Stabilization of Nonlinear Systems via Designed Center Manifold

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***Abstract*—This paper addresses the problem of the local state feedback stabilization of a class of nonlinear systems with nonminimum phase zero dynamics. A new technique, namely, the *Lyapunov function with homogeneous derivative along solution curves* has been developed to test the approximate stability of the dynamics on the center manifold. A set of convenient sufficient conditions are provided to test the negativity of the homogeneous derivatives. Using these conditions and assuming the zero dynamics has stable and center linear parts, a method is proposed to design controls such that the dynamics on the designed center manifold of the closed-loop system is approximately stable. It is proved that using this method, the first variables in each of the integral chains of the linearized part of the system do not affect the approximation order of the dynamics on the center manifold. Based on this fact, the concept of injection degree is proposed. According to different kinds of injection degrees certain sufficient conditions are obtained for the stabilizability of the nonminimum phase zero dynamics. Corresponding formulas are presented for the design of controls.**

***Index Terms*—Approximate stability, center manifold, injection degree, Lyapunov function with homogeneous derivative, zero dynamics.**

I. Introduction

STABILIZATION is one of the basic tasks in control design. The asymptotic stability and stabilization of nonlinear systems have received significant attention [18]–[24]. The center manifold approach has been developed to solve the problem [1], [2], [12], [18], [24]. In [1], [2], some special nonlinear controls are designed to stabilize some particular control systems. The method used there is basically a case-by-case study. For control systems in normal form, assume the center manifold has minimum phase, then a *quasi-linear* feedback can be used to stabilize linearly controllable variables. We refer to [3]–[6] for minimum phase method and its applications.

Based on these pioneer works, this paper proposes a procedure to produce a state feedback to stabilize nonminimum phase zero dynamics. The designed state feedback control ensures that the dynamics on the designed center manifold of the closed-loop system is approximately stable. To obtain the desirable properties, we combine the center manifold method with Lyapunov function method.

Motivated by the works on stabilization of homogeneous vector field [13]–[17], we propose a new method, namely, that of a *Lyapunov function with homogeneous derivatives along solution curves*. This Lyapunov function is used to test the approximate stability of a dynamics with odd degree approximating systems, where degree means the polynomial degree. It is particularly suitable for testing the dynamics on a designed center manifold of a closed-loop system, because the degrees of the approximate system of the dynamics on the center manifold may be converted by certain state feedback controls to have odd degree. In this way, the method is applicable to a large class of nonlinear systems with stable and center zero dynamics.

To avoid counting the order of smoothness, through this paper the systems and all other objects involved are assumed to be C^∞, or as smooth as required, on a neighborhood of the origin.

We motivate this work by means of a practical problem: consider the stabilization of an airplane via a designed center manifold. We may find some useful observations from this example for design of both the center manifold and the stabilizing controls. The following example is basically taken from [7], with a modification that the speed is assumed to be dependent on altitude when the atmospheric resistance is taken into consideration.

Example 1.1 [7]: Denote an airplane's altitude in meters by h. Assume that the body of the plane is slanted ϕ radians with respect to the horizontal and that the ground speed is $c(h)$. Also, assume the flight path forms an angle of α radians with the horizontal and α is small. The system is described as

$$\begin{cases}\dot{\alpha} = \alpha(\phi - \alpha)\\ \ddot{\phi} = -\omega(\phi - \alpha - bu)\\ \dot{h} = c(h)\alpha\end{cases} \tag{1.1}$$

where $\omega > 0$ is a constant representing a natural oscillation frequency and a and b are positive constants. The problem we address is altitude tracking: i.e., a target altitude ξ, where $\dot{\xi} = c(h_0)\alpha$.

Set $x_1 = \alpha, x_2 = a(\phi - \alpha), x_3 = a\dot{\phi} - a^2\phi + a^2\alpha$ and assume $dz/dh|_{h_0} \neq 0$. We have $\dot{z} = (c(h) - c(h_0))\alpha := q(z)\alpha$ with $q(0) = 0$. Then the system (1.1) is transformed into a standard form as

$$\begin{cases}\dot{x_1} = x_2\\ \dot{x_2} = x_3\\ \dot{x_3} = -\omega x_2 - a x_3 + a\omega b u\\ \dot{z} = q(z)x_1.\end{cases} \tag{1.2}$$

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We assume that $q_z(0) := (d/dz)q(0) \neq 0$ and denote $f(x,z) = -\omega x_2 - a x_3$, $g(x,z) = a\omega b$. Then system (1.2) is in a normal form for affine nonlinear systems [4]. The zero dynamics (with $y = x_1$) becomes $\dot{z} = 0$, which is not asymptotically stable. Therefore, a quasi-linear control can not make the origin asymptotically stable, and a nonlinear state feedback control should be considered.

Motivated by the early works [1], [2], we may try the following control:

$$u = -\frac{f(x,z)}{g(x,z)} + \frac{1}{g(x,z)}(a_1 x_1 + a_2 x_2 + a_3 x_3 + bz^2). \quad (1.3)$$

To get a stabilizing control, we can first choose a_1, a_2, a_3 to stabilize the linearly controllable variables x_1, x_2, x_3, and then choose b to stabilize the central variable z. To determine a possible value of b, let

$$\phi(z) = \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \\ \phi_3(z) \end{pmatrix} = 0\left(\|z\|^2\right)$$

be used to approximate the center manifold. We refer to [9] for the notation $0\left(\|x\|^k\right)$ and the following operator M. Then we have

$$\begin{aligned} M\phi(z) =& D\phi(z)\left(q(z)\phi_1(z)\right) \\ & - \begin{pmatrix} \phi_2(z) \\ \phi_3(z) \\ a_1\phi_1(z) + a_2\phi_2(z) + a_3\phi_3(z) + bz^2 \end{pmatrix} \\ =& 0\left(\|z\|^4\right) \\ & - \begin{pmatrix} \phi_2(z) \\ \phi_3(z) \\ a_1\phi_1(z) + a_2\phi_2(z) + a_3\phi_3(z) + bz^2 \end{pmatrix}. \end{aligned}$$

Choose

$$\begin{cases} \phi_1 = -\frac{b}{a_1} z^2 \\ \phi_i = 0, \quad i = 2, 3. \end{cases}$$

Then $M\phi(z) = 0\left(\|z\|^4\right)$. According to the approximation theorem [9], the center manifold can be expressed as

$$\begin{cases} x_1 = h_1(z) = -\frac{b}{a_1} z^2 + 0\left(\|z\|^4\right) \\ x_i = h_i(z) = 0\left(\|z\|^4\right), \quad i = 2, 3. \end{cases} \quad (1.4)$$

The dynamics on the center manifold is

$$\dot{z} = \left(q_z(0)z + 0\left(\|z\|^2\right)\right) h_1(z) = -\frac{b}{a_1} q_z(0) z^3 + 0\left(\|z\|^4\right). \quad (1.5)$$

Choose $\{a_1, a_2, a_3, b\}$ such that the linear part is Hurwitz and $-(b/a_1)q_z(0) < 0$, say $a_1 = -1$, $a_2 = a_3 = -3$, $b = -q_z(0)$. The feedback control becomes

$$u = -\frac{f(x,z)}{g(x,z)} + \frac{1}{g(x,z)}\left(-x_1 - 3x_2 - 3x_3 - q_z(0)z^2\right).$$

It follows that (1.5) is asymptotically stable at origin, and then so is the closed-loop system. $\square$

Some observations from this example follow.

1) The higher degree $(\deg \geq 2)$ state feedback doesn't affect the local stability of the linearly controllable variables but it may affect the center part variables by changing the structure of the center manifold.
2) Higher order feedback can be "injected" into the dynamics on center manifold through the first variable, x_1, of the integral chain. The variable x_1 doesn't affect the order of approximation of the center manifold. This component of the linear part can be employed to modify the nonlinear dynamics.
3) Since the center manifold is approximated up to a certain degree the approximated dynamics on the center manifold should be asymptotically stable up to certain degree uncertainties to assure the stability of the original system.

The paper is organized as follows. Section II defines the concept of *Lyapunov function with homogeneous derivative* along solution curves and gives some fundamental properties. Section III provides several sufficient conditions for testing the approximate stability of vector fields. Sections IV–VII discuss design methods for affine nonlinear systems with zero center. The general result is in Section IV. Then according to the injection degrees, the classified testing conditions and formulas for odd, even and mixed injection degrees are presented in Sections V–VII, respectively. Section VIII contains some concluding remarks.

II. Lyapunov Function With Homogeneous Derivative

Since in general we can only obtain an approximation of the center manifold, it is necessary to have some convenient tools to verify the stability of the dynamics on center manifold through its approximated dynamics. For this purpose a new concept, Lyapunov function with homogeneous derivative, is proposed in this section.

Consider a dynamical system

$$\dot{x} = f(x), \quad x \in R^n \quad (2.1)$$

with $f(0) = 0$.

We use Z_+ for the set of nonnegative integers. For a multi-index $S = (s_1, \ldots, s_n) \in Z_+^n$ and $x = (x_1, \ldots, x_n) \in R^n$, we denote

$$|S| = \sum_{i=1}^n s_i; \quad x^S = \prod_{i=1}^n (x_i)^{s_i}; \quad S! = \prod_{i=1}^n (s_i)!.$$

Note that $0! = 1$, so $S! \neq 0$. For a smooth function $F(x)$, we denote

$$\frac{\partial^{|S|} F(x)}{\partial x^S} = \frac{\partial^{|S|} F(x)}{\partial x_1^{s_1} \partial x_2^{s_2} \cdots \partial x_n^{s_n}}.$$

Then we can give the following definition.

Definition 2.1:

1) Let k_i be the lowest degree of nonvanishing terms of the Taylor expansion of $f_i(x)$, $i = 1, \ldots, n$. A system consisted of only the lowest degrees' (k_i) terms of (2.1) is said

to be the (lowest degrees') approximate system of (2.1). It can be expressed formally as

$$\dot{x}_i = g_i(x) := \sum_{|S|=k_i} \frac{1}{S!} \frac{\partial^{|S|} f_i(x)}{\partial x^S}(0)x^S, \quad i = 1, \ldots, n. \tag{2.2}$$

2) System (2.2) is said to be an odd approximation of (2.1) if all k_i are odd.
3) System (2.1) is said to be approximately stable if

$$\dot{x}_i = f_i(x) + 0\left(\|x\|^{k_i+1}\right), \quad i = 1, \ldots, n$$

is locally asymptotically stable at origin.

Remark 1:

1) In (2.2) g_i is a homogeneous polynomial of degree k_i. So $g = (g_1, \ldots, g_n)^{\mathrm{T}}$ is a component-wise homogeneous vector field.
2) When $k_1 = \cdots = k_n := k$, the approximate stability defined above coincides with the conventional one [24]. Otherwise, it is coordinate-depending. It is clear that approximate stability implies asymptotic stability, but the inverse is not true.

Definition 2.2: Given a component-wise homogeneous polynomial vector field $g = (g_1, \ldots, g_n)^{\mathrm{T}}$, a positive definite polynomial $V > 0$ is said to be a Lyapunov function with homogeneous derivative (LFHD) along g, if the Lie derivative L_gV is homogeneous with

$$\deg(L_gV) = \deg\left(\frac{\partial V}{\partial x_i}\right) + \deg(g_i), \quad i = 1, \ldots, n.$$

The following example provides two typical LFHD, which will be used later.

Example 2.3: Let $g = (g_1, \ldots, g_n)^{\mathrm{T}}$ be a component-wise homogeneous vector field with odd degrees, $\deg(g_i) = k_i$, $i = 1, \ldots, n$, and m be a given integer satisfying

$$2m \geq \max\{k_1, \ldots, k_n\} + 1.$$

1) Set $2m_i = 2m - k_i + 1$, $i =, \ldots, n$, then

$$V = \sum_{i=1}^{n} p_i x_i^{2m_i} \tag{2.3}$$

is a LFHD along g if $p_i > 0$, $\forall\, i$.

2) Assume $k_1 = \cdots = k_{n_1} := k^1$; $k_{n_1+1} = \cdots = k_{n_1+n_2} := k^2$; $\cdots$ $k_{n_1+\cdots+n_{r-1}+1} = \cdots = k_{n_1+\cdots+n_r} := k^r$, where k^i are odd and $\sum_{i=1}^{r} n_i = n$. Denote $x = (x^1, \ldots, x^r)$, with $\dim(x^i) = n_i$ and set $2m^i = 2m - k^i + 1$, $i = 1, \ldots, r$, then

$$V = \sum_{i=1}^{r} \left(\left(x_1^i\right)^{m^i}, \ldots, \left(x_{n_i}^i\right)^{m^i}\right) \times P_i \left(\left(x_1^i\right)^{m^i}, \ldots, \left(x_{n_i}^i\right)^{m^i}\right)^{\mathrm{T}} \tag{2.4}$$

is a LFHD along g if P_i, $i = 1, \ldots, r$ are positive–definite matrices with dimensions $n_i \times n_i$.

□

Note that the derivative of V in either (2.3) or (2.4) along g is then a homogeneous polynomial of degree $2m$.

The following example shows that LFHD is a new concept because both V and g are not homogeneous but the derivative is. Since the approximate system of a smooth system is always component-wise homogeneous, method of LFHD can be used for testing the stability of the odd-degree approximated systems. It is particularly useful in testing the stability of the dynamics on center manifold of the closed-loop systems, because the leading degree of the dynamics may be converted to odd by suitable state feedback.

Example 2.4: Consider the following system:

$$\begin{cases} \dot{x} := f_1(x, y) = -x\sin(x^2 - y^2) \\ \dot{y} := f_2(x, y) = y^4 \ln(1 - 2y + x). \end{cases} \tag{2.5}$$

Using Taylor expansion, the approximate system of (2.5) is obtained as

$$\begin{cases} \dot{x} := g_1(x, y) = -x^3 + xy^2 \\ \dot{y} := g_2(x, y) = -2y^5 + xy^4. \end{cases} \tag{2.6}$$

First of all, we show that the vector field g in (2.6) is not homogeneous with respect to any group of dilations of the form $\Delta_t : (x_1, x_2) \mapsto (t^{r_1}x_1, t^{r_2}x_2)$ [12]. Assume (2.6) is k-th homogeneous with dilation (r_1, r_2), that is

$$\begin{cases} g_1(t^{r_1}x, t^{r_2}y) = t^{k+r_1} g_1(x, y) \\ g_2(t^{r_1}x, t^{r_2}y) = t^{k+r_2} g_2(x, y). \end{cases} \tag{2.7}$$

From the first equations of (2.6) and (2.7) we have $r_1 = r_2$ and $k = 2r_1$ and from the second equations of (2.6) and (2.7) we have $r_1 = r_2$ and $k = 4r_1$. It follows that $k = r_1 = r_2 = 0$. So (2.6) is not homogeneous with any dilation. However, we can construct a LFHD as $V = x^4 + y^2$, which is not homogeneous. Then the derivative of V along (2.6) is

$$\dot{V} = -4x^6 + 4x^4y^2 - 4y^6 + 2xy^5 \leq -x^6 - y^6.$$

(The last inequality can be shown by using the inequality (3.1) in the next section.) So the derivative is homogeneous and negative definite. The following proposition will show that (2.5) is asymptotically stable at origin. □

The following proposition is fundamental for LFHD.

Proposition 2.5: System (2.1) is approximately stable at origin if there exists a LFHD of its approximate system (2.2) such that its derivative along (2.2) is negative–definite.

Proof: Assume L_gV is negative definite, then it should be of even degree, say $\deg(L_gV) = 2m$. We claim that there exists a real number $b > 0$ such that

$$L_gV(x) \leq -b \sum_{i=1}^{n} (x_i)^{2m}. \tag{2.8}$$

Since L_gV is negative definite, on the compact "sphere"

$$S = \left\{ z \,\middle|\, \sum_{i=1}^{n} (z_i)^{2m} = 1 \right\}$$

$L_gV(x)$ attains its maximum value $-b < 0$. That is

$$L_gV(z) \leq -b < 0, \quad z \in S.$$

Now any $x \in R^n$ can be expressed as $x = kz$ for some $z \in S$. Then

$$L_g V(x) = L_g V(kz) = k^{2m} L_g V(z) \le -bk^{2m} = -b \sum_{i=1}^{n} (x_i)^{2m}$$

which proves the claim.

Using (2.8), the derivative of the LFHD becomes

$$\dot{V}|_f = \dot{V}|_{g+0(\|x\|^{K+1})} \le -b \sum_{i=1}^{n} (x_i)^{2m} + 0\left(\|x\|^{2m+1}\right) \quad (2.9)$$

where $g + 0\left(\|x\|^{K+1}\right)$ is a shorthand for $\left(g_1(x) + 0\left(\|x\|^{k_1+1}\right), \ldots, g_n(x) + 0\left(\|x\|^{k_n+1}\right)^{\mathrm{T}}\right)$. □

For the homogeneous vector fields [11] gives (with slightly different statement) the following.

Theorem 2.6 [11]: Assume (2.1) has $k_1 = \cdots = k_n = k$ and its approximate system (2.2) is asymptotically stable. then (2.1) is asymptotically stable.

The Proposition 2.5 and Theorem 2.6 will be our major tools for testing approximate stability.

III. Some Sufficient Conditions for Negativity

This section investigates some sufficient conditions for testing approximate stability of systems with odd approximate systems.

We need the following inequality, which is based on the fact that the algebraic average is greater than or equal to the geometric average.

Lemma 3.1: Let $S \in Z_+^n$ and $x \in R^n$. The following inequality holds:

$$\left|x^S\right| \le \sum_{j=1}^{n} \frac{s_j}{|S|} |x_j|^{|S|}. \quad (3.1)$$

Given a component-wise homogeneous polynomial vector field $g = \mathrm{col}(g_1, \ldots, g_n)$ with $\dim(g_i) = k_i$ $i = 1, \ldots, n$. We express g_i as

$$g_i(x) = a_{d_i}^i x_i^{k_i} + \sum_{S \ne d_i} a_S^i x^S, \quad i = 1, \ldots, n \quad (3.2)$$

where the index $d_i = k_i \delta_i = (0, \ldots, k_i, \ldots, 0)$, which indicates the *diagonal term*. Then we have

Theorem 3.2: Cross Row Diagonal Dominating Principle (CRDDP): The vector field g, given in above, is asymptotically stable at origin, if there exists an integer m with $2m > \max\{k_1, \ldots, k_n\}$, such that

$$-a_{d_i}^i > \sum_{|S|=k_i, S \ne d_i} \left|a_S^i\right| \left(\frac{s_i + 2m - k_i}{2m}\right) + \sum_{j=1, j\ne i}^{n} \sum_{|S|=k_j} \left|a_S^j\right| \left(\frac{s_i}{2m}\right), \quad i = 1, \ldots, n \quad (3.3)$$

where $S = (s_1, \ldots, s_n) \in Z_+^n$.

Proof: Choose a LFHD as

$$V = \sum_{i=1}^{n} \left(\frac{1}{2m - k_i + 1}\right) x_i^{2m-k_i+1},$$

Then we have

$$\dot{V}|_g = \sum_{i=1}^{n} \sum_{|S|=k_i} a_S^i x^S x_i^{2m-k_i}. \quad (3.4)$$

This is a homogeneous polynomial of degree $2m$. Now using (3.1) to split each term in (3.4) and collecting terms, (3.3) yields that

$$\dot{V}|_g < -\sum_{i=1}^{n} \epsilon_i x_i^{2m}, \text{ for some } \epsilon_i > 0.$$

The conclusion follows immediately. □

One obvious improvement for this estimation can be done as the following: Negative semidefinite nondiagonal terms can be eliminated from the estimation. Formally, for each g_i define a set of its terms by their exponents as

$$Q_i = \left\{|S| = k_i | s_j (j \ne i) \text{ are even and } a_S^i < 0\right\}.$$

Terms with exponents in Q_i are negative–semidefinite in $(\partial V/\partial x_i) g_i$. Moving such terms from (3.3) yields

$$-a_{d_i}^i > \sum_{|S|=k_i, S \notin Q_i} \left|a_S^i\right| \left(\frac{s_i + 2m - k_i}{2m}\right) + \sum_{j=1, j\ne i}^{n} \sum_{|S|=k_j, S\notin Q_j} \left|a_S^j\right| \left(\frac{s_i}{2m}\right), \quad i = 1, \ldots, n. \quad (3.5)$$

Later on we will simply use (3.5) as CRDDP.

Next, we give a simpler form, which deals with each row independently.

Corollary 3.3 Diagonal Dominating Principle (DDP):: Given a polynomial vector field g as in Theorem 3.2. It is asymptotically stable at origin if

$$-a_{d_i}^i > \sum_{|S|=k_i, S\notin Q_i} \left|a_S^i\right|, \quad i = 1, \ldots, n. \quad (3.6)$$

Proof: Since in (3.5) m can be arbitrary large, let $m \to \infty$, the right-hand side of (3.5) becomes right-hand side of (3.6). Hence the strict inequality (3.6) implies (3.5) for large enough m. □

In fact, DDP is an analog of Gersgorin's theorem [25]. Considering linear systems, they provide same stability results. However, CRDDP does not have its linear analog.

Using inequality (3.1), we can reduce the homogeneous polynomial of $\deg = 4k$ into a "dominating" quadratic form with variables $x_i^{2k}, i = 1, \ldots, n$.

Algorithm 3.4: Quadratic Form Reducing Algorithm (QFRA): Let $g = \mathrm{col}(g_1, \ldots, g_n)$ and $\deg(g_i) = k_i$ $i = 1, \ldots, n$ with odd k_i.

Step 1) Choose smallest even number $m = 2k$ such that $2m > \max\{k_1, \ldots, k_n\}$. Construct a $2m$ homogeneous polynomial $q(x)$ as

$$q(x) = \sum_{i=1}^{n} x_i^{2m-k_i} g_i.$$

Step 2) Find all terms in $q(x)$, for which the index S of x^S has component s_i less than $2k$. Split it into two equal exponent groups in the alphabetical order of x_i.

e.g., for $ax_1^2x_2x_3^5$ we have $S = (2, 1, 5)$, $m = 4$ and $k = 2$. It is split as $ax_1^2x_2x_3^5 = ax_1^2x_2x_3 \times x_3^4$. For $ax_1^2x_2^5x_3$ we have $S = (2, 5, 1)$, and it is split as $ax_1^2x_2^5x_3 = ax_1^2x_2^2 \times x_2^3x_3$.

Step 3) Using (3.1) to convert them into several $2k$ exponent terms, e.g.,

$$\begin{aligned} ax_1^2x_2x_3 \times x_3^4 \le& |a| \left(\frac{x_1^4}{2} + \frac{x_2^4}{4} + \frac{x_3^4}{4}\right) x_3^4 \\ =& |a| \left(\frac{1}{2}x_1^4x_3^4 + \frac{1}{4}x_2^4x_3^4 + \frac{1}{4}x_3^8\right). \end{aligned}$$

Replace the original terms in $q(x)$ by their splitting terms.

The algorithm produces a quadratic form of $x_i^{2k}, i = 1, \ldots, n$. Then the following can be proved by constructing a suitable LFHD.

Proposition 3.5: If the resulting quadratic form produced by the above algorithm is negative definite, then $q(x)$ is negative definite. Consequently, $g(x)$ is asymptotically stable at zero.

The following example is used to describe the notations and results in the above Theorem 3.2 through Proposition 3.5.

Example 3.6: Find a region for parameter λ, such that the following system is asymptotically stable at origin:

$$\begin{cases} \dot{x}_1 = \sin(x_1) - x_1 \cos(2\lambda x_2) \\ \dot{x}_2 = x_2^2 \ln(1 - x_2 - x_3) + 0.5x_2^2x_3 \\ \dot{x}_3 = 2x_3^3(1 - \cosh(x_3 - x_2)) - 1.1x_3^5. \end{cases} \tag{3.7}$$

Using Taylor expansion on (3.7), its approximate system is

$$\begin{cases} \dot{x}_1 = -\frac{1}{6}x_1^3 + 2\lambda^2 x_1 x_2^2 \ (:= g_1) \\ \dot{x}_2 = -x_2^3 - \frac{1}{2}x_2^2x_3 \ (:= g_2) \\ \dot{x}_3 = -2.1x_3^5 + 2x_3^4x_2 - x_3^3x_2^2 \ (:= g_3). \end{cases} \tag{3.8}$$

We figure out all the parameters in Theorem 3.2 and Corollary 3.3 as follows: For $g_1(x)$, $k_1 = 3$. Denote by $G_1 = \{S \mid |S| = k_1 = 3\}$, then

$$\begin{aligned} G_1 =& \{(300), (210), (201), (120), (111), (102), (030), \\ & (021), (012), (003)\} := \{S_1, S_2, \ldots, S_{10}\}. \end{aligned}$$

Note that $d_1 = (300) = S_1$, $a_{d_1}^1 = -1/6$, $a_{S_4}^1 = 2\lambda^2$, $a_{S_i}^1 = 0$, ($i \ne 1$, and $i \ne 4$). It is easy to check that there is no term in Q_1, so $Q_1 = \phi$.

For $g_2(x)$, $k_3 = 3$. Hence, $G_2 = G_1$. Then $d_2 = (030) = S_7$, $a_{d_2}^2 = -1$, $a_{S_8}^2 = -1/2$, $a_{S_i}^1 = 0$, ($i \ne 7$, and $i \ne 8$). We also have $Q_2 = \phi$.

For $g_3(x)$, $k_3 = 5$. Hence, $G_3 = \{(500), (410), (401), \ldots, (005)\}$. Then $d_3 = (005)$, $a_{d_3}^3 = -2.1$, $a_{(014)}^3 = 2$, $a_{(023)}^3 = -1$. For the other $S \in G_3$, $a_S^3 = 0$. Since the last term is in Q_3, so $Q_3 = \{(023)\}$.

Now we are ready to test the negativity of the derivative. We first check DDP. For second and third equations, the dominating condition (3.6) is satisfied. For first equation, (3.6) yields $1/6 > 2\lambda^2$. So

$$|\lambda| < 0.288\,675\,134\,6.$$

Next we check CRDDP. Let $m = 3$. Then (3.5) yields

$$\begin{cases} \frac{1}{6} > 2\lambda^2 \left(\frac{4}{6}\right) \\ 1 > \frac{2}{6}(2\lambda^2) + \frac{1}{2}\left(\frac{5}{6}\right) + 2\left(\frac{1}{6}\right) \\ 2.1 > 2\left(\frac{5}{6}\right) + \frac{1}{2}\left(\frac{1}{6}\right). \end{cases}$$

The solution is

$$|\lambda| < 0.353\,553\,390\,6.$$

Finally, let us use QFRA. The smallest even m should be 4. Then

$$\begin{aligned} q(x) = & -\frac{1}{6}x_1^8 + 2\lambda^2 x_1^6 x_2^2 - x_2^8 - \frac{1}{2}x_2^7x_3 \\ & -2.1x_3^8 + 2x_3^7x_2 - x_3^6x_2^2. \end{aligned}$$

The algorithm produces a quadratic form as

$$\begin{pmatrix} -\frac{1}{6} + \lambda^2 & \frac{1}{2}\lambda^2 & 0 \\ \frac{1}{2}\lambda^2 & -\frac{5}{8} & \frac{5}{16} \\ 0 & \frac{5}{16} & -0.6 \end{pmatrix}.$$

To make it negative–definite we have

$$|\lambda| < 0.392\,253\,521\,8.$$

□

In fact, we can prove that in general QFRA is stronger than CRDDP and CRDDP is stronger than DDP. However, DDP is the easiest one in use, while QFRA is the most difficult one. Later on, according to the problems one or more of these three methods are used for testing the negative–definiteness of the derivatives of LFHD.

IV. Stabilization of Systems With Zero Center

Consider an affine nonlinear system with the following Byrnes-Isidori canonical form [4]:

$$\begin{cases} \dot{x}^i = A^i(\xi) + B^i(\xi)u_i, \ x^i \in R^{n_i}, \ i = 1, \ldots, m, \\ \qquad \sum_{i=1}^m n_i = n; \ \xi = (x, w, z) \\ \dot{w} = Sw + p(\xi), \ w \in R^s; \ \Re\sigma(S) < 0 \\ \dot{z} = Cz + q(\xi), \ z \in R^t; \ \Re\sigma(C) = 0 \end{cases} \tag{4.1}$$

where

$A^i(\xi)$ $= \mathrm{col}(x_2^i, \ldots, x_{n_i}^i, f_i(\xi))$, $f_i(0) = 0$;
$B^i(\xi)$ $= \mathrm{col}(0, \ldots, 0, g_i(\xi))$, $g_i(0) \ne 0$;
$p(\xi)$ and $q(\xi)$ vanish at origin with their first derivatives.

Since the first variables in each integral chain play a particular role, we adopt the following notations:

$$\begin{aligned} x =& (x_1, \bar{x}_1), \text{ where } x_1 = \left(x_1^1, \ldots, x_1^m\right) \\ \bar{x}_1 =& \left(x_2^1, \ldots, x_{n_1}^1, \ldots, x_2^m, \ldots, x_{n_m}^m\right). \end{aligned}$$

System (4.1) is said to have *zero center* if $C = 0$. Only this case is considered in this paper.

Let $\psi_i^{(r)}(z)$, $r = 2, \ldots, h$, $i = 1, \ldots, m$, be a set of polynomials of z with degree r. Define

$$\begin{cases} p_j^c(z) := p_j(x(z), 0, z), \text{ with } x_1^i = \sum_{r=2}^h \psi_i^{(r)}(z), \\ \quad \overline{x}_1^i = 0, \; i = 1, \ldots, m, \; j = 1, \ldots, s; \\ q_k^c(z) := q_k(x(z), 0, z), \text{ with } x_1^i = \sum_{r=2}^h \psi_i^{(r)}(z), \\ \quad \overline{x}_1^i = 0, \; i = 1, \ldots, m, \; k = 1, \ldots, t; \\ \tilde{q}_k^c(z) := q_k(x(z), w(z), z), \text{ with} \\ \quad x_1^i = \sum_{r=2}^h \psi_i^{(r)}(z) + E_1^i(z), \\ \qquad (\overline{x}_1^i)_j = E_j^i(z), \; j = 2, \ldots, n_i, \; i = 1, \ldots, m \end{cases}$$

where $E_j^i(z)$, $i = 1, \ldots, m$, $j = 1, \ldots, n_i$, and $w(z)$ are uncertain functions, and they will be specified later. We denote $p^c(z) = (p_1^c(z), \ldots, p_s^c(z))^{\mathrm{T}}$ etc.

The following theorem shows a general design idea. Polynomials of degrees 2 to h are used for the nonlinear control design.

Theorem 4.1: Assume $C = 0$ and there exists a set of polynomials $\psi_i^{(r)}(z)$, $r = 2, \ldots, h$; $i = 1, \ldots, m$, $\deg\left(\psi_i^{(r)}(z)\right) = r$, and an integer $e \geq 2$, such that the following conditions C1)–C4) hold:

C1) $p^c(z) = 0(\|z\|^{e+1})$;

C2) $q^c(z) = 0(\|z\|^{e})$;

C3) if $E_j^i(z) = 0(\|z\|^{e+1})$, and $w(z) = 0(\|z\|^{e+1})$, then

$$\dot{z} = \tilde{q}^c(z) \text{ and } \dot{z} = q^c(z) \tag{4.2}$$

have same approximate system;

C4) $\dot{z} = q^c(z)$ is approximately stable.

Then system (4.1) is (locally) asymptotically stabilizable (at origin). Moreover, if C1–C4 are satisfied, a suitable feedback control, which stabilizes system (4.1) is

$$u_i = -\frac{f_i(\xi)}{g_i(\xi)} + \frac{1}{g_i(\xi)}\left(\sum_{j=1}^{n_i} a_j^i x_j^i - a_1^i\left(\sum_{r=2}^{h} \psi_i^{(r)}(z)\right)\right), \quad i = 1, \ldots, m. \tag{4.3}$$

where $\lambda^{n_i} - \sum_{j=1}^{n_i} a_j^i \lambda^{j-1}$ is Hurwitz.

Proof: Choose

$$\Phi(z) = \begin{cases} x_1^i(z) = \sum_{r=2}^h \psi_i^{(r)}(z) \\ \overline{x}_1^i(z) = 0, \quad i = 1, \ldots, m \\ w(z) = 0 \end{cases}$$

to approximate the center manifold of the closed-loop system with control (4.3). Using C1), C2), and control (4.3), we have

$$M\Phi(z) = \begin{pmatrix} \begin{pmatrix} \sum_{r=2}^h \frac{\partial \psi_i^{(r)}(z)}{\partial z} \\ 0 \end{pmatrix}, \quad i = 1, \ldots, m \\ 0 \end{pmatrix} q^c(z) - \begin{pmatrix} 0 \\ 0 \\ p^c(z) \end{pmatrix} = 0\left(\|z\|^{e+1}\right). \tag{4.4}$$

The dynamics on the center manifold is

$$\dot{z} = q(x(z), w(z), z). \tag{4.5}$$

According to the approximation Theorem [9], (4.4) ensures than in (4.5) the functions $x(z)$ and $w(z)$ have the following forms:

$$x_1^i = \sum_{r=2}^h \psi_i^{(r)}(z) + 0\left(\|z\|^{e+1}\right), \quad \overline{x}_1^i = 0\left(\|z\|^{e+1}\right), \quad i = 1, \ldots, m; \quad w = 0\left(\|z\|^{e+1}\right).$$

Now (4.5) is of the type of the first equation of (4.2). So conditions C3) and C4) ensure the approximate stability of (4.5). Hence, the closed-loop form of system (4.1) is asymptotically stable. □

It is clear from above proof that $e + 1$ is the order of the approximation error.

In Section I, it has been pointed out that the higher order feedback can be injected into the dynamics on the center manifold through x_1. To distinct different injection types we define the injection degrees as

Definition 4.2: For system (4.1) the injection-degree, d_k, is defined as

$$d_k = \min\left\{2|T| + |S| \;\middle|\; |T| > 0, \quad \frac{\partial^{|\mathrm{T}|+|S|} q_k}{\partial (x_1)^{\mathrm{T}} \partial z^S}(0) \neq 0\right\}, \quad k = 1, \ldots, t.$$

In fact, the d_k are the lowest degrees of the nonvanishing terms in the dynamics on center manifold which contains $x_1(z)$.

Given system (4.1) the approximation order e can be estimated from (4.4). Let l_j be the lowest degree of the nonvanishing terms in p_j^c. Then we have

$$e = \min\{d_i, \; i = 1, \ldots, t; \; l_j - 1, \; j = 1, \ldots, s\}. \tag{4.6}$$

It can be seen intuitively that, (e.g., refer to some examples in [1], [2]) an even-degree leading system can hardly be homogeneously stable. Our design idea is: When the injection-degree, d_k, is odd, use it as lowest degree of the resulting system, i.e., for the dynamics on the center manifold, let $L_k = d_k$. Otherwise, choose control to eliminate d_k degree terms and turn the lowest degree of the resulting system to odd, i.e., $L_k = d_k + 1$. In such a way, we finally make the dynamics on the center manifold to have an odd approximate system. L_k will be called *leading degree*.

Remark: Even in Theorem 4.1 $e + 1 > h$ is not claimed, it is required implicitly. Otherwise some terms of $\psi_i^{(r)}(z)$ in the designed approximation of the center manifold will be meaningless.

Using e, d_k, and L_k, conditions C1)–C3) in Theorem 4.1 is computable.

Proposition 4.3: In Theorem 4.1, for arbitrary chosen $\psi_i^{(r)}$, $i = 1, \ldots, m$; $r = 2, \ldots, h$

i) condition C1) holds, iff

$$\frac{\partial^{|\mathrm{T}|+|S|} p_k}{\partial (x_1)^{\mathrm{T}} \partial z^S}(0) = 0, \quad 2|T| + |S| \leq e, \quad k = 1, \ldots, s; \tag{4.7}$$

ii) condition C2) holds, if

$$\frac{\partial^{|\mathrm{T}|+|S|} q_k}{\partial (x_1)^{\mathrm{T}} \partial z^S}(0) = 0, \quad 2|T| + |S| \leq d_k - 1; \quad k = 1, \ldots, t; \tag{4.8}$$

iii) condition C3) holds, iff (4.8) holds and when $|U|+|V|>0$

$$\begin{aligned}\frac{\partial^{|T|+|S|+|U|+|V|}q_k}{\partial(x_1)^{T}\partial z^S\partial\overline{x}_1^U\partial w^V}(0)=&0,\\ 2|T|+|S|+(e+1)(|U|+|V|)\le& L_k;\quad k=1,\ldots,t.\end{aligned}\tag{4.9}$$

Proof: In $p_k(\xi)$ set $\overline{x}_1=0$ and $w=0$, then use Taylor expansion on x_1 and z. Note that $x_1^k(z)=0\left(\|z\|^2\right)$. Then (4.7) means all terms in $p_k^c(z)$ of degree less than or equal to e are zero. Since $e\le d_k$, (4.7) holds for $2|T|+|S|\le e-1$, which means all terms in $q_k^c(z)$ of degree less than e are zero. As for C3), note that $\overline{x}_1^k(z)=0\left(\|z\|^{e+1}\right)$ and $w(z)=0\left(\|z\|^{e+1}\right)$. Then it is easily seen that (4.9) holds, iff, both $\overline{x}_1$ and w don't appear in the approximate system of the dynamics on the center manifold. Hence the two equations in (4.2) have same approximate system. □

Equation (4.8) is sufficient for C2). But it is necessary for the required leading degrees. So we call (4.7)–(4.9) the *degree matching conditions*. They are always assumed in the following sections for center manifold design.

We use an example to give a detailed description for all the objects in this section.

Example 4.4: Consider the following system:

$$\begin{cases}\dot{x}_1=x_2,\ \dot{x}_2=x_3,\ \dot{x}_3=u\\ \dot{w}_1=-w+x_1z_1^2z_2\\ \dot{z}_1=az_1^2+z_1x_1+z_2x_2\\ \dot{z}_2=z_1z_2x_1.\end{cases}\tag{4.10}$$

For this system $m=1$, $\overline{x}_1=(x_2,x_3)$, $s=1$, $t=2$, $p(x,w,z)=x_1z_1^2z_2$, $q_1(x,w,z)=az_1^2+z_1x_1+z_2x_2$, and $q_2(x,w,z)=z_1z_2x_1$. Consequently, we have

$$\begin{cases}p^c(z)=\sum_{r=2}^h\psi_i^{(r)}(z)z_1^2z_2\\ q_1^c(z)=az_1^2+z_1\sum_{r=2}^h\psi_i^{(r)}(z)\\ q_2^c(z)=z_1z_2\sum_{r=2}^h\psi_i^{(r)}(z)\end{cases}\tag{4.11}$$

and

$$\begin{cases}\tilde{q}_1^c(z)=az_1^2+z_1\left(\sum_{r=2}^h\psi_i^{(r)}(z)+E_1^1(z)\right)+z_2E_2^1(z)\\ \tilde{q}_2^c(z)=z_1z_2\left(\sum_{r=2}^h\psi_i^{(r)}(z)+E_1^2(z)\right)\end{cases}\tag{4.12}$$

where $\psi_i^{(r)}(z)$, $r=2,\ldots,h$ will be chosen to design control, E_1^1, E_2^1, and E_1^2 are some uncertain terms of $0\left(\|z\|^{e+1}\right)$.

According to Definition 4.2, the injection degrees are $d_1=3$ and $d_2=4$. Hence we choose $L_1=3$ and $L_2=5$. From $p^c(z)$ we have $l=5$. Then

$$e=\min\{d_1=3,d_2=4,l-1=4\}=3.$$

It is ready to check that (4.7) holds.

Consider (4.8). For q_1, when $S=(2,0)$ and $T=(0)$, $|S|+|T|\le d_1-1$. However

$$\frac{\partial^{|S|}q_1}{\partial z^S}=\frac{\partial^2q_1}{\partial z_1^2}=2a.$$

So (4.8) holds for q_1 iff $a=0$. It is easy to check that (4.8) is true for q_2.

For (4.9), only q_1 has a term involving $\overline{x}_1$ and/or w, which is z_2x_2. For this term $T=(0)$, $S=(0,1)$, $U=(1,0)$, and $V=(0)$. So

$$2|T|+|S|+(e+1)(|U|+|V|)=5>L_1=3.$$

equation (4.9) is, therefore, satisfied.

We conclude that the degree matching conditions are satisfied iff $a=0$.

Next, from (4.11) and (4.12) it is clear that the two equations in (4.2) have same approximate system.

V. Stabilization for Odd Injection-Degree Center

This section considers the case when all the injection degrees equal to a same odd number. Then, we have the following.

Theorem 5.1: Assume system (4.1) with $C=0$ has an odd universal injection degree, say $d_k=L\le 3$, $\forall\ k$. The system is state feedback stabilizable, if it satisfies the degree matching conditions (4.7)–(4.9) with $h=2$, $2\le e\le L$ and $L_k=L$, $\forall\ k$ and there exists a quadratic homogeneous vector $\psi(z)=\mathrm{col}(\psi_1,\ldots,\psi_m)$, such that

$$\begin{aligned}\dot{z}_k=&\sum_{2|T|+|S|=L}\frac{1}{T!S!}\frac{\partial^{|T|+|S|}q_k}{\partial(x_1)^{T}\partial z^S}(0)z^S\left(\prod_{i=1}^m(\psi_i)^{T_i}(z)\right),\\ k=&1,\ldots,t\end{aligned}\tag{5.1}$$

is asymptotically stable at origin. Moreover, if the above conditions are satisfied, (4.3) with $\psi_i^{(2)}=\psi_i$ is a suitable feedback control, which stabilizes system (4.1).

Proof: Using control (4.3), conditions (4.8) and (4.9) assure the lowest degree of the dynamics of the closed-loop system on the center manifold is L. Note that in this case $d_i=e=L$, $i=1,\ldots m$. Conditions (4.7) and (4.8) assure the center manifold is described as

$$\begin{aligned}x_l^i=&\psi_i(z)+R_i,\ \text{where}\ R_i=0\left(\|z\|^{L+1}\right)\\ \overline{x}_l^i=&0\left(\|z\|^{L+1}\right),\quad i=1,\ldots,m;\quad w=0\left(\|z\|^{L+1}\right).\end{aligned}$$

Using (4.8) and (4.9), R_i, $\overline{x}_1^i$, and w will not appear into the degree L terms. Hence the degree L terms of the dynamics are exactly the right side of (5.1). That is, (5.1) is the approximate system of the dynamics on the center manifold. Since (5.1) is homogeneous and asymptotically stable at origin, Theorem 2.4 assures the approximate stability of the dynamics on the center manifold of the closed-loop system. Then the asymptotical stability of the closed-loop system follows from Theorem 4.1. □

When $d_k=3$, $k=1,\ldots t$ it is an interesting case [1]. Now set $h=2$, and $L=3$. The previous result leads to the following simpler one.

Corollary 5.2: System (4.1) with $C=0$ is state feedback stabilizable if

C1) $\partial p/\partial x_1'(0)=0$; $\partial^2p/\partial x_1'\partial z(0)=0$; $i=1,\ldots,m$; $\partial p/\partial z(0)=0$; $\partial^2p/\partial z^2(0)=0$; $\partial^3p/\partial z^3(0)=0$;
C2) $\partial q/\partial x(0)=0$; $\partial q/\partial z(0)=0$; $\partial^2q/\partial z^2(0)=0$;
C3) there exists quadratic homogeneous vector field $\psi(z)=\mathrm{col}(\psi_1,\ldots,\psi_m)$, such that

$$\dot{z}=D(z)\psi(z)+E(z)\tag{5.2}$$

is asymptotically stable. Where $D(z)$ and $E(z)$ are $t\times m$ and $t \times 1$ matrices with entries as

$$D_{ij} = \sum_{k=l}^{t} \frac{\partial^2 q_i}{\partial x_l^j \partial z_k}(0) z_k: \quad i=1,\dots,t; \quad j=1,\dots m;$$
$$E_i = \sum_{|S|=3} \frac{\partial^3 q_i}{S! \partial z^S}(0) z^S; \quad i=1,\dots,t$$

respectively. Moreover, (4.3) with $\psi_i^{(2)} = \psi_i$ is a suitable feedback control, which stabilizes system (4.1).

The following example shows that when the injection-degree is 3 we have only to solve a set of algebraic inequalities to obtain the required control.

Example 5.3: Consider the following system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x,z) + g_1(x,z)u_1 \\ \dot{x}_3 = f_2(x,z) + g_2(x,z)u_2 \\ \dot{z}_1 = q_1(x,z) \\ \dot{z}_2 = q_2(x,z) \end{cases} \tag{5.3}$$

where $f_i(0)=0$, $g_i(0) \neq 0$ satisfies the condition C2 in Corollary 5.2, $i = 1.2$. Our goal is to find a sufficient condition for system (5.3) to be feedback stabilizable. Denote by

$$c_{ij}^k = \frac{\partial^2 q_k}{\partial z_i \partial x_j}(0), \quad k=1,2 \quad i=1,2 \quad j=1,3$$
$$d_{ij}^k = \frac{\partial^3 q_k}{i! j! \partial (z_1)^i \partial (z_2)^j}(0), \quad k=1,2; \quad i+j=3.$$
$$\psi_1 = a z_1^2 + b z_1 z_2 + c z_2^2 \quad \psi_2 = d z_1^2 + e z_1 z_2 + f z_2^2.$$

Then (5.2) leads to the following:

$$\begin{pmatrix} c_{11}^1 z_1 + c_{21}^1 z_2 & c_{13}^1 z_1 + c_{23}^1 z_2 \\ c_{11}^2 z_1 + c_{21}^2 z_2 & c_{13}^2 z_1 + c_{23}^2 z_2 \end{pmatrix} \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} + \begin{pmatrix} \sum_{t=0}^3 d_{t\,3-t}^1 z_1^t z_2^{3-t} \\ \sum_{t=0}^3 d_{t\,3-t}^2 z_1^t z_2^{3-t} \end{pmatrix}. \tag{5.4}$$

Thus the sufficient conditions developed in Section III may be used to design a stabilizing control.

To specify the above general form, we consider the following system as a special case of (5.3):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x,z) + g_1(x,z)u_1 \\ \dot{x}_3 = f_2(x,z) + g_2(x,z)u_2 \\ \dot{z}_1 = \sin\left(x_1 z_1 - z_1^2 z_2\right) - 3\tan\left(z_1^2 z_2\right) \\ \dot{z}_2 = z_1 - z_2 + z_2^3 - z_1 e^{x_1} + z_2 e^{x_3}. \end{cases} \tag{5.5}$$

Then the coefficients are computed as

$$\begin{cases} c_{11}^1 = 1;\ c_{13}^1 = 0;\ c_{21}^1 = 0;\ c_{23}^1 = 0 \\ c_{11}^2 = -1;\ c_{13}^2 = 0;\ c_{21}^2 = 0;\ c_{23}^2 = 1 \\ d_{30}^1 = 0;\ d_{21}^1 = -4;\ d_{12}^1 = 0;\ d_{03}^1 = 0 \\ d_{30}^2 = 0;\ d_{21}^2 = 0;\ d_{12}^2 = 0;\ d_{03}^2 = 1. \end{cases}$$

(5.4) becomes

$$\begin{pmatrix} z_1 & 0 \\ -z_1 & z_2 \end{pmatrix} \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} + \begin{pmatrix} -4 z_1^2 z_2 \\ z_2^3 \end{pmatrix}$$
$$= \begin{pmatrix} a z_1^3 + b z_1^2 z_2 + c z_1 z_2^2 - 4 z_1^2 z_2 \\ -a z_1^3 + (d-b) z_1^2 z_2 + (e-c) z_1 z_2^2 + (f+1) z_2^3 \end{pmatrix}. \tag{5.6}$$

Using CRDDP with $m=2$, we have

$$\begin{cases} -a > \frac{3}{4}|b| + \frac{2}{4}|c| + 3 + \frac{3}{4}|a| + \frac{2}{4}|d-b| + \frac{1}{4}|e-c| \\ -f-1 > \frac{1}{4}|b| + \frac{2}{4}|c| + 1 + \frac{1}{4}|a| + \frac{2}{4}|d-b| + \frac{3}{4}|e-c|. \end{cases} \tag{5.7}$$

One particular solution of (5.7) is $a=-25$; $b=4$; $c=0$; $d=4$; $e=0$; $f=-10$. Then

$$\phi = \begin{pmatrix} -25 z_1^2 + 4 z_1 z_2 \\ 4 z_1^2 - 10 z_2^2 \end{pmatrix}.$$

To stabilize linear part, one may choose $a_1^1 = -1$; $a_2^1 = -2$; $a_1^2 = -1$. Then (4.3) yields:

$$\begin{cases} u_1 = -\frac{f_1(x,z)}{g_1(x,z)} + \frac{1}{g_1(x,z)}\left(-x_1 - 2x_2 - 25 z_1^2 + 4 z_1 z_2\right) \\ u_2 = -\frac{f_2(x,z)}{g_2(x,z)} + \frac{1}{g_2(x,z)}\left(-x_3 + 4 z_1^2 - 10 z_2^2\right). \end{cases}$$

□

VI. Stabilization for Even Injection Degree Center

In this section we consider the even injection degree case. Assume there exists a positive even number c such that the injection degrees are either c or $c+1$. Then we design a suitable control to turn the leading degree of the dynamics on the center manifold to $L_k = c+1$ for all k.

For system (4.1) with $C=0$. Assume there exists a positive even number c such that the injection degrees are either $d_k = c$ or $d_k = c+1$. Set $L_k := L = c+1$, $\forall\, k$. Let if it satisfies the degree matching conditions (4.7)–(4.9) with $h=3$, $3 \le e \le L$. (4.3) be used to form the closed-loop system. Then

Lemma 6.1: A necessary condition for L to be the leading degree of the dynamics on the center manifold of the closed-loop system is: there exists a quadratic homogeneous vector $\psi(z) = \mathrm{col}(\psi_1,\dots,\psi_m)$, such that

$$\sum_{2|T|+|S|=c} \frac{1}{T! S!} \frac{\partial^{|\mathrm{T}|+|S|} q_k}{\partial (x_1)^{\mathrm{T}} \partial z^S}(0) z^S \left(\prod_{i=1}^m \psi_i^{\mathrm{T}_i}(z)\right) = 0, \quad k = 1,\dots,t. \tag{6.1}$$

Proof: According to condition (4.8), when we calculate the approximate system of the systems in (4.1), only the quadratic terms, $\phi_i^{(2)}(z)$ in x_1^i, $i=1,\dots,m$ are chosen to form the degree c terms. This turns out to be the left side of (6.1). To make the leading degree $L = c+1$, the control should be chosen to eliminate degree c terms, which leads to the (6.1). □

Then we have the following sufficient condition:

Theorem 6.2: Given system (4.1) as described in the above, and (6.1) is assumed for certain $\psi(z)$. If there exists a cubic homogeneous vector $\phi(z) = \mathrm{col}(\phi_1, \ldots, \phi_m)$, such that

$$\begin{aligned}\dot{z}_k = &\sum_{\substack{2|T|+|S|=c\\|T|\neq 0}} \frac{1}{T!S!}\frac{\partial^{|\mathrm{T}|+|S|}q_k}{\partial(x_1)^{\mathrm{T}}\partial z^S}(0)z^S\\ &\times\left(\sum_{i=1}^m T_i\phi_i(z)\psi_i^{\mathrm{T}_i-1}(z)\prod_{j\neq i}\psi_j^{\mathrm{T}_j}(z)\right)\\ &+\sum_{2|T|+|S|=c+1}\frac{1}{T!S!}\frac{\partial^{|\mathrm{T}|+|S|}q_k}{\partial(x_1)^{\mathrm{T}}\partial z^S}(0)z^S\\ &\times\left(\prod_{i=1}^m\psi_i^{\mathrm{T}_i}(z)\right), \quad k=1,\ldots,t \end{aligned} \tag{6.2}$$

is asymptotically stable at origin, then (4.1) is stabilizable.

Moreover, if the above conditions are satisfied, a suitable feedback control, which stabilizes system (4.1) is (4.3) with $\psi_i^{(2)} = \psi_i$ and $\psi_i^{(3)} = \phi_i$.

Proof: Conditions (4.7), (4.8) and (6.1) assure that the center manifold is defined by

$$\begin{cases} x_1^i = \psi_i(z) + \phi_i(z) + R_i, \text{ where } R_i = 0\left(\|z\|^{L+1}\right);\\ \overline{x}_1^i = 0\left(\|z\|^{L+1}\right), \quad i=1,\ldots,m; \quad w = 0\left(\|z\|^{L+1}\right).\end{cases}$$

According to Theorem 4.1, we have only to construct the approximate system of the dynamics on the center manifold of the closed-loop system and show that it is approximately stable. Conditions (4.8) and (4.9) assure that $R_i, \overline{x}_1^i$, and w will not appear in the degree L terms. Hence the degree L terms can only from terms of $2|T|+|S| = c$ and $|T| \neq 0$. Consider the product

$$\left(x_1^1\right)^{\mathrm{T}_1}\cdots\left(x_m^1\right)^{\mathrm{T}_m} = \prod_{i=1}^m\left(\psi_i + \phi_i + 0\left(\|z\|^{L+1}\right)\right)^{\mathrm{T}_i}.$$

To raise the power by 1, there should be exactly one i and one of the T_i factors, which is chosen to provide a cubic term ϕ_i, and from all other $T_i - 1$ factors and the $j \neq i$ factors the quadratic terms should be chosen. A careful calculation shows that (6.2) is the required approximate system. Now Theorem 2.4 assures the asymptotically stability of the dynamics on the center manifold and thus the stability of the closed-loop system. □

Example 6.2: Consider the following system:

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = f(x,z) + g(x,z)u\\ \dot{z}_1 = \sin\left(x_1z_1^2\right) + z_1^2z_2^2 + x_2^2\\ \dot{z}_2 = z_2^2\tan(x_1) - z_2^4\left(e^{z_2}-2\right) + 0.5z_1^2z_2^3.\end{cases} \tag{6.3}$$

Since the injection degrees are 4, 4, we choose leading degree $L = 5$. Now, since we use quadratic and cubic feedback, so $h = 3$. There is no $p(\xi)$, we check (4.7) first. Since $q^e(z) = 0\left(\|z\|^4\right)$, so $e = 4$. It is obvious that (4.7) is satisfied. To check (4.8), only $q_1(\xi)$ contains $\overline{x}_1 = x_2$, so we have only to check it. Since in $q_1(\xi)$ the degree of x_2 is 2, it is easy to see that when $2|T|+|S|+5(|U|+|V|) \le L = 5$, we have $|U| \le 1$. So (4.8) holds, (4.9) is obviously true.

Now denote

$$\begin{aligned}\psi &= az_1^2 + bz_1z_2 + cz_2^2;\\ \phi &= \alpha z_1^3 + \beta z_1^2z_2 + \gamma z_1z_2^2 + \delta z_2^3.\end{aligned}$$

Then (6.1) becomes

$$\begin{cases} z_1^2z_2^2 + z_1^2\left(az_1^2 + bz_1z_2 + cz_2^2\right) = 0\\ z_2^4 + z_2^2\left(az_1^2 + bz_1z_2 + cz_2^2\right) = 0\end{cases}$$

Set $a = 0$, $b = 0$, $c = -1$. Equation (6.1) is satisfied.

Equation (6.2) yields a fifth degree homogeneous vector field as

$$\begin{pmatrix} z_1^2\left(\alpha z_1^3 + \beta z_1^2z_2 + \gamma z_1z_2^2 + \delta z_2^3\right)\\ z_2^2\left(\alpha z_1^3 + \beta z_1^2z_2 + \gamma z_1z_2^2 + \delta z_2^3\right) - z_2^5 + 0.5z_1^2z_2^3\end{pmatrix}.$$

Simply choose $\alpha = -1$, $\beta = 0$, $\gamma = 0$, and $\delta = 0$. It is ready to check that CRDDP is satisfied with $m = 3$. Chosen $a_1 = -1$, $a_2 = -2$, then (4.3) provides a stabilizing control as

$$u = -\frac{f(x,z)}{g(x,z)} + \frac{1}{g(x,z)}\left(-x_1 - 2x_2 - \left(z_2^2 + z_1^3\right)\right).$$

□

VII. Stabilization for Mixed Injection Degree Center

This section considers a general case when the injection degrees differ. By reordering z we may, without loss of generality, assume that the injection degrees are $(d_1, \ldots, d_\alpha, \ldots, d_t)$ and d_i is odd for $i < \alpha$, even for $i > \alpha$. Then, we have the following.

Theorem 7.1: System (4.1) with $C = 0$ is feedback stabilizable if it satisfies the degree matching conditions (4.7)–(4.9) with $e \ge 3$ and the following conditions.

C1) There exists a quadratic homogeneous vector $\psi(z) = \mathrm{col}(\psi_1, \ldots, \psi_m)$, such that

$$\begin{aligned}&\sum_{2|T|+|S|=d_k}\frac{1}{T!S!}\frac{\partial^{|\mathrm{T}|+|S|}q_k}{\partial(x_1)^{\mathrm{T}}\partial z^S}(0)z^S\\ &\qquad\times\left(\prod_{i=1}^m(\psi_i)^{\mathrm{T}_i}(z)\right) = 0, \quad k > \alpha.\end{aligned} \tag{7.1}$$

C2) There exists a cubic homogeneous vector $\phi(z) = \mathrm{col}(\phi_1, \ldots, \phi_m)$, such that

$$\begin{cases}\dot{z}_k = \sum_{2|T|+|S|=d_k}\frac{1}{T!S!}\frac{\partial^{|\mathrm{T}|+|S|}q_k}{\partial(x_1)^{\mathrm{T}}\partial z^S}(0)z^S\\ \qquad\times\left(\prod_{j=1}^m(\psi_j)^{\mathrm{T}_j}(z)\right); \quad k\le\alpha\\ \dot{z}_k = \sum_{\substack{2|T|+|S|=d_k\\|T|\neq 0}}\frac{1}{T!S!}\frac{\partial^{|\mathrm{T}|+|S|}q_k}{\partial(x_1)^{\mathrm{T}}\partial z^S}(0)z^S\\ \qquad\times\left(\sum_{i=1}^m T_i\phi_i(z)\psi_i^{\mathrm{T}_i-1}(z)\prod_{j\neq i}\psi_j^{\mathrm{T}_j}(z)\right)\\ \qquad+\sum_{2|T|+|S|=d_k+1}\frac{1}{T!S!}\frac{\partial^{|\mathrm{T}|+|S|}q_k}{\partial(x_1)^{\mathrm{T}}\partial z^S}(0)z^S\\ \qquad\times\left(\prod_{j=1}^m(\psi_j)^{\mathrm{T}_j}(z)\right); \quad k>\alpha\end{cases} \tag{7.2}$$

is approximately stable at origin.

Moreover, if above conditions are satisfied, the control (4.3) with $\psi_i^{(2)} = \psi_i$ and $\psi_i^{(3)} = \phi_i$ stabilizes (4.1).

Proof: The proof is basically the combination of the proofs of Theorem 5.1 and Theorem 6.2, therefore, we will just sketch it.

1) Equation (7.1) implies that the lowest degree terms in even injection degree subsystems are eliminated by suitable feedback $(\psi(z))$.
2) Equation (7.2) is the approximate system of the dynamics on center manifold.

Using Theorems 4.1 and 4.2, the degree matching conditions and the approximate stability of (7.2) ensure the stability of the closed-loop system. □

Remark: Unlike Theorem 5.1 and Theorem 6.2, since in Theorem 7.1 the approximate system of the dynamics on the center manifold of the closed-loop system is not homogenous and so approximate stability is required to assure the asymptotically stable of the dynamics on the center manifold. Meanwhile, the results in Theorem 5.1 and Theorem 6.2 are topologically invariant. The result in Theorem 7.1 is coordinate-dependent and so it is not topologically invariant.

Example 7.2: Consider the following system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x,z) + g(x,z)u \\ \dot{z}_1 = \sin(z_1 x_1) + z_1 z_2^2 + \lambda z_1^3 \\ \dot{z}_2 = z_2^2 \tan(x_1 + x_2) + z_1 z_2^3 \end{cases} \tag{7.3}$$

where

$f(0,0) \quad = 0;$
$g(0,0) \quad \neq 0;$
$\lambda \qquad$ parameter.

The injection degrees are $d_1 = 3$, $d_2 = 4$, so we choose the designed lowest degrees as $L_1 = 3$ and $L_2 = 5$. Let $h = 3$, and $\psi = az_1^2 + bz_1z_2 + cz_2^2$, $\phi = \alpha z_1^3 + \beta z_1^2 z_2 + \gamma z_1 z_2^2 + \delta z_2^3$. Since $q^c(z) = 0\left(\|z\|^3\right)$ the approximation order $e = 3$. We don't need to consider (4.7) because there is no $p(\xi)$. Consider (4.8), it is obviously true. As for (4.9), only $q_2(\xi)$ contains x_2. Since $e = 3$ as $x_2(z) = 0\left(\|z\|^{e+1}\right)$ the lowest term involving x_2 is $z_2^2 x_2 \sim 0\left(\|z\|^6\right)$. Hence (4.9) holds for $2|T| + |S| + 4(|U| + |V|) < 6$. However, $L_2 = 5$, so (4.9) is true.

Next, we check that (7.1) renders $a = 0$, $c = 0$, $b = -1$. Moreover, (7.2) turns out to be

$$\begin{cases} \dot{z}_1 = \lambda z_1^3 - z_1^2 z_2 + z_1 z_2^2 \\ \dot{z}_2 = z_2^2 \left(\alpha z_1^3 + \beta z_1^2 z_2 + \gamma z_1 z_2^2 + \delta z_2^3\right). \end{cases} \tag{7.4}$$

To find a possible solution, we may simply set $\alpha = \beta = \gamma = 0$, then test (7.4) by DDP, CRDDP, and QFRA, respectively.

DDP yields: $\delta < 0$, $\lambda < -2$.

Set $m = 3$, CRDDP yields $\delta < -0.5$, $\lambda < -1.5$.

Choosing $m = 4$, QFRA yields a quadratic form as $(\lambda + 1.25)z_1^8 + 0.75 z_1^4 z_2^4 + \delta z_2^8$. To make it negative–definite we need $\lambda < -1.25$, $\delta < 9/(64\lambda + 80)$.

It is clear that the last method provides the best estimation. Using (4.3) and choosing linear feedback with $a_1 = -1$, $a_2 = -2$, we conclude that when $\lambda < -1.25$ system (7.3) is stabilized by the following control

$$u = -\frac{f(x,z)}{g(x,z)} + \frac{1}{g(x,z)}\left(-x_1 - 2x_2 - z_1 z_2 + \delta z_2^3\right).$$

□

In fact, Theorem 4.1 and Proposition 4.2 allow us to use any nonlinear polynomial state feedback. The formulas in Theorem 5.1, Corollary 5.2, Theorem 6.2, and Theorem 7.1 use only quadratic and cubic polynomials. The next example shows that sometimes even higher degree terms are necessary.

Example 7.3: Consider the system

$$\begin{cases} \dot{x}_1 = x_2; \ldots; \dot{x}_{n_1-1} = x_{n_1}; \dot{x}_{n_1} = u \\ \dot{y}_1 = y_2; \ldots; \dot{y}_{n_2-1} = y_{n_2}; \dot{y}_{n_2} = v \\ \dot{z}_1 = x_1 y_1 \\ \dot{z}_2 = 2z_2 x_1 + z_1^2 z_2 \\ \dot{z}_3 = z_3 y_1 + \mu_1 z_1 z_3 x_1 + \mu_2 z_1 z_3 x_1^2, \quad \mu_1 > 0, \end{cases} \tag{7.5}$$

where μ_1 and μ_2 are parameters.

It is easily checked that the degree matching conditions (4.7)–(4.9) are trivially satisfied because $p^c(z) = q^c(z) = 0$, and $q(\xi)$ doesn't contain either x_k or y_k, $k > 1$. So if only we can find the approximate system of the dynamics on the center manifold, which is approximately stable, we are done.

Case 1) Consider a subsystem of (7.5) where the differential equations about z_2 and z_3 are removed.

Now $d_1 = 4$. So we choose $L_1 = 5$. Using Theorem 6.1, (6.1) implies either $x_1 = 0\left(\|z\|^3\right)$ or $y_1 = 0\left(\|z\|^3\right)$. An obvious solution for (6.2) is

$$\begin{cases} \psi_1(z) = az_1^2, & \phi_1(z) = 0; \\ \psi_2(z) = 0, & \phi_2(z) = \alpha z_1^3, \quad a\alpha < 0 \end{cases}$$

which leads to a set of controls

$$\begin{cases} u = \sum_{i=1}^{n_1} a_i^1 x_i + az_1^2 \\ v = \sum_{i=1}^{n_2} a_i^2 y_i + \alpha z_1^3 \end{cases} \tag{7.6}$$

where a_i^j, $i = 1, \ldots, n_j$, $j = 1, 2$ are coefficients of Hurwitz polynomials. If $n_1 = n_2 = 1$, this is a system discussed in Brockett [8] and the solution (7.6) is in Aeyels [1].

Case 2) Consider subsystem of (7.5), where the differential equation about z_3 is removed.

Now $d_1 = 4$ and $d_2 = 3$. We set $L_1 = 5$ and $L_2 = 3$. Using Theorem 7.1, (7.1) leads to the same conclusion as in Case 1: i.e., one of x_1 or y_1 should be $0\left(\|z\|^3\right)$. To avoid notational mess, we simply choose $x_1 = az_1^2 + bz_2^2$ and $y_1 = az_1^3$. Then (7.2) turns out to be

$$\begin{cases} \dot{z}_1 = a\alpha z_1^5 + b\alpha z_1^3 z_2^2 \\ \dot{z}_2 = 2az_1^2 z_2 + 2bz_2^3 + z_1^2 z_2. \end{cases}$$

We use CRDDP and choose $m = 3$. Then $b < 0$. For finding a particular set of solutions we assume:

$a < 0, \alpha = 1, 2|a| - 1 \geq 0$. By some algebraic computations, we finally obtain the following condition:

$$\begin{cases} |a| > 2|b| - 1 \\ |a| < 1.25|b| + 0.5. \end{cases} \tag{7.7}$$

Then the control becomes

$$\begin{cases} u = \sum_{i=1}^{n_1} a_i^1 x_i - a_1^1 \left(az_1^2 + bz_2^2\right) \\ v = \sum_{i=1}^{n_2} a_i^2 y_i - a_1^2 z_1^3 \end{cases}$$

where $a < 0$, $b < 0$ and they satisfy (7.7).

Case 3) Consider overall system. Since $d_1 = 4$, $d_2 = 3$ and $d_3 = 3$, trying Theorem 7.1, we set $L_1 = 5$ and $L_2 = L_3 = 3$. But it doesn't work because either x_1 or y_1 should be $0\left(\|z\|^3\right)$. Then $L_3 = 3$ means for z_3 the injection terms don't affect the approximate system of the dynamics on the center manifold. So, we have to raise L_3 to at least 5. That is, we have to assume

$$y_1 = \phi^{(3)}(z) + \phi^{(4)}(z).$$

Now a new problem occurs. Since the fourth degree terms, $\phi^{(4)}(z)$, are used, to keep it as a meaningful term in the real dynamics on the center manifold the order of the approximation error should be $e+1 \geq 5$. (It was mentioned in Section IV). Choose $e = 4$ then $L_2 = 3$ will violate the approximation error. So we are forced to choose, at least, $L_2 = 5$. Motivated by the previous results, we choose the designed center manifold as

$$\begin{cases} x_1(z) = az_1^2 + bz_2^4 + 0\left(\|z\|^5\right) \\ y_1(z) = cz_1^3 + \phi^{(4)}(z) + 0\left(\|z\|^5\right). \end{cases} \tag{7.8}$$

Then the approximate system of becomes

$$\begin{cases} \dot{z}_1 = acz_1^5 + 0\left(\|z\|^6\right) \\ \dot{z}_2 = 2z_2\left(az_1^2 + bz_1^4\right) + z_1^2 z_2 \\ \dot{z}_3 = cz_1^3 z_3 + z_3\phi^{(4)}(z) + a\mu_1 z_1^3 z_3 + 0\left(\|z\|^6\right). \end{cases}$$

To make it approximately stable we may choose $ac < 0$, $b < 0$, $a = -0.5$, $c = -a\mu_1$ and $\phi^{(4)} = -z_3^4$. Since $\mu_1 > 0$, then a feasible choice is: $a = -0.5$, $b = -1$, $c = 0.5\mu_1$. It follows that the following control, as a particular case of (4.3), stabilizes the system (7.5).

$$\begin{cases} u = \sum_{i=1}^{n_1} a_i^1 x_i + a_1^1 \left(-0.5z_1^2 - z_2^4\right) \\ v = \sum_{i=1}^{n_2} a_i^2 y_i + a_1^2 \left(-0.5\mu_1 z_1^3 - z_3^4\right). \end{cases}$$

□

VIII. Conclusion

The stabilization problem for affine nonlinear systems with nonminimum phase zero dynamics was considered in the paper. The major results of the paper are the followings: First, a new tool, the Lyapunov function with homogeneous derivative along solution curves was proposed. Based on this, three independent sufficient conditions (Cross Row Diagonal Dominating Principle, Diagonal Dominating Principal, Quadratic Form Reducing Algorithm) were developed to test the negative definiteness of the homogeneous polynomials. It was shown that this new tool is particularly suitable for testing the approximate stability of the dynamics with odd lowest nonvanishing terms.

Secondly, it was shown that under certain designed state feedback controls, the first variables of each integral chains of the linearized part of the system could be used as the "controls" of the dynamics on the center manifold of the closed-loop systems. This followed because the choice of the approximation functions for them does not affect the approximation accuracy of the dynamics on the center manifold.

In the light of the above two results, a systematic design technique was developed to provide a set of sufficient conditions for designing controls which stabilize the dynamics on the designed center manifold, and then stabilize the overall system.

Only the systems with zero center were discussed in this paper. However, the method can also be used for affine nonlinear systems with oscillatory center [26] or the case of a center with multiple zero eigenvalues [27].

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Clyde Martin, photograph and biography not available at the time of publication.