Abstract—This paper addresses the problem of the local state feedback stabilization of a class of nonlinear systems with nonminimum phase zero dynamics. A new technique, namely, the Lyapunov function with homogeneous derivative along solution curves has been developed to test the approximate stability of the dynamics on the center manifold. A set of convenient sufficient conditions are provided to test the negativity of the homogeneous derivatives. Using these conditions and assuming the zero dynamics has stable and center linear parts, a method is proposed to design controls such that the dynamics on the designed center manifold of the closed-loop system is approximately stable. It is proved that using this method, the first variables in each of the integral chains of the linearized part of the system do not affect the approximation order of the dynamics on the center manifold. Based on this fact, the concept of injection degree is proposed. According to different kinds of injection degrees certain sufficient conditions are obtained for the stabilizability of the nonminimum phase zero dynamics. Corresponding formulas are presented for the design of controls.

Index Terms—Approximate stability, center manifold, injection degree, Lyapunov function with homogeneous derivative, zero dynamics.

I. INTRODUCTION

STABILIZATION is one of the basic tasks in control design. The asymptotic stability and stabilization of nonlinear systems have received significant attention [18]–[24]. The center manifold approach has been developed to solve the problem [1], [2], [12], [18], [24]. In [1], [2], some special nonlinear controls are designed to stabilize some particular control systems. The method used there is basically a case-by-case study. For control systems in normal form, assume the center manifold has minimum phase, then a quasi-linear feedback can be used to stabilize linearly controllable variables. We refer to [3]–[6] for minimum phase method and its applications.

Based on these pioneer works, this paper proposes a procedure to produce a state feedback to stabilize nonminimum phase zero dynamics. The designed state feedback control ensures that the dynamics on the designed center manifold of the closed-loop system is approximately stable. To obtain the desirable properties, we combine the center manifold method with Lyapunov function method.

Motivated by the works on stabilization of homogeneous vector field [13]–[17], we propose a new method, namely, that of a Lyapunov function with homogeneous derivatives along solution curves. This Lyapunov function is used to test the approximate stability of a dynamics with odd degree approximating systems, where degree means the polynomial degree. It is particularly suitable for testing the dynamics on a designed center manifold of a closed-loop system, because the degrees of the approximate system of the dynamics on the center manifold may be converted by certain state feedback controls to have odd degree. In this way, the method is applicable to a large class of nonlinear systems with stable and center zero dynamics.

To avoid counting the order of smoothness, through this paper the systems and all other objects involved are assumed to be $C^\infty$, or as smooth as required, on a neighborhood of the origin.

We motivate this work by means of a practical problem: consider the stabilization of an airplane via a designed center manifold. We may find some useful observations from this example for design of both the center manifold and the stabilizing controls. The following example is basically taken from [7], with a modification that the speed is assumed to be dependent on altitude when the atmospheric resistance is taken into consideration.

Example 1.1 [7]: Denote an airplane’s altitude in meters by $h$. Assume that the body of the plane is slanted $\phi$ radians with respect to the horizontal and that the ground speed is $c(h)$. Also, assume the flight path forms an angle of $\alpha$ radians with the horizontal and $c$ is small. The system is described as

$$
\begin{align*}
\dot{h} &= c(h)\alpha, \\
\dot{\phi} &= -\omega(h - \alpha - bh), \\
\dot{\alpha} &= \omega.
\end{align*}
$$

where $\omega > 0$ is a constant representing a natural oscillation frequency and $a$ and $b$ are positive constants. The problem we address is altitude tracking: i.e., a target altitude $\xi$, where $\xi = h_0\alpha$. Set $x_1 = \alpha, x_2 = c(h - \alpha), x_3 = \omega c(h - \alpha)$ and assume $\dot{h}/dh|_{h_0} \neq 0$. We have $\dot{z} = (c(h) - c(h_0))\alpha + q(z)\alpha$ with $q(0) = 0$. Then the system (1.1) is transformed into a standard form as

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -\omega x_2 - ax_3 + \alpha c(h_0)h_0, \\
\dot{z} &= q(z)x_1,
\end{align*}
$$

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We assume that \( q_z(0) := (d/dz)q(0) \neq 0 \) and denote \( f(x, z) = -\omega x_2 - \alpha_3 x_3, g(x, z) = \alpha_0 \). Then system (1.2) is in a normal form for affine nonlinear systems [4]. The zero dynamics (with \( y = x_1 \)) becomes \( \dot{z} = 0 \), which is not asymptotically stable. Therefore, a quasi-linear control cannot make the origin asymptotically stable, and a nonlinear state feedback control should be considered.

Motivated by the early works [1], [2], we may try the following control:

\[
\dot{u} = -\frac{f(x, z)}{g(x, z)} + \frac{1}{g(x, z)} (a_1x_1 + a_2x_2 + a_3x_3 + b_0z^2). \tag{1.3}
\]

To get a stabilizing control, we can first choose \( a_1, a_2, a_3 \) to stabilize the linearly controllable variables \( x_1, x_2, x_3 \), and then choose \( b \) to stabilize the central variable \( z \). To determine a possible value of \( b \), let

\[
\Phi(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \\ \phi_3(z) \\ \phi_4(z) \end{pmatrix} = 0 (||z||^2)
\]

be used to approximate the center manifold. We refer to [9] for the notation \( 0 (||z||^k) \) and the following operator \( M \). Then we have

\[
M\Phi(z) = D\Phi(z) (g(z)\phi_2(z))
\]

\[
- \begin{pmatrix} \phi_2(z) \\ \phi_3(z) \\ \phi_4(z) \end{pmatrix}
\]

\[
= 0 (||z||^4)
\]

\[
- \begin{pmatrix} \phi_2(z) \\ \phi_3(z) \end{pmatrix}
\]

\[
= 0 (||z||^2).
\]

Choose

\[
\begin{cases}
\phi_1 = -\frac{b}{a_1} z^2 \\
\phi_i = 0, & i = 2, 3.
\end{cases}
\]

Then \( M\Phi(z) = 0 (||z||^4) \). According to the approximation theorem [9], the center manifold can be expressed as

\[
\begin{pmatrix} x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \end{pmatrix} = h_1(z) = -\frac{b}{a_1} z^2 + 0 (||z||^4), \quad i = 2, 3.
\]

The dynamics on the center manifold is

\[
\dot{z} = (q_z(0) z + 0 (||z||^2)) h_1(z) = \frac{b}{a_1} q_z(0) z^3 + 0 (||z||^4). \tag{1.5}
\]

Choose \( \{a_1, a_2, a_3, b\} \) such that the linear part is Hurwitz and \(-b/a_1 q_z(0) < 0\), say \( a_1 = -1, a_2 = a_3 = -3, b = -q_z(0) \). The feedback control becomes

\[
\dot{u} = -\frac{f(x, z)}{g(x, z)} + \frac{1}{g(x, z)} (-x_1 - 3x_2 - 3x_3 - q_z(0) z^2).
\]

It follows that (1.5) is asymptotically stable at origin, and then so is the closed-loop system.

Some observations from this example follow.

1) The higher degree (\( \text{deg} \geq 2 \)) state feedback doesn’t affect the local stability of the linearly controllable variables but it may affect the center part variables by changing the structure of the center manifold.

2) Higher order feedback can be “injected” into the dynamics on center manifold through the first variable, \( x_1 \), of the integral chain. The variable \( x_1 \) doesn’t affect the order of approximation of the center manifold. This component of the linear part can be employed to modify the nonlinear dynamics.

3) Since the center manifold is approximated up to a certain degree the approximated dynamics on the center manifold should be asymptotically stable up to certain degree uncertainties to assure the stability of the original system.

The paper is organized as follows. Section II defines the concept of Lyapunov function with homogeneous derivative along solution curves and gives some fundamental properties. Section III provides several sufficient conditions for testing the approximate stability of vector fields. Sections IV–VII discuss design methods for affine nonlinear systems with zero center. The general result is in Section IV. Then according to the injection degrees, the classified testing conditions and formulas for odd, even and mixed injection degrees are presented in Sections V–VII, respectively. Section VIII contains some concluding remarks.

II. LYAPUNOV FUNCTION WITH HOMOGENEOUS DERIVATIVE

Since in general we can only obtain an approximation of the center manifold, it is necessary to have some convenient tools to verify the stability of the dynamics on center manifold through its approximated dynamics. For this purpose a new concept, Lyapunov function with homogeneous derivative, is proposed in this section.

Consider a dynamical system

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{2.1}
\]

with \( f(0) = 0 \).

We use \( Z_+^n \) for the set of nonnegative integers. For a multi-index \( S = (s_1, \ldots, s_n) \in Z_+^n \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we denote

\[
|S| = \prod_{i=1}^n s_i, \quad x^S = \prod_{i=1}^n (x_i)^{s_i}, \quad S! = \prod_{i=1}^n (s_i)!.
\]

Note that \( 0! = 1 \), so \( S! \neq 0 \). For a smooth function \( F(x) \), we denote

\[
\frac{\partial^{|S|} F(x)}{\partial x^S} = \frac{\partial^{|S|} F(x)}{\partial x_1^{s_1} \partial x_2^{s_2} \cdots \partial x_n^{s_n}}.
\]

Then we can give the following definition.

**Definition 2.1:**

1) Let \( k_0 \) be the lowest degree of nonvanishing terms of the Taylor expansion of \( f_i(x_i), \ i = 1, \ldots, n \). A system consisted of only the lowest degrees’ \((k_i)\) terms of (2.1) is said...
to be the (lowest degrees’) approximate system of (2.1). It can be expressed formally as

\[ \dot{x}_i = g_i(x) := \sum_{|S|=k_i} \frac{1}{S!} \frac{\partial^{|S|} f(x)}{\partial x^S}(0)x^S, \quad i = 1, \ldots, n. \]  

(2.2)

2) System (2.2) is said to be an odd approximation of (2.1) if all \( k_i \) are odd.

3) System (2.1) is said to be approximately stable if

\[ \dot{x}_i = f_i(x) + 0(|x|)^{k_i+1}, \quad i = 1, \ldots, n \]

is locally asymptotically stable at origin.

Remark 1:

1) In (2.2) \( g_i \) is a homogeneous polynomial of degree \( k_i \). So \( g = (g_1, \ldots, g_n)^T \) is a component-wise homogeneous vector field.

2) When \( k_1 = \cdots = k_n := k \), the approximate stability defined above coincides with the conventional one [24]. Otherwise, it is coordinate-depending. It is clear that approximate stability implies asymptotic stability, but the inverse is not true.

Definition 2.2: Given a component-wise homogeneous polynomial vector field \( g = (g_1, \ldots, g_n)^T \), a positive definite polynomial \( p \) is said to be a Lyapunov function with homogeneous derivative (LFHD) along \( g \), if the Lie derivative \( L_g V \) is homogeneous with

\[ \text{deg}(L_g V) = \text{deg} \left( \frac{\partial V}{\partial x_i} \right) + \text{deg}(g_i), \quad i = 1, \ldots, n. \]

The following example provides two typical LFHD, which will be used later.

Example 2.3: Let \( g = (g_1, \ldots, g_n)^T \) be a component-wise homogeneous vector field with odd degrees, \( \text{deg}(g_i) = k_i, i = 1, \ldots, n \), and \( m \) be a given integer satisfying

\[ 2m \geq \max \{k_1, \ldots, k_n\} + 1. \]

1) Set \( 2m_i = 2m - k_i + 1, i = 1, \ldots, n \), then

\[ V = \sum_{i=1}^{n} p_{i} x_i^{2m_i}; \]  

(2.3)

is a LFHD along \( g \) if \( p_i > 0, \forall i \).

2) Assume \( k_1 = \cdots = k_m := k^1; k_1, k_1+1 = \cdots = k_{m_1}, k_{m_1}+1 = \cdots = k_{m_1+m_2} := k^2; \cdots \) \( k_1, k_1+1 = \cdots = k_{m_1+m_2} := k^r \), where \( k^j \) are odd and \( \sum_{i=1}^{r} m_i = n \). Denote \( x = (x_1^1, \ldots, x_r^i) \), with \( \text{dim}(x^i) = n_i \) and set \( 2m^i = 2m - k^i + 1, i = 1, \ldots, r \). Then

\[ V = \sum_{i=1}^{r} \left( \begin{pmatrix} x_1^i \end{pmatrix}^{m^i}, \ldots, \begin{pmatrix} x_n^i \end{pmatrix}^{m^i} \right) \times P_i \ \left( \begin{pmatrix} x_1^i \end{pmatrix}^{m^i}, \ldots, \begin{pmatrix} x_n^i \end{pmatrix}^{m^i} \right)^T; \]

(2.4)

is a LFHD along \( g \) if \( P_i, i = 1, \ldots, r \) are positive–definite matrices with dimensions \( n_i \times n_i \).

Note that the derivative of \( V \) in either (2.3) or (2.4) along \( g \) is then a homogeneous polynomial of degree \( 2m \).

The following example shows that LFHD is a new concept because both \( V \) and \( g \) are not homogeneous but the derivative is. Since the approximate system of a smooth system is always component-wise homogeneous, method of LFHD can be used for testing the stability of the odd-degree approximated systems. It is particularly useful in testing the stability of the dynamics on center manifold of the closed-loop systems, because the leading degree of the dynamics may be converted to odd by suitable state feedback.

Example 2.4: Consider the following system:

\[ \begin{cases} \dot{x} = f_1(x, y) = -x \sin(x^2 - y^2) \\ \dot{y} = f_2(x, y) = y^4 \ln(1 - 2y + x). \end{cases} \]

(2.5)

Using Taylor expansion, the approximate system of (2.5) is obtained as

\[ \begin{cases} \dot{x} = g_1(x, y) = -x^3 + xy^2 \\ \dot{y} = g_2(x, y) = -2y^6 + xy^4. \end{cases} \]

(2.6)

First of all, we show that the vector field \( g \) in (2.6) is not homogeneous with respect to any group of dilations of the form \( \Delta_1 : (x_1, x_2) \mapsto (\ell^{r_1}x_1, \ell^{r_2}x_2) \) [12]. Assume (2.6) is \( k \)-th homogeneous with dilation \((r_1, r_2)\), that is

\[ \begin{cases} g_1(\ell^{r_1}x_1, \ell^{r_2}x_2) = \ell^{k+r_1}g_1(x, y) \\ g_2(\ell^{r_1}x_1, \ell^{r_2}x_2) = \ell^{k+r_2}g_2(x, y). \end{cases} \]

(2.7)

From the first equations of (2.6) and (2.7) we have \( r_1 = r_2 \) and \( k = 2r_1 \) and from the second equations of (2.6) and (2.7) we have \( r_1 = r_2 \) and \( k = 4r_1 \). It follows that \( k = r_1 = r_2 = 0 \). So (2.6) is not homogeneous with any dilation. However, we can construct a LFHD as \( V = x^4 + y^2 \), which is not homogeneous. Then the derivative of \( V \) along (2.6) is

\[ \dot{V} = -4x^6 + 4x^4y^2 - 4y^6 + 2xy^5 \leq -x^6 - y^6. \]

(The last inequality can be shown by using the inequality (3.1) in the next section.) So the derivative is homogeneous and negative definite. The following proposition will show that (2.5) is asymptotically stable at origin.

The following proposition is fundamental for LFHD.

Proposition 2.5: System (2.1) is approximately stable at origin if there exists a LFHD of its approximate system (2.2) such that its derivative along (2.2) is negative–definite.

Proof: Assume \( L_g V \) is negative definite, then it should be of even degree, say \( \text{deg}(L_g V) = 2m \). We claim that there exists a real number \( b > 0 \) such that

\[ L_g V(x) \leq -b \sum_{i=1}^{n} (x_i)^{2m}. \]

(2.8)

Since \( L_g V \) is negative definite, on the compact "sphere"

\[ S = \left\{ z \mid \sum_{i=1}^{n} (z_i)^{2m} = 1 \right\} \]

\( L_g V(z) \) attains its maximum value \(-b < 0\). That is

\[ L_g V(z) \leq -b < 0, \quad z \in S. \]
Now any \( x \in \mathbb{R}^n \) can be expressed as \( x = k z \) for some \( z \in S \).
Then
\[
L_g V(x) = L_g V(kz) = k^{2m} L_g V(z) \leq -\mu k^{2m} = -b \sum_{i=1}^{n} (x_i)^{2m}
\]
which proves the claim.

Using (2.8), the derivative of the LFHD becomes
\[
\dot{V}_f = \dot{V}_{f+\alpha\|e\|^{K+1}} \leq -b \sum_{i=1}^{n} (x_i)^{2m} + 0 (\|e\|^{2m+1}) (2.9)
\]
where \( g + 0 (\|e\|^{K+1}) \) is a shorthand for \( (g_1(x) + 0 (\|e\|^{K+1})) \ldots, g_n(x) + 0 (\|e\|^{K_n+1})^T \).

For the homogeneous vector fields \cite{11} gives (with slightly different statement) the following.

**Theorem 2.6 [11]:** Assume (2.1) has \( k_1 = \ldots = k_n = k \) and its approximate system (2.2) is asymptotically stable. then (2.1) is asymptotically stable.

The Proposition 2.5 and Theorem 2.6 will be our major tools for testing approximate stability.

### III. SOME SUFFICIENT CONDITIONS FOR NEGATIVITY

This section investigates some sufficient conditions for testing approximate stability of systems with odd approximate systems.

We need the following inequality, which is based on the fact that the algebraic average is greater than or equal to the geometric average.

**Lemma 3.1:** Let \( S \in \mathbb{Z}_4^n \) and \( x \in \mathbb{R}^n \). The following inequality holds:
\[
|x^S| \leq \sum_{j=1}^{n} s_j |x_j|^{|S|}.
\] (3.1)

Given a component-wise homogeneous polynomial vector field \( g = \text{col}(g_1, \ldots, g_n) \) with \( \text{dim}(g_i) = k_i \ i = 1, \ldots, n \).
We express \( g_i \) as
\[
g_i(x) = a_{d_i}^i x^{k_i} + \sum_{S \neq k_i} a_S^i x^S, \quad i = 1, \ldots, n
\] (3.2)
where the index \( d_i = k_i + (0, \ldots, k_i, \ldots, 0) \), which indicates the diagonal term. Then we have

**Theorem 3.2: Cross Row Diagonal Dominating Principle (CRDDP):** The vector field \( g \) given in above, is asymptotically stable at origin, if there exists an integer \( m \) with \( 2m > \max\{k_1, \ldots, k_n\} \), such that
\[
-a_{d_i}^i > \sum_{|S| = k_i, S \neq k_i} |a_S^i| \left( \frac{s_i + 2m - k_i}{2m} \right) + \sum_{j=1, j \neq i}^{n} \sum_{|S| = k_j} |a_S^j| \left( \frac{s_j}{2m} \right), \quad i = 1, \ldots, n
\] (3.3)
where \( S = (s_1, \ldots, s_n) \in Z_4^n \).

**Proof:** Choose a LFHD as
\[
V = \sum_{i=1}^{n} \left( \frac{1}{2m - k_i + 1} \right) x_i^{2m-k_i+1},
\]
Then we have
\[
\dot{V}_f = \sum_{i=1}^{n} \sum_{|S| = k_i} a_S^i x_i^{2m-k_i}.
\] (3.4)
This is a homogeneous polynomial of degree \( 2m \). Now using (3.1) to split each term in (3.4) and collecting terms, (3.3) yields that
\[
\dot{V}_f < -\sum_{i=1}^{n} \epsilon_i x_i^{2m}, \quad \text{for some} \ \epsilon_i > 0,
\]
The conclusion follows immediately.

One obvious improvement for this estimation can be done as the following: Negative semidefinite nondiagonal terms can be eliminated from the estimation. Formally, for each \( g_i \) define a set of its terms by their exponents as
\[
Q_i = \{ |S| = k_i | s_j \neq i \} \text{ are even and } a_S^i < 0 \}.
\]
Terms with exponents in \( Q_i \) are negative–semidefinite in \( (\partial V / \partial x_i) g_i \). Moving such terms from (3.3) yields
\[
-a_{d_i}^i > \sum_{S \neq k_i, S \notin Q_i} |a_S^i| \left( \frac{s_i + 2m - k_i}{2m} \right) + \sum_{j=1, j \neq i}^{n} \sum_{|S| = k_j} |a_S^j| \left( \frac{s_j}{2m} \right), \quad i = 1, \ldots, n.
\] (3.5)
Later on we will simply use (3.5) as CRDDP.
Next, we give a simpler form, which deals with each row independently.

**Corollary 3.3 Diagonal Dominating Principle (DDP):**
Given a polynomial vector field \( g \) as in Theorem 3.2. It is asymptotically stable at origin if
\[
-a_{d_i}^i > \sum_{|S| = k_i, S \notin Q_i} |a_S^i|, \quad i = 1, \ldots, n.
\] (3.6)

**Proof:** Since in (3.5) \( m \) can be arbitrary large, let \( m \to \infty \), the right-hand side of (3.5) becomes right-hand side of (3.6). Hence the strict inequality (3.6) implies (3.5) for large enough \( m \).

In fact, DDP is an analog of Gersgorin’s theorem [25]. Considering linear systems, they provide same stability results. However, CRDDP does not have its linear analog.

Using inequality (3.1), we can reduce the homogeneous polynomial of \( \deg = 4k \) into a “dominating” quadratic form with variables \( x_i^{2k}, i = 1, \ldots, n \).

**Algorithm 3.4: Quadratic Form Reducing Algorithm (QFRA):** Let \( g = \text{col}(g_1, \ldots, g_n) \) and \( \deg(g_i) = k_i \ i = 1, \ldots, n \) with odd \( k_i \).
Step 1) Choose smallest even number $m = 2k$ such that $2m > \max\{k_1, \ldots, k_n\}$. Construct a $2m$ homogeneous polynomial $q(x)$ as

$$q(x) = \sum_{i=1}^{n} x_i^{2m-k_i} g_k.$$

Step 2) Find all terms in $q(x)$, for which the index $S$ of $x^S$ has component $s_j$ less than $2k$. Split it into two equal exponent groups in the alphabetical order of $x_i$.

For $x_2 x_3$, we have $S = (2, 5, 1)$, and it is split as $\alpha_2 x_2^2 x_3^1 = \alpha_2 x_2 x_3^2 x_3^1$. We also use $\alpha_2 x_2 x_3^4$. For $x_2 x_3^3$, we have $S = (2, 5, 1)$, and it is split as $\alpha_2 x_2 x_3^4 = \alpha_2 x_2^3 x_3^1$.

Step 3) Using (3.1) to convert them into several $2k$ exponent terms, e.g.,

$$\alpha_2 x_2 x_3^4 \leq |a_i| \left( \begin{array}{c}
\frac{x_4}{2} + \frac{x_4}{4} + \frac{x_4}{4} \end{array} \right) x_3^4
= |a_i| \left( \frac{1}{2} x_4^2 x_3^4 + \frac{1}{4} x_4^2 x_3^4 + \frac{1}{4} x_4^2 \right).$$

Replace the original terms in $q(x)$ by their splitting terms.

The algorithm produces a quadratic form of $x_i^{2k}, i = 1, \ldots, n$. Then the following can be proved by constructing a suitable LFHHD.

**Proposition 3.5:** If the resulting quadratic form produced by the above algorithm is negative definite, then $q(x)$ is negative definite. Consequently, $q(x)$ is asymptotically stable at zero.

The following example is used to describe the notations and results in the above Theorem 3.2 through Proposition 3.5.

**Example 3.6:**

Find a region for parameter $\lambda$, such that the following system is asymptotically stable at origin:

$$\begin{align*}
\dot{x}_1 &= \sin(x_1) - x_1 \cos(2x_2) \\
\dot{x}_2 &= x_2^2 \ln(1 - x_2 - x_3) + 0.5 x_2^2 x_3 \\
\dot{x}_3 &= 2 x_3^2 (1 - \cos(x_3 - x_2)) - 1.1 x_3^2.
\end{align*}
$$

Using Taylor expansion on (3.7), its approximate system is

$$\begin{align*}
\dot{x}_1 &= -\frac{1}{6} x_1^3 + 2 x_2 x_2^2 (\approx g_1) \\
\dot{x}_2 &= x_2^2 - \frac{1}{6} x_2 x_3^3 (\approx g_2) \\
\dot{x}_3 &= -2.1 x_3^2 + 2 x_3^2 x_2 - x_3 x_2^2 (\approx g_3).
\end{align*}
$$

We note that $d_1 = (300, 210, 201, 120, 111, 102, 200, 301, 021, 201, 003) = \{S_1, S_2, \ldots, S_{10}\}$.

Using Taylor expansion on (3.7), its system is

$$\begin{align*}
\dot{x}_1 &= -\frac{1}{6} x_1^3 + 2 x_2 x_2^2 (\approx g_1) \\
\dot{x}_2 &= x_2^2 - \frac{1}{6} x_2 x_3^3 (\approx g_2) \\
\dot{x}_3 &= -2.1 x_3^2 + 2 x_3^2 x_2 - x_3 x_2^2 (\approx g_3).
\end{align*}
$$

We note that $d_1 = (300, 210, 201, 120, 111, 102, 200, 301, 021, 201, 003) = \{S_1, S_2, \ldots, S_{10}\}$.

Note that $d_1 = (300, 210, 201, 120, 111, 102, 200, 301, 021, 201, 003) = \{S_1, S_2, \ldots, S_{10}\}$.

For $g_2(x), k_3 = 3$. Hence, $G_2 = G_4$. Then $d_2 = (000) = S_7, a_{23} = -1/6, a_{25} = 1/2, a_{26} = 0$. We also have $Q_2 = \phi$.

For $g_3(x), k_3 = 5$. Hence, $G_3 = \{(500), (410), (401), \ldots, (005)\}$. Then $d_3 = (005), a_{30} = -2.1, a_{34} = 2, a_{35} = 0$. For the other $S \in S_9, a_{30}^\top = 0$. Since the last term is in $Q_3$, so $Q_3 = \{(005)\}$.

Now we are ready to test the negativity of the derivative. We first check DDP. For second and third equations, the dominating condition (3.6) is satisfied. For first equation, (3.6) yields $1/6 > 2\lambda_2$. So $|\lambda| < 0.2886751346$.

Next we check CRDDP. Let $m = 3$. Then (3.5) yields

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{6} > 2 \lambda_2^2 \\
1 > \frac{1}{6} (2 \lambda_2^2) + \frac{1}{2} (\frac{1}{6}) + 2 \left( \frac{1}{6} \right) \\
2.1 > 2 \left( \frac{1}{6} \right) + \frac{1}{2} (\frac{1}{6})
\end{array} \right.
\end{align*}$$

The solution is $|\lambda| < 0.353533906$.

Finally, let us use QFRA. The smallest even $m$ should be 4.

Then

$$q(x) = -\frac{1}{6} x_1^8 + 2 \lambda_2 x_1 x_2^2 - x_3^8 - \frac{1}{2} x_2 x_3
- 2.1 x_3^2 + 2 x_3 x_2 - x_3 x_2^2.$$

The algorithm produces a quadratic form as

$$\begin{align*}
\left( \begin{array}{cc}
-\frac{1}{6} + \lambda_2^2 & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{6} \\
-\frac{1}{6} & 0
\end{array} \right)
\end{align*}$$

To make it negative–definite we have $|\lambda| < 0.3922535218$.

In fact, we can prove that in general QFRA is stronger than CRDDP and CRDDP is stronger than DDP. However, DDP is the easiest one in use, while QFRA is the most difficult one. Later on, according to the problems one or more of these three methods are used for testing the negative–definiteness of the derivatives of LFHHD.

**IV. STABILIZATION OF SYSTEMS WITH ZERO CENTER**

Consider an affine nonlinear system with the following Byrnes-Isidori canonical form [4]:

$$\begin{align*}
\dot{x}^i &= A^i(x)(\xi) + B^i(\xi)u_i, x^i \in R^{n_i}, i = 1, \ldots, m, \\
\sum_{i=1}^{m} n_i \xi &= \eta \in \xi, x^i \in \xi, w \in \xi; \\
\Phi(S) < 0
\end{align*}$$

where

$$\begin{align*}
A^i(\xi) &= \text{col}(f_i, \ldots, f_i, f_i), f_i(0) = 0; \\
B^i(\xi) &= \text{col}(0, \ldots, 0, g_i), g_i(0) \neq 0; \\
\Phi(\xi) \text{ and } q(\xi) \text{ vanish at origin with their first derivatives.}
\end{align*}$$

Since the first variables in each integral chain play a particular role, we adopt the following notations:

$$x = (x_1, x_1), \quad x_1 = (x_1^1, \ldots, x_1^{n_1})$$

$$\bar{x}_1 = (x_2, x_3, \ldots, x_n).$$
System (4.1) is said to have zero center if \( C = 0 \). Only this case is considered in this paper.

Let \( \psi^{(r)}(x), r = 2, \ldots, h; i = 1, \ldots, m \), be a set of polynomials of \( x \) with degree \( r \). Define

\[
\begin{align*}
\{ p^i_j(z) := p_j(x(z), 0, z), \text{ with } & x^i_1 = \sum_{r=2}^h \psi^{(r)}_i(z), \\
& \nabla x^i_1 = 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, s_i, \\
\{ q^i_k(z) := q_k(x(z), 0, z), \text{ with } & x^i_1 = \sum_{r=2}^h \psi^{(r)}_i(z), \\
& \nabla x^i_1 = 0, \quad i = 1, \ldots, m, \quad k = 1, \ldots, s_i, \\
\{ \tilde{q}^i_k(z) := \tilde{q}_k(x(z), w(z), z), \text{ with } & x^i_1 = \sum_{r=2}^h \psi^{(r)}_i(z), \\
& (\nabla x^i_1)^j = E^i_j(z), \quad j = 2, \ldots, n_i, \quad i = 1, \ldots, m
\end{align*}
\]

where \( E^i_j(z), i = 1, \ldots, m, j = 1, \ldots, n_i, \) and \( w(z) \) are uncertain functions, and they will be specified later. We denote \( p^i_j(z) = (p^i_1(z), \ldots, p^i_s(z))^T \) etc.

The following theorem shows a general design idea. Polynomials of degrees 2 to \( h \) are used for the nonlinear control design.

**Theorem 4.1**: Assume \( C = 0 \) and there exists a set of polynomials \( \psi^{(r)}_i(z), r = 2, \ldots, h; i = 1, \ldots, m, \deg \psi^{(r)}_i(z) = r \), and an integer \( c \geq 2 \), such that the following conditions (C1)–(C4) hold:

\begin{enumerate}
\item [C1)] \( p^i_j(z) = O(||z||^{c+1}) \);
\item [C2)] \( q^i_k(z) = O(||z||^{c}) \);
\item [C3)] If \( E^i_j(z) = O(||z||^{c+1}) \), and \( w(z) = O(||z||^{c+1}) \), then

\[
\dot{z} = \tilde{q}^i_j(z) \quad \text{and} \quad \dot{z} = q^i_k(z)
\]

(C.2)

have same approximate system;
\item [C4)] \( \dot{z} = q^i_j(z) \) is approximately stable.
\end{enumerate}

Then system (4.1) is (locally) asymptotically stabilizable (at origin). Moreover, if C1–C4 are satisfied, a suitable feedback control, which stabilizes system (4.1) is

\[
u_i = -\frac{f_i(x)}{g_i(x)} + \frac{1}{g_i(x)} \left( \sum_{j=1}^{n_i} a_j \dot{x}^j - a_1 \left( \sum_{r=2}^h \psi^{(r)}(z) \right) \right),
\]

\[
i = 1, \ldots, m.
\]

where \( \lambda^{n_i} = \sum_{j=1}^{n_i} a_j \lambda^{j-1} \) is Hurwitz.

**Proof**: Choose

\[
\Phi(z) = \left\{
\begin{array}{ll}
x^i_1(z) = \sum_{r=2}^h \psi^{(r)}_i(z) & \text{if } i = 1, \ldots, m \\
\nabla x^i_1 = 0 & \text{if } i = 1, \ldots, m \\
w(z) = 0
\end{array}
\right.
\]

to approximate the center manifold of the closed-loop system with control (4.3). Using C1, C2, and control (4.3), we have

\[
M\Phi(z) = \left( \sum_{r=2}^h \frac{\partial \psi^{(r)}(z)}{\partial z} \right), \quad i = 1, \ldots, m \quad q^i_j(z)
\]

\[
- \left( \begin{array}{c}
0 \\
0 \\
p^i_j(z)
\end{array} \right) = 0 \left( ||z||^{c+1} \right).
\]

The dynamics on the center manifold is

\[
\dot{z} = q(x(z), w(z), z).
\]

According to the approximation Theorem [9], (4.4) ensures that in (4.5) the functions \( x(z) \) and \( w(z) \) have the following forms:

\[
x^i_1(z) = \sum_{r=2}^h \psi^{(r)}_i(z) = 0 \left( ||z||^{c+1} \right), \quad p^i_1(z) = 0 \left( ||z||^{c+1} \right),
\]

\[
i = 1, \ldots, m; \quad w = 0 \left( ||z||^{c+1} \right).
\]

Now (4.5) is of the type of the first equation of (4.2). So conditions C3 and C4 ensure the approximate stability of (4.5). Hence, the closed-loop form of system (4.1) is asymptotically stable.

It is clear from above proof that \( c + 1 \) is the order of the approximation error.

In Section I, it has been pointed out that the higher order feedback can be injected into the dynamics on the center manifold through \( \nu_i \). To distinct different injection types we define the injection degrees as

**Definition 4.2**: For system (4.1) the injection-degree, \( d_k \), is defined as

\[
d_k = \min \left\{ \left[ T \right] + \left[ S \right] \mid \left[ T \right] > 0, \quad \frac{\partial \left[ T \right] + \left[ S \right] q_k}{\partial (x_1)^T \partial z^S} = 0 \neq 0 \right\},
\]

\[
k = 1, \ldots, t.
\]

In fact, the \( d_k \) are the lowest degrees of the nonvanishing terms in the dynamics on center manifold which contains \( x^i_1(z) \).

Given system (4.1) the approximation order \( c \) can be estimated from (4.4). Let \( l_j \) be the lowest degree of the nonvanishing terms in \( p^i_1(z) \). Then we have

\[
c = \min \{ d_k; \quad i = 1, \ldots, t; \quad l_j - 1, \quad j = 1, \ldots, s \}.
\]

It can be seen intuitively that, (e.g., refer to some examples in [1], [2]) an even-degree leading system can hardly be homogeneous stable. Our design idea is: When the injection-degree, \( d_k \), is odd, use it as lowest degree of the resulting system, i.e., for the dynamics on the center manifold, let \( L_k = d_k \). Otherwise, choose control to eliminate \( d_k \) degree terms and turn the lowest degree of the resulting system to odd, i.e., \( L_k = d_k + 1 \). In such a way, we finally make the dynamics on the center manifold to have an odd approximate system. \( L_k \) will be called leading degree.

**Remark**: Even in Theorem 4.1 \( c + 1 > h \) is not claimed, it is required implicitly. Otherwise some terms of \( \psi^{(r)}_i(z) \) in the designed approximation of the center manifold will be meaningless.

Using \( c, d_k, \) and \( L_k \), conditions C1–C3) in Theorem 4.1 is computable.

**Proposition 4.3**: In Theorem 4.1, for arbitrary chosen \( \psi^{(r)}_i(z), \) \( i = 1, \ldots, m; \quad r = 2, \ldots, h \)

i) condition C1) holds, if

\[
\frac{\partial \left[ T \right] + \left[ S \right] q_k}{\partial (x_1)^T \partial z^S} = 0, \quad 2 \left[ T \right] + \left[ S \right] \leq c, \quad k = 1, \ldots, s
\]

ii) condition C2) holds, if

\[
\frac{\partial \left[ T \right] + \left[ S \right] q_k}{\partial (x_1)^T \partial z^S} = 0, \quad 2 \left[ T \right] + \left[ S \right] \leq d_k - 1, \quad k = 1, \ldots, t
\]
iii) condition C3) holds, iff (4.8) holds and when $|U| + |V| > 0$
\[
\frac{\partial |T| + |S| + |U| + |V|}{\partial \psi_k} q_k = 0,
\]
\[
2|T| + |S| + (e + 1)(|U| + |V|) \leq L_k; \quad k = 1, \ldots, t.
\]

Proof: In $p_k(x)$ set $\tau_1 = 0$ and $w = 0$, then use Taylor expansion on $x_1$ and $z$. Note that $x_1^k(0) = 0 (||z||^2)$. Then (4.7) means all terms in $p_k^e(z)$ of degree less than or equal to $e$ are zero. Since $e \leq d_k$, (4.7) holds for $2|T| + |S| \leq e - 1$, which means all terms in $q_k^e(z)$ of degree less than $e$ are zero. As for C3), note that $\tau_1(z) = 0 (||z||^{e+1})$ and $w(z) = 0 (||z||^{e+1})$. Then it is easily seen that (4.9) holds, iff, both $\tau_1$ and $w$ don't appear in the approximate system of the dynamics on the center manifold. Hence the two equations in (4.2) have same approximate system.

Equation (4.8) is sufficient for C2). But it is necessary for the required leading degrees. So we call (4.7)–(4.9) the degree matching conditions. They are always assumed in the following sections for center manifold design.

We use an example to give a detailed description for all the objects in this section.

Example 4.4: Consider the following system:
\[
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u
\end{aligned}
\]
\[
\begin{aligned}
\dot{y}_1 &= -w + x_2 x_2, \\
\dot{y}_2 &= a x_2^2 + z_1 x_1 + z_2 x_2, \\
\dot{y}_3 &= z_2 z_2 x_1.
\end{aligned}
\]

For this system $m = 1$, $\tau_1 = (x_2, x_3)$, $s = 1$, $t = 2$, $p(x, u, z) = x_1 x_2 x_2 x_2$, $q_1(x, u, w, z) = a x_2^2 + z_1 x_1 + z_2 x_2$, and $q_2(x, w, z) = z_2 z_2 x_1$. Consequently, we have
\[
\begin{aligned}
f(z) &= \sum_{r=2}^{h} p_{ij}^{(r)}(z) z_1 z_2 z_2, \\
q_1(z) &= a x_2^2 + z_1 \sum_{r=2}^{h} p_{ij}^{(r)}(z) \\
q_2(z) &= a x_2^2 + z_1 \sum_{r=2}^{h} p_{ij}^{(r)}(z) + E_1(z) + z_2 E_2(z)
\end{aligned}
\]

and
\[
\begin{aligned}
q_1^e(z) &= a x_2^2 + z_1 \left( \sum_{r=2}^{h} p_{ij}^{(r)}(z) + E_1(z) \right), \\
q_2^e(z) &= z_2 z_2 \left( \sum_{r=2}^{h} p_{ij}^{(r)}(z) + E_2(z) \right)
\end{aligned}
\]

where $p_{ij}^{(r)}(z)$, $r = 2, \ldots, h$ will be chosen to design control, $E_1$, $E_2$, and $E_3$ are some uncertain terms of $0 (||z||^{e+1})$.

According to Definition 4.2, the injection degrees are $d_1 = 3$ and $d_2 = 4$. Hence we choose $L_1 = 3$ and $L_2 = 5$. From $\dot{p}(z)$ we have $l = 5$. Then
\[
e = \min\{d_1 = 3, d_2 = 4, l - 1 = 4\} = 3.
\]

It is ready to check that (4.7) holds.

Consider (4.8). For $q_1$, when $S = (2, 0)$ and $T = (0)$, $|S| + |T| \leq d_1 - 1$. However
\[
\frac{\partial |S| q_1}{\partial z_2} = \frac{\partial q_1}{\partial z_1} = 2a.
\]

So (4.8) holds for $q_1$ iff $a = 0$. It is easy to check that (4.8) is true for $q_2$.

For (4.9), only $q_1$ has a term involving $\tau_1$ and/or $w$, which is $z_2^2 z_2$. For this term $T = (0)$, $S = (0, 1)$, $U = (1, 0)$, and $V = (0)$. So
\[
2|T| + |S| + (e + 1)(|U| + |V|) = 5 > L_1 = 3
\]
equation (4.9) is, therefore, satisfied.

We conclude that the degree matching conditions are satisfied iff $a = 0$.

Next, from (4.11) and (4.12) it is clear that the two equations in (4.2) have same approximate system.

V. STABILIZATION FOR ODD INJECTION-DEGREE CENTER

This section considers the case when all the injection degrees equal to a same odd number. Then, we have the following.

Theorem 5.1: Assume system (4.1) with $C = 0$ has an odd universal injection degree, say $d_0 = L \leq 3, \forall k$. The system is state feedback stabilizable, if it satisfies the degree matching conditions (4.7)–(4.9) with $\delta h = 2, 2 \leq e \leq L$ and $L_k = L, \forall k$ and there exists a quadratic homogeneous vector field $\psi(z) = \text{col}(\psi_1, \ldots, \psi_m)$, such that
\[
\dot{z}_k = \frac{1}{2|T| + |S| + |U| + |V|} q_k(0) z^S \left( \prod_{i=1}^{m} (\psi_i)^{T_i}(z) \right),
\]
\[
k = 1, \ldots, t.
\]

is asymptotically stable at origin. Moreover, if the above conditions are satisfied, (4.3) with $\psi_i(0) = \psi_i$ is a suitable feedback control, which stabilizes system (4.1).

Proof: Using control (4.3), conditions (4.8) and (4.9) assure the lowest degree of the dynamics of the closed-loop system on the center manifold is $L$. Note that in this case $d_i = e = L, i = 1, \ldots, m$. Conditions (4.7) and (4.8) assure the center manifold is described as
\[
\tau_1(z) = (||z||^{L+1}) = 0 (||z||^{L+1}),
\]
\[
\tau_1(z) = \psi(z) + R_k, \quad R_k = (||z||^{L+1}) = 0 (||z||^{L+1}),
\]
\[
\psi_i(z) = \psi_i(0) = \psi_i(z) = \psi_i(z).
\]

Using (4.8) and (4.9), $R_k, \tau_1, \text{ and } w$ will not appear into the degree $L$ terms. Hence the degree $L$ terms of the dynamics are exactly the right side of (5.1). That is, (5.1) is the approximate system of the dynamics on the center manifold. Since (5.1) is homogeneous and asymptotically stable at origin, Theorem 2.4 assures the approximate stability of the dynamics on the center manifold of the closed-loop system. Then the asymptotical stability of the closed-loop system follows from Theorem 4.1.

When $d_k = 3, k = 1, \ldots, t$ it is an interesting case [1]. Now set $h = 2$, and $L = 3$. The previous result leads to the following simpler one.

Corollary 5.2: System (4.1) with $C = 0$ is state feedback stabilizable if
\[
\begin{aligned}
\delta_i p_i(x, t, x) = 0; & \quad \delta_i p_i(x, t, x) = 0; \quad i = 1, \ldots, m; \\
\delta_i x_i(t, x) = 0; & \quad \delta_i p_i(x, 0) = 0; \quad \delta_i x_i(t, x) = 0;
\end{aligned}
\]
\[
\begin{aligned}
\delta q_i(x, t, x) = 0; & \quad \delta q_i(t, x) = 0; \quad \delta q_i(x, 0) = 0; \quad \delta q_i(t, x) = 0;
\end{aligned}
\]
\[
\begin{aligned}
\delta q_i(x, t, x) = 0; & \quad \delta q_i(t, x) = 0; \quad \delta q_i(x, 0) = 0; \quad \delta q_i(t, x) = 0.
\end{aligned}
\]

there exists quadratic homogeneous vector field $\psi(z) = \text{col}(\psi_1, \ldots, \psi_m)$, such that
\[
\dot{z}_k = D(z)\psi(z) + E(z),
\]
\[
(k = 1, \ldots, t).
\]
is asymptotically stable. Where \( D(z) \) and \( E(z) \) are \( t \times m \) and \( t \times 1 \) matrices with entries as

\[
D_{ij} = \frac{\partial^2 q_j}{\partial z_i \partial z_i}(0)z_k; \quad i = 1, \ldots, t; \quad j = 1, \ldots, m;
\]

\[
E_i = \frac{\partial^2 q_j}{\partial z_i \partial z_i}(0)z_k; \quad i = 1, \ldots, t
\]

respectively. Moreover, (4.3) with \( \psi^{(2)} = \psi \) is a suitable feedback control, which stabilizes system (4.1).

The following example shows that when the injection-degree is 3 we have only to solve a set of algebraic inequalities to obtain the required control.

**Example 5.3:** Consider the following system:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1(x, z) + g_1(x, z)u_1 \\
\dot{x}_3 &= f_2(x, z) + g_2(x, z)u_2 \\
\dot{z}_1 &= g_1(x, z) \\
\dot{z}_2 &= g_2(x, z)
\end{align*}
\]

(5.3)

where \( f_i(0) = 0, \) \( g_k(0) \neq 0 \) satisfies the condition C2 in Corollary 5.2, \( i = 1, 2. \) Our goal is to find a sufficient condition for system (5.3) to be feedback stabilizable. Denote by

\[
\begin{align*}
\psi_1 &= a_k^2 + b_k^2 + c_k^2 \\
\psi_2 &= d_k^2 + e_k^2 + f_k^2
\end{align*}
\]

Then (5.2) leads to the following:

\[
\psi_1 = \frac{\partial^2 q_k}{\partial z_i \partial z_i}(0), \quad k = 1, 2 \quad i = 1, 2 \quad j = 1, 3
\]

\[
\psi_2 = \frac{\partial^2 q_k}{\partial z_i \partial z_i}(0), \quad k = 1, 2 \quad i = 1, 2 \quad j = 1, 3
\]

(5.4)

Then (5.2) becomes

\[
\begin{align*}
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -z_2 \end{pmatrix} \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} + \begin{pmatrix} -4z_1^2z_2 \\ \alpha z_1^2 + b_2^2z_2 + c_2z_2 \end{pmatrix} \\
&= \begin{pmatrix} -z_1^2 + (d - b)z_2^2 + (e - c)z_1z_2 + (f + 1)z_2^3 \end{pmatrix}
\end{align*}
\]

(5.6)

Using CRDDP with \( m = 2, \) we have

\[
\begin{align*}
-a &> \frac{1}{2}[d + b] + \frac{3}{4}|c| + 3 + \frac{1}{2}|a| + \frac{3}{4}|d| - \frac{1}{2}|e| - \frac{1}{4}|c| \\
f &> \frac{1}{2}[d + b] + \frac{3}{4}|c| + 1 + \frac{1}{2}|a| + \frac{3}{4}|d| - \frac{1}{2}|e| - \frac{3}{4}|c|, \\
(5.7)
\end{align*}
\]

One particular solution of (5.7) is \( a = -25; \) \( b = 4; \) \( c = 0; \) \( d = 4; \) \( e = 0; \) \( f = -10. \) Then

\[
\phi = \begin{pmatrix} -25z_1^2 + 4z_1z_2 \\ 4z_1^2 - 10z_2^3 \end{pmatrix}
\]

To stabilize linear part, one may choose \( a_1^2 = -1; \) \( a_2^2 = -2; \) \( a_3^2 = -1. \) Then (4.3) yields:

\[
\begin{align*}
u_1 &= -\frac{f_1(x, z)}{g_1(x, z)} + \frac{1}{g_1(x, z)} (-x_1 - 2x_2 - 25z_1^2 + 4z_1z_2) \\
u_2 &= -\frac{f_2(x, z)}{g_2(x, z)} + \frac{1}{g_2(x, z)} (-x_2 + 4z_1^2 - 10z_2^3)
\end{align*}
\]

VI. STABILIZATION FOR EVEN INJECTION DEGREE CENTER

In this section we consider the even injection degree case. Assume there exists a positive even number \( c \) such that the injection degrees are either \( c \) or \( e + 1. \) Then we design a suitable control to turn the leading degree of the dynamics on the center manifold to \( L_k = c + 1 \) for all \( k. \)

For system (4.1) with \( C = 0, \) assume there exists a positive even number \( c \) such that the injection degrees are either \( d_k = c \) or \( d_k = c + 1. \) Set \( L_k := L = c + 1, \forall k. \) Let if it satisfies the degree matching conditions (4.7)-(4.9) with \( b = 3, 3 \leq e \leq L. \) (4.3) be used to form the closed-loop system. Then

**Lemma 6.1:** A necessary condition for \( L \) to be the leading degree of the dynamics on the center manifold of the closed-loop system is: there exists a quadratic homogeneous vector \( \psi(z) = \text{col}(\psi_1, \ldots, \psi_m), \) such that

\[
\sum_{2|T|+|S|=c} 1
\]

\[
\prod_{i=1}^{m} \psi_i^T(z) = 0
\]

(6.1)

**Proof:** According to condition (4.8), when we calculate the approximate system of the systems in (4.1), only the quadratic terms, \( \phi^{(2)}_i(z) \) in \( x_i, i = 1, \ldots, m \) are chosen to form the degree \( c \) terms. This turns out to be the left side of (6.1). To make the leading degree \( L = c + 1, \) the control should be chosen to eliminate degree \( c \) terms, which leads to the (6.1). □

Then we have the following sufficient condition:
Theorem 6.2: Given system (4.1) as described in the above, and (6.1) is assumed for certain \( \psi(z) \). If there exists a cubic homogeneous vector \( \phi(z) = \text{col}(\phi_1, \ldots, \phi_m) \), such that

\[
2[T]+|S|+5(|U|+|V|) \leq L = 5,
\]

we have \(|U| \leq 1\). So (4.8) holds, (4.9) is obviously true.

Now denote

\[
\psi = a_2^2 + b_2 z_2 + c_2 z_2^2; \\
\phi = \alpha_2^2 + \beta_2 z_2 + \gamma_2 z_2^2 + \delta_2 z_2^3.
\]

Then (6.1) becomes

\[
\begin{align*}
\frac{2}{T!} a_2^2 + b_2 (a_2^2 + b_2 z_2 + c_2 z_2^2) = 0 \\
\frac{2}{T!} z_2^2 + b_2 (a_2^2 + b_2 z_2 + c_2 z_2^2) = 0
\end{align*}
\]

Set \( a = 0, b = 0, c = -1 \). Equation (6.1) is satisfied.

Equation (6.2) yields a fifth degree homogeneous vector field as

\[
\left( z_2^2 \left( \alpha_2^2 + \beta_2 z_2 + \gamma_2 z_2^2 + \delta_2 z_2^3 \right) - z_2^2 + 0.5 z_2^2 z_2^3 \right).
\]

Simply choose \( \alpha = -1, \beta = 0, \gamma = 0, \) and \( \delta = 0 \). It is easy to check that CRD is satisfied with \( m = 3 \). Chosen \( \alpha_2 = -1, \alpha_2 = -2 \), then (4.3) provides a stabilizing control as

\[
u = \frac{-f(x, z) + 1}{g(x, z)} \left( -x_1 - 2x_2 - (z_2^3 + z_2^4) \right).
\]

\[\square\]

VII. STABILIZATION FOR MIXED INJECTION DEGREE CENTER

This section considers a general case when the injection degrees differ. By reordering \( z \) we may, without loss of generality, that the injection degrees are \( d_1, d_2, \ldots, d_k \) and \( d_k \) is odd for \( i < \alpha \), even for \( i \geq \alpha \). Then, we have the following.

Theorem 7.1: System (4.1) with \( C = 0 \) is feedback stabilizable if it satisfies the degree matching conditions (4.7)–(4.9) with \( e \geq 3 \) and the following conditions.

C1) There exists a quadratic homogeneous vector \( \psi(z) = \text{col}(\psi_1, \ldots, \psi_m) \), such that

\[
\sum_{2[T]+|S| = h_0} \frac{1}{T! S!} \partial^{[T]+[S]} q_{h_0} \partial^{[T]} \partial^{[S]} (0) z^S \times \left( \prod_{i=1}^m (\psi_i)^{T_i}(z) \right) = 0, \quad k > \alpha.
\]

C2) There exists a cubic homogeneous vector \( \phi(z) = \text{col}(\phi_1, \ldots, \phi_m) \), such that

\[
\begin{align*}
\sum_{2[T]+|S| = h_0} \frac{1}{T! S!} & \partial^{[T]+[S]} q_{h_0} \partial^{[T]} \partial^{[S]} (0) z^S \times \left( \prod_{i=1}^m (\psi_i)^{T_i}(z) \right) \\
& + \sum_{2[T]+|S| = h_0+1} \frac{1}{T! S!} \partial^{[T]+[S]} q_{h_0+1} \partial^{[T]} \partial^{[S]} (0) z^S \times \left( \prod_{i=1}^m (\psi_i)^{T_i}(z) \right)
\end{align*}
\]

Since the injection degrees are 4, 4, we choose leading degree \( L = 5 \). Now, since we use quadratic and cubic feedback, so \( h = 3 \). There is no \( \mathbf{p}(\xi) \), we check (4.7) first. Since \( q_0^d(\xi) = 0 \left[ (\xi)^4 \right] \), so \( e = 4 \). It is obvious that (4.7) is satisfied. To check (4.8), only \( q_1(\xi) \) contains \( T_2 = x_2 \), so we have only to check it. Since in \( q_1(\xi) \) the degree of \( x_2 \) is 2, it is easy to see that when
is approximately stable at origin.

Moreover, if above conditions are satisfied, the control (4.3) with \( \psi^{(2)}_i = \psi_i \) and \( \psi^{(3)}_i = \phi_i \) stabilizes (4.1).

**Proof:** The proof is basically the combination of the proofs of Theorem 5.1 and Theorem 6.2, therefore, we will just sketch it.

1) Equation (7.1) implies that the lowest degree terms in even injection degree subsystems are eliminated by suitable feedback \( \psi(x, z) \).

2) Equation (7.2) is the approximate system of the dynamics on center manifold.

Using Theorems 4.1 and 4.2, the degree matching conditions and the approximate stability of (7.2) ensure the stability of the closed-loop system.

**Remark:** Unlike Theorem 5.1 and Theorem 6.2, since in Theorem 7.1 the approximate system of the dynamics on the center manifold of the closed-loop system is not homogeneous and so approximate stability is required to assure the asymptotically stable of the dynamics on the center manifold. Meanwhile, the results in Theorem 5.1 and Theorem 6.2 are topologically invariant. The result in Theorem 7.1 is coordinate-dependent and so it is not topologically invariant.

**Example 7.2:** Consider the following system:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(x, z) + g(x, z)u \\
\dot{z}_1 &= \sin(z_1x_1) + z_1z_2 + \lambda z_3^2 \\
\dot{z}_2 &= z_2^2 \tan(x_1 + x_2) + z_1z_3^2
\end{align*}
\]

(7.3)

where

\[
\begin{align*}
f(0, 0) &= 0; \\
g(0, 0) &\neq 0; \\
\lambda &\text{ parameter.}
\end{align*}
\]

The injection degrees are \( d_1 = 3, d_2 = 4 \), so we choose the designed lowest degrees as \( L_1 = 3 \) and \( L_2 = 5 \). Let \( h = 3 \) and \( \psi = ax_1^2 + bx_2x_2 + c_2x_2^2 + d_2x_2^3 \), \( \phi = \alpha z_1^2 + \beta z_2^2 + \gamma z_1z_2 + \delta z_3^2 \).

Since \( g(z) = 0 (||z||^3) \) the approximation order \( c = 3 \). We don’t need to consider (4.7) because there is no \( p(z) \). Consider (4.8), it is obviously true. As for (4.9), only \( g_2(z) \) contains \( x_2 \).

Since \( c = 3 \) as \( x_2(z) = 0 (||z||^{c+1}) \) the lowest term involving \( x_2 \) is \( z_2^2 \). Hence (4.9) holds for \( 2||T|| + ||S|| + 4(||U|| + ||V||) \) < 6. However, \( L_2 = 5 \), so (4.9) is true.

Next, we check that (7.1) renders \( a = 0, \ c = 0, \ b = -1 \).

Moreover, (7.2) turns out to be

\[
\begin{align*}
\dot{z}_1 &= \lambda \dot{z}_2^2 - z_1z_2 + z_1z_3^2 \\
\dot{z}_2 &= \alpha z_1^2 + \beta z_2^2 + \gamma z_1z_2 + \delta z_3^2
\end{align*}
\]

(7.4)

To find a possible solution, we may simply set \( \alpha = \beta = \gamma = 0 \), then test (7.4) by DDP, CRDDP, and QFRA, respectively.

DDP yields: \( \delta < 0, \ \lambda < -2 \).

Set \( m = 3 \), CRDDP yields \( \delta < -0.5, \ \lambda < -1.5 \).

Choosing \( m = 4 \), QFRA yields a quadratic form as \( (\lambda + 1.25)z_1^2 + 0.75z_1z_2 + 0.5z_2^2 \). To make it negative–definite we need \( \lambda < -1.25, \ \delta < 9/(64\lambda + 80) \).

It is clear that the last method provides the best estimation.

Using (4.3) and choosing linear feedback with \( a_1 = -1, a_2 = -2 \), we conclude that when \( \lambda < -1.25 \) system (7.3) is stabilized by the following control

\[
u = -\frac{f(x, z)}{g(x, z)} + \frac{1}{g(x, z)} (-x_1 - 2x_2 - z_1z_2 + \delta z_3^2) .
\]

\[\square\]

In fact, Theorem 4.1 and Proposition 4.2 allow us to use any nonlinear polynomial state feedback. The formulas in Theorem 5.1, Corollary 5.2, Theorem 6.2, and Theorem 7.1 use only quadratic and cubic polynomials. The next example shows that sometimes even higher degree terms are necessary.

**Example 7.3:** Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2; \cdots; \dot{x}_m = x_m; \dot{x}_m = u \\
\dot{y}_1 &= y_2; \cdots; \dot{y}_{n-1} = y_{n-1}; \dot{y}_{n} = v \\
\dot{z}_1 &= x_1y_1 \\
\dot{z}_2 &= 2y_2x_1 + z_2^2 \\
\dot{z}_3 &= z_3y_1 + \mu_1z_1z_3x_1 + \mu_2z_1z_3x_1^2, \ \mu_1 > 0,
\end{align*}
\]

(7.5)

where \( \mu_1 \) and \( \mu_2 \) are parameters.

It is easily checked that the degree matching conditions (4.7)–(4.9) are trivially satisfied because \( p'(z) = q'(z) = 0 \), and \( q(z) \) doesn’t contain either \( x_k \) or \( y_k, k > 1 \). So if only we can find the approximate system of the dynamics on the center manifold, which is approximately stable, we are done.

**Case 1** Consider a subsystem of (7.5) where the differential equations about \( z_2 \) and \( z_3 \) are removed.

Now \( d_2 = 4 \). So we choose \( L_1 = 5 \). Using Theorem 6.1, (6.1) implies either \( x_1 = 0 (||z||^0) \) or \( y_1 = 0 (||z||^3) \). An obvious solution for (6.2) is

\[
\begin{align*}
\psi_1(z) &= az_1^2, \ \phi_1(z) = 0; \\
\psi_2(z) &= 0, \ \phi_2(z) = \alpha z_1^2, \ \alpha > 0
\end{align*}
\]

which leads to a set of controls

\[
\begin{align*}
u &= \sum_{i=1}^{n_1} a_{i1}^2x_i + a_{i2}^2y_i \\
v &= \sum_{i=1}^{n_2} a_{i1}^2x_i + a_{i2}^2y_i
\end{align*}
\]

(7.6)

where \( a_{i1}, i = 1, \ldots, n_1, j = 1, 2 \) are coefficients of Hurwitz polynomials. If \( n_1 = n_2 = 1 \), this is a system discussed in Brockett [8] and the solution (7.6) is in Aeyels [1].

**Case 2** Consider subsystem of (7.5), where the differential equation about \( z_3 \) is removed.

Now \( d_1 = 4 \) and \( d_2 = 3 \). We set \( L_1 = 5 \) and \( L_2 = 3 \). Using Theorem 7.1, (7.1) leads to the same conclusion as in Case 1: i.e., one of \( x_1 \) or \( y_1 \) should be \( 0 (||z||^3) \). To avoid notational mess, we simply choose \( x_1 = ax_1^2 + bx_1 \) and \( y_1 = az_1^2 \). Then (7.2) turns out to be

\[
\begin{align*}
\dot{z}_1 &= \alpha z_1^2 + \beta z_1^2 \\
\dot{z}_2 &= 2z_1^2 + 3z_1^2 + z_2^2
\end{align*}
\]

(7.7)

We use CRDDP and choose \( m = 3 \). Then \( b < 0 \). For finding a particular set of solutions we assume:
\[ \alpha < 0, \alpha = 1, 2|\alpha| - 1 \geq 0. \] By some algebraic computations, we finally obtain the following condition:
\[ \begin{cases} |\alpha| > 2|\beta| - 1 \\ |\alpha| < 1.25|\beta| + 0.5. \end{cases} \tag{7.7} \]

Then the control becomes
\[ \begin{cases} u = \sum_{i=1}^{n_1} a_i^1 x_i - a_i^1 \left( a_i^2 + b_i^2 \right) \\ v = \sum_{i=1}^{n_2} a_i^2 y_i - a_i^2 \left( a_i^3 + c_i^3 \right) \end{cases} \]

where \( \alpha < 0, \beta < 0 \) and they satisfy (7.7).

Case 3) Consider overall system. Since \( d_1 = 4, d_2 = 3 \) and \( d_3 = 3 \). Then the approximate system of becomes
\[ \begin{cases} x_1(z) = a_1 x_1 + b_1 z_1 + 0 \left( ||z||^5 \right) \\ y_1(z) = c_1 z_1 + \phi^{(3)}(z) + 0 \left( ||z||^5 \right). \tag{7.8} \]

Then the approximate system of becomes
\[ \begin{cases} z_1 = a c_1 \phi^{(4)} + 0 \left( ||z||^6 \right) \\ z_2 = a c_2 \phi^{(4)} + b z_4 + c z_2 \phi^{(4)} \\ z_3 = a c_3 z_3 + (b z_4 + c z_3) \phi^{(4)} \phi^{(4)}(z) + a \mu_1 z_1^2 z_3 + 0 \left( ||z||^6 \right). \end{cases} \]

To make it approximately stable we may choose \( ac < 0, b < 0, a = -0.5, c = -a \mu_1 \) and \( \phi^{(4)} = -z_3 \). Since \( \mu_1 > 0 \), then a feasible choice is: \( a = -0.5, b = -1, c = 0.5 \mu_1 \). It follows that the following control, as a particular case of (4.3), stabilizes the system (7.5).
\[ \begin{cases} u = \sum_{i=1}^{n_1} a_i^1 x_i + a_i^1 \left( -0.5 \phi^{(4)} - z_3 \right) \\ v = \sum_{i=1}^{n_2} a_i^2 y_i + a_i^2 \left( -0.5 \phi^{(4)} - z_3 \right). \end{cases} \]

\[ \Box \]

VIII. Conclusion

The stabilization problem for affine nonlinear systems with nonminimum phase zero dynamics was considered in the paper. The major results of the paper are the followings:

First, a new tool, the Lyapunov function with homogeneous derivative along solution curves was proposed. Based on this, three independent sufficient conditions (Cross Row Dominating Principle, Diagonal Dominating Principal, Quadratic Form Reducing Algorithm) were developed to test the negative definiteness of the homogeneous polynomials. It was shown that this new tool is particularly suitable for testing the approximate stability of the dynamics with odd lowest nonvanishing terms.

Secondly, it was shown that under certain designed state feedback controls, the first variables of each integral chains of the linearized part of the system could be used as the "controls" of the dynamics on the center manifold of the closed-loop systems. This followed because the choice of the approximation functions for them does not affect the approximation accuracy of the dynamics on the center manifold.

In the light of the above two results, a systematic design technique was developed to provide a set of sufficient conditions for designing controls which stabilize the dynamics on the designed center manifold, and then stabilize the overall system.

Only the systems with zero center were discussed in this paper. However, the method can also be used for affine nonlinear systems with oscillatory center [26] or the case of a center with multiple zero eigenvalues [27].

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References


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Clyde Martin, photograph and biography not available at the time of publication.