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H_∞ control, stabilization, and input–output stability of nonlinear systems with homogeneous properties☆

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New relations are established between H_∞ control and stabilization, and between input–output stability and Lyapunov stability of nonlinear systems with homogeneous properties. These results can be applied to H_∞ controller design and stability analysis.

Abstract

In this paper, we discuss the problems about H_∞ control, stabilization and input–output stability of nonlinear systems, which may not satisfy known regularity conditions related to smoothness. To this end, a class of homogeneous systems and systems that can be approximated by homogeneous systems, are concentrated on. New relationships are established between H_∞ control and stabilization, and between L_p stability and Lyapunov stability. At first, with Hamilton–Jacobi–Isaacs inequality, the nonlinear H_∞ control problem of the systems with homogeneous properties is discussed. The results show that their stabilizability via homogeneous feedback, in the case without exogenous input signals, implies the solvability of their H_∞ control problem. Then, simply formulated results on input–output stability are obtained, based on the relations among the homogeneity degrees concerned with the considered systems. The conclusions hold globally for homogeneous systems and locally for those that can be homogeneously approximated.

Keywords: Nonlinear systems; H_∞ control; Stabilization; Stability analysis; Nonsmoothness

1. Introduction

This paper tackles the H_∞ control and input–output stability problems for nonlinear systems, especially for the systems without regular assumptions to guarantee the smoothness of the solutions to the problems. To deal with the situation, the paper concentrates on the systems with homogeneous properties.

With solid application backgrounds (referring to M'Closkey & Murray, 1997; Rui, Reyhanoglu, Kolmanovosky, Cho, & McClamroch, 1997), many problems about homogeneous systems have been studied, such as stabilization (Coron & Praly, 1991; Kawski, 1989; Hermes, 1991a; Sepulchre & Aeyels, 1996), optimal control (Hermes, 1996), and input-to-state stability (Ryan, 1995). Further results have been obtained (locally) for the systems that can be approximated by homogeneous systems (Hermes, 1991a,b; Celikovsky & Nijmeijer, 1997; Celikovsky, 1997; Celikovsky & Aranda-Bricaire, 1999). The results showed that homogeneity may be a useful tool in the nonsmooth feedback design for the systems having uncontrollable unstable mode of its approximate linearization, which cannot be stabilized smoothly. In addition, homogeneous techniques can also be applied to nonsmooth finite-time design of control systems (Hong, Huang, & Xu, 2001).

Recent years have witnessed an increasing interest in nonlinear H_∞ control and input–output properties of nonlinear systems. With the formulation based on differential game theory and dissipativity theory, nonlinear H_∞ control problem becomes a problem to solve Hamilton–Jacobi–Isaacs equations or inequalities (Isidori & Atolfi,

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Nomenclature

$\lvert\cdot\rvert$	absolute value of real numbers
$\lVert\cdot\rVert$	Euclidean norm of vectors, e.g. $\lVert x\rVert = \sqrt{x^{\mathrm{T}}x}$
$\lVert\cdot\rVert_p$	L_p norm, $p \geqslant 1$
r_0	$r_0 = \min\{r_1, \dots, r_n\} > 0$
$\mathbf{D}V$	shorter form for $\partial V/\partial x$; and $\mathbf{D}V^{\mathrm{T}}$ for the transpose of $\mathbf{D}V$.
S	generalized unit sphere, i.e., $\{x \in R^n: \Gamma(x) = 1\}$

1992; van der Schaft, 1992, 1993, 1996; Ball, Helton, & Walker, 1994; Baramov & Kimura, 1996). Unfortunately, in most cases, it is still very difficult to construct control laws for nonlinear H_∞ problems because it is not easy to analyze and solve the induced Hamilton–Jacobi–Isaacs equations or inequalities, and even to check the existence of their smooth solutions. Therefore, it is usually assumed beforehand that the equations or inequalities have smooth solutions. To remove the assumptions about smoothness of the solutions, the analysis of viscosity solutions was given (Soravia, 1996; Hong, Yung, Mei, & Qin, 1997). In addition, the construction of H_∞ controllers for special nonlinear systems was studied. For example, using the well-known backstepping idea, a design approach to H_∞ control was proposed for the systems in the parametric-strict-feedback form (Pan & Basar, 1998). Moreover, the problem was discussed for a class of homogeneous systems with trivial dilation via smooth feedback laws (Hong & Li, 1998).

It is well known that the problem of H_∞ control is related to input-output properties and L_2 gain problem (Basar & Bernhard, 1991; van der Schaft, 1992). Fruitful work has been done for input–output stability and input-to-state stability (Desoer & Vidyasagar, 1975; Vidyasagar & Vannelli, 1982; Sontag, 1995; Ryan, 1995). A quite comprehensive survey about the research topic was given in (Sontag, 1995).

The paper is organized as follows. The formulation of the considered problems and related concepts are introduced in Section 2, followed by some preliminary results for homogeneous systems in Section 3. Then nonlinear H_∞ control is studied in Sections 4 and 5. The studied nonlinear systems and the designed H_∞ controllers may not be differentiable, different from those in (Hong & Li, 1998). Global results on the H_∞ control of homogeneous systems are shown, and local results are obtained correspondingly for the systems that can be homogeneously approximated. In Section 6, the results on L_p stability are proposed, both globally and locally in different cases. Finally, concluding remarks are given.

2. Definitions and problem formulation

2.1. Homogeneous properties

First of all, some basic concepts about homogeneity are introduced (referring to Rosier (1992), M'Closkey & Murray (1997) and Celikovsky and Aranda-Bricaire (1999)).

Definition 2.1. Dilation $\Delta_\varepsilon^{(r_1,\dots,r_n)}$ is a mapping, depending on dilation coefficients $(r_1, \dots, r_n)$, which assigns to every real $\varepsilon > 0$ a global diffeomorphism

$$\Delta_\varepsilon^{(r_1,\dots,r_n)}(x_1, \dots, x_n) = (\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n)$$

where $x_1, \dots, x_n$ are suitable coordinates on R^n and $r_1, \dots, r_n$ are positive real numbers.

Definition 2.2. A function $V(x_1, \dots, x_n)$ is called homogeneous of degree $q \in R$ with respect to the dilation $\Delta_\varepsilon^{(r_1,\dots,r_n)}$, if there exists $q \in R$ such that

$$V(\Delta_\varepsilon^{(r_1,\dots,r_n)}(x_1, \dots, x_n)) = \varepsilon^q V(x_1, \dots, x_n). \tag{1}$$

A vector field $f(x) = (f_1(x), \dots, f_n(x))^{\mathrm{T}}$ is called homogeneous of degree $k \in R$ with respect to the dilation $\Delta_\varepsilon^{(r_1,\dots,r_n)}$ if there exists $k \in R$ such that

$$f_i(\Delta_\varepsilon^{(r_1,\dots,r_n)}(x_1, \dots, x_n)) = \varepsilon^{k+r_i} f_i(x_1, \dots, x_n), \quad i = 1, \dots, n. \tag{2}$$

A system, $\dot{x} = f(x)$, is called homogeneous if its vector field $f(x)$ is homogeneous.

In the sequel, where no confusion arises, we will simply use "with respect to the dilation $(r_1, \dots, r_n)$" instead of "with respect to the dilation $\Delta_\varepsilon^{(r_1,\dots,r_n)}$".

Remark 2.1. Set $r_0 = \min\{r_1, \dots, r_n\} > 0$. If $r_1 = \cdots = r_n = r_0$, the dilation is called trivial. Without loss of generality, r_0 is taken as 1 (Hermes, 1996). For instance, function $V(x)$ is homogeneous of degree q with respect to the dilation $(r_1, \dots, r_n)$ if and only if it is homogeneous of degree q/r_0 with respect to the dilation $(r_1/r_0, \dots, r_n/r_0)$.

Definition 2.3. A continuous map $\Gamma: R^n \to R$ is called a homogeneous norm (with respect to the dilation $(r_1, \dots, r_n)$), if it is a strictly positive-definite function homogeneous of degree 1 with respect to the dilation $(r_1, \dots, r_n)$.

In the paper, the homogeneous norm is taken in the form of

$$\Gamma(x) = (|x_1|^{c/r_1} + \cdots + |x_n|^{c/r_n})^{1/c}, \tag{3}$$

where $c > \max\{r_i,\ i = 1, \dots, n\}$.

Remark 2.2. Let $S = \{x \in R^n: \Gamma(x) = 1\}$, which may be viewed as a generalized unit sphere. Notice that equations or inequalities consisting of homogeneous (with respect to the same dilations) functions are valid if and only if they are valid on S. This is a crucial property facilitating the analysis of homogeneous systems.

The next lemma is adapted from Theorem 2 and Remark 1 of Rosier (1992).

Lemma 2.1. *Suppose that system $\dot{x} = f(x)$ is homogeneous of degree k with respect to the dilation $(r_1, \dots, r_n)$, where f is continuous and $x = 0$ is the asymptotically stable equilibrium of the system. Then, the equilibrium is globally asymptotically stable, and for any positive integer j and any real number $\sigma_0 > j \max\{r_1, \dots, r_n\}$, there is a function $V \in C^j(R^n) \cap C^\infty(R^n \backslash \{0\})$ homogeneous of degree σ_0 with respect to the same dilation as that of f such that V is positive definite, radially unbounded (i.e., $V(x) \to +\infty$ as $\|x\| \to +\infty$), and $\dot{V}(x)|_{\dot{x} = f(x)} < 0$ for all $x \neq 0$.*

Definition 2.4. A function $\tilde{h}(x)$ is called higher degree with respect to $\Gamma(x)^q$ whose dilation coefficients are $(r_1, \dots, r_n)$, denoted by $\tilde{h} = o(\Gamma(x)^q)$, if

$$\lim_{\varepsilon \to 0} \frac{\tilde{h}(\varepsilon^{r_1} x_1, \dots, \varepsilon^{r_n} x_n)}{\varepsilon^q} = 0 \tag{4}$$

for any fixed $x \neq 0$. A vector field $\tilde{f}(x)$ is called higher degree with respect to $\Gamma(x)^k$ whose dilation coefficients are $(r_1, \dots, r_n)$, denoted by $\tilde{f} = o(\Gamma(x)^k)$ if, for all $i = 1, \dots, n$,

$$\lim_{\varepsilon \to 0} \frac{\tilde{f}_i(\varepsilon^{r_1} x_1, \dots, \varepsilon^{r_n} x_n)}{\varepsilon^{k + r_i}} = 0 \tag{5}$$

for any fixed $x \neq 0 \in R^n$. Moreover, consider a nonlinear system

$$\dot{x} = f(x) + \tilde{f}(x), \quad x \in R^n, \tag{6}$$

where the system

$$\dot{x} = f(x), \quad x \in R^n \tag{7}$$

is homogeneous of degree k with respect to some dilation $(r_1, \dots, r_n)$ and $\tilde{f}(x)$ is higher degree of $\Gamma(x)^k$ with $(r_1, \dots, r_n)$ as its dilation coefficients. Then system (6) is said to be approximated by a homogeneous system (7), or system (7) is called a leading homogeneous system of system (6).

Remark 2.3. Usually, for a nonlinear system, its leading homogeneous system may not be unique (Hermes, 1991a). In addition, if its leading homogeneous system is asymptotically stable, then the system itself is asymptotically stable locally (Hermes, 1991b; Rosier, 1992).

In the sequel, where no confusion arises, "homogeneous" means "homogeneous with respect to the dilation $(r_1, \dots, r_n)$".

2.2. H_∞ control and stabilization

The problem of nonlinear H_∞ control problem of affine nonlinear system via state feedback can be described briefly as follows. Consider an affine nonlinear control system

$$\begin{aligned} &\dot{x} = f(x) + G_1(x)u + G_2(x)w, \quad x \in R^n, \ w \in R^\mu, \\ &z = h(x) + d(x)u, \quad z \in R^l, \ u \in R^m, \end{aligned} \tag{8}$$

where f, g, h, d are continuous and $f(0) = 0$, $h(0) = 0$, $G_1(0) \neq 0$. Here $w(t)$ denotes exogenous input variables, including all bounded disturbance (so $\|w\|_\infty < \infty$); u denotes the control signal; z denotes the (penalty) output variable; x denotes the state variable of the system. In addition, $d^T(x)d(x)$ is of full rank, which is a basic assumption in the study of H_∞ control to guarantee that the (local) saddle solution exists uniquely. As usual (see Doyle, Glover, Khargonekar, & Francis, 1989; Isidori & Atolfi, 1992), we assume

$$d^T d = \theta^2 I, \quad h^T d = 0, \tag{9}$$

where θ is a positive number and I is the identity matrix.

Following the formulation of van der Schaft (1992, 1993), or Isidori and Atolfi (1992), nonlinear H_∞ control can be viewed as nonlinear L_2-gain control with internal stability.

Definition 2.5. Nonlinear H_∞ control problem of (8) can be solved via static state feedback if, for a positive number $\lambda > 0$, there exists a static state feedback $u = u(x)$ such that for any $T > 0$, $w(\cdot) \in L_2(R^+, R^\mu)$, the system has

(1) finite L_2-gain ($\leqslant \lambda$): namely, when the initial condition $x(0) = 0$, we have,

$$\int_0^T z^T(t) z(t) \, dt \leqslant \lambda^2 \int_0^T w^T(t) w(t) \, dt; \tag{10}$$

(2) internal stability: namely, when $w(t) \equiv 0$, the closed loop system is asymptotically stable.

Remark 2.4. As a matter of fact, condition (2) can usually be derived from condition (1) under some regular conditions as mentioned in the references such as (van der Schaft, 1993). However, in this paper, we will show that, under certain assumptions, condition (2) implies condition (1).

In the paper, we study a class of homogeneous nonlinear systems with respect to the dilation $(r_1, \dots, r_n)$,

$$\begin{aligned} &\dot{x} = f(x) + G_1(x)u + G_2(x)w, \quad x \in R^n, \ w \in R^\mu, \\ &z = h(x) + Du, \quad z \in R^l, \ u \in R^m, \end{aligned} \tag{11}$$

where f, G_1, G_2, h are continuous and $C^1(R^n\backslash\{0\})$ with $f(0)=0, h(0)=0, G_1(0)\neq 0$. D is a constant matrix with $D^TD=\theta^2 I$ as assumed in (9). Moreover,

1. $f(x)$ is a homogeneous vector field of degree $k > -r_0$ with respect to the dilation $(r_1,\ldots,r_n)$.
2. $G_j(x)=(G_{j1}^T(x),\ldots,G_{jn}^T(x))^T$ $(j=1,2)$ and $G_{ji}(x)$, $j=1,2;\ i=1,\ldots,n$ are homogeneous vector fields of degree $s=-r_0$ with respect to the dilation $(r_1,\ldots,r_n)$.
3. $h(x)$ is a continuous homogeneous function of degree $q=k+r_0>0$ with respect to the dilation $(r_1,\ldots,r_n)$.

Definition 2.6. Consider system (11) when $w\equiv 0$. In this case, the system can be written as

$$\dot{x}=f(x)+G_1(x)u, \quad x\in R^n,\ u\in R^m \tag{12}$$

where $f(x)$ and G_1 are homogeneous as defined above. We shall say that system (12) is homogeneously stabilizable if there is a feedback law $u(x)$ such that the equilibrium $x=0$ of the closed loop system $\dot{x}=f(x)+G_1(x)u(x)$ is asymptotically stable and homogeneous of degree k.

Remark 2.5. In Definition 2.6, the feedback law $u=u(x)$ should be of homogeneity degree $k+r_0$, that is, $u(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n)=\varepsilon^{k+r_0}u(x)$.

2.3. Input–output stability

Then let us go to input–output stability problem. Consider a nonlinear system

$$\begin{aligned}&\dot{x}=f(x)+G(x)w, \quad x\in R^n,\\ &z=h(x), \quad z\in R^l,\ w\in R^u,\end{aligned} \tag{13}$$

where $f(0)=0, h(0)=0$ and $w(t)$ denotes bounded input variables.

Definition 2.7. System (13) is said to be L_p stable or have a finite L_p-gain $(p\geqslant 1)$ if there exists a finite number $\lambda>0$ such that when $w(\cdot)\in L_p(R^+,R^u)$, we have $z(\cdot)\in L_p(R^+,R^l)$, and

$$\|z\|_p \leqslant \lambda\|w\|_p + C \tag{14}$$

where constant C depends on $x(0)$ and $\|\cdot\|_p$ denotes the norm on $L_p(R^+,R^l)$ or $L_p(R^+,R^u)$ as appropriate.

Definition 2.8. System (13) is said to be small signal L_p stable if there exist positive constants λ, r_*, and ε such that when $\|x(0)\|\leqslant r_*, w(\cdot)\in L_p(R^+,R^u)$ and $\|w(\cdot)\|\leqslant\varepsilon$, we have $z(\cdot)\in L_p(R^+,R^l)$, and

$$\|z\|_p \leqslant \lambda\|w\|_p + C \tag{15}$$

where C depends only on $x(0)$.

Remark 2.6. Sometimes, the solutions of nonlinear dynamics may blow up in finite time and therefore, may not exist on $t\in[0,\infty)$. Thus, let $[0,T^*)$ be the maximal existence interval of solution $x(t)$ of system (8), or (13), with initial condition $x(0)$ and given $(u(t),w(t))$ of (8), or $w(t)$ of (13). T^* may be some positive real number or ∞. In what follows, the discussions are assumed to be held for $t\in[0,T^*)$.

3. Preliminary results

In the section, preliminary results are given for homogeneous systems.

If homogeneous system (12) is homogeneously stabilizable by $u(x)$ (Definition 2.6), then, from Lemma 2.1, we can obtain a radially unbounded Lyapunov function $V\in C^\infty(R^n\backslash\{0\})$ for the closed loop system, which is continuous on R^n and homogeneous with degree $k+2r_0$ such that

$$\mathbf{D}V[f(x)+G_1(x)u(x)]<0, \quad x\neq 0 \tag{16}$$

Lemma 3.1. *If homogeneous system* (12) *is homogeneously stabilizable with V defined as above, then there is a constant* α^0 *such that, for* $\alpha>\alpha^0$,

$$u^*(x)=\begin{cases}-\alpha G_1^T(x)\mathbf{D}V^T(x), & x\neq 0,\\ 0, & x=0,\end{cases} \tag{17}$$

is a continuous stabilizing feedback law for system (12).

Proof. Define the following sets:

$$S^+=\{x\in R^n\colon x\neq 0,\ \mathbf{D}V(x)f(x)\geqslant 0\},$$

$$S^-=\{x\in R^n\colon \mathbf{D}V(x)f(x)<0\},$$

$$S^0=\{x\in R^n\colon x\neq 0,\ \mathbf{D}V(x)G_1(x)=0\}.$$

Note that $u^*(x)$ is homogeneous of degree $k+r_0>0$ (Remark 2.5), and it is C^1, may be except the origin. Since $\lim_{\varepsilon\to 0}u^*(\varepsilon^{r_1}x_1,\ldots,\varepsilon^{r_n}x_n)=0$, control law (17) is also continuous at the origin.

From (16), we obtain that $S^0\subset S^-$. This exactly means that $V(x)$ is a control Lyapunov function, a concept introduced by Artstein (1983) and Sontag (1983), for the system in question. Let

$$M=\max_{x\in S^+\cap S}\{\mathbf{D}V(x)f(x)\}$$

and

$$L=\min_{x\in S^+\cap S}\{\mathbf{D}V(x)G_1(x)G_1^T(x)\mathbf{D}V^T(x)\}.$$

Take $\alpha^0=M/L$. Then, for any $\alpha>\alpha^0$, when $x\neq 0$, by homogeneity, there is a suitable ε such that

$$\begin{aligned}&\mathbf{D}V(x)f(x)+\mathbf{D}V(x)G_1(x)u^*(x)\\ &\quad=\{-\alpha\mathbf{D}VG_1G_1^T\mathbf{D}V^T+\mathbf{D}Vf\}(\varepsilon^{r_1}y_1,\ldots,\varepsilon^{r_n}y_n)<0,\end{aligned}$$

where $y = (y_1, \dots, y_n)^T \in S$ (Remark 2.2). This implies that the homogeneous function $V(x)$ decreases strictly when $x \neq 0$. Therefore, system (12) under the feedback law (17) is globally asymptotically stable. □

Remark 3.1. Different methods have been proposed to construct (homogeneous) stabilizing feedback laws for homogeneous systems, such as Hermes (1991a,b, 1996) and Kawski (1989). Among them, Hermes (1996) proposed a homogeneous stabilizing control in the form of $u(x) = -\alpha[\mathbf{D}V_0(x)]^{1/(\sigma-1)}$, where V_0 is smooth with homogeneity degree $\sigma > 1$.

The following lemma is useful in the study of H_∞ control of homogeneous systems.

Lemma 3.2. *If homogeneous system* (12) *is stabilizable via homogeneous feedback with V as defined in Lemma* 3.1, *then for any continuous homogeneous function ϕ of degree $k + r_0$, there are positive constants γ_1 and γ_2, such that, when $x \neq 0$, at least one of the following inequalities should be satisfied*:

$$\begin{aligned} &\mathbf{D}V(x)f(x) + \frac{1}{\gamma_1}\|\phi(x)\|^2 \leqslant 0, \\ &-\|\mathbf{D}V(x)G_1(x)\| + \frac{1}{\gamma_2}\|\phi(x)\| \leqslant 0 \end{aligned} \tag{18}$$

Proof. Note that the terms on the left-hand sides of inequalities (18) are homogeneous because the degree of $\mathbf{D}Vf$ is $k + 2r_0 - r_i + k + r_i = 2(k + r_0)$ and the degree of $\mathbf{D}VG_1$ is $k + 2r_0 - r_i + s + r_i = k + r_0$, respectively. From Remark 2.2, the discussion can be held sufficiently when $x \in S$.

From the proof of Lemma 3.1, $S^0 \subset S^-$. Then there are an open set E_0, its closure E and its complement set $\bar{E}_0$ (a closed set), such that $S^0 \cup \{0\} \subset E_0 \cup \{0\}$ and $E \subset S^- \cup \{0\}$.

For any $x \in S$, we can study the problem in the two cases: (1) $x \in S \cap E$ and (2) $x \in S \cap \bar{E}_0$.

(1) Since $E \subset S^- \cup \{0\}$ and $E \cap S$ are closed, there are real numbers κ_1 and κ_2:

$$0 < \kappa_1 \leqslant -\max_{S \cap E} \mathbf{D}V(x)f(x),$$

$$\kappa_2 \geqslant \max_{S \cap E} \phi^T(x)\phi(x) \geqslant 0$$

Then taking $\gamma_1 \geqslant \kappa_2/\kappa_1$ yields the first inequality of (18).

(2) Since $S^0 \cup \{0\} \subset E_0 \cup \{0\}$ and $S \cap \bar{E}_0$ are closed, $\|\mathbf{D}V(x)G_1(x)\|$ has a nonzero minimum and $\phi^T(x)\phi(x)$ has a maximum on $S \cap \bar{E}_0$. Therefore, there are positive constants κ_3, κ_4 such that

$$0 < \kappa_3 \leqslant \min_{x \in S \cap \bar{E}_0} \|\mathbf{D}V(x)G_1(x)\|,$$

$$\kappa_4 \geqslant \max_{x \in S \cap \bar{E}_0} \|\phi(x)\| \geqslant 0$$

Then taking $\gamma_2 \geqslant \kappa_4/\kappa_3$ yields the second inequality of (18). □

Before discussing H_∞ control problem, we, as usual, define a Hamiltonian function for system (11)

$$H(V, x, u, w) = \begin{cases} \mathbf{D}V(f + G_1 u + G_2 w) + z^T z - \lambda^2 w^T w, & x \neq 0, \\ \theta^2 u^T u - \lambda^2 w^T w, & x = 0. \end{cases} \tag{19}$$

Then a saddle solution to $\min_u \max_w H(V, x, u, w)$ can be written in the following form because $D^T D = \theta^2 I$:

$$u^* = \begin{cases} -\dfrac{1}{2\theta^2} G_1^T \mathbf{D}V^T, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

$$w^* = \begin{cases} \dfrac{1}{2\lambda^2} G_2^T \mathbf{D}V^T, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Lemma 3.3. *Consider the homogeneous system* (11) *with $G_1 = G_2$. If the system when $w \equiv 0$ is homogeneously stabilizable, with V defined as in Lemma* 3.1, *then, for any $\lambda > \theta$ with θ defined in* (9), *there is a suitable constant $\alpha > 0$ such that, $H(\alpha V, x, u^*, w^*) \leqslant 0$, that is*,

$$\alpha \mathbf{D}Vf - \frac{\alpha^2}{4\theta^2}\mathbf{D}VG_1G_1^T\mathbf{D}V^T + \frac{\alpha^2}{4\lambda^2}\mathbf{D}VG_2G_2^T\mathbf{D}V^T + h^T h \leqslant 0. \tag{20}$$

Proof. Since (20) is homogeneous, our discussion can be mainly limited on "sphere" S, analogously as discussed in Remark 2.2. Define

$$L_\alpha(x) = \alpha \mathbf{D}Vf - \left(\frac{\alpha^2}{4\theta^2} - \frac{\alpha^2}{4\lambda^2}\right)\mathbf{D}VG_1G_1^T\mathbf{D}V^T + h^T h.$$

Also, two cases are discussed separately to obtain (20), that is, $L_\alpha(x) \leqslant 0$.

(1) $x \in S \cap E$. According to Lemma 3.2 with taking $\phi = h$, γ_1 can be got. Take $\alpha^- = \gamma_1$, then, when $\alpha > \alpha^-$, we have

$$L_\alpha(x) < L_{\alpha^-}(x) \leqslant \gamma_1 \mathbf{D}Vf + \|h\|^2 \leqslant 0. \tag{21}$$

(2) $x \in S \cap \bar{E}_0$. Set κ_3, κ_4 as in Lemma 3.2, and take $\kappa \geqslant \max_{x \varepsilon S} \mathbf{D}Vf(x)$, then

$$L_\alpha(x) \leqslant L(\alpha) \overset{\text{def}}{=} \alpha\kappa - \frac{\alpha^2}{4}\left(\frac{1}{\theta^2} - \frac{1}{\lambda^2}\right)\kappa_3^2 + \kappa_4^2.$$

To yield the inequality $L(\alpha) \leqslant 0$, the solutions for $L(\alpha) = 0$ should be found at first:

$$\alpha_1 = \frac{2\kappa + 2\sqrt{\kappa^2 + (1/\theta^2 - 1/\lambda^2)\kappa_3^2\kappa_4^2}}{(1/\theta^2 - 1/\lambda^2)\kappa_3^2},$$

$$\alpha_2 = \frac{2\kappa - 2\sqrt{\kappa^2 + (1/\theta^2 - 1/\lambda^2)\kappa_3^2\kappa_4^2}}{(1/\theta^2 - 1/\lambda^2)\kappa_3^2}.$$

Then taking $\alpha \geqslant \max\{\alpha_1, \alpha_2\}$ or $\alpha \leqslant \min\{\alpha_1, \alpha_2\}$ yields $L(\alpha) \leqslant 0$. However, only the positive solution α_1 makes sense in the problem. Thus, we select $\alpha^+ \geqslant \alpha_1$ and when $\alpha > \alpha^+$, we have

$$L_\alpha(x) < L_{\alpha^+}(x) \leqslant 0. \tag{22}$$

Take $\alpha^* = \max\{\alpha^-, \alpha^+\}$, and if $\alpha > \alpha^*$,

$$H(\alpha V, x, u^*, w^*) = L_\alpha(x) < 0 \tag{23}$$

holds for any $x \in S$, and therefore, for any $0 \neq x \in R^n$.

Furthermore, when $x = 0$, we have $w^* = 0$ and $u^* = 0$. Thus, inequality (20) holds for any $x \in R^n$. □

$V(x)$ may not be smooth, so the following lemma may be needed.

Lemma 3.4.[2] *$V(x(t))$, defined in Lemma 3.3, is absolutely continuous in the maximal existence interval of the solution $x(t)$ (for system (11) with $u = u^*(x(t))$): $[0, T^*)$.*

Proof. Note that $x \neq 0$ if and only if $V \neq 0$. Since V is continuous with respect to x and $x(t)$ is (absolutely) continuous, V is continuous, and when $x \neq 0$, V is differentiable and its derivative is

$$\dot{V} = \mathbf{D}V(f + G_1 u^* + G_2 w) \overset{\text{def}}{=} F_1(x) + F_2(x)w, \tag{24}$$

where $F_1(x)$ and $F_2(x)$ are homogeneous of degree $2(k + r_0) > 0$ and $k + r_0 > 0$, respectively. Because $w(t)$ is bounded for any $t > 0$ (so $\|w\|_\infty < \infty$), $V(x(t))$ is locally Lipschitz continuous on $[0, T^*)$, which implies the absolute continuity of $V(x(t))$ with respect to t. □

4. Nonlinear H_∞ control

In this section, we will show that, if homogeneous system (12) can be stabilized via homogeneous feedback, then the H_∞ control problem of system (11) can be solved globally. Moreover, the H_∞ control problem of a class of systems, whose leading homogeneous systems can be homogeneously stabilized, can also be solved locally.

First of all, we discuss (11) in the case of $G_1 = G_2$.

[2] The presented proofs of Lemma 3.4 and Theorem 4.1, based on the suggestion of an anonymous reviewer, are much simpler than the original ones.

Theorem 4.1. *Consider the homogeneous system (11) with $G_1 = G_2$. If system (11) in the case of $w \equiv 0$ is homogeneously stabilizable, then for any given $\lambda > \theta$ with θ defined in (9), the global H_∞ control problem of system (11) is solvable (with finite L_2-gain λ) and its control law can be given in the form of (17).*

Proof. From homogeneous stabilizability, there is a Lyapunov function $V(x)$ and a control law $u^*(x)$ defined as in Lemma 3.1. Take $V_\alpha = \alpha V$ with $\alpha > \max\{\alpha^*, \alpha^0\}$.

It is easy to obtain $H(V_\alpha, x, u^*, w) \leqslant H(V_\alpha, x, u^*, w^*) \leqslant 0$. According to Lemma 3.4, $V_\alpha(x(t))$ is absolutely continuous, and therefore, it is differentiable almost everywhere and is the integral of its derivative. Therefore, by noting that F_1 and F_2 in (24) are continuous at $x = 0$, $\forall T \in [0, T^*)$,

$$\begin{aligned} 0 &\geqslant \int_0^T H(V_\alpha, x, u^*, w)\,\mathrm{d}\tau \\ &= \int_0^T \dot{V}_\alpha(\tau)\,\mathrm{d}\tau + \int_0^T (z^{\mathrm{T}}z - \lambda^2 w^{\mathrm{T}}w)\,\mathrm{d}\tau \\ &= V_\alpha(x(T)) - V_\alpha(x(0)) + \int_0^T (z^{\mathrm{T}}z - \lambda^2 w^{\mathrm{T}}w)\,\mathrm{d}\tau, \end{aligned}$$

which yields $\int_0^T (z^{\mathrm{T}}z - \lambda^2 w^{\mathrm{T}}w)\,\mathrm{d}\tau \leqslant 0$ since $V_\alpha(x(T)) \geqslant 0$ and $V_\alpha(x(0)) = 0$.

When $w \equiv 0$, the asymptotic stability of closed loop system with the control law $u^*(x)$, can be obtained immediately from Lemma 3.1. □

In fact, the results in this case can be extended easily to the case when the matching condition holds, that is, $G_2 = G_1 X$, where the matrix X may not be of full rank. In this case, λ can be taken to be greater than $\theta\|X\|$. The analysis procedure of the H_∞ control problem in the matching-condition case is almost the same as the above one.

Then, we consider a general case, where $G_1(x)$ may have no direct relation with $G_2(x)$.

Theorem 4.2. *Consider the homogeneous system (11). If system (11) in the case of $w \equiv 0$ is homogeneously stabilizable, then there is a positive constant $\bar{\lambda}$ such that, for any $\lambda > \bar{\lambda}$, the H_∞ control problem of system (11) can be solved (with L_2-gain λ).*

Proof. At first, we prove

$$\begin{aligned} &H(\alpha V, x, u^*, w^*) \\ &\quad = \alpha \mathbf{D}Vf - \frac{\alpha^2}{4\theta^2}\mathbf{D}VG_1G_1^{\mathrm{T}}\mathbf{D}V^{\mathrm{T}} + \frac{\alpha^2}{4\lambda^2}\mathbf{D}VG_2G_2^{\mathrm{T}}\mathbf{D}V^{\mathrm{T}} \\ &\qquad + h^{\mathrm{T}}h \leqslant 0, \end{aligned} \tag{25}$$

where V is the homogeneous Lyapunov function as defined above and $\alpha > 0$ is a constant to be determined. Fix $\alpha/\lambda = l_0 > 0$, a constant. For convenience, we take $l_0 = 1$ (namely, we take $\lambda = \alpha$ for (25)). Then the remaining task is to determine the value α.

To obtain inequality $H(\alpha V, x, u^*, w^*) \leqslant 0$, two cases are discussed as in Lemma 3.3

(1) $x \in S \cap E$. With taking $\phi = \|h\| + \|\mathbf{D} V G_2/2\|$ in Lemma 3.2 and taking $\alpha^- = \gamma_1$, $\alpha > \alpha^-$, we have $H(\alpha V, x, u^*, w^*) \leqslant 0$ for $l = \alpha$;

(2) $x \in S \cap \bar{E}_0$. Set $\kappa_3, \kappa_4, \kappa$ as defined in Lemma 3.2. Then we get the positive solution α^+ to $H(\alpha V, x, u^*, w^*) = 0$ as follows:

$$\alpha^+ = \frac{2\kappa\theta^2 + 2\theta^2\sqrt{\kappa^2 + \kappa_3^2\kappa_4^2/\theta^2}}{\kappa_3^2},$$

Taking $\alpha > \alpha^+$ gives $H(\alpha V, x, u^*, w^*) \leqslant 0$ for $l = \alpha$.

Take $\alpha^* = \max\{\alpha^-, \alpha^+\}$ and $\bar{\lambda} = \alpha^*$. Then, for any L_2-gain $\lambda > \bar{\lambda}$, (25) holds by taking $\alpha = \lambda > \alpha^*$ on S. Therefore, it holds for any $x \in R^n$. Similar to the proof of Theorem 4.1, we integrate both sides of (25) and inequality (10) can be obtained.

The asymptotic stability of the closed loop system with $w \equiv 0$ follows from Lemma 3.1, too. □

As van der Schaft (1992) discussed the H_∞ control problem of nonlinear systems and their linear approximation systems, we study relations between nonlinear systems and their leading homogeneous systems.

Consider

$$\begin{aligned} \dot{x} &= \hat{f}(x) + \hat{G}_1(x)u + \ + \hat{G}_2(x)w, \\ z &= \hat{h}(x) + \hat{D}(x)u, \end{aligned} \tag{26}$$

where

$$\hat{f}(x) = f(x) + \tilde{f}(x), \quad \hat{G}_1(x) = G_1(x) + \tilde{G}_1(x),$$

$$\hat{G}_2(x) = G_2(x) + \tilde{G}_2(x), \quad \hat{h}(x) = h(x) + \tilde{h}(x),$$

$$\hat{D}(x) = D + \tilde{D}(x)$$

with $f(x)$, $h(x)$, G_1, G_2, and D defined as in system (11), and

$$\tilde{f}(x) = o(\Gamma(x)^k), \quad \tilde{h}(x) = o(\Gamma(x)^q), \quad \tilde{D}(x) = o(1)$$

$$\tilde{G}_j(x) = (\tilde{G}_{j1}, \ldots, \tilde{G}_{jn})^{\mathrm{T}}, \quad \tilde{G}_j(x) = o(\Gamma(x)^s), \ j = 1,2.$$

Therefore, system (26) can be approximated by the systems in the form of (11), or in other words, system (11) is a leading homogeneous system of (26).

If, for suitable $\lambda > 0$, system (26) has a local solution to its H_∞ control problem, then the H_∞ control problem can be solved for system (11) when $\|w(t)\|$ is small enough to keep the state in its valid neighborhood of $x = 0$. Conversely, as suggested in (23) and the proofs of the above theorems, for any given $\lambda > \lambda^*$ as discussed above, there is V_α such that the H_∞ control problem of (11) can be solved with $H(V, x, u^*, w^*)|_{(11)} < 0$. In this way,

$$\begin{aligned} H(V, x, u^*, w^*)|_{(26)} &= H(V, x, u^*, w^*)|_{(11)} \\ &\quad + o(\Gamma(x)^{2k + 2r_0}) < 0, \end{aligned}$$

holds near the origin.

Also, when $w \equiv 0$, the equilibrium $x = 0$ of (26) is locally asymptotically stable since the origin of its leading homogeneous system (11) is asymptotically stable (Hermes, 1991b; Rosier, 1992). This means the robustness of homogeneous system (11) against "higher degree perturbations" in a local sense. Therefore, the local H_∞ control problem of system (26) can be solved if the corresponding problem of system (11) is solvable.

Thus, the main result of the section can be described as follows: if a leading homogeneous system of a given nonlinear system is homogeneously stabilizable when $w \equiv 0$, then the H_∞ control problem of the nonlinear system in question can be solved (locally).

5. Examples and remarks

At first, a special form of Young's inequality (Beckenbach & Bellman, 1971) is given:

Lemma 5.1. *When a, b, y are all positive numbers, the following inequality holds*:

$$ya \leqslant y^{1+b} + a^{1+1/b}.$$

With the lemma, two examples are discussed to illustrate the procedure of H_∞ control design.

Example 5.1. Consider a nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1 - 14x_2^3, \\ \dot{x}_2 &= u + w, \\ z &= \begin{bmatrix} x_2 \\ u \end{bmatrix}. \end{aligned} \tag{27}$$

System (27) is homogeneous of degree 0 with respect to the dilation (3,1). Namely, it can be rewritten in the form of (11) with $G_1 = G_2 = (0,1)^{\mathrm{T}}$, $f(x) = (x_1 - 14x_2^3, 0)^{\mathrm{T}}$, $h(x) = (x_2, 0)^{\mathrm{T}}$ and $k = 0$, $s = -1$, $q = 1$.

As well known, the system in the form of

$$\dot{x}_1 = x_1 - 14x_2^3,$$

$$\dot{x}_2 = u$$

cannot be stabilized via smooth feedback, which has been studied widely (Coron & Praly, 1991; Kawski, 1989; Celikovsky & Aranda-Bricaire, 1999). The following procedure also provides a way for its stabilization design.

According to Lemma 5.1, a function, of homogeneity degree 2, taken as

$$V = 4(3x_1^{4/3} - 2x_1x_2 + 4x_2^4)^{1/2} \tag{28}$$

is positive definite and C^1 when $x \neq 0$ and can be viewed as a candidate Lyapunov function. Then $S^0 \subset S^-$, where S^0 and S^- were defined in the proof of Lemma 3.1, because, when $x \neq 0$, $\partial V/\partial x_2 = 0$ leads to $x_1 = 8x_2^3 \neq 0$, which implies

$$\frac{\partial V}{\partial x_4}(x_1 - 14x_2^3) = -12x_2^2 < 0.$$

Therefore, from Lemma 3.1 and Theorem 4.1, the problems of stabilization and H_∞ control can be solved. Here set the L_2-gain value $\lambda = \sqrt{2}$ for the H_∞ control problem of (27).

Take $E = \{x: 3x_2^4 \leqslant x_1 x_2 \leqslant 13x_2^4\}$ as in Lemma 3.2. After estimation, we obtain $\alpha^- = \gamma_1 \geqslant 135/4$ and $\alpha^+ = 2835$. Then, when $\alpha > \max\{\alpha^-, \alpha^+\} = 2835$, the control law

$$u^* = \begin{cases} \dfrac{\alpha 8(x_1 - 8x_2^3)}{V}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

is an H_∞ controller with L_2-gain $\sqrt{2}$.

Example 5.2. Consider

$$\begin{aligned} \dot{x}_1 &= x_1 - 8x_3^3 + |x_2|^{3/2}\operatorname{sgn}(x_2) + x_3^2 w, \\ \dot{x}_2 &= x_3 u_1 + x_1 x_3, \\ \dot{x}_3 &= u_2 + x_2, \\ z &= \left(x_1 + \frac{x_3}{2} \quad u_1 \quad u_2\right)^{\mathrm{T}}. \end{aligned} \tag{29}$$

System (29) is not homogeneous, but it can be approximated by a homogeneous one with respect to the dilation (3, 2, 1):

$$\begin{aligned} \dot{x}_1 &= x_1 - 8x_3^3 + |x_2|^{3/2}\operatorname{sgn}(x_2) + x_3^2 w, \\ \dot{x}_2 &= x_3 u_1, \\ \dot{x}_3 &= u_2, \\ z &= \left(\frac{x_3}{2} \; u_1 \; u_2\right)^{\mathrm{T}}, \end{aligned} \tag{30}$$

where $k = 0$, $s = -1$ and $\operatorname{sgn}(\cdot)$ denotes the sign function.

To solve the H_∞ control problem of (29), we consider (30) at first. By Lemma 5.1,

$$V = (3x_1^{8/3} - x_1^{7/3}x_3 + 2x_3^8 - x_1|x_2|^{5/2}\operatorname{sgn}(x_2) + 4x_2^4)^{1/4} \tag{31}$$

is a positive-definite homogeneous function. Then take

$$E = \Big\{x: x_1^2 + x_3^6 \leqslant |x_2|^3;\ |x_2|^3 + (x_1 - 4^{6/7}x_3^3)^2 \leqslant x_1^2;\ \left[x_1 - \frac{32}{5}|x_2|^{5/2}\operatorname{sgn}(x_2)\right]^2 + (x_1 - 4^{6/7}x_3^3)^2 \leqslant |x_2|^3\Big\}.$$

Clearly, $S^0 \subset S^-$, where S^0 and S^- were defined in the proof of Lemma 3.1. Therefore, as discussed in Lemma 3.1 and then according to Theorem 4.2, the H_∞ control problem is solvable and the feedback law is $u^* = (u_1^* u_2^*)^{\mathrm{T}}$ with

$$u_1^* = \begin{cases} -\dfrac{\alpha x_3}{2}\dfrac{\partial V}{\partial x_2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$u_2^* = \begin{cases} -\dfrac{\alpha}{2}\dfrac{\partial V}{\partial x_3}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

with $\alpha = 75575$ after estimation (the procedure proposed in Theorem 4.1), is a global H_∞ controller of system (30) for L_2-gain $\lambda = 152$. Thus, a local solution of H_∞ control problem for (29) is obtained around the origin.

Remark 5.1. The construction of the proposed H_∞ controllers is Lyapunov-based. Unfortunately, we have no systematic way to construct such Lyapunov functions.

Remark 5.2. Here we did not discuss how to find the smallest L_2-gain λ^*, which may not be a constant in nonlinear case. However, with the conditions in Theorem 4.1 and following the discussion ideas proposed in Remark 4 of (Hong & Li, 1998), we have $\lambda^* = \theta$. In fact, the results here are consistent with those in the case of $r_1 = r_2 = \cdots = r_n$ in (Hong & Li, 1998).

Remark 5.3. Theorems 4.1 and 4.2 also suggest a method to estimate the gain α in feedback. However, the estimation, directly depending on the choice of the function V and the set E, is usually quite conservative.

6. L_p stability

In this section, input–output stability in L_p sense is studied with the same homogeneous techniques.

Suppose that system (13) is homogeneous, i.e.,

$$\begin{aligned} \dot{x} &= f(x) + G(x)w, \quad x \in R^n, \\ z &= h(x), \quad z \in R^l, \ w \in R^u, \end{aligned} \tag{32}$$

where $f(x)$, $h(x)$ and $G(x)$ are continuous, $C^1(R^n \backslash \{0\})$ and homogeneous of degrees $k > -r_0$, $q > 0$, and $s \geqslant -r_0$, respectively. $f(0)=0$ and $h(0)=0$. Moreover, we consider the system corresponding to (32) with $w \equiv 0$:

$$\dot{x}=f(x), \quad x \in R^n \tag{33}$$

which is homogeneous since (32) is.

The following is the main result of the section.

Theorem 6.1. *Suppose that system* (32) *and* (33) *are defined as above. If $q+s=k$ and the equilibrium $x=0$ of* (33) *is asymptotically stable, then* (32) *is globally L_p-stable when $p \geqslant (k+r_0)/q$.*

Proof. Since (33) is homogeneous and asymptotically stable, a radially unbounded Lyapunov function V of homogeneity degree $\sigma - k$ with $\sigma \geqslant k + r_0$ can be constructed for system (33), according to Lemma 2.1. Obviously, $\sigma \geqslant q+s+r_0 \geqslant q > 0$. Note that $V \in C^1(R^n \backslash \{0\})$ and is continuous in R^n. Then, with the help of homogeneous properties, the following inequalities hold globally:

$$c_1 \Gamma(x)^{\sigma-k} \leqslant V(x) \leqslant c_2 \Gamma(x)^{\sigma-k},$$

$$\dot{V}|_{(33)} \leqslant -c_0 \Gamma(x)^{\sigma},$$

$$\left\| \frac{\partial V}{\partial x} G(x) \right\| \leqslant c_3 \Gamma(x)^{\sigma-k+s}, \ x \neq 0$$

and $\|h(x)\| \leqslant c_5 \Gamma(x)^q$ for some suitable positive constants c_0, c_1, c_2, c_3, and c_5.

Consider

$$\dot{V}|_{(5.1)} = \frac{\partial V}{\partial x}[f(x) + G(x)w]$$
$$\leqslant -c_0 \Gamma(x)^{\sigma} + c_3 \Gamma(x)^{\sigma-k+s}\|w\|, \tag{34}$$

when $x \neq 0$. Then two cases are studied separately.

(1) $\sigma > q$. Note that $-\Gamma(x)^{\sigma} \leqslant -c_5^{-\sigma/q}\|z\|^{\sigma/q}$. According to Lemma 5.1 by setting $p = \sigma/q$, (34) leads to

$$\dot{V} \leqslant -\frac{c_0}{2}\Gamma(x)^{\sigma} + c_3\left(\frac{c_0}{2c_3}\right)^{1-\sigma/q}\|w\|^p$$
$$\leqslant -\frac{c_0}{2c_5^p}\|z\|^p + \left(\frac{2c_3}{c_0}\right)^{p-1} c_3\|w\|^p, \quad x \neq 0.$$

(2) $\sigma = q$. Then (34) becomes

$$\dot{V} \leqslant -\frac{c_0}{c_5^p}\|z\|^p + c_3\|w\|^p, \quad x \neq 0.$$

The inequalities in the above two cases can be rewritten in a unified form

$$\dot{V} \leqslant -c_6\|z\|^p + c_4\|w\|^p, \quad x \neq 0 \tag{35}$$

for suitable positive constants c_4 and c_6.

Similar to Lemma 3.4, V can be proved to be absolutely continuous in the maximal existence interval of solution $x(t)$ of (32), then integrating both sides of (35), we have, for any $T > 0$,

$$V(x(T)) - V(x(0)) \leqslant -c_6 \int_0^T \|z(\tau)\|^p \, \mathrm{d}\tau + c_4 \int_0^T \|w(\tau)\|^p \, \mathrm{d}\tau, \tag{36}$$

which leads to

$$\int_0^T \|z(\tau)\|^p \, \mathrm{d}\tau \leqslant \frac{c_4}{c_6}\int_0^T \|w(\tau)\|^p \, \mathrm{d}\tau + \frac{V(x(0))}{c_6}.$$

Since $p = \sigma/q \geqslant (k+r_0)/q \geqslant 1$, there are positive constants λ and $C(x(0))$, such that

$$\|z\|_p \leqslant \lambda \|w\|_p + C(x(0)).$$

Consider the case of $p = \infty$. From Lemma 2.1, there is a smooth homogeneous Lyapunov function V_0 with its degree $\sigma_0 > \max_{1 \leqslant i \leqslant n} r_i$. Similar to the above procedure, it is not difficult to obtain

$$c_1 \Gamma(x)^{\sigma_0} \leqslant V_0(x) \leqslant c_2 \Gamma(x)^{\sigma_0}$$

and

$$\dot{V}_0 \leqslant -[c_7 \Gamma(x)]^{\sigma_0 + k} + [c_8 \|w\|]^{(\sigma_0+k)/q}$$

for some suitable positive numbers c_1, c_2, c_7, c_8. Then, a result similar to the inequality of input-to-state stability can be obtained (with the techniques used in the related references such as Ryan (1995) and Sontag (1995)):

$$\Gamma(x(t))^q \leqslant \beta_0(\Gamma(x(0))^q) + \gamma_0(\|w\|_\infty),$$

where $\gamma_0(\|w\|_\infty) = (c_1^{-1}c_2)^{q/\sigma_0} c_7^{-q} c_8 \|w\|_\infty$. Recalling that $\|z\| \leqslant c_5 \Gamma(x)^q$, we have $\|z\|_\infty \leqslant \beta_*(x(0)) + c_\infty \|w\|_\infty$ for some suitable function β_* and positive number c_∞. Therefore, L_∞ stability of the system can be obtained. □

Remark 6.1. Ryan (1995) studied the input-to-state stability for the systems of the following form:

$$\dot{x} = f(x, w) \tag{37}$$

with $f(\varepsilon x, \varepsilon w) = \varepsilon^v f(x, w)$ for $v \geqslant 1$ and $\varepsilon > 0$, and showed that, if the origin is an asymptotically stable equilibrium of the unforced system $\dot{x} = f(x, 0)$, then (37) is input-to-state stable in L_p sense for all $p \geqslant v$. Note that Theorem 6.1 is consistent with the result by taking $v = k+1$ and $q = r_0 = 1$.

Furthermore, consider a class of systems in the form of

$$\dot{x} = f(x) + \tilde{f}(x) + [G(x) + \tilde{G}(x)]w,$$
$$z = h(x) + \tilde{h}(x), \tag{38}$$

where f, G, and h are continuous and homogeneous of degree $k > -r_0$, $s \geqslant -r_0$, and $q > 0$, respectively, as defined above. Moreover, $\tilde{f} = o(\Gamma(x)^k)$,

$$\tilde{G} = (\tilde{g}_1, \ldots, \tilde{g}_n)^{\mathrm{T}}, \quad \tilde{g}_i(x) = o(\Gamma(x)^s) \in R^n, \quad i = 1, \ldots, n$$

and $\tilde{h}(x) = o(\Gamma(x)^q)$. In other words, system (38) can be approximated by a system in the form of (32).

A local result on L_p stability for system (38) can be obtained with the techniques used in the proof of Theorem 6.1 and the analysis ideas proposed in (Vidyasagar & Vannelli, 1982).

Theorem 6.2. *Suppose that* (38) *and* (33) *are defined as above. If $q + s \geqslant k$ and the equilibrium $x = 0$ of* (33) *is asymptotically stable, then system* (38) *is small signal L_p-stable when $p \geqslant \max\{1,(k + r_0)/q\}$.*

7. Conclusions

In the paper, the problems of H_∞ control and L_p stability of a class of nonlinear systems, including some systems that cannot be stabilized by smooth feedback or nonsmooth systems. To deal with that difficult situation, we focus on the case that the systems in question possess some homogeneous properties. Relations between the homogeneous stabilization of the considered systems and their H_∞ control problem are found. Furthermore, simple relations between input-output stability and homogeneity degrees are shown, too.

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