive algorithms. The strictly positive realness condition is frequently used in convergence analysis for identification algorithms.

The main contributions of this paper are in the following aspects. First, we establish nonuniqueness and a complete characterization of the MFDs of a given system. While normalized coprime factorization of a scalar ARMA system is unique, its MIMO counterpart is far more complicated and has distinct features that are important for practical identification design.

Second, this paper presents two possible approaches in identifying the MFDs. One is of deterministic nature by using impulse responses; and the other employs stochastic probing signals and correlation analysis. Algorithms and their convergence analysis lay a foundation for practical and reliable implementation of the identification process. In addition, we have shown that the algorithms remain viable under noisy observations. Finally, in comparison to the classical AIC and BIC, the algorithms here do not require prior bounds on the system orders.

These findings have several important implications. For example, since the same impulse response can have multiple MFDs, the original physical parameters may be lost since the input-output data cannot uniquely determine the MFDs. This implies that additional structural information should be used to pinpoint the physical-system-based model. On the other hand, if one is allowed to choose model structures to explain the input-output behavior, then our characterization can be used to guide model selections. It will become clear that some MFDs will have less complexity than others, leading to potential complexity reduction in treating such systems.

The paper is organized into the following sections. In section 2, the equivalent conditions of the MFD with a fixed order which represents uniquely the impulse response of the system are presented. An example illustrates that for a given impulse response there may exist multiple MFDs with different orders. In section 3, a testing criterion is derived to determine if a given MFD is unique. When the given MFD is not unique, a method is devised to detect the order ranges of the other MFDs. Algorithms for finding all additional MFDs are also presented. Section 4 provides a complete characterization of all MFDs for the impulse response of the given system. Parallel results are derived by the correlation method in section 5. In addition, parameter estimation of all the MFDs, by using either the input-output data or the output data only, is included in section 6. When the system observations are corrupted by noise, our methods are extended to ARMAX systems in section 7. Finally, section 8 concludes the paper with some remarks on specifics of results given in the paper.

2. Preliminaries.

Given a system in (1.1), stability of $A(z)$ allows us to represent the system by the transfer matrix $H(z)$ from $\{A(z)\ B(z)\}$ as

$$H(z) \triangleq A^{-1}(z)B(z) = \sum_{i=0}^{\infty} H_i z^i, \tag{2.1}$$

where $\{H_i, i = 0, 1, \ldots\}$ is the impulse response with $H_0 = B_0$, $\|H_i\| = O(\lambda^i), i > 1$, for some $0 < \lambda < 1$. Then, the output y_k in (1.1) can be expressed in a convolution sum form

$$y_k = \sum_{i=0}^{\infty} H_i u_{k-i}. \tag{2.2}$$

Conversely, given a transfer matrix $H(z) = \sum_{i=0}^{\infty} H_i z^i$, the matrix pair $\{A(z)\ B(z)\}$ satisfying (2.1) is called a matrix fraction description of $H(z)$. The system coefficients $\{A_1, \ldots, A_p, B_0, B_1, \ldots, B_q\}$ are related to the impulse response $\{H_i, i \geq 0\}$ by the equation

(2.3)
$$B_0 + B_1 z + \cdots + B_q z^q = (I + A_1 z + \cdots + A_p z^p)(H_0 + H_1 z + \cdots + H_i z^i + \cdots).$$

Identifying coefficients for the same orders of z on both sides of (2.3) implies

$$B_i = \sum_{j=0}^{p \wedge i} A_j H_{i-j}, \quad 0 \leq i \leq q, \tag{2.4}$$

$$H_i = -\sum_{j=1}^{p \wedge i} A_j H_{i-j}, \ i \geq q+1, \tag{2.5}$$

where $A_0 = I$ and $a \wedge b = \min(a, b)$. Sweeping the index $i \in \{q+1, \ldots, q+np\}$ in (2.5), we obtain

$$[A_1\ A_2\ \ldots\ A_p] L(p, q) = -[H_{q+1}\ H_{q+2}\ \ldots\ H_{q+np}], \tag{2.6}$$

where $L(p, q)$ is a Toeplitz matrix

$$L(p, q) \triangleq \begin{bmatrix} H_q & H_{q+1} & \ldots & H_{q+np-1} \\ H_{q-1} & H_q & \ldots & H_{q+np-2} \\ \vdots & \vdots & & \vdots \\ H_{q-p+1} & H_{q-p+2} & \ldots & H_{q+(n-1)p} \end{bmatrix}, \tag{2.7}$$

and the notation $H_i \triangleq 0$ for $i < 0$ is used. Define

$$\theta_A^T \triangleq [A_1\ A_2\ \ldots\ A_p], \quad W^T \triangleq -[H_{q+1}\ H_{q+2}\ \ldots\ H_{q+np}].$$

Then, it follows from (2.6) that

$$\theta_A = \left(L(p, q) L(p, q)^T\right)^{-1} L(p, q) W, \tag{2.8}$$

if $L(p, q)$ is full row rank. By solving (2.8) and (2.4) under fixed (p, q), an MFD of $H(z)$ can be obtained.

A primary question is, is there another MFD $\{X(z)\, Y(z)\}$ for the impulse response $H(z)$, which is generated from $\{A(z)\ B(z)\}$? In other words, is $\{A(z)\ B(z)\}$ unique for a given $H(z)$? This problem has been studied over several decades [21, 13, 5, 18], but a complete answer remains elusive.

Starting from the matrix pair $\{A(z)\ B(z)\}$ of orders (p, q) in (1.1) with its corresponding impulse response $H(z)$ from (2.1), denote by $\mathcal{M}(p, q)$ the set of all matrix polynomial pairs $\{X(z)\ Y(z)\}$ satisfying

(i) $X^{-1}(z)Y(z) = H(z)$, where $X(z) \in \mathbb{R}^{n\times n}$ is stable with $X(0) = I$ and $Y(z) \in \mathbb{R}^{n\times m}$;

(ii) $\deg X(z) \leq p$ and $\deg Y(z) \leq q$.

Obviously, $\{A(z)\ B(z)\} \in \mathcal{M}(p, q)$.

Without coprimeness constraints, $\mathcal{M}(p, q)$ contains infinitely many pairs. However, under coprime conditions, fixed orders, and constraint on the coefficients of the highest orders of $A(z)$ and $B(z)$, the pair becomes unique. The following conditions are equivalent in characterizing the uniqueness of $\{A(z)\ B(z)\}$ in $\mathcal{M}(p, q)$.

PROPOSITION 2.1 (see [18]). *The following statements are equivalent:*

H1: $\{A(z)\ B(z)\}$ *in* $\mathcal{M}(p, q)$ *is unique.*

H2: $A(z)$ *and* $B(z)$ *have no common left factor and the composite matrix* $[A_p\ B_q]$ *is of full row rank,*

H3: *There exist no nonzero* n*-vector polynomial* $d(z)$ *and* m*-vector polynomial* $c(z)$ *with orders strictly less than* p *and* q*, respectively, such that* $d^T(z)H(z)+c^T(z) = 0$.

H4: *The matrix* $L(p, q)$ *is full row rank.*

The main question this paper intends to answer is, if we remove the restrictions $\deg X(z) \leq p$ and $\deg Y(z) \leq q$, are there other MFDs of $H(z)$? In other words, we are seeking $\{X(z)\ Y(z)\} \neq \{A(z)\ B(z)\}$ with orders (s, t),

$$X(z) = I + X_1 z + \cdots + X_s z^s,$$
$$Y(z) = Y_0 + Y_1 z + \cdots + Y_t z^t$$

such that

(1) $X^{-1}(z)Y(z) = H(z)$,

(2) and $X(z)$ and $Y(z)$ have no common left factor with the composite matrix $[X_s\ Y_t]$ being full row rank.

The following example confirms nonuniqueness of MFDs when the order restriction is removed.

Example 2.1. Consider $\{A(z)\ B(z)\} \in \mathcal{M}(3, 1)$:

$$y_k + A_1 y_{k-1} + A_2 y_{k-2} + A_3 y_{k-3} = B_0 u_k + B_1 u_{k-1},$$

where

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 1.2 & 0.8 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.2 & 0 \\ 0.8 & -0.4 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

and $\{X(z)\ Y(z)\} \in \mathcal{M}(4, 0)$:

$$y_k + X_1 y_{k-1} + X_2 y_{k-2} + X_3 y_{k-3} + X_4 y_{k-4} = Y_0 u_k,$$

where

$$X_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & -0.5 \end{bmatrix},\ X_2 = \begin{bmatrix} 0.75 & -0.5 \\ 0.95 & 0.3 \end{bmatrix},\ X_3 = \begin{bmatrix} 0.3 & 0.4 \\ 0.9 & 0 \end{bmatrix},$$
$$X_4 = \begin{bmatrix} 0.3 & -0.2 \\ 0.3 & -0.2 \end{bmatrix},\ Y_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

It is straightforward to verify that they have the same $H(z)$ and satisfy conditions (1) and (2).

Let us denote by $\mathcal{M}$ the set of all matrix pairs $\{X(z)\ Y(z)\}$ satisfying conditions (1) and (2) if $\{A(z)\ B(z)\}$ is unique in $\mathcal{M}(p,q)$. It is clear that $\{A(z)\ B(z)\} \in \mathcal{M}$, and hence $\mathcal{M}$ is nonempty. We call (s,t) the orders of the pair $\{X(z)\ Y(z)\}$ if $\{X(z)\ Y(z)\} \in \mathcal{M}$ with $\deg X(z) = s, \deg Y(z) = t$. It is clear that each element $\{X(z)\ Y(z)\} \in \mathcal{M}$ of the orders (s,t) is also unique in $\mathcal{M}(s,t)$ via Proposition 2.1.

Remark 2.1. Both $\{A(z)\ B(z)\}$ and $\{X(z)\ Y(z)\}$ belong to $\mathcal{M}$ in Example 2.1. The complexity of a model is defined as the total number of parameters it contains. For example, the complexity of a model with orders (p,q) and input-output dimensions (n,m) is equal to $n^2p + nmq$. Therefore, the complexities of $\{A(z), B(z)\}$ and $\{X(z), Y(z)\}$ in Example 2.1 are $2^2 \times 3 + 2 \times 1 \times 2 = 16$ and $2^2 \times 4 + 2 \times 1 \times 1 = 18$, respectively. As a result, $\{A(z), B(z)\}$ is less complex than $\{X(z), Y(z)\}$.

3. Order characterization of MFDs. We consider the ranges of orders of all pairs belonging to $\mathcal{M}$, and the necessary and sufficient conditions for existence of more than one pair in $\mathcal{M}$. We first state the following lemma.

LEMMA 3.1. *Assume that $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$. Then $\{X(z)\ Y(z)\}$ of orders $(p,q+j)$ for any $j \geq 1$ also belongs to $\mathcal{M}$ if and only if $X(z) \equiv A(z)$, $Y(z) \equiv B(z)$, $Y_{q+j} = 0$, and A_p is full rank. Similarly, when $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$, then $\{X(z)\ Y(z)\}$ of orders $(p+j,q)$ for any $j \geq 1$ is also in $\mathcal{M}$ if and only if $X(z) \equiv A(z)$, $Y(z) \equiv B(z)$, $X_{p+j} = 0$, and B_q is of rank n.*

Proof. Necessity: Since both $\{A(z)\ B(z)\}$ of orders (p,q) and $\{X(z)\ Y(z)\}$ of orders $(p,q+j)$ belong to $\mathcal{M}$, $\{A(z)\ B(z)\}$ is unique in $\mathcal{M}(p,q)$ and $\{X(z)\ Y(z)\}$ is also unique in $\mathcal{M}(p,q+j)$ via the definition of $\mathcal{M}$. Noting $\mathcal{M}(p,q) \subset \mathcal{M}(p,q+j)$, we see that both $\{A(z)\ B(z)\}$ and $\{X(z)\ Y(z)\}$ are in $\mathcal{M}(p,q+j)$. However, $\{X(z)\ Y(z)\}$ is unique in $\mathcal{M}(p,q+j)$, which means that $A(z) \equiv X(z)$ and $B(z) \equiv Y(z)$. Denote the coefficient of the highest order of $Y(z)$ by Y_{q+j}. Thus $Y_{q+j} = 0$ since $B(z)$ is a polynomial of order q. The statement that $\{X(z)\ Y(z)\}$ is unique in $\mathcal{M}(p,q+j)$ derives that $[X_p\ Y_{q+j}]$ is full row rank by Proposition 2.1. As a result, $A_p = X_p$ is full rank.

Sufficiency: Suppose that $\{A(z)\ B(z)\} \in \mathcal{M}$ and is of orders (p,q). Then $A(z)$ and $B(z)$ have no common left factor. If the coefficient A_p of the highest order of $A(z)$ is full rank, then $\{A(z)\ \widetilde{B}(z)\}$ of orders $(p,q+j)$ also belongs to $\mathcal{M}$ for any $j \geq 1$ by Proposition 2.1 since $[A_p\ 0]$ is still full row rank and $A(z)$ and $\widetilde{B}(z)$ have no common left factor, where $\widetilde{B}(z) = B(z) + 0z^{q+1} + \cdots + 0z^{q+j}$.

The proof for the case B_q, the coefficient of the highest order of $B(z)$, is of rank n is similar. ∎

Based on Lemma 3.1, we call $\{A(z)\ B(z)\}$ of orders (p,q) a distinct pair in $\mathcal{M}$, if both the coefficients A_p and B_q of the highest orders of $A(z)$ and $B(z)$ are nonzero. Therefore, the number of the distinct elements in $\mathcal{M}$ is always finite and is determined by the system orders of any pair in $\mathcal{M}$. This is illustrated by the following theorem.

THEOREM 3.2. *Assume $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$.*

(i) *There exists a pair $\{X(z)\ Y(z)\} \in \mathcal{M}$ with orders (s,t) and $\{X(z)\ Y(z)\} \neq \{A(z)\ B(z)\}$ only in the cases $s > p,\ t < q$ or $s < p,\ t > q$. The total number of distinct pairs $\{X(z)\ Y(z)\} \in \mathcal{M}$ is no more than $p+q$.*

(ii) *There exists a pair $\{X(z)\ Y(z)\} \in \mathcal{M}$ of orders (s,t) with $s > p,\ t < q$ if and only if there is an $n \times n(s-p+1)$ full-row-rank matrix*

$$K \triangleq [K_{s-p}\ K_{s-p-1}\ \ldots\ K_0]$$

such that $KB(s,t) = 0$, the rows of K constitute a basis of the left kernel space of $B(s,t)$, i.e., any $n(s-p+1)$-row-vector α^T satisfying $\alpha^T B(s,t) = 0$

must be a linear combination of the rows of the matrix K, *where*

$$B(s,t) = \begin{bmatrix} B_q & B_{q-1} & \dots & B_{q-(s-p)} & \dots & B_{t-(s-p)+1} \\ 0 & B_q & \dots & B_{q-(s-p)+1} & \ddots & B_{t-(s-p)+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & B_q & \dots & B_{t+1} \end{bmatrix}, \tag{3.1}$$

and the $n \times n$ *matrix polynomial*

$$K(z) = K_0 + K_1 z + \cdots + K_{s-p} z^{s-p} \tag{3.2}$$

is unimodular, that is, $\det K(z)$ *equals a nonzero constant for all* z *on the complex plane.*

(iii) *There exists a pair* $\{X(z)\ Y(z)\} \in \mathcal{M}$ *of orders* (s,t) *with* $s < p,\ t > q$ *if and only if there is an* $n \times n(t-q+1)$ *full-row-rank matrix*

$$K' \triangleq [K'_{t-q}\ K'_{t-q-1}\ \dots\ K'_0]$$

such that $K'A(s,t) = 0$, *the rows of* K' *compose a basis of the left kernel space of* $A(s,t)$, *i.e., any* $n(t-q+1)$*-row-vector* β^T *satisfying* $\beta^T A(s,t) = 0$ *must be a linear combination of the rows of the matrix* K', *where*

$$A(s,t) = \begin{bmatrix} A_p & A_{p-1} & \dots & A_{p-(t-q)} & \dots & A_{s-(t-q)+1} \\ 0 & A_p & \dots & A_{p-(t-q)+1} & \ddots & A_{s-(t-q)+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_p & \dots & A_{s+1} \end{bmatrix}, \tag{3.3}$$

and the matrix polynomial

$$K'(z) = K'_0 + K'_1 z + \cdots + K'_{t-q} z^{t-q} \tag{3.4}$$

is unimodular.

Proof. (i) By the uniqueness of $\{X(z)\ Y(z)\} \in \mathcal{M}(s,t)$, it is impossible to have $s \geq p$ and $t \geq q$, and by the uniqueness of $\{A(z)\ B(z)\} \in \mathcal{M}(p,q)$ it is not possible to have $s < p, t \leq q$ or $s \leq p,\ t < q$. Therefore, only the cases $s > p,\ t < q$ and $s < p,\ t > q$ are possible. In the case $s > p,\ t < q$ by the uniqueness of $\{X(z)\ Y(z)\} \in \mathcal{M}(s,t)$ there is only one $X(z)$ with order s corresponding to a $Y(z)$ with order t so that $\{X(z)\ Y(z)\} \in \mathcal{M}$. Noting $0 \leq t < q$, there are at most q different pairs of $\{X(z)\ Y(z)\} \in \mathcal{M}$. Similarly, there are at most $p-1$ different pairs of $\{X(z)\ Y(z)\} \in \mathcal{M}$ in the case $s < p,\ t > q$. As a result, the number of the whole distinct matrix pairs in $\mathcal{M}$ is no more than $q + (p-1) + 1 = p + q$.

(ii) *Necessity:* Assume that $\{X(z)\ Y(z)\} \in \mathcal{M}$ with orders $(s,t) : s > p,\ t < q$, where

$$X(z) = I + X_1 z + \cdots + X_s z^s,$$
$$Y(z) = Y_0 + Y_1 z + \cdots + Y_t z^t.$$

Set $K(z) \triangleq X(z)A^{-1}(z)$. Then, we have

$$X(z) = K(z)A(z), \tag{3.5}$$

$$Y(z) = X(z)H(z) = X(z)A^{-1}(z)B(z) = K(z)B(z). \tag{3.6}$$

Since both $A(z)$ and $X(z)$ are stable, $\det K(z)$ cannot identically equal zero. Hence, we have $\operatorname{rank} K(z) = n$. Let us express $K(z)$ in the Smith–McMillan canonical form [12, 22]

$$K(z) = U(z)\mathrm{diag}\left[\frac{q_1(z)}{p_1(z)}, \frac{q_2(z)}{p_2(z)}, \dots, \frac{q_n(z)}{p_n(z)}\right] V(z)$$
$$= U(z)P^{-1}(z)Q(z)V(z), \tag{3.7}$$

where $U(z)$ and $V(z)$ are $n \times n$ unimodular matrices,

$$P(z) = \mathrm{diag}\left[p_1(z), p_2(z), \dots, p_n(z)\right],$$
$$Q(z) = \mathrm{diag}\left[q_1(z), q_2(z), \dots, q_n(z)\right]$$

with $p_i(z)$ and $q_i(z)$ being coprime $\forall i = 1, \dots, n$. Putting the expression of $K(z)$ given by (3.7) into (3.5) and (3.6) leads to

$$Q^{-1}(z)P(z)U^{-1}(z)X(z) = V(z)A(z),\ Q^{-1}(z)P(z)U^{-1}(z)Y(z) = V(z)B(z). \tag{3.8}$$

Noting that the right-hand sides of both equalities in (3.8) are matrix polynomials, we find that the ith rows of both $P(z)U^{-1}(z)X(z)$ and $P(z)U^{-1}(z)Y(z)$ must be divided by q_i $\forall i = 1, \dots, n$. Noting that q_i and p_i are coprime $\forall i = 1, \dots, n$, we find that $Q(z)$ must be a common left factor of $U^{-1}(z)X(z)$ and $U^{-1}(z)Y(z)$. In other words, both $Q^{-1}(z)U^{-1}(z)X(z)$ and $Q^{-1}(z)U^{-1}(z)Y(z)$ are matrix polynomials. Noticing that $Q^{-1}(z)$ and $P(z)$ in (3.5) are commutative, we find that $P(z)$ is a common left factor of $V(z)A(z)$ and $V(z)B(z)$. Since $A(z)$ and $B(z)$ have no common left factor, there exist matrix polynomials $M(z)$ and $N(z)$ such that $A(z)M(z) + B(z)N(z) = I$, and hence $V(z)A(z)M(z)V^{-1}(z) + V(z)B(z)N(z)V^{-1}(z) = I$. This means that $V(z)A(z)$ and $V(z)B(z)$ also have no common left factor. Consequently, $P(z)$ is unimodular. Then, from (3.7) it is seen that $K(z)$ is a matrix polynomial. Let us denote it by $K(z) \triangleq I + K_1 z + \cdots + K_r z^r$. Likewise, set $\overline{K}(z) \triangleq A(z)X^{-1}(z)$. By carrying out similar treatment as above, $\overline{K}(z)$ is also a matrix polynomial. Noting $\overline{K}(z) = K^{-1}(z)$, $K(z)$ is unimodular.

We now show $\deg K(z) = s-p$. If the highest order r of $K(z)$ is greater than $s-p$, then by comparing the coefficients of the highest order on both sides of (3.5) we have $K_r A_p = 0$. Noticing $t < q$, from (3.6), we have $K_r B_q = 0$. Since $[A_p\ B_q]$ is of full row rank, we obtain $K_r = 0$. Similarly, we derive that $K_i = 0,\ s-p+1 \le i \le r-1$. Thus we obtain that $\deg K(z) \le s-p$. By (3.5), we have $\deg X(z) \le \deg K(z) + \deg A(z)$, which implies that $\deg K(z) \ge s-p$. Therefore, we obtain $\deg K(z) = s-p$.

Denote the coefficients of $K(z)$ by $K \triangleq [K_{s-p}\ K_{s-p-1}\ \dots\ I]$. Noting $\deg Y(z) = t < q$, and comparing the coefficients of the orders $t+1 \le i \le q$ on both sides of (3.6), we have $KB(s,t) = 0$. It is clear that the n rows of K are linearly independent. It remains to show that the n rows of K are a basis of the left kernel space of $B(s,t)$ for the necessary part. Let α be a nonzero $n(s-p+1)$-vector satisfying $\alpha^T B(s,t) = 0$. Assume the converse that α^T is linearly independent of rows of K. Form an $n \times n(s-p+1)$-matrix

$$\widetilde{K} = \left[\widetilde{K}_{s-p}\ \widetilde{K}_{s-p-1}\ \dots\ \widetilde{K}_0\right],$$

which is obtained from K by replacing its any row with α^T. Since K is of full row rank and α^T is linearly independent of rows of K, the matrix $\widetilde{K}$ is also of full row rank. It is clear that $\widetilde{K}B(s,t) = 0$, $\widetilde{K} \neq K$, and $\widetilde{K}(z) \neq K(z)$, where

$$\widetilde{K}(z) \triangleq \widetilde{K}_0 + \widetilde{K}_1 z + \cdots + \widetilde{K}_{s-p} z^{s-p}.$$

We now show that $\operatorname{rank}\widetilde{K}(z) = n$. If $\operatorname{rank}\widetilde{K}(z)$ were less than n, then there would exist an n-vector $\gamma \neq 0$ such that $\gamma^T \widetilde{K}(z) = 0 \ \forall z$, and we would have $\gamma^T \widetilde{K} = 0$. But, this is impossible since $\widetilde{K}$ is of full row rank. Define $\widehat{K}(z) = K(z) + \epsilon \widetilde{K}(z) = K(z)(I + \epsilon K^{-1}(z)\widetilde{K}(z))$. Since $K(z)$ is unimodular, one can select a sufficiently small $\epsilon > 0$ such that $\widehat{K}(z)$ is stable and $\widehat{K}(0)$ is nonsingular. Set

$$\widehat{A}(z) \triangleq \widehat{K}^{-1}(0)\widehat{K}(z)A(z), \quad \widehat{B}(z) \triangleq \widehat{K}^{-1}(0)\widehat{K}(z)B(z).$$

From the above definition it is seen that $\deg \widehat{A}(z) \leq s$, and by $\widetilde{K}B(s,t) = 0$ it follows that $\deg \widehat{B}(z) \leq t$. Since $\widehat{K}(z)$ is of full rank, we have

$$\widehat{A}^{-1}(z)\widehat{B}(z) = (\widehat{K}^{-1}(0)\widehat{K}(z)A(z))^{-1}\widehat{K}^{-1}(0)\widehat{K}(z)B(z) = A^{-1}(z)B(z).$$

This means $\{\widehat{A}(z)\ \widehat{B}(z)\} \in \mathcal{M}(s,t)$. Noticing $\widetilde{K}(z) \neq K(z)$, by (3.5) and (3.6) we see that $\{\widehat{A}(z)\ \widehat{B}(z)\} \neq \{X(z)\ Y(z)\}$. However, by assumption, $\{X(z)\ Y(z)\} \in \mathcal{M}$ of order (s,t). In other words, $\{X(z)\ Y(z)\}$ is unique in $\mathcal{M}(s,t)$. The contradiction shows that there does not exist any nonzero α^T linearly independent of rows of K and satisfying $\alpha^T B(s,t) = 0$.

Sufficiency: Since $K(z)$ is unimodular, $\det K(z)$ is a nonzero constant. Thus we have $\det K_0 = \det K(z) \neq 0$ by setting $z = 0$. Define

$$X(z) \triangleq K_0^{-1}K(z)A(z), \ Y(z) \triangleq K_0^{-1}K(z)B(z). \tag{3.9}$$

We now show that $\{X(z)\ Y(z)\} \in \mathcal{M}$ of orders (s,t).

It is clear that $X(0) = I$, $\deg X(z) \leq s$, and $\deg Y(z) \leq t$ since $KB(s,t) = 0$. It is also clear that the matrix pair $\{X(z)\ Y(z)\}$ has the same impulse responses as those for $\{A(z)\ B(z)\}$. Since $A(z)$ and $B(z)$ have no common left factor, there exist matrix polynomials $M(z)$ and $N(z)$ such that $A(z)M(z) + B(z)N(z) = I$, and hence

$$K_0^{-1}K(z)A(z)M(z)K^{-1}(z)K_0 + K_0^{-1}K(z)B(z)N(z)K^{-1}(z)K_0 = I.$$

This means that neither $X(z)$ nor $Y(z)$ have a common left factor. It remains to show that the matrix $[X_s\ Y_t]$ is full row rank, where X_s and Y_t are the matrix coefficients of the highest order of $X(z)$ and $Y(z)$, respectively. Assume the converse: there exists a nonzero column vector $\alpha \in \mathbb{R}^n$ such that $\alpha^T[X_s\ Y_t] = 0$. Let S be an orthogonal matrix such that the first element of $S\alpha$ is zero. Choose β so that the first element of $S\beta$ is one and zero elsewhere. Then, the matrix polynomial $F(z) \triangleq I + \beta\alpha^T z$ is unimodular since $\det(SF(z)S^T) = 1$. Define

$$\widetilde{A}(z) \triangleq F(z)X(z) = F(z)K_0^{-1}K(z)A(z), \ \widetilde{B}(z) \triangleq F(z)Y(z) = F(z)K_0^{-1}K(z)B(z). \tag{3.10}$$

Noticing $\alpha^T X_s = 0$, we see that $\deg \widetilde{A}(z) \leq s$. Similarly, by $\alpha^T Y_t = 0$ we have $\deg \widetilde{B}(z) \leq t (< q)$.

Thus, $\{\widetilde{A}(z)\ \widetilde{B}(z)\}$ has the same impulse responses as those of $\{X(z)\ Y(z)\}$ or $\{A(z)\ B(z)\}$, and $\{\widetilde{A}(z)\ \widetilde{B}(z)\} \neq \{X(z)\ Y(z)\}$. We note that $\deg \widetilde{A}(z) > p$, because the converse assumption $\deg \widetilde{A}(z) \le p$ would lead to $\{\widetilde{A}(z)\ \widetilde{B}(z)\} \in \mathcal{M}(p,q)$. This violates the uniqueness of $\{A(z)\ B(z)\}$ in $\mathcal{M}(p,q)$. Set $\widetilde{K}(z) \triangleq F(z)K_0^{-1}K(z)$. We now show $\deg \widetilde{K}(z) = \deg \widetilde{A}(z) - p$. Let $\widetilde{K}_r$ be the corresponding coefficient of the highest order z^r in $\widetilde{K}(z)$. If $r > \deg \widetilde{A}(z) - p$, then comparing coefficients of z^r in (3.10) by noticing $\deg \widetilde{B}(z) \le t < q$ leads to $\widetilde{K}_r[A_p\ B_q] = 0$. Since $[A_p\ B_q]$ is full row rank, we conclude that $\widetilde{K}_r = 0\ \forall r > \deg \widetilde{A}(z) - p$, and hence $\deg \widetilde{K}(z) \le \deg \widetilde{A}(z) - p$. On the other hand, from (3.10) it is seen that $\deg \widetilde{K}(z) \ge \deg \widetilde{A}(z) - p$. Consequently, $\deg \widetilde{K}(z) = \deg \widetilde{A}(z) - p \le s - p$.

Denote the matrix coefficients of $\widetilde{K}(z)$ by $\widetilde{K} \triangleq [\widetilde{K}_{s-p}\ \widetilde{K}_{s-p-1}\ \dots\ I]$. By (3.9) and (3.10), we see that all row vectors of the matrix

$$N(s,t) \triangleq \begin{bmatrix} K_0^{-1}K_{s-p} & K_0^{-1}K_{s-p-1} & \dots & I \\ \widetilde{K}_{s-p} & \widetilde{K}_{s-p-1} & \dots & I \end{bmatrix}$$

belong to the left kernel space of the matrix $B(s,t)$. By the assumption of the theorem the rows of $[K_0^{-1}K_{s-p}, K_0^{-1}K_{s-p-1}, \dots, I]$ compose a basis of the left kernel space of $B(s,t)$. Then there is an $n \times n$ matrix Γ such that

$$\Gamma[K_0^{-1}K_{s-p}\ K_0^{-1}K_{s-p-1}\ \dots\ I] = [\widetilde{K}_{s-p}\ \widetilde{K}_{s-p-1}\ \dots\ I].$$

Comparing the last $n \times n$ matrix at both sides of the equality, we find Γ must be an identity matrix. However, this is impossible as can be seen from (3.9) and (3.10), since $\{\widetilde{A}(z)\ \widetilde{B}(z)\} \ne \{X(z)\ Y(z)\}$. The contradiction shows that $[X_s\ Y_t]$ is full row rank, and the proof of sufficiency is completed for the case $s > p$ and $t < q$.

(iii) For the case $s < p, t > q$ we can similarly verify that $\deg K'(z) = t - q$, $K'A(s,t) = 0$, and any nonzero $n(t-q+1)$-row-vector β^T for which $\beta^T A(s,t) = 0$ must be a linear combination of rows of K'.

The sufficiency for this case can be proved similarly to that for case (ii). ∎

THEOREM 3.3. *Assume that both the matrix pairs $\{A(z)\ B(z)\}$ of orders (p,q) and $\{X(z)\ Y(z)\}$ of orders (s,t) belong to $\mathcal{M}$, and $\{A(z)\ B(z)\} \ne \{X(z)\ Y(z)\}$. Then the orders (p,q) and (s,t) satisfy the following inequalities:*

$$(n-1)s + t \ge (n-1)p + q, \tag{3.11}$$

$$s \le p + (n-1)q \tag{3.12}$$

if $s > p$ and $t < q$; and

$$(n-1)t + s \ge (n-1)q + p, \tag{3.13}$$

$$t \le q + (n-1)p \tag{3.14}$$

if $s < p$ and $t > q$.

Proof. In the case $s > p$ and $t < q$, as proved in Theorem 3.2 (in the necessity part of (ii)), $K(z) \triangleq X(z)A^{-1}(z)$ is unimodular and $\deg K(z) = s - p$. Therefore, $\deg K^{-1}(z) = \deg Adj K(z) \le (n-1)(s-p)$. It follows from (3.6) that $B(z) = K^{-1}(z)Y(z)$, and hence we have $\deg B(z) \le \deg K^{-1}(z) + \deg Y(z)$. Thus we have $q \le (n-1)(s-p) + t$, which means $(n-1)s + t \ge (n-1)p + q$.

We rewrite (3.5) and (3.6) as

$$A(z) = K^{-1}(z)X(z), \tag{3.15}$$

$$B(z) = K^{-1}(z)Y(z). \tag{3.16}$$

Since $K(z)$ is unimodular, we have $K^{-1}(z) = I + K'_1 z + \cdots + K'_r z^r$. We now show $\deg K^{-1}(z) = q - t$. Assume the converse: $r > q - t$. From (3.16) it is seen that the coefficient of z^{r+t} is $K'_r Y_t = 0$, where Y_t is the coefficient of z^t in $Y(z)$. Similarly, the coefficient of z^{r+s} on the right-hand side of (3.15) is $K'_r X_s = 0$, because $r+s \geq s > p$. By the full row rankness of $[X_s\ Y_t]$ we have $K'_r = 0$. Therefore, $\deg K^{-1}(z) \leq q - t$. On the other hand, from (3.16) it follows that $q \leq \deg K^{-1}(z) + t$. Thus, we conclude that $\deg K^{-1}(z) = q - t$. This implies that $\deg K(z) \leq (n-1)(q-t)$, since $K(z)$ is unimodular. Noticing $\deg K(z) = s - p$, we have $s \leq p + (n-1)(q-t) \leq p + (n-1)q$. For the case $s < p$ and $t > q$, the assertions (3.13) and (3.14) can be proved in a similar way. □

COROLLARY 3.4. *Assume that both the matrix pairs $\{A(z)\ B(z)\}$ of orders (p, q) and $\{X(z)\ Y(z)\}$ of orders (s, t) belong to $\mathcal{M}$, and $\{A(z)\ B(z)\} \neq \{X(z)\ Y(z)\}$.*

(i) *If A_p is nonsingular, then the orders (s, t) of $\{X(z)\ Y(z)\}$ must be located in $\{p < s \leq p + (n-1)q,\ 0 \leq t < q\}$ and the total number of distinct pairs in $\mathcal{M}$ is no more than $q + 1$.*

(ii) *Similarly, if B_q is of rank n, then the only possible orders (s, t) of $\{X(z)\ Y(z)\}$ satisfy $\{1 \leq s < p, q < t \leq q + (n-1)p\}$ and the total number of distinct pairs in $\mathcal{M}$ is no more than p.*

(iii) *As a consequence of* (i) *and* (ii), *if both A_p and B_q are of rank n, then we cannot have that both the matrix pairs $\{A(z)\ B(z)\}$ of orders (p, q) and $\{X(z)\ Y(z)\}$ of orders (s, t) belong to $\mathcal{M}$, and $\{A(z)\ B(z)\} \neq \{X(z)\ Y(z)\}$. In other words, $\{A(z)\ B(z)\}$ is the unique distinct pair in $\mathcal{M}$. In particular, $\{A(z)\ B(z)\}$ is always unique in $\mathcal{M}$ if the system* (1.1) *is scalar.*

Proof. We first show the assertion (1). If A_p is nonsingular, then $A(s, t)$ defined by (3.3) is full row rank. By Theorem 3.2(iii) the orders (s, t) of $\{X(z)\ Y(z)\}$ cannot fit the case $s < p,\ t > q$. So, by Theorem 3.2(i) we must have $s > p,\ t < q$. Then, the total number of distinct pairs in $\mathcal{M}$ is no more than $q + 1$ via the range of the index $0 \leq t < q$, and by (3.12) $s \leq p + (n-1)q$. The proof for (ii) and (iii) is similar. □

Remark 3.1. When the number of distinct pairs in $\mathcal{M}$ is greater than 1, the user has a choice to take, for example, the one with the least number of system parameters, the one with the lowest order of the AR-part, or with the lowest order of the MA-part, etc.

Assume that $\{A(z)\ B(z)\}$ of orders (p, q) belongs to $\mathcal{M}$ and is available. Theorems 3.2 and 3.3 together with Corollary 3.4 provide an access to find all the distinct pairs in $\mathcal{M}$. By Theorem 3.2 the total number of distinct pairs in $\mathcal{M}$ is no more than $p + q$, and Theorem 3.3 and Corollary 3.4 give the range of possible orders. To be specific, either the possible orders are null or their range is located in $\{p < s \leq p + (n-1)q, 0 \leq t < q\}$, $\{1 \leq s < p, q < t \leq q + (n-1)p\}$, or $\{p < s \leq p + (n-1)q, 0 \leq t < q$ and $1 \leq s < p, q < t \leq q + (n-1)p\}$ by judging if A_p and B_q are of rank n. If the range of the possible orders is located in $\{p < s \leq p + (n-1)q, 0 \leq t < q\}$, then whether a pair of orders (s, t) belongs to $\mathcal{M}$ or not depends on whether the dimension of the left kernel space of the matrix $B(s, t)$ defined by (3.1) is n or is not. If a pair of orders (s, t) satisfies the condition, then it belongs to $\mathcal{M}$ and is equal to $\{K_0^{-1} K(z) A(z), K_0^{-1} K(z) B(z)\}$, where $K(z)$ is calculated by (3.2) in Theorem 3.2. Clearly, the possible pairs with orders (s, t) in the range $\{1 \leq s < p, q < t \leq q + (n-1)p\}$ belonging to $\mathcal{M}$ can be found in a similar way.

Example 3.1. Assume the pair $\{A(z)\ B(z)\}$ of orders $(3, 1)$ in Example 2.1 is known. Then the range of possible orders (s, t) is $3 < s \leq 3 + (2-1)1, 0 \leq t < 1$

by Corollary 3.4 since A_3 is of full rank. As a result, the orders of the unique pair other than $\{A(z)\ B(z)\}$ are $s=4, t=0$. By a straightforward calculation, the row vectors of the matrix $\left[\begin{smallmatrix} -0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{smallmatrix}\right]$ is a basis of the left kernel space of the matrix $B(4,0) = \left[\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{smallmatrix}\right]^T$ defined by (3.1).

Define $K(z) \triangleq I + K_1 z$, where $K_1 = \left[\begin{smallmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{smallmatrix}\right]$. Thus, another pair in $\mathcal{M}$ is equal to $[K(z)A(z)\ K(z)B(z)]$, which is identical to $\{X(z)\ Y(z)\}$ in Example 2.1. Therefore, the set $\mathcal{M}$ containing $\{A(z)\ B(z)\}$ has only two distinct pairs: $\{A(z)\ B(z)\}$ and $\{X(z)\ Y(z)\}$.

4. Determining MFD from impulse responses. In the last section we have discussed how to derive the pairs of $\mathcal{M}$ if $\{A(z)\ B(z)\} \in \mathcal{M}$ is available. In this section, we assume that $\{A(z)\ B(z)\} \in \mathcal{M}$ but not necessarily available, and then proceed to find all pairs in $\mathcal{M}$ only on the basis of impulse responses $\{H_i, i \geq 0\}$ of $\{A(z)\ B(z)\}$.

THEOREM 4.1. *Assume that $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$. Then, in the case $s \geq p,\ t \geq q$, the ranks of the Toeplitz matrix defined by* (2.7) *are as follows:*

$$\operatorname{rank} L(s,t) = \begin{cases} np & \text{if } s-t = p-q, \\ n(s-t+q) + \operatorname{rank} \Lambda(t-s-(q-p)) & \text{if } s-t < p-q, \\ np + \operatorname{rank} \Theta(s-t-(p-q)) & \text{if } s-t > p-q, \end{cases}$$

where

$$\Lambda(t) \triangleq \begin{bmatrix} A_p & A_{p-1} & \dots & A_{p-t+1} \\ 0 & A_p & \dots & A_{p-t+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_p \end{bmatrix} \tag{4.1}$$

is an $nt \times nt$ matrix with $A_0 = I$ and $A_i = 0$ for $i < 0$, and

$$\Theta(t) \triangleq \begin{bmatrix} B_q & B_{q-1} & \dots & B_{q-t+1} \\ 0 & B_q & \dots & B_{q-t+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B_q \end{bmatrix} \tag{4.2}$$

is an $nt \times mt$ matrix with $B_i = 0$ for $i < 0$.

Proof. (i) We first show $\operatorname{rank} L(s,t) = np$ if $s-t = p-q$. It is clear that the matrix $L(p,q)$ is a part of the last np rows of $L(s,t)$, and is full row rank by Proposition 2.1. This means that $\operatorname{rank} L(s,t) \geq np$. On the other hand, since the row vectors of the $n(s-p) \times ns$ matrix

$$\begin{bmatrix} I & A_1 & \dots & A_p & 0 & \dots & 0 \\ 0 & I & A_1 & \dots & A_p & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & A_1 & \dots & A_p \end{bmatrix}$$

all belong to the left kernel space of $L(s,t)$ by (2.5) and they are linearly independent, we have $\operatorname{rank} L(s,t) \leq ns - n(s-p) = np$. So, $\operatorname{rank} L(s,t) = np$ when $s-t = p-q$.

(ii) We prove that $\operatorname{rank} L(p,t) = n(p-(t-q)) + \operatorname{rank} \Lambda(t-q)$ for $t \geq q+1$. For this it suffices to show that the dimension of the left kernel space of $L(p,t)$ is equal to that of the left kernel space of $\Lambda(t-q)$, i.e., $np - \operatorname{rank} L(p,t) = n(t-q) - \operatorname{rank}\Lambda(t-q)$, because from here it follows that $\operatorname{rank} L(p,t) = np - (n(t-q) - \operatorname{rank}\Lambda(t-q)) = n(p-(t-q)) + \operatorname{rank}\Lambda(t-q)$. We show that there is a bijection between the left kernel spaces of $L(p,t)$ and $\Lambda(t-q)$. As a consequence, these kernel spaces have the same dimension.

We first show that there exists an injection from the left kernel space of $L(p,t)$ to that of $\Lambda(t-q)$. Assume $x = [x_1^T\ x_2^T \dots\ x_p^T]^T \neq 0$ with $x_i \in \mathbb{R}^n$ is any point of the left kernel space of $L(p,t)$. Define

$$d(z) \triangleq \sum_{i=1}^{p} x_i z^{i-1}, \tag{4.3}$$

$$\eta^T(z) \triangleq d^T(z) A^{-1}(z). \tag{4.4}$$

We want to show the following:

(a) $\eta^T(z)$ is a vector polynomial with $\deg \eta^T(z) \leq t-q-1$. If this is true, let us denote $\eta^T(z) \triangleq \eta_0^T + \eta_1^T z + \cdots + \eta_{t-q-1}^T z^{t-q-1}$ and combine the coefficients of $\eta^T(z)$ into a vector $\eta \triangleq [\eta_{t-q-1}^T\ \eta_{t-q-2}^T\ \dots\ \eta_0^T]^T$.
Define the linear mapping $f : x \to \eta$;

(b) η belonging to the left kernel space of $\Lambda(t-q)$;

(c) any x belonging to the left kernel space of $L(p,t)$ and satisfying $f(x) = 0$ must be zero.

Let us prove these assertions.

(a) We prove that $\eta^T(z)$ is a vector polynomial with $\deg \eta^T(z) \leq t-q-1$. We first show

$$\sum_{j=1}^{p} x_j^T H_{t-j+l} = 0 \quad \forall l \geq 1. \tag{4.5}$$

By the assumption that $x = [x_1^T\ x_2^T\ \dots\ x_p^T]^T \neq 0$ belongs to the left kernel space of $L(p,t)$, we have (4.5) for $1 \leq l \leq np$. Noting

$$H(z) = A^{-1}(z)B(z) = \frac{A^*(z)B(z)}{a(z)} = \frac{B^*(z)}{a(z)}, \tag{4.6}$$

where $a(z) \triangleq \det A(z) = \sum_{i=0}^{np} a_i z^i$, $A^*(z)$ is the adjoint matrix of $A(z)$, and $B^*(z) \triangleq A^*(z)B(z) = \sum_{j=0}^{(n-1)p+q} B_j^* z^j$, we have

$$\begin{aligned}(1 + a_1 z + \cdots + a_{np} z^{np})(H_0 + H_1 z + \cdots + H_i z^i + \cdots)\\ = B_0^* + B_1^* z + \cdots + B_{(n-1)p+q}^* z^{(n-1)p+q}.\end{aligned} \tag{4.7}$$

Identifying coefficients for the same degrees of z on both sides of (4.7), we obtain

$$H_j = -\sum_{i=1}^{np} a_i H_{j-i} \quad \forall\ j > q + (n-1)p. \tag{4.8}$$

In the case that $l = np+1$, we have

$$\begin{aligned}\sum_{j=1}^{p} x_j^T H_{t-j+np+1} &= -\sum_{j=1}^{p} x_j^T \sum_{i=1}^{np} a_i H_{t-j+np+1-i}\\ &= -\sum_{i=1}^{np} a_i \sum_{j=1}^{p} x_j^T H_{t-j+np+1-i} = 0.\end{aligned} \tag{4.9}$$

Thus (4.5) holds for $l = np+1$. As a result, we can show inductively that (4.5) holds. From (4.5) it follows that

$$\begin{aligned} d^T(z)H(z) &= \sum_{i=1}^{p}\left(x_i^T z^{i-1}\left(\sum_{j=0}^{t-i} H_j z^j + \sum_{j=t-i+1}^{\infty} H_j z^j\right)\right) \\ &= \sum_{i=1}^{p}\sum_{j=0}^{t-i} x_i^T H_j z^{i+j-1} + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{p} x_i^T H_{t-i+k}\right) z^{t+k-1} \\ &= \sum_{i=1}^{p}\sum_{j=0}^{t-i} x_i^T H_j z^{i+j-1} \triangleq c^T(z), \end{aligned} \tag{4.10}$$

and the orders of $d(z)$ and $c(z)$ are strictly less than p and t, respectively. Consequently, we have

$$d^T(z)A^{-1}(z)B(z) = c^T(z). \tag{4.11}$$

Since $A(z)$ and $B(z)$ have no common left factor and $c^T(z)$ is a vector polynomial, the zeros of $A(z)$ must be canceled with $d^T(z)$. Therefore, $\eta^T(z)$ is a vector polynomial denoted by $\eta^T(z) \triangleq \eta_0^T + \eta_1^T z + \cdots + \eta_r^T z^r$.

By (4.4) and (4.11) we have

$$\eta^T(z)A(z) = d^T(z), \tag{4.12}$$

$$\eta^T(z)B(z) = c^T(z). \tag{4.13}$$

Comparing the coefficients of the highest order on both sides of (4.12) and (4.13), respectively, we have $\eta_r^T A_p = 0$ and $\eta_r^T B_q = 0$. Since $[A_p \ B_q]$ has full row rank, it derives that $\eta_r^T = 0$. Noting $\deg d(z) < p$ and $\deg c(z) < t$, it follows that $\eta_i^T = 0 \ \forall t-q \le i \le r-1$ inductively, and hence we conclude that $\deg \eta^T(z) \le t-q-1$.

(b) We show that η belongs to the left kernel space of $\Lambda(t-q)$. Since $\deg d^T(z) < p$, from (4.12) we see the coefficients of z^i in $\eta^T(z)A(z)$ equal zero for $i \ge p$. Consequently we have

$$\begin{cases} \eta_{t-q-1}^T A_p = 0, \\ \eta_{t-q-1}^T A_{p-1} + \eta_{t-q-2}^T A_p = 0, \\ \qquad\qquad \vdots \\ \eta_{t-q-1}^T A_{p-t+q+1} + \eta_{t-q-2}^T A_{p-t+q+2} + \cdots + \eta_0^T A_p = 0, \end{cases} \tag{4.14}$$

which can be rewritten in the following matrix form:

$$\eta^T \Lambda(t-q) = 0. \tag{4.15}$$

This means that η^T belongs to the left kernel space of $\Lambda(t-q)$.

(c) Using $\eta = f(x) = 0$ we obtain $d^T(z)A^{-1}(z) = \eta^T(z) = 0$ by (4.4). This implies that $d^T(z) = 0$, and hence $x = 0$. Therefore, the mapping f from the left kernel space of $L(p,t)$ to the left kernel space of $\Lambda(t-q)$ is an injection.

We show that there exists an injection from the left kernel space of $\Lambda(t-q)$ to that of $L(p,t)$. Let $\eta^T \triangleq [\eta_{t-q-1}^T \ \eta_{t-q-2}^T \ \ldots \ \eta_0^T]$ belong to the left kernel space of

$\Lambda(t-q)$. Define the vector polynomials

$$\eta^T(z) \triangleq \sum_{i=1}^{t-q} \eta_{i-1}^T z^{i-1}, \tag{4.16}$$

$$d^T(z) \triangleq \eta^T(z) A(z). \tag{4.17}$$

It is clear that the row vector η^T satisfies the matrix equation (4.15), which implies that the coefficients of z^i in $d^T(z)$ are zero for $i \geq p$. Write $d^T(z) \triangleq \sum_{i=1}^{p} x_i^T z^{i-1}$, and form the vector $x = [x_1^T \; x_2^T \; \ldots \; x_p^T]^T \in \mathbb{R}^{np}$ from the coefficients of $d^T(z)$. We are ready to show that $x^T L(p,t) = 0$, and the linear mapping $\widetilde{f} : \eta \to x$ is an injection. From (4.17) it follows that

$$d^T(z) H(z) = d^T(z) A^{-1}(z) B(z) = \eta^T(z) B(z). \tag{4.18}$$

The left side of (4.18) can be expressed as

$$d^T(z) H(z) = \sum_{i=1}^{p} \sum_{j=0}^{\infty} x_i^T H_j z^{i+j-1} = \sum_{j=0}^{\infty} \left(\sum_{i=1}^{p} x_i^T H_{j-i+1} \right) z^j. \tag{4.19}$$

It is clear that $\deg d^T(z) H(z) = \deg \eta^T(z) B(z) < t$, which implies the coefficients of z^i in $d^T(z)H(z)$ equal zero for $i \geq t$. In other words, we have

$$\sum_{i=1}^{p} x_i^T H_{t-i+l} = 0 \;\forall\; l \geq 1, \tag{4.20}$$

which means that $x^T L(p,t) = 0$, or x^T belongs to the left kernel space of $L(p,t)$.

To prove that the linear mapping $\widetilde{f}$ is an injection, it suffices to show that $\eta = 0$ for any η belonging to the left kernel space of $\Lambda(t-q)$ and $\widetilde{f}(\eta) = 0$. Since $x = \widetilde{f}(\eta) = 0$, we see that $\eta^T(z) A(z) = d^T(z) = 0$. This means that $\eta^T(z) = 0$, and hence $\eta = 0$.

Thus, we have shown that the left kernel space of $L(p,t)$ and the left kernel space of $\Lambda(t-q)$ are in one-to-one correspondence, and hence $\operatorname{rank} L(p,t) = n(p-(t-q)) + \operatorname{rank} \Lambda(t-q)$ for $t \geq q+1$. The proof of (ii) is completed.

(iii) We show $\operatorname{rank} L(s,t) = n(s-t+q) + \operatorname{rank} \Lambda(t-s-(q-p))$ when $s-t < p-q$, while the special case $s = p$ has been proved in (ii). Since the matrix $L(p, t-(s-p))$ is the first nmp columns of the last np rows of $L(s,t)$, $\operatorname{rank} L(s,t) \geq \operatorname{rank} L(p, t-(s-p))$ by (ii).

On the other hand, let the rows of the $[np - \operatorname{rank} L(p, t-(s-p))] \times np$ matrix $\mathcal{G} = [\mathcal{G}_1 \; \ldots \; \mathcal{G}_p]$ be linearly independent and compose a basis of the left kernel space of the matrix $L(p, t-(s-p))$. Noticing that $L(p, t-(s-p))$ is the first nmp columns of the last np rows of $L(s,t)$, it is seen that all rows of $\mathcal{G}$ belong to the left kernel space of the last np rows of the matrix $L(s,t)$ in terms of (4.8).

Consequently, by (2.5) all rows of the $[n(s-p) + (np - \operatorname{rank} L(p, t-(s-p)))] \times ns$ matrix

$$\begin{bmatrix} I & A_1 & \ldots & A_p & 0 & \ldots & 0 \\ 0 & I & A_1 & \ldots & A_p & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & I & A_1 & \ldots & A_p \\ 0 & \ldots & 0 & 0 & \mathcal{G}_1 & \ldots & \mathcal{G}_p \end{bmatrix}$$

TABLE 1
The ranks of the Toeplitz matrices.

rank $L(s,t)$ (s \ t)	q	$q+1$	$q+2$	$q+3$	$\dots$
p	np	$n(p-1)+\operatorname{rank}A_p$	$n(p-2)+\operatorname{rank}\Lambda(2)$	$n(p-3)+\operatorname{rank}\Lambda(3)$	$\dots$
$p+1$	$np+\operatorname{rank}B_q$	np	$n(p-1)+\operatorname{rank}A_p$	$n(p-2)+\operatorname{rank}\Lambda(2)$	$\ddots$
$p+2$	$np+\operatorname{rank}\Theta(2)$	$np+\operatorname{rank}B_q$	np	$n(p-1)+\operatorname{rank}A_p$	$\ddots$
$p+3$	$np+\operatorname{rank}\Theta(3)$	$np+\operatorname{rank}\Theta(2)$	$np+\operatorname{rank}B_q$	np	$\ddots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

belong to the left kernel space of $L(s,t)$, and they are linearly independent. Therefore, $\operatorname{rank}L(s,t)\le ns-\big(n(s-p)+np-\operatorname{rank}L(p,t-(s-p))\big)=\operatorname{rank}L(p,t-(s-p))$.

Thus, we have shown that $\operatorname{rank}L(s,t)=\operatorname{rank}L(p,t-(s-p))=n(s-t+q)+\operatorname{rank}\Lambda(t-s-(q-p))$ when $s-t<p-q$.

(iv) The assertion of the theorem for the case $s-t>p-q$ can be proved similarly to the case $s-t<p-q$. ∎

Remark 4.1. The results of Theorem 4.1 are summarized in Table 1 when $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$.

THEOREM 4.2. *Assume that $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$. Then, a matrix pair $\{X(z)\,Y(z)\}$ of orders (s,t) belongs to $\mathcal{M}$ if and only if* $\operatorname{rank}L(s+i,t+i)=ns$ *for $i=0,1$, (or, equivalently, the dimension of the left kernel space of $L(s+i,t+i)$ equals ni for $i=0,1$).*

Proof. Necessity: Assume the matrix pair $\{X(z)\ Y(z)\}$ of orders (s,t) belongs to $\mathcal{M}$. By Proposition 2.1, $\{X(z)\ Y(z)\}$ is unique in $\mathcal{M}(s,t)$ and $\operatorname{rank}L(s,t)=ns$, and by Theorem 4.1 $\operatorname{rank}L(s+i,t+i)=ns\ \forall i>0$.

Sufficiency: By assumption $\{A(z)\ B(z)\}$ of orders (p,q) belongs to $\mathcal{M}$, it is seen that $\mathcal{M}$ is nonempty, and hence we denote the impulse response of $\{A(z)\ B(z)\}$ by $\{H_i,i\ge 0\}$. Without loss of generality, we may assume that p is the smallest order of $X(z)$ for all matrix pairs $\{X(z)\ Y(z)\}$ belonging to $\mathcal{M}$. In other words, $p\le s$ for any matrix pair $\{X(z)\ Y(z)\}$ of orders (s,t) belonging to $\mathcal{M}$. Under the assumption $\operatorname{rank}L(s+i,t+i)=ns$ for $i=0,1$, we want to construct a matrix pair $\{X(z)\ Y(z)\}$ of orders (s,t) based on the impulse responses $\{H_i\}$ so that $\{X(z)\ Y(z)\}\in\mathcal{M}$. Since $\operatorname{rank}L(s+1,t+1)=ns$ and $L(s+1,t+1)$ is an $n(s+1)\times mn(s+1)$ matrix, there exists an $n\times n(s+1)$ partitioned matrix $X\triangleq[X_0\ X_1\ \dots\ X_s]$ with rank n such that $XL(s+1,t+1)=0$.

We show that $X_0\in\mathbb{R}^{n\times n}$ is nonsingular. If X_0 were singular, then there would a nonsingular matrix S such that the last row of SX_0 would be zero. Denote the last row of the matrix $SX=S[X_0\ X_1\ \dots\ X_s]$ by $x=[0\ x_1\dots x_s]$. Thus we would have that $[x_1\dots x_s]L(s,t)=0$ with $[x_1\dots x_s]\ne 0$ since $\operatorname{rank}(X)=n$. This contradicts the assumption of full row rankness: $\operatorname{rank}L(s,t)=ns$. Therefore, X_0 is nonsingular.

Without loss of generality, we use $X=[I\ X_1\ \dots\ X_s]$ to denote the matrix $X_0^{-1}X$.

From $XL(s+1,t+1)=0$ it follows that

$$\sum_{j=0}^{s}X_jH_{t+l-j}=0\ \forall\ 1\le l\le n(s+1). \tag{4.21}$$

We show that (4.21) holds for $l\ge 1$.

Let us first show for the case $l = n(s+1)+1$. By (3.11) and (4.8), we have

$$\begin{aligned}\sum_{j=0}^{s} X_j H_{t+n(s+1)+1-j} &= -\sum_{j=0}^{s} X_j \sum_{i=1}^{np} a_i H_{t+n(s+1)+1-j-i} \\ &= -\sum_{i=1}^{np} a_i \left(\sum_{j=0}^{s} X_j H_{t+n(s+1)+1-j-i} \right) = 0,\end{aligned} \tag{4.22}$$

where the last equation is obtained by (4.21). Thus, inductively we are convinced that (4.21) is true for $l \geq 1$. Therefore,

$$\operatorname{rank} L(s+i, t+i) = ns \quad \forall\, i \geq 0.$$

Using the impulse responses $\{H_i, i \geq 0\}$ and $\{X_i,\ 1 \leq i \leq s\}$ define the matrices $\{Y_i,\ 0 \leq i \leq t\}$ as follows:

$$Y_i = \sum_{j=0}^{s} X_j H_{i-j} \quad \forall\, 0 \leq i \leq t, \tag{4.23}$$

where $X_0 = I$. Then, form the matrix pair $\{X(z)\ Y(z)\}$, where

$$X(z) \triangleq I + X_1 z + \cdots + X_s z^s, \tag{4.24}$$

$$Y(z) \triangleq Y_0 + Y_1 z + \cdots + Y_t z^t. \tag{4.25}$$

Noting that (4.23) is similar to (2.4), we find that the impulse responses $\{\widetilde{H}_i, i \geq 0\}$ of $\{X(z)\ Y(z)\}$ are equal to

$$\widetilde{H}_i = Y_i - \sum_{j=1}^{s} X_j \widetilde{H}_{i-j} \quad \forall\, 0 \leq i \leq t, \tag{4.26}$$

$$\widetilde{H}_i = -\sum_{j=1}^{s} X_j \widetilde{H}_{i-j} \ \forall\, i \geq t+1. \tag{4.27}$$

We proceed to show that $\{\widetilde{H}_i, i \geq 0\}$ coincides with the impulse responses $\{H_i, i \geq 0\}$ of $\{A(z)\ B(z)\}$. We first prove that $\widetilde{H}_i = H_i$ for $0 \leq i \leq t$ by induction. By (4.23) and (4.26) for $i = 0$, we have $\widetilde{H}_0 = Y_0 = H_0$. Inductively, assume that $\widetilde{H}_i = H_i$ for $0 \leq i \leq r\ (< t)$. By (4.23) and (4.26), we have

$$\widetilde{H}_{r+1} = Y_{r+1} - \sum_{j=1}^{s} X_j \widetilde{H}_{r+1-j} = Y_{r+1} - \sum_{j=1}^{s} X_j H_{r+1-j} = Y_{r+1} - (Y_{r+1} - H_{r+1}) = H_{r+1}.$$

Therefore, $\widetilde{H}_i = H_i$ for $0 \leq i \leq t$. It remains to show $\widetilde{H}_i = H_i$ for $i \geq t+1$. We complete this by induction.

In the case that $i = t+1$, we have

$$\widetilde{H}_{t+1} = -\sum_{j=1}^{s} X_j \widetilde{H}_{t+1-j} = -\sum_{j=1}^{s} X_j H_{t+1-j} = H_{t+1},$$

Table 2
Search way.

$s \backslash t$	0	1	2	3	4	...
1	$\xrightarrow{1}\uparrow$	$4\uparrow$	$9\uparrow$	$16\uparrow$	$25\uparrow$	$\vdots$
2	$\xrightarrow{2}$	$\xrightarrow{3}\uparrow$	$8\uparrow$	$15\uparrow$	$24\uparrow$	$\vdots$
3	$\xrightarrow{5}$	$\xrightarrow{6}$	$\xrightarrow{7}\uparrow$	$14\uparrow$	$23\uparrow$	$\vdots$
4	$\xrightarrow{10}$	$\xrightarrow{11}$	$\xrightarrow{12}$	$\xrightarrow{13}\uparrow$	$22\uparrow$	$\vdots$
5	$\xrightarrow{17}$	$\xrightarrow{18}$	$\xrightarrow{19}$	$\xrightarrow{20}$	$\xrightarrow{21}\uparrow$	$\vdots$
$\vdots$	...	...	...	...	...	...

where the first equality is by (4.27), the second equality is because we have proved that $\widetilde{H}_i = H_i$ for $0 \le i \le t$, while the last equality is because (4.21) is valid for all $l \ge 1$. Inductively, assume that $\widetilde{H}_i = H_i$ for $t+1 \le i \le r$. We have

$$\widetilde{H}_{r+1} = -\sum_{j=1}^{s} X_j \widetilde{H}_{r+1-j} = -\sum_{j=1}^{s} X_j H_{r+1-j} = H_{r+1},$$

where the first equality is by (4.27), the second equality is by the inductive assumption, while the last equality is because (4.21) is valid for all $l \ge 1$. Thus, we have shown that the impulse responses of $\{X(z)\ Y(z)\}$ are the same as those of $\{A(z)\ B(z)\}$. Since $L(s,t)$ is full row rank, by Proposition 2.1, $\{X(z)\ Y(z)\}$ is unique in $\mathcal{M}(s,t)$, and hence $\{X(z)\ Y(z)\} \in \mathcal{M}$. The proof is completed. $\square$

Remark 4.2. If $\mathcal{M}$ is nonempty, Theorem 3.3 characterizes the possible orders of the pairs contained in $\mathcal{M}$. All the pairs in $\mathcal{M}$ can be obtained by the available parameters $\{A_1, \cdots, A_p, B_0, \cdots, B_q\}$ of $\{A(z)\ B(z)\}$ by Theorem 3.2. In the proof of Theorem 4.2 a concrete method of finding a pair in $\mathcal{M}$ is described. Therefore, all the pairs in $\mathcal{M}$ can be found by using the impulse responses $\{H_i\}$. According to Theorem 4.2, to determine if a pair of orders (s,t) with $s \ge 1, t \ge 0$ is contained in $\mathcal{M}$ or not, we need only to check the rank conditions $\operatorname{rank} L(s,t) = ns$ and $\operatorname{rank} L(s+1,t+1) = ns$. If the conditions are satisfied, then the pair of orders (s,t) belongs to $\mathcal{M}$. Since the upper bound for orders of the pairs in $\mathcal{M}$ is unknown, we may try to first find a pair with orders (s,t), by searching along the lower and right edges of expanding squares as shown in Table 2: (1,0); (2,0), (2,1) (1,1); (3,0), (3,1), (3,2), (2,2), (1,2); (4,0), (4,1), (4,2), (4,3), (3,3), (2,3), (1,3); and so on. The searching process continues until the rank conditions are satisfied by a pair, which will serve as the first pair in $\mathcal{M}$. Then, the range of possible orders of other distinct pairs in $\mathcal{M}$ is obtained by Theorem 3.3 and Corollary 3.4. Thus, other distinct pairs can be found by checking the rank conditions given by Theorem 4.2 within the finite range.

In the following, pairs in $\mathcal{M}$ are numbered in the order of the above searching sequence; and the number of distinct pairs in $\mathcal{M}$ will be denoted by n_a, and the ith distinct pair in $\mathcal{M}$ by $\{X^i(z), Y^i(z)\}$ of orders (p^i, q^i).

Example 4.1. Consider the model in Example 2.1. The ranks of the Toeplitz matrices consisting of the impulse responses are illustrated in Table 3, and orders of the pairs in $\mathcal{M}$ are circled in Table 3. The pairs of orders $(3, 1+j)$ for any $j \ge 1$ are in $\mathcal{M}$, and so are the pairs of orders $(3,1)$ and $(4,0)$. This is consistent with our

TABLE 3
The ranks of the Toeplitz matrices.

rank $L(s,t)$ (s \ t)	0	1	2	3	4	5	6	7	$\dots$
1	2	2	2	2	2	2	2	2	$\dots$
2	4	4	4	4	4	4	4	4	$\ddots$
3	6	6	6	6	6	6	6	6	$\ddots$
4	8	7	6	6	6	6	6	6	$\ddots$
5	9	8	7	6	6	6	6	6	$\ddots$
6	10	9	8	7	6	6	6	6	$\ddots$
7	11	10	9	8	7	6	6	6	$\ddots$
8	12	11	10	9	8	7	6	6	$\ddots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

analysis. By the searching process given above, the orders of the first found pair are $(3,1)$, followed by $(4,0)$.

5. Determining MFD from correlation functions. Let us impose the following conditions on system (1.1):

B1: $\{u_k\}$ is a sequence of independent and identically distributed (i.i.d.) random vectors with $Eu_k = 0$, $E\|u_k\|^{2+\delta} < \infty$ for some $\delta > 0$, and $Eu_k u_k^T = I$;

B2: $A(z)$ and $B(z)B^T(z^{-1})z^q$ have no common left factor, $[A_p \ B_q]$ is full row rank, B_0 is full column rank, and $\det A(z) \neq 0 \ \forall |z| \leq 1$.

Under these conditions, without loss of generality, we may assume the output $\{y_k\}$ is a sequence of zero mean stationary random vectors with the correlation functions $R_i \triangleq Ey_k y_{k-i}^T$. Multiplying $y_{k-t}^T, t \geq q+1$ on both sides of (1.1) from the right and taking the expectation, we obtain

$$\begin{aligned} &E(y_k + A_1 y_{k-1} + \cdots + A_p y_{k-p}) y_{k-t}^T \\ &\qquad = E(B_0 u_k + B_1 u_{k-1} + \cdots + B_q u_{k-q}) y_{k-t}^T = 0 \ \ \forall\, t \geq q+1, \end{aligned}$$

which yields

$$\sum_{i=0}^{p} A_i R_{q-i+l} = 0 \ \forall\ l \geq 1. \tag{5.1}$$

Choosing the indices $1 \in \{1 \leq 1 \leq np\}$ in (5.1), we have the following linear algebraic equation called the Yule–Walker equation:

$$[A_1 \ A_2 \ \dots \ A_p]\Gamma(p,q) = -[R_{q+1} \ R_{q+2} \ \dots \ R_{q+np}], \tag{5.2}$$

where

$$\Gamma(p,q) \triangleq \begin{bmatrix} R_q & R_{q+1} & \dots & R_{q+np-1} \\ R_{q-1} & R_q & \dots & R_{q+np-2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{q-p+1} & R_{q-p+2} & \dots & R_{q+(n-1)p} \end{bmatrix}. \tag{5.3}$$

Similarly to (2.8) we can rewrite (5.2) as

$$\theta_A = (\Gamma(p,q)\Gamma(p,q)^T)^{-1}\Gamma(p,q)W \tag{5.4}$$

whenever $\Gamma(p,q)$ is full row rank, where

$$\theta_A^T \triangleq [A_1\ A_2\ \ldots\ A_p] \text{ and } W^T \triangleq -[R_{q+1}\ R_{q+2}\ \ldots\ R_{q+np}].$$

From (2.2) it follows that

$$\begin{aligned} R_j = Ey_k y_{k-i}^T &= E\left(\sum_{i=0}^{\infty} H_i u_{k-i}\right)\left(\sum_{l=0}^{\infty} H_l u_{k-j-l}\right)^T \\ &= \left(\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} H_i E u_{k-i} u_{k-j-l}^T H_l^T\right) = \sum_{l=0}^{\infty} H_{j+l} H_l^T. \end{aligned} \tag{5.5}$$

Set $\xi_k \triangleq B(z)u_k$. Then the spectral density of ξ_k is

$$\Phi^\xi(z) = B(z)B^T(z^{-1}),$$

while the spectral density of $\{y_k\}$ given by (1.1)–(1.3) is

$$\Phi(z) \triangleq \sum_{j=-\infty}^{\infty} R_j z^j = A^{-1}(z)B(z)B^T(z^{-1})A^{-T}(z^{-1}), \tag{5.6}$$

which implies

$$\Phi^\xi(z) = B(z)B^T(z^{-1}) = A(z)\Phi(z)A^T(z^{-1}). \tag{5.7}$$

Since the right-hand side of (5.7) is equal to

$$\begin{aligned} A(z)\Phi(z)A^T(z^{-1}) &= \sum_{i=0}^{p} A_i z^i \sum_{k=-\infty}^{\infty} R_k z^k \sum_{j=0}^{p} A_j^T z^{-j} \\ &= \sum_{i=0}^{p}\sum_{k=-\infty}^{\infty}\sum_{j=0}^{p} A_i R_k A_j^T z^{i+k-j} \\ &= \sum_{k=-\infty}^{\infty}\left(\sum_{i=0}^{p}\sum_{j=0}^{p} A_i R_{k+j-i} A_j^T\right) z^k, \end{aligned} \tag{5.8}$$

we have

$$B(z)B^T(z^{-1}) = \sum_{k=-q}^{q}\left(\sum_{i=0}^{p}\sum_{j=0}^{p} A_i R_{k+j-i} A_j^T\right) z^k. \tag{5.9}$$

Therefore, to derive the coefficients of $B(z)$ it is a matter of factorizing the right-hand-side of (5.9). Similarly to Proposition 2.1, we have the following result.

PROPOSITION 5.1 (see [18]). *Assume that* $\{A(z)\ B(z)\} \in \mathcal{M}(p,q)$ *and* B_0 *is full column rank. Then, the following* H5 *and* H6 *are equivalent.*

H5: *The matrix* $\Gamma(p,q)$ *defined by* (5.3) *is full row rank.*

H6: *The matrix polynomials* $A(z)$ *and* $B(z)B^T(z^{-1})z^q$ *have no common left factor and the matrix* $[A_p\ B_q]$ *is full row rank.*

This proposition gives the necessary and sufficient condition for recovering the parameters of the AR-part of the system (1.1) by use of the correlation functions if the orders (p, q) are available. The condition that B_0 is of full column rank implies that the dimension n of the output y_k is greater than or equal to the dimension m of the input u_k, i.e., $(m \leq n)$, but this is not a restriction. Let us explain this. In the case $m > n$, the spectral density $B(z)B^T(z^{-1})$ of ξ_k is nonnegative definite on the unit circle $|z| = 1$. Let us denote the rank of $B(z)B^T(z^{-1})$ by $\widetilde{m}$. It is clear that $\widetilde{m} \leq n$. By the innovation representation[4],[18, Lemmas 4 and 5], there exists an $n \times \widetilde{m}$ matrix polynomial $\widetilde{B}(z)$ and the $\widetilde{m}$ dimensional innovation process $\{\widetilde{u}_k\}$ with $E\widetilde{u}_k\widetilde{u}_k^T = I_{\widetilde{m}}$ such that $B(z)B^T(z^{-1}) = \widetilde{B}(z)\widetilde{B}^T(z^{-1})$ and the constant term $\widetilde{B}_0$ of $\widetilde{B}(z)$ is of full column rank. As a result, the spectral densities of these two systems $\{A(z)y_k = B(z)u_k\}$ and $\{A(z)y_k = \widetilde{B}(z)\widetilde{u}_k\}$ are the same, and hence their correlation functions are also identical. This means that these two systems cannot be distinguished by the correlation functions of the output. Therefore, we can think of these two systems as equivalent from the point of correlation functions. Denote the highest nonzero coefficient of $\widetilde{B}(z)$ by $\widetilde{B}_{\widetilde{q}}$ with $\widetilde{q} \leq q$. Noticing $\widetilde{m} \leq n$ and that the constant term $\widetilde{B}_0$ of $\widetilde{B}(z)$ is of full column rank, by referring to [18, Theorem 2′], we see that the AR-part $A(z)$ of $\{A(z)y_k = B(z)u_k\}$ or $\{A(z)y_k = \widetilde{B}(z)\widetilde{u}_k\}$ can be uniquely determined if $[A_p\ \widetilde{B}_{\widetilde{q}}]$ is of full rank and the matrix polynomials $A(z)$ and $B(z)B^T(z^{-1})z^q$ have no common left factor. However, in general, one has no method to derive the original $B(z)$ only by using correlation functions without additional information about $B(z)$.

Likewise, there may also exist other matrix pairs which shares the same correlation functions with $\{A(z)\ B(z)\}$. Let us by $\mathcal{N}$ denote the totality of the matrix pairs $\{X(z)\ Y(z)\}$ of orders (s, t) which satisfy the following conditions:

(1) The correlation functions of $\{X(z)\ Y(z)\}$ and those of $\{A(z)\ B(z)\}$ are identical;

(2) $X(z)$ and $Y(z)Y^T(z^{-1})z^t$ have no common left factor, $[X_s\ Y_t]$ is full row rank, and Y_0 is full column rank, where

$$X(z) = I + X_1 z + \cdots + X_s z^s,$$
$$Y(z) = Y_0 + Y_1 z + \cdots + Y_t z^t.$$

Remark 5.1. In section 3 we have shown that if $\{X(z)\ Y(z)\}$ of orders (s, t) belongs to $\mathcal{M}$, then there is no other pair of orders (s, t) in $\mathcal{M}$ other than $\{X(z)\ Y(z)\}$. However, this is not the case for $\mathcal{N}$. To see this, let us take the following example. Assume $A(z)$ is an arbitrary one-dimensional stable polynomial. Let us take $B(z) = B_1(z) \stackrel{\Delta}{=} 1 + 2z$ and $B(z) = B_2(z) \stackrel{\Delta}{=} 2(1 + 0.5z)$. Then, it is straightforward to verify $B_1(z)B_1(z^{-1}) = B_2(z)B_2(z^{-1})$. This means that the matrix pairs $\{A(z)\ B_1(z)\}$ and $\{A(z)\ B_2(z)\}$ share the same spectral function, and hence the same correlation functions. Therefore, if $\{X(z)\ Y(z)\}$ of orders (s, t) belonging to $\mathcal{N}$, there may exist a distinct pair $\{X(z)\ Y_1(z)\}$ of orders (s, t) belonging to $\mathcal{N}$. However, their $X(z)$ is unique.

Theorem 3.3 and Corollary 3.4 are concerned with $\mathcal{M}$; for $\mathcal{N}$, we have similar results formulated as follows.

THEOREM 5.1. *Assume that both the matrix pairs $\{A(z)\ B(z)\}$ of orders (p, q) and $\{X(z)\ Y(z)\}$ of orders (s, t) belong to $\mathcal{N}$ and $X(z) \neq A(z)$. Then the orders (p, q) and (s, t) satisfy the following inequalities:*

$$(n-1)s + t \geq (n-1)p + q, \tag{5.10}$$
$$s \leq p + (n-1)q \tag{5.11}$$

if $s > p$ *and* $t < q$*;*

$$(n-1)t + s \geq (n-1)q + p, \tag{5.12}$$
$$t \leq q + (n-1)p \tag{5.13}$$

if $s < p$ *and* $t > q$*. Moreover,* $1 \leq s < p$ *and* $q < t \leq q + (n-1)p$ *if* A_p *is nonsingular, and* $p < s \leq p + (n-1)q$ *and* $0 \leq t < q$ *if* $\operatorname{rank} B_q = n$*. Finally, if both the ranks of* A_p *and* B_q *equal* n*, then* $X(z) \equiv A(z)$ *and* $Y(z)Y^T(z^{-1})$ *are identical for all* $\{X(z)\ Y(z)\} \in \mathcal{N}$*. In particular, this is the case if* $n = m = 1$*.*

The following propositions can be proved similarly to Theorems 4.1 and 4.2.

THEOREM 5.2. *Assume that* $\{A(z)\ B(z)\}$ *of orders* (p, q) *belongs to* $\mathcal{N}$*. Then, in the case* $s \geq p, t \geq q$ *it holds that*

$$\operatorname{rank}\Gamma(s,t) = \begin{cases} np & \text{if } \ s-t = p-q, \\ n(s-t+q) + \operatorname{rank}\Lambda(t-s-(q-p)) & \text{if } \ s-t < p-q, \\ np + \operatorname{rank}\Theta(s-t-(p-q)) & \text{if } \ s-t > p-q, \end{cases}$$

where $A(t)$ *and* $B(t)$ *are given by* (4.1) *and* (4.2)*, respectively.*

THEOREM 5.3. *Assume that* $\{A(z)\ B(z)\}$ *of orders* (p, q) *belongs to* $\mathcal{N}$*. Then* $\{X(z)\ Y(z)\}$ *of orders* (s, t) *belongs to* $\mathcal{N}$ *if and only if* $\operatorname{rank}\Gamma(s+i, t+i) = ns$ *for* $i = 0, 1$*.*

In case $\mathcal{N}$ is nonempty, similarly to the previous section, all pairs in $\mathcal{N}$ can be found by using the correlation function $\{R_i\}$ on the basis of Theorems 5.1–5.3.

Remark 5.2. It is clear that the order estimation problem considered in this section contains as a special case the order estimation of the multivariate ARMA model in time series analysis, where $n = m$, $B_0 = I$, and $B(z)$ is stable.

6. Estimating MFD. In sections 4 and 5 we have seen that all MFD in $\mathcal{M}$ and in $\mathcal{N}$ can be found by using the impulse responses $\{H_i\}$ of the system and the correlation functions of the system output, respectively. We now discuss how to estimate MFD by using the input-output data or by the output data only.

6.1. Estimation by use of impulse responses. In order to derive all pairs in $\mathcal{M}$ from the input-output data we face two problems: (1) how to estimate $\{H_i\}$ on the basis of input-output data; (2) how to estimate $\operatorname{rank} L(s,t)$ on the basis of estimated $H_{i,k}$. The first problem can be solved by using the existing results [17]. We formulate them as propositions.

PROPOSITION 6.1. *Assume the following conditions hold:*

- A1: $\{u_k\}$ *is a sequence of i.i.d. zero mean random vectors with* $E\|u_k\|^{2(2+\delta)} < \infty$ *for some* $\delta > 0$ *and* $Eu_k u_k^T = I$*.*
- A2: *The matrix polynomials* $A(z)$ *and* $B(z)$ *have no common left factor,* $[A_p\ B_q]$ *is full row rank, and* $\det A(z) \neq 0\ \forall |z| \leq 1$*.*

Then,

$$Ey_k u_{k-i}^T = \sum_{j=0}^{\infty} H_j E(u_{k-j} u_{k-i}^T) = H_i. \tag{6.1}$$

Motivated by (6.1)*, the stochastic approximation algorithm with expanding truncations*

[6] *is used to recursively estimate* $\{H_i, i \geq 0\}$*:*

$$H_{i,k} = \left[H_{i,k-1} - \frac{1}{k}(H_{i,k-1} - y_k u_{k-i}^T)\right] \cdot I_{\left[\|H_{i,k-1} - \frac{1}{k}(H_{i,k-1} - y_k u_{k-i}^T))\| \leq M_{\delta_{i,k}}\right]}, \tag{6.2}$$

$$\delta_{i,k} = \sum_{j=1}^{k-1} I_{\left[\|H_{i,j-1} - \frac{1}{j}(H_{i,j-1} - y_j u_{j-i}^T))\| > M_{\delta_{i,j}}\right]}, \tag{6.3}$$

where $\{M_k\}$ *is an arbitrarily chosen sequence of positive real numbers increasingly diverging to infinity,* $H_{i,0}$ *is an arbitrary initial value, and* I_A *denotes the indicator function of a set* A*.*

Then by [17], *the estimates for the impulse responses* $\{H_i, i \geq 0\}$ *have the following convergence rate:*

$$\|H_{i,k} - H_i\| = o(k^{-\nu}) \quad a.s. \quad i \geq 0 \quad \forall\, \nu \in (0, 1/2). \tag{6.4}$$

Having obtained the convergence rate (6.4), we then can apply the method proposed in [10] to estimate the rank $L(s,t) = \operatorname{rank} L(s,t)L^T(s,t) \triangleq r(s,t)$. Let us formulate the rank estimation method used in [10] as a lemma.

LEMMA 6.1 (see [10]). *Let* P *be an* $l \times l$*-symmetric and nonnegative definite matrix with rank equal to* $\mu \leq l$ *and with characteristic polynomial* $h(z) \triangleq \det(zI - P) = z^l + h_1 z^{l-1} + \cdots + h_l$*. Let* $h_{j,k}$ *be the estimates for* h_j $\forall j = 0, 1, \ldots, l$ $(h_0 \triangleq 1)$ *with convergence rate* $|h_{j,k} - h_j| = o(k^{-\nu})$ *for some* $\nu > 0$ $\forall j = 0, 1, \cdots, l$*. Then the estimate* μ_k *for* μ *defined by*

$$\mu_k \triangleq \max\{j | Q_{j,k} \geq \varepsilon, j = 0, 1, \ldots, l\}$$

is strongly consistent: $\mu_k \xrightarrow[k\to\infty]{} \mu$ *a.s., where the decision numbers* $Q_{j,k}$ *are given as*

$$Q_{j,k} \triangleq \frac{|h_{j,k}| + \frac{1}{\log k}}{|h_{j+1,k}| + \frac{1}{\log k}}, k \geq 1, j = 0, 1, \ldots, l,$$

$h_{l+1,k} \triangleq 0$ *and any number greater than* 1 *may serve as the threshold* ε*.*

For estimating the orders and the parameters of the pairs in $\mathcal{M}$ based on the approximate impulse responses $\{H_{i,k}\}$ given by (6.2)–(6.3) at time k, it suffices to carry out the same process given in Remark 4.2 but with the following changes: the rank $r(s,t)$ of $L(s,t)$ is replaced by its estimate $r_k(s,t)$ and the search range for finding the first pair is restricted to $1 \leq s \leq \log(k) + 1, 0 \leq t \leq \log(k)$. For any given orders (s,t), the matrix $L(s,t)L^T(s,t)$ serves as P in Lemma 6.1, and the coefficients of $\det(zI - L(s,t)L^T(s,t))$ are estimated by replacing H_i in $L(s,t)$ with $H_{i,k}$ given by (6.2)–(6.3). The obtained estimate $r_k(s,t)$ converges to $r(s,t)$ by Lemma 6.1 and (6.4). As a result, the pairs in $\mathcal{M}$ can always be found by Theorem 4.2 when k is sufficiently large. Denote the estimates for the orders and the parameters of the ith pair in $\mathcal{M}$ by (p_k^i, q_k^i) and $\{X_k^i(z)\ Y_k^i(z)\}$, respectively. Thus, we have $(p_k^i, q_k^i) \xrightarrow[k\to\infty]{} (p^i, q^i)$, and $\{X_k^i(z)\ Y_k^i(z)\}$ converges to $\{X^i(z)\ Y^i(z)\}$ with the convergence rate $o(k^{-\nu}),\ \nu \in (0, 1/2)$.

6.2. Estimation by use of correlation functions. Under the condition B1 and B2, the correlation functions $R_i \triangleq E y_k y_{k-i}^T$ can recursively be estimated by the output $\{y_k\}$:

$$R_{i,k} = R_{i,k-1} - \frac{1}{k}(R_{i,k-1} - y_k y_{k-i}^T), \tag{6.5}$$

where $R_{i,k}$ denote the estimates for R_i for any $i \geq 0$ at time k, and the following rate of convergence takes place [7, 11]

$$\|R_{i,k} - R_i\| = o(k^{-\nu}) \quad \forall\ \nu \in \left(0, \min\left(\frac{1}{2}, \frac{\delta}{2+\delta}\right)\right). \tag{6.6}$$

The steps of estimating the pairs in $\mathcal{N}$ by correlation functions is the same as by impulse responses given above. The only difference is that the estimated impulse response (6.2)–(6.3) is replaced by the estimated correlation function (6.5). Denote the estimates for the orders of the ith pair in $\mathcal{N}$ by (p_k^i, q_k^i). The estimate $X_k^i(z)$ for $X^i(z)$ is given by (5.2) in which the correlation function R_i and the orders (p^i, q^i) are replaced by their estimates $R_{i,k}$ and (p_k^i, q_k^i), respectively. Thus, we have $(p_k^i, q_k^i) \xrightarrow[k\to\infty]{} (p^i, q^i)$, and $X_k^i(z)$ converges to $X^i(z)$ with the convergence rate $o(k^{-\nu})$, $\nu \in (0, \min(\frac{1}{2}, \frac{\delta}{2+\delta}))$. It is clear that the rate given by (6.6) plays the same role as (6.4) in the case where estimation is based on impulse responses. The parameter estimation of $B(z)$ is described in the next section.

7. Extension to ARMAX systems.

Consider the following multivariable ARMAX model:

$$A(z)y_k = B(z)u_k + C(z)w_k, \tag{7.1}$$

which, in comparison with (1.1), has an additional term $C(z)w_k$ with $w_k \in R^n$, where

$$C(z) = I + C_1 z + \cdots + C_r z^r \quad \text{with} \quad C_r \neq 0. \tag{7.2}$$

7.1. Estimation of ARMAX by impulse responses.

We now list conditions to be used.

C1: $\{u_k\}$ is a sequence of i.i.d. random vectors with $Eu_k = 0$, $E\|u_k\|^{2(2+\delta)} < \infty$ for some $\delta > 0$, and $Eu_k u_k^T = I$, and is independent of $\{w_k\}$.

C2: $\{w_k\}$ is a sequence of i.i.d. zero mean random vectors with $E\|w_k\|^{2+\delta} < \infty$ for some $\delta > 0$ and $Ew_k w_k^T = R_w > 0$, where R_w is unknown.

C3: $A(z)$ and $B(z)$ have no common left factor, and $[A_p\ B_q]$ is full row rank.

C4: $\det A(z) \neq 0\ \forall |z| \leq 1$, and $\det C(z) \neq 0\ \forall |z| \leq 1$.

C5: An upper bound r^* for r is available.

Set $y_k^u \triangleq A^{-1}(z)B(z)u_k$ and $y_k^w \triangleq A^{-1}(z)C(z)w_k$. Thus, by the stability of $A(z)$, we have

$$y_k = A^{-1}(z)B(z)u_k + A^{-1}(z)C(z)w_k = y_k^u + y_k^w.$$

Similarly to system (1.1), the triple $\{A(z)\ B(z)\ C(z)\}$ in (7.1) may not be unique. By $\mathcal{W}$ denote the totality of the matrix triples $\{X(z)\ Y(z)\ Z(z)\}$ of orders (s, t, ς) satisfying the following conditions:

(1) $\{X(z)\ Y(z)\}$ shares the same impulse responses with $\{A(z)\ B(z)\}$;
(2) $X(z)$ and $Y(z)$ have no common left factor, and $[X_s\ Y_t]$ is full row rank;
(3) $Z(z) = X(z)A^{-1}(z)C(z)$, where

$$\begin{aligned} X(z) &= I + X_1 z + \cdots + X_s z^s, \\ Y(z) &= Y_0 + Y_1 z + \cdots + Y_t z^t, \\ Z(z) &= I + Z_1 z + \cdots + Z_\varsigma z^\varsigma. \end{aligned}$$

It is clear that $\{A(z)\ B(z)\ C(z)\}$ in (7.1) belongs to $\mathcal{W}$ if the conditions C3–C4 hold.

Remark 7.1. The different triples if they exist in $\mathcal{W}$ produce the identical output $\{y_k\}$ under the same input $\{u_k\}$ and innovation process $\{w_k\}$. Assume that triples $\{A(z)\ B(z)\ C(z)\}$ of orders (p,q,r) and $\{X(z)\ Y(z)\ Z(z)\}$ of orders (s,t,ς) belong to $\mathcal{W}$, and $\{A(z)\ B(z)\ C(z)\} \neq \{X(z)\ Y(z)\ Z(z)\}$. Then the orders (p,q,r) and (s,t,ς) satisfy the following inequalities:

$$p < s \le p + (n-1)q,\ t < q,\ \varsigma \le r + s - p < r + s \tag{7.3}$$

or

$$s < p,\ q < t < q + (n-1)p,\ \varsigma \le r + t - q \le r + t. \tag{7.4}$$

Assume that the number of distinct elements in $\mathcal{W}$ is equal to n_a. Then the ith distinct element of orders (p^i, q^i, r^i) in $\mathcal{W}$ is represented as

$$X^i(z)y_k = Y^i(z)u_k + Z^i(z)w_k, \tag{7.5}$$

where

$$X^i(z) = I + X_1^i z + \cdots + X_{p^i}^i z^{p^i}, \tag{7.6}$$

$$Y^i(z) = Y_0^i + Y_1^i z + \cdots + Y_{q^i}^i z^{q^i}, \tag{7.7}$$

$$Z^i(z) = I + Z_1^i z + \cdots + Z_{r^i}^i z^{r^i}. \tag{7.8}$$

For simplicity let us call $X^i(z)y_k$, $Y^i(z)u_k$, and $Z^i(z)w_k$ the AR-part, X-part, and MA-part, respectively. First, the AR-part and X-part are estimated by use of the impulse responses of $\{A(z)\ B(z)\}$ on the basis of the input-output data $\{u_k, y_k\}$, and then the MA-part is estimated by the input-output data $\{u_k, y_k\}$ and the estimated AR-part and X-part.

7.1.1. Estimation of AR-part and X-part. Under C1–C4, we have

$$Ey_k u_{k-i}^T = Ey_k^u u_{k-i}^T = \sum_{j=0}^{\infty} H_j E(u_{k-j} u_{k-i}^T) = H_i. \tag{7.9}$$

Similarly to the estimation steps using impulse response in section 6, the impulse response $\{H_i, i \ge 0\}$ of $\{A(z)\ B(z)\}$ is still estimated by (6.2) and (6.3) based on (7.9), and under C1–C4 the convergence rate is again given by (6.4). As a result, the estimated orders (p_k^i, q_k^i) converge to the true order (p^i, q^i). By the estimated impulse response $\{H_{i,k}\}$, we derive the strongly consistent estimates $X_k^i(z)$ and $Y_k^i(z)$ for $X^i(z)$ and $Y^i(z)$ via (2.8) and (2.4).

7.1.2. Estimation of MA-part. Set $\chi_k^i \triangleq Z^i(z)w_k$, $1 \le i \le n_a$. Thus we have $\chi_k^i = X^i(z)y_k - Y^i(z)u_k$, which can be estimated as follows:

$$\widehat{\chi}_k^i = X_k^i(z)y_k - Y_k^i(z)u_k, \tag{7.10}$$

where

$$X_k^i(z) = I + X_{1,k}^i z + \cdots + X_{p_k^i,k}^i z^{p_k^i},$$

$$Y_k^i(z) = Y_{0,k}^i + Y_{1,k}^i z + \cdots + Y_{q_k^i,k}^i z^{q_k^i}.$$

Thus, the correlation functions $S^i_j \triangleq E\chi^i_k\chi^{iT}_{k-j}, j \geq 0$ of the sequence $\{\chi^i_k\}$ are estimated by the following algorithm:

$$S^i_{j,k} = S^i_{j,k-1} - \frac{1}{k}\left(S^i_{j,k-1} - \widehat{\chi}^i_k\widehat{\chi}^{iT}_{k-j}\right), \tag{7.11}$$

where $S^i_{j,k}$ is the estimate for S^i_j at time k. Further, define

$$Q^i_{j,k} \triangleq \frac{\|S^i_{j,k}\| + \frac{1}{\log k}}{\|S^i_{j+1,k}\| + \frac{1}{\log k}}, \quad 0 \leq j \leq r^* + \max(p^i_k, q^i_k). \tag{7.12}$$

Choose a fixed $\varepsilon > 1$. The estimate for r^i at time k is given by

$$r^i_k \triangleq \max\{j | Q^i_{j,k} \geq \varepsilon,\ 0 \leq j \leq r^* + \max(p^i_k, q^i_k)\} \tag{7.13}$$

if there exists some $j : 0 \leq j \leq r^* + \max(p^i_k, q^i_k)$ such that $Q^i_{j,k} \geq \varepsilon$. Otherwise, set $r^i_k \triangleq r^* + \max(p^i_k, q^i_k)$.

Assume that C1–C5 hold. Then

$$r^i_k \xrightarrow[k\to\infty]{} r^i,\ 1 \leq i \leq n_a. \tag{7.14}$$

After the consistent order estimation of the MA-part is obtained, by the method proposed in [8] we can obtain the strongly consistent estimate for the parameters of the MA-part and the covariance of the innovation process.

7.2. Estimation of ARMAX by correlation functions. In this subsection, we use the correlation function method to estimate the ARMAX system (7.1). In this case, instead of C1–C5 we need the following set of conditions:

D1: $\{u_k\}$ is a sequence of i.i.d. random vectors with $Eu_k = 0$, $E\|u_k\|^{2+\delta} < \infty$ for some $\delta > 0$, and $Eu_ku^T_k = I$, and is independent of $\{w_k\}$.

D2: $\{w_k\}$ is a sequence of i.i.d. zero mean random vectors with $E\|w_k\|^{2+\delta} < \infty$ for some $\delta > 0$ and $Ew_kw^T_k = R_w > 0$, where R_w is unknown.

D3: $A(z)$ and $B(z)B^T(z^{-1}) + C(z)R_wC^T(z^{-1})$ have no common left factor, and the matrix $[A_p\ B_vB^T_0 + C_vR_w]$ is full row rank, where $v \triangleq \max(q, r)$, $B_j \triangleq 0$ if $j > q$, and $C_l \triangleq 0$ if $l > r$.

D4: $\det A(z) \neq 0\ \forall|z| \leq 1$ and $\det C(z) \neq 0\ \forall|z| \leq 1$.

By the innovation representation [4, 18], under D1–D4 the system (7.1) can be represented as

$$A(z)y_k = D(z)\psi_k, \quad D(z) = I + D_1z + \cdots + D_vz^v \tag{7.15}$$

with $v \triangleq \max(q, r)$, where we have the following:

(i) ψ_k is an n-dimensional random process, $E\psi_k = 0$, $E\psi_k\psi^T_j = R_\psi\delta_{k,j}$ with $\delta_{k,j} = 1$ if $k = j$ and $\delta_{k,j} = 0$ if $k \neq j$, and $R_\psi > 0$.

(ii) $D(z)$ is a $n \times n$ matrix polynomial with $\det D(z) \neq 0\ \forall|z| < 1$.

(iii) $A(z)$ and $D(z)$ have no common left factor, and $[A_p\ D_v]$ is full row rank.

(iv) $A(z)$ and $D(z)$ are uniquely defined.

As pointed out before, there may exisit other triples $\{X(z)\ Y(z)\ Z(z)\}$ of orders (s, t, ς) which can also model the sytem (7.1) other than $\{A(z)\ B(z)\ C(z)\}$, and $\{X(z)\ Y(z)\ Z(z)\}$ also has the innovation representation $\{X(z)y_k = G(z)\phi_k\}$ of orders (s, g) having the corresponding property, where $G(z) = I + G_1z + \cdots + G_gz^g$ and $g \triangleq \max(t, \varsigma)$. Similarly to $\mathcal{W}$, denote by $\mathcal{V}$ the totality of the matrix triples $\{X(z)\ Y(z)\ Z(z)\}$ satisfying the following conditions:

(1) $\{A(z)y_k = D(z)\psi_k\}$ and $\{X(z)y_k = G(z)\phi_k\}$ have the same correlation function;

(2) $X(z)$ and $G(z)R_\phi G^T(z^{-1})$ have no common left factor, and $[X_s\ G_g]$ is full row rank, where $R_\phi \triangleq E\phi_k\phi_k^T$;

(3) $Z(z) = X(z)A^{-1}(z)C(z)$.

It is clear that $\{A(z)\ B(z)\ C(z)\}$ in (7.1) belongs to $\mathcal{V}$ if the conditions D1–D4 hold.

Remark 7.2. Assume $\{A(z)\ B(z)\ C(z)\}$ of orders (p, q, r) and $\{X(z)\ Y(z)\ Z(z)\}$ of orders (s, t, ς) belong to $\mathcal{V}$, and $\{A(z)\ B(z)\ C(z)\} \neq \{X(z)\ Y(z)\ Z(z)\}$. Then the orders (p, q, r) and (s, t, ς) satisfy the following inequalities:

$$(n-1)s + g \leq (n-1)p + v, \quad p < s \leq p + (n-1)v, \quad g < v, \tag{7.16}$$

or

$$(n-1)g + s \leq (n-1)v + p, \quad v < g \leq v + (n-1)p, \quad s < p. \tag{7.17}$$

Denote the number of distinct pairs in $\mathcal{V}$ by n_b. For simplicity of notation, the ith distinct pair of orders (p^i, q^i, r^i) in $\mathcal{V}$ is still represented as (7.5)–(7.8), and $v^i \triangleq \max(q^i, r^i)$. Under D1–D4, the estimate $\{R_{i,k}\}$ for the correlation function $\{R_i, i \geq 0\}$ of the output y_k in (7.1) has the convergence rate (6.6). Similarly to the estimation steps using the correlation function in section 6, the estimated orders (p_k^i, v_k^i) converge to the true orders (p^i, v^i). By the estimated correlation function $\{R_{i,k}\}$, we derive the strongly consistent estimates $X_k^i(z)$ for $X^i(z)$ in terms of (5.4).

Set $\varphi_k^i \triangleq X^i(z)y_k,\ 1 \leq i \leq n_b$. Then the estimate $\widehat{\varphi}_k^i$ for φ_k^i is given as follows:

$$\widehat{\varphi}_k^i = X_k^i(z)y_k, \tag{7.18}$$

where $X_k^i(z) = I + X_{1,k}^i z + \cdots + X_{p_k^i,k}^i z^{p_k^i}$. By D1, we have

$$E\varphi_k^i u_{k-j}^T = E(Y^i(z)u_k + Z^i(z)w_k)u_{k-j}^T = EY^i(z)u_k u_{k-j}^T = Y_j^i,\ 0 \leq j \leq v^i. \tag{7.19}$$

The estimates for the parameters of the X-part can be given by the following algorithm:

$$Y_{j,k}^i = Y_{j,k-1}^i - 1/k\left(Y_{j,k-1}^i - \widehat{\varphi}_k^i u_{k-j}^T\right),\ 0 \leq j \leq v_k^i. \tag{7.20}$$

Define

$$Q_{j,k}^i \triangleq \frac{\|Y_{j,k}^i\| + \frac{1}{\log k}}{\|Y_{j+1,k}^i\| + \frac{1}{\log k}}, \quad 0 \leq j \leq v_k^i. \tag{7.21}$$

Choose a fixed $\varepsilon > 1$. Then the estimate for q^i at time k is given by

$$q_k^i \triangleq \max\{j | Q_{j,k}^i \geq \varepsilon,\ 0 \leq j \leq v_k^i\} \tag{7.22}$$

if there exists some $j : 0 \leq j \leq v_k^i$ such that $Q_{j,k}^i \geq \varepsilon$. Otherwise, set $q_k^i \triangleq v_k^i$.

For the estimation of the MA-part, the consistent estimate for the order r^i is obtained by (7.10)–(7.13) in which only the upper bound $r^* + \max(p_k^i, q_k^i)$ on the index j in (7.12) and (7.13) is replaced by v_k^i, while the strongly consistent estimates for the parameters of the MA-part and the covariance of the innovation process can also be produced by the method given in [8].

Remark 7.3. The set $\mathcal{W}$ may be different from $\mathcal{V}$. The conditions on the system structure, input signal, and innovation process for the two kinds of estimation methods have some distinctions. The main different points are C3 and D3 on the system structure, and there is no requirement on a prior upper bound on the order of the MA-part for the correlation function method, while this is needed for the impulse response method.

8. Concluding remarks. We have shown how to determine all possible MFDs for a linear multivariable linear system. It is worth noting the following points:

1. There is no restriction on the dimensions of input and output of the system.
2. Determining MFDs of a given system includes finding both the orders and coefficients. No upper bound for orders is required.
3. MFDs may be determined by the impulse responses of the system or by the correlation functions of the system output.
4. MFDs can consistently be estimated by using either the input-output data or the output data only.

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