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RIESZ BASIS APPROACH TO THE STABILIZATION OF A FLEXIBLE BEAM WITH A TIP MASS*

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Abstract. Using an abstract condition of Riesz basis generation of discrete operators in the Hilbert spaces, we show, in this paper, that a sequence of generalized eigenfunctions of an Euler–Bernoulli beam equation with a tip mass under boundary linear feedback control forms a Riesz basis for the state Hilbert space. In the meanwhile, an asymptotic expression of eigenvalues and the exponential stability are readily obtained. The main results of [*SIAM J. Control Optim.*, 36 (1998), pp. 1962–1986] are concluded as a special case, and the additional conditions imposed there are removed.

Key words. beam equation, boundary control, stability, eigenvalues, Riesz basis

AMS subject classifications. 93C20, 93D15, 35B35, 35P10

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1. Introduction.

When a tip mass is attached to the free end, the vibration of a flexible beam that is clamped at one end and controlled at the free end can be described by the following Euler–Bernoulli beam equation (Conrad and Morgül, 1998):

$$(1)\qquad \begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = 0, \quad t \geq 0, \\ -y_{xxx}(1,t) + my_{tt}(1,t) = u(t), \quad t \geq 0, \end{cases}$$

where y is the amplitude of the vibration, m is the tip mass, and u is the boundary control force applied at the free end. In order to achieve uniform stability for this system, one has to employ "higher" derivative controllers. The following linear feedback control law is proposed in Conrad and Morgül (1998):

$$u(t) = -\alpha y_t(1,t) + \beta y_{xxxt}(1,t), \quad t \geq 0,$$

where α and β are real constants. The closed-loop system then becomes

$$(2)\qquad \begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = 0, \\ y_{xxx}(1,t) = \alpha y_t(1,t) + my_{tt}(1,t) - \beta y_{xxxt}(1,t). \end{cases}$$

The energy multiplier method is used in Conrad and Morgül (1998) to show that system (2) is exponentially stable for any $\alpha, \beta > 0$. It is further proved for a special case where $m = \alpha\beta$ that a set of generalized eigenfunctions of system (2) forms a Riesz basis for the state Hilbert space, usually referred to as the Riesz basis (generation) property, and that the spectrum-determined growth condition holds, both for almost

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every $\alpha > 0$. Systems with the Riesz basis property are usually referred to as Riesz spectral systems (Xu and Sallet, 1996).

Verification of the Riesz basis property is a very important problem both theoretically and practically. Usually, the property will lead to the establishment of such results as the spectrum-determined growth condition and the exponential stability of the system. However, such verification is usually difficult because the associated system operator is non-self-adjoint. For one-dimensional string equations with general variable coefficients under linear boundary feedback control, successful treatments have been made for the basis property in last two decades; we refer to Cox and Zuazua (1994), Shubov (1996, 1997), and the references therein. The case of a string equation with a tip mass was investigated in Morgül, Conrad, and Rao (1994). An abstract treatment of general Riesz spectral system with one rank perturbation can be found in Sun (1981), Rebarber (1989), and Xu and Sallet (1996), to name just a few. In Rao (1997) and, recently, Li et al. (1999), the beam equations with "low order" perturbations were considered. Since the model of serially connected Euler–Bernoulli beams under joint linear feedback control was proposed in Chen et al. (1987), many efforts have been made to study the asymptotic distribution of the eigenvalues (Chen et al., 1989) and the exponential stability (Rebarber, 1995). However, the spectrum-determined growth condition had not been reported until Conrad (1990), where a cantilevered beam equation was shown to have the Riesz basis property for small feedback gain, and hence the spectrum-determined growth condition is concluded for this special case. The general case for this cantilevered beam equation was resolved partly by Conrad and Morgül (1998).

In these works mentioned above, the verification of Riesz basis generation relies upon Bari's theorem (see, for example, Gohberg and Krein, 1969): if $\{\phi_n\}_1^\infty$ is a Riesz basis for a Hilbert space H, and $\{\psi_n\}_1^\infty$, an ω-linearly independent sequence in H, is quadratically close to $\{\phi_n\}_1^\infty$ in the sense that

$$\sum_{n=1}^{\infty} \| \phi_n - \psi_n \|^2 < \infty,$$

then $\{\psi_n\}_1^\infty$ is also a Riesz basis itself for H. In order to use Bari's theorem, the following steps are required:

(i) to estimate "high" eigenfrequencies by asymptotic analysis technique;

(ii) to find a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ (where N is a large integer) such that $\{\psi_n\}_{N+1}^\infty$ is quadratically close to a given Riesz basis $\{\phi_n\}_1^\infty$: $\sum_{n=N+1}^\infty \| \phi_n - \psi_n \|^2 < \infty$; and

(iii) to show that the number of linearly independent "low" eigenvectors is exactly N, or, more generally, as in Rao (1997) and Shubov (1996), to show that the root subspace of the system is complete in the state space.

While steps (i) and (ii) are relatively easy, step (iii) has been very difficult, in general, so far. Toward easing this difficulty, Guo (to appear) recently establishes an abstract condition under which steps (i) and (ii) automatically imply step (iii) for discrete operators in general Hilbert spaces. This greatly simplifies the verification of the Riesz basis property in applications. In this paper, we shall use this result (a simplified proof is presented in the appendix of the present paper) to show that a sequence of generalized eigenfunctions of system (2) forms a Riesz basis for the state Hilbert space for any real parameters $\alpha, \beta \neq 0$ and m. This covers the main results of Conrad and Morgül (1998) as a special case and removes the additional conditions imposed there. The exponential stability of the system is then readily established from an asymptotic expression of the eigenvalues, which is also obtained in the process of verification of the Riesz basis generation property.

The paper is organized as follows. In sections 2 and 3, some asymptotic expressions of eigenvalues and eigenfunctions are presented. Section 4 is devoted to the Riesz basis generation. Concluding remarks are given in section 5. Finally, in the appendix, we present a much simplified proof of the abstract result obtained in Guo (to appear) about the Riesz basis property of discrete operators in general Hilbert spaces.

2. Asymptotic expressions of eigenvalues and eigenfunctions. Throughout the paper, we always assume that $\beta \neq 0$. As in Conrad and Morgül (1998), the state Hilbert space for system (2) is $\mathbf{H} = H_E^2(0,1) \times L^2(0,1) \times \mathbb{C}$, where $H_E^2(0,1) = \{f \in H^2(0,1) \mid f(0) = f'(0) = 0\}$, with the inner product induced norm defined as

$$\| (f,g,\eta) \|^2 = \int_0^1 [| f''(x) |^2 + | g(x) |^2] dx + K | \eta |^2,$$

where $K > 0$ is any constant. Equation (2) can be written as an evolutionary equation in $\mathbf{H}$:

$$\frac{dY(t)}{dt} = \mathcal{A}Y(t), \tag{3}$$

where $Y(t) = (y(\cdot,t), y_t(\cdot,t), -y_{xxx}(1,t) + m\beta^{-1}y_t(1,t))$ and the operator $\mathcal{A} : D(\mathcal{A})(\subset \mathbf{H}) \to \mathbf{H}$ is defined as follows:

$$\begin{cases} \mathcal{A}(f,g,\eta) = (g, -f^{(4)}, -\beta^{-1}\eta - \beta^{-1}(\alpha - m\beta^{-1})g(1)) \quad \forall (f,g,\eta) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \{(f,g,\eta) \in (H^4 \cap H_E^2) \times H_E^2 \times \mathbb{C}, f''(1) = 0, \eta = -f'''(1) + m\beta^{-1}g(1)\}. \end{cases}$$

Now, we present the following lemma on the spectrum of the operator $\mathcal{A}$.

LEMMA 2.1. *$\mathcal{A}^{-1}$ exists and is compact on $\mathbf{H}$. Hence the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of isolated eigenvalues only: $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ denotes the set of eigenvalues of $\mathcal{A}$. Moreover, each $\lambda = i\tau^2 \in \sigma(\mathcal{A}), \lambda \neq -\beta^{-1}$, is geometrically simple and satisfies the characteristic equation*

$$\tau(i\tau^2 + \beta^{-1})(1 + \cosh\tau\cos\tau) + (m\tau^2 - \alpha i)(\sinh\tau\cos\tau - \cosh\tau\sin\tau) = 0. \tag{4}$$

An eigenfunction (f,g,η) corresponding to $\lambda = i\tau^2 \in \sigma(\mathcal{A}) (\lambda \neq -\beta^{-1})$ is given by

$$\begin{cases} f(x) &= \sinh\tau(1-x) - \sin\tau(1-x) - (\sinh\tau\cos\tau x + \sinh\tau x\cos\tau) \\ & \quad + (\cosh\tau x\sin\tau + \cosh\tau\sin\tau x), \\ g &= \lambda f, \\ \eta &= -\dfrac{\lambda\beta^{-1}}{\lambda + \beta^{-1}}(\alpha - m\beta^{-1})f(1). \end{cases} \tag{5}$$

Proof. A simple calculation shows that

$$\mathcal{A}^{-1}(f,g,\eta) = (f_1, g_1, \eta_1) \quad \forall (f,g,\eta) \in \mathbf{H},$$

$$f_1 = \int_x^1 \frac{(x-\tau)^3}{6} g(\tau)d\tau + \int_0^1 \left(\frac{\tau^3}{6} - x\frac{\tau^2}{2}\right) g(\tau)d\tau + \frac{(x-1)^3 - 3x + 1}{6}[\beta\eta + \alpha f(1)],$$

$$g_1 = f, \eta_1 = -\beta\eta - (\alpha - m\beta^{-1})f(1).$$

Since $f_1^{(4)} = -g, g_1 = f, |\eta_1| \le |\beta||\eta| + |\alpha - m\beta^{-1}|\|f\|_{H^2}$, it follows that

$$\|\mathcal{A}^{-1}(f, g, \eta)\|_{H^4 \times H^2 \times \mathbb{C}} \le M\|(f, g, \eta)\|_{\mathbf{H}}$$

for some constant $M > 0$. By the Sobolev embedding theorem, $\mathcal{A}^{-1}$ is compact on $\mathbf{H}$. This is the first part.

Second, for any $\lambda \in \sigma_p(\mathcal{A}), \lambda \ne -\beta^{-1}$, solving eigenvalue problem

$$\mathcal{A}(f, g, \eta) = (g, -f^{(4)}, -\beta^{-1}\eta - \beta^{-1}(\alpha - m\beta^{-1})g(1)) = \lambda(f, g, \eta),$$

one gets

$$g = \lambda f, \eta = -\frac{\lambda\beta^{-1}}{\lambda + \beta^{-1}}(\alpha - m\beta^{-1})f(1),$$

where f satisfies

$$\left\{ \begin{array}{l} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ f(0) = f'(0) = f''(1) = 0, \\ (\lambda + \beta^{-1})f'''(1) = \beta^{-1}\lambda(\alpha + m\lambda)f(1). \end{array} \right. \tag{6}$$

If (6) has two linearly independent solutions f_1, f_2, then there are constants c, d ($|c| + |d| \ne 0$) such that $cf_1(1) + df_2(1) = 0$. It follows from (6) that $f = cf_1 + df_2$ satisfies

$$\left\{ \begin{array}{l} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ f(0) = f'(0) = f(1) = f''(1) = f'''(1) = 0. \end{array} \right.$$

A simple calculation shows that the above equation has only zero solution. Hence $cf_1 + df_2 \equiv 0$. This contradicts the assumption that f_1, f_2 are linearly independent. Therefore, each $\lambda = i\tau^2 \in \sigma(\mathcal{A}), \lambda \ne -\beta^{-1}$, is geometrically simple.

Now, let $\lambda = i\tau^2$. By the first equation in (6) and the conditions $f(0) = f'(0) = 0$, we have

$$f(x) = c_1(\cosh\tau x - \cos\tau x) + c_2(\sinh\tau x - \sin\tau x),$$

where c_1 and c_2 are constants. Since $f''(1) = 0$, we can set

$$c_1 = \sinh\tau + \sin\tau, c_2 = -\cosh\tau - \cos\tau.$$

Obviously c_1, c_2 can not be zero simultaneously. Hence

$$\begin{aligned} f(x) = {} & \sinh\tau(1 - x) - \sin\tau(1 - x) - (\sinh\tau\cos\tau x + \sinh\tau x\cos\tau) \\ & + (\cosh\tau x\sin\tau + \cosh\tau\sin\tau x), \end{aligned}$$

which satisfies

$$\left\{ \begin{array}{l} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ f(0) = f'(0) = f''(1) = 0 \end{array} \right. \tag{7}$$

for all $\lambda = i\tau^2$.

Finally, from the last condition $(\lambda + \beta^{-1})f'''(1) = \beta^{-1}\lambda(\alpha + m\lambda)f(1)$, one can obtain (4) (see also Conrad and Morgül, 1998). The proof is complete. □

LEMMA 2.2. *There is a family of eigenvalues* $\{\lambda_n, \overline{\lambda}_n\}, \lambda_n = i\tau_n^2$ *of* $\mathcal{A}$ *satisfying*

$$\lambda = \lambda_n = i\tau_n^2 = -2m + i(k\pi)^2 + \mathcal{O}(n^{-1}), \tag{8}$$

where $k = n - 1/2$ *and* n *is a sufficiently large positive integer. An eigenfunction* (f_n, g_n, η_n) *of* $\mathcal{A}$ *corresponding to* λ_n *satisfies*

$$\begin{aligned} F_n(x) &= 2\tau_n^{-2}e^{-\tau_n}\begin{pmatrix} f_n''(x) \\ g_n(x) \\ \eta_n \end{pmatrix}^T \\ &= \begin{pmatrix} e^{-k\pi x} + (-1)^n e^{-k\pi(1-x)} + (\cos k\pi x - \sin k\pi x) \\ i[e^{-k\pi x} + (-1)^n e^{-k\pi(1-x)} - (\cos k\pi x - \sin k\pi x)] \\ 0 \end{pmatrix}^T + \mathcal{O}(n^{-1}), \end{aligned} \tag{9}$$

which holds uniformly for $x \in [0,1]$. *Consequently,*

$$\| F_n(x) \|^2_{L^2\times L^2\times\mathbb{C}} = \| 2\tau_n^{-2}e^{-\tau_n}(f_n, g_n, \eta_n) \|^2_{\mathbf{H}} \to 2 \text{ as } n \to \infty. \tag{10}$$

Proof. Let $k = n - 1/2$, with n as a sufficiently large positive integer. Noting that as $n \to \infty, \|e^{-k\pi x}\|^2_{L^2} \to 0, \|e^{-k\pi(1-x)}\|^2_{L^2} \to 0, \|\cos k\pi x - \sin k\pi x\|^2_{L^2} \to 1$, we can conclude (10) from (9) immediately. So only (9) should be verified. First, in a small neighborhood of $k\pi$, the following estimates are valid uniformly for all $n > 0$:

$$2e^{-\tau}\sinh\tau = 1 + \mathcal{O}(e^{-2|\tau|}), 2e^{-\tau}\cosh\tau = 1 + \mathcal{O}(e^{-2|\tau|}), \sin\tau = \mathcal{O}(1), \cos\tau = \mathcal{O}(1).$$

Second, multiplying $-2i\tau^{-3}e^{-\tau}$ on both sides of (4) yields

$$\cos\tau = \mathcal{O}(|\ \tau\ |^{-1}) \text{ or } \cos\tau = \frac{m}{\tau}i(\cos\tau - \sin\tau) + \mathcal{O}(|\ \tau\ |^{-2}), \tag{11}$$

which is valid uniformly in a small neighborhood of $k\pi$ for all $n > 0$. Since $\cos k\pi = 0$, we can apply Rouche's theorem to the functions $f(\tau) = \cos\tau$ and $g(\tau) = -\mathcal{O}(|\ \tau\ |^{-1})$ in a small neighborhood of $k\pi$ to find a solution to the first equation of (11) to be

$$\tau = \tau_n = k\pi + \mathcal{O}(n^{-1}). \tag{12}$$

Note that

$$\begin{cases} e^{-\tau_n y} = e^{-k\pi y} + \mathcal{O}(n^{-1}), \\ \sin\tau_n x = \sin k\pi x + \mathcal{O}(n^{-1}), \cos\tau_n x = \cos k\pi x + \mathcal{O}(n^{-1}), \end{cases} \tag{13}$$

which holds uniformly for bounded $y > 0$ and $x \in [0,1]$. Upon substituting (12) into the second equation of (11), the term $\mathcal{O}(n^{-1})$ in the expression (12) satisfies

$$-\sin k\pi\mathcal{O}(n^{-1}) = -\frac{mi}{k\pi}\sin k\pi + \mathcal{O}(n^{-2}),$$

so

$$\mathcal{O}(n^{-1}) = \frac{mi}{k\pi} + \mathcal{O}(n^{-2}).$$

This, together with (12), gives

$$\tau_n = k\pi + \frac{mi}{k\pi} + \mathcal{O}(n^{-2}).$$

Then (8) readily follows.

Now let $\tau = \tau_n$ and $(f_n, g_n, \eta_n) = (f, g, \eta)$ be defined by (5). Since

$$\tau^{-2} f_n''(x) = \sinh \tau(1-x) + \sin \tau(1-x) + (\sinh \tau \cos \tau x - \sinh \tau x \cos \tau) \\ +(\cosh \tau x \sin \tau - \cosh \tau \sin \tau x),$$

it follows from (13) that

$$\begin{aligned} 2\tau^{-2} e^{-\tau} f_n''(x) &= e^{-\tau x} + \cos \tau x - e^{-\tau(1-x)} \cos \tau + e^{-\tau(1-x)} \sin \tau - \sin \tau x + \mathcal{O}(e^{-k\pi}) \\ &= e^{-k\pi x} + e^{-k\pi(1-x)} \sin k\pi + \cos k\pi x - \sin k\pi x + \mathcal{O}(n^{-1}) \\ &= e^{-k\pi x} + (-1)^n e^{-k\pi(1-x)} + (\cos k\pi x - \sin k\pi x) + \mathcal{O}(n^{-1}), \\ 2\tau^{-2} e^{-\tau} g_n(x) &= i e^{-\tau} f_n(x) = i[e^{-\tau x} - \cos \tau x - e^{-\tau(1-x)} \cos \tau + e^{-\tau(1-x)} \sin \tau \\ &\quad + \sin \tau x] + \mathcal{O}(e^{-k\pi}) \\ &= i[e^{-k\pi x} + (-1)^n e^{-k\pi(1-x)} - (\cos k\pi x - \sin k\pi x)] + \mathcal{O}(n^{-1}), \\ 2\tau^{-2} e^{-\tau} \eta_n &= -2\tau^{-2} e^{-\tau} \tfrac{\lambda \beta^{-1}}{\lambda + \beta^{-1}} (\alpha - m\beta^{-1}) f_n(1) = \mathcal{O}(n^{-2}). \end{aligned}$$

The above estimates are valid uniformly for $x \in [0, 1]$. (9) is established. ∎

It should be pointed out that we are not sure at this stage that (8) is an asymptotic expression for all eigenvalues of $\mathcal{A}$. This will be cleared up after the verification of the Riesz basis generation in section 4.

3. Results of an auxiliary system. In this section, we consider an auxiliary system which is composed of a conservative system and an ordinary differential equation coupled. This system will produce a reference Riesz basis of $\mathbf{H}$, required in Theorem 6.3 in the appendix in verification of basis generation. The principle of constructing this system is based on an observation of the characteristic equation (4) that the "dominant" equation of (4) is $i\tau^3(1+\cosh \tau \cos \tau) = 0$, which can be obtained by letting $\alpha = m = \beta^{-1} = 0$. In this case, system (2) becomes

$$\begin{cases} y_{tt} + y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = y_{xxxt}(1,t) = 0. \end{cases}$$

Naturally, we consider the well-posed conservative system

$$\begin{cases} y_{tt} + y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = y_{xxx}(1,t) = 0, \end{cases}$$

which has the same nonzero eigenvalues as that of the system above. In order to get a state space the same as that of the system (2), i.e., $\mathbf{H}$, we complete the conservative system with another ordinary differential equation; then the auxiliary system is obtained, which is described by the following equation:

$$\begin{cases} y_{tt} + y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = y_{xxx}(1,t) = 0, \\ \dot{\eta}(t) = 0. \end{cases}$$

Alternatively, we can describe the auxiliary system in the form of an evolutionary equation in $\mathbf{H}$,

$$\frac{dY(t)}{dt} = \mathcal{A}_0 Y(t), \tag{14}$$

where the operator $\mathcal{A}_0 : D(\mathcal{A}_0)(\subset \mathbf{H}) \to \mathbf{H}$ is defined as follows:

$$\begin{cases} \mathcal{A}_0(f,g,\eta) = (g, -f^{(4)}, 0) \quad \forall (f,g,\eta) \in D(\mathcal{A}_0), \\ D(\mathcal{A}_0) = \{(f,g,\eta) \in (H^4 \cap H_E^2) \times H_E^2 \times \mathbb{C}, f''(1) = f'''(1) = 0\}. \end{cases}$$

It is easy to show that $1 \in \rho(\mathcal{A}_0), (I - \mathcal{A}_0)^{-1}$ is compact on $\mathcal{H}$ and $\mathcal{A}_0^* = -\mathcal{A}_0$. That is, $\mathcal{A}_0$ is a skew-adjoint operator with compact resolvent on $\mathbf{H}$ (and hence $i\mathcal{A}_0$ is self-adjoint with compact resolvent). It then follows from a well-known result in functional analysis that (i) there is a sequence of normalized eigenfunctions of $\mathcal{A}_0$ which forms an orthonormal basis of $\mathbf{H}$; (ii) for each eigenvalue of $\mathcal{A}_0$, its geometric multiplicity equals its algebraic multiplicity; and (iii) all eigenvalues of $\mathcal{A}_0$ lie on the imaginary axis. (ii) is actually a consequence of (i). (iii) comes directly from the skew-adjointness of $\mathcal{A}_0$. These are advantages of the construction of $\mathcal{A}_0$.

All the analysis of the operator $\mathcal{A}$ in the preceding section is true for the operator $\mathcal{A}_0$. In particular, each $\mu \in \sigma(\mathcal{A}_0), \mu \neq 0$, is geometrically simple and hence algebraically simple. And the characteristic equation for $\mu = i\omega^2 (\neq 0) \in \sigma(\mathcal{A}_0)$ is

$$1 + \cosh\omega \cos\omega = 0. \tag{15}$$

Since all eigenvalues of $\mathcal{A}_0$ lie on the imaginary axis, we need consider only the positive solutions to (15) in order to find all nonzero eigenvalues of $\mathcal{A}_0$.

For $\omega > 0$, writing (15) as $\cos\omega = \mathcal{O}(e^{-\omega})$, we can get the positive solutions of (15) being

$$\omega = \omega_n = k\pi + \mathcal{O}(e^{-k\pi}), \tag{16}$$

where $k = n - 1/2$ for all sufficiently large positive integers n.

Therefore, the spectrum of $\mathcal{A}_0$ consists of all pairs $\{\mu_n, \overline{\mu}_n\}$ together with possibly another finite set, where $\mu_n = i\omega_n^2$ with ω_n given in (16). This is unlike $\mathcal{A}$; $\mu_n = i\omega_n^2 = i(k\pi)^2 + \mathcal{O}(k\pi e^{-k\pi})$ is now indeed an asymptotic expression for all eigenvalues of $\mathcal{A}_0$.

Now, letting $\alpha = m = \beta^{-1} = 0$ and $\tau_n = \omega_n$, in Lemma 2.2, we get an eigenvector (u_n, v_n, ν_n) of $\mathcal{A}_0$ corresponding to $\mu_n = i\omega_n^2 (\neq 0)$ given below:

$$\begin{cases} u_n(x) &= \sinh\omega_n(1-x) - \sin\omega_n(1-x) - (\sinh\omega_n \cos\omega_n x + \sinh\omega_n x \cos\omega_n) \\ &\quad +(\cosh\omega_n x \sin\omega_n + \cosh\omega_n \sin\omega_n x), \\ v_n &= \mu_n u_n, \\ \nu_n &= 0. \end{cases} \tag{17}$$

Clearly, the asymptotic expression (12) is also valid for ω_n defined in (16). Noting that only the expression (12) is used in the proof of Lemma 2.2, we have the following counterpart of Lemma 2.2 for $\mathcal{A}_0$. Similar results were also obtained in Lancaster and Shkalikov (1994).

LEMMA 3.1. *The spectrum of $\mathcal{A}_0$ consists of all $\{\mu_n, \overline{\mu}_n\}$ but possibly a finite number of the other eigenvalues, where $\mu_n = i\omega_n^2, \omega_n$ is determined by* (16). *And the eigenvalues μ_n ($\overline{\mu}_n$) are algebraically simple for all large n. In addition, an eigenfunction (u_n, v_n, ν_n) of $\mathcal{A}_0$ corresponding to μ_n satisfies*

$$\begin{aligned} G_n(x) &= 2\omega_n^{-2} e^{-\omega_n} \begin{pmatrix} u_n''(x) \\ v_n(x) \\ \nu_n \end{pmatrix}^T \\ &= \begin{pmatrix} e^{-k\pi x} + (-1)^n e^{-k\pi(1-x)} + (\cos k\pi x - \sin k\pi x) \\ i[e^{-k\pi x} + (-1)^n e^{-k\pi(1-x)} - (\cos k\pi x - \sin k\pi x)] \\ 0 \end{pmatrix}^T + \mathcal{O}(n^{-1}), \end{aligned} \tag{18}$$

which holds uniformly for all $x \in [0,1]$.

4. Riesz basis generation. In this section, we shall apply Theorem 6.3 in the appendix to get the basis property of $\mathcal{A}$. To do this, we need a reference Riesz basis of $\mathbf{H}$ first. This is accomplished by collecting the eigenfunctions of $\mathcal{A}_0$. As we can conclude from Lemma 3.1 that a "maximal" set (see appendix) of ω-linearly independent eigenfunctions of $\mathcal{A}_0$ consists of all (u_n, v_n, ν_n) defined by (17) but a finite number of the other eigenfunctions, we may assume, without loss of generality, that such a set is

$$\{2\omega_n^{-2}e^{-\omega_n}(u_n, v_n, \nu_n)\}_1^\infty \cup \{\text{their conjugates}\}.$$

Since $\mathcal{A}_0$ is skew-adjoint, the set $\{2\omega_n^{-2}e^{-\omega_n}(u_n, v_n, \nu_n)\}_1^\infty \cup \{\text{their conjugates}\}$ forms an orthogonal basis of $\mathbf{H}$. Because they are approximately normalized (that is, they are upper and lower bounded) according to (10), the set is indeed a Riesz basis of $\mathbf{H}$ by a well-known fact that all approximately normalized Riesz bases in a separate Hilbert space are equivalent.

From (9) and (18), it follows that there is a large positive integer N such that

$$\begin{aligned}(19)\qquad &\sum_{n>N}^{\infty} \| 2\tau_n^{-2}e^{-\tau_n}(f_n, g_n, \eta_n) - 2\omega_n^{-2}e^{-\omega_n}(u_n, v_n, \nu_n) \|_{\mathbf{H}}^2 \\ &= \sum_{n>N}^{\infty} \| F_n - G_n \|_{L^2\times L^2\times\mathbb{C}}^2 = \sum_{n>N}^{\infty} \mathcal{O}(n^{-2}) < \infty.\end{aligned}$$

The same is true for their conjugates. Note that all $\lambda_n = i\tau_n^2$ are different for large n; we can now apply Theorem 6.3 in the appendix to obtain the main results of the present paper.

THEOREM 4.1. *Let the operator* $\mathcal{A}$ *be defined as in* (3). *Then*

(i) *there is a sequence of generalized eigenfunctions of* $\mathcal{A}$ *which forms a Riesz basis for the state space* $\mathbf{H}$*;*

(ii) *all of the eigenvalues of* $\mathcal{A}$ *have the asymptotic expression* (8)*; and*

(iii) *all* $\lambda \in \sigma(\mathcal{A})$ *with sufficiently large modulus are algebraically simple.*

Therefore, $\mathcal{A}$ *generates a* C_0*-group on* $\mathbf{H}$ *for any real constants* $m, \alpha,$ *and* β. *Moreover, for the semigroup* $e^{\mathcal{A}t}$ *generated by* $\mathcal{A}$*, the spectrum-determined growth condition holds. And the growth rate of* $e^{\mathcal{A}t}$ *is not less than* $-2m$.

The stability result for the system (2) is given in the following corollary.

COROLLARY 4.2. *The semigroup* $e^{\mathcal{A}t}$ *is exponentially stable for any* $m, \alpha, \beta > 0$.

Proof. Taking the inner product of $\mathbf{H}$ as in the beginning of the section 2 with $K = \beta^2/(m + \alpha\beta)$, it is calculated in Conrad and Morgül (1998) that

$$\mathrm{Re}\langle \mathcal{A}Y, Y\rangle = -\frac{K}{\beta} \mid f'''(1) \mid^2 - \frac{Km\alpha}{\beta^2} \mid g(1) \mid^2 \leq 0 \quad \forall\ Y = (f, g, \eta) \in D(\mathcal{A}).$$

That is, $\mathcal{A}$ is dissipative and hence no eigenvalues of $\mathcal{A}$ lie on the open right half complex plane. Now, if $\mathcal{A}Y = \lambda Y$, $Y = (f, g, \eta)$, and $\mathrm{Re}\lambda = 0$, then $f'''(1) = g(1) = 0$. It follows from (6) that

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ f(0) = f'(0) = f''(1) = 0, \\ f'''(1) = f(1) = 0. \end{cases}$$

As it is indicated in the proof of Lemma 2.1, the above equation has a zero solution only. Hence $f \equiv 0$ and so $g = \eta = 0$ by (5). Therefore,

$$\text{Re}\lambda < 0 \quad \forall \lambda \in \sigma(\mathcal{A}). \tag{20}$$

Finally, since $\mathcal{A}$ is of compact resolvent, there are only finitely many eigenvalues of $\mathcal{A}$ in any bounded region of the complex plane, which, together with Theorem 4.1 (ii), shows that there is a constant $\omega > 0$ such that

$$S(\mathcal{A}) = \sup_{\lambda \in \sigma(\mathcal{A})} \text{Re}\lambda < -\omega. \tag{21}$$

The exponential stability then follows from the spectrum-determined growth condition. The proof is complete. □

Before ending the section, we indicate that Theorem 4.1 can be used to obtain the basis property and the spectrum-determined growth condition of a beam equation without tip mass under linear boundary feedback control. Let $\mathcal{A}_1$ be defined by setting $m = \theta\beta$, where θ is real, in the definition of the operator $\mathcal{A}$; that is,

$$\begin{cases} \mathcal{A}_1(f, g, \eta) = (g, -f^{(4)}, -\beta^{-1}\eta - \beta^{-1}(\alpha - \beta^{-1})g(1)) \quad \forall (f, g, \eta) \in D(\mathcal{A}_1), \\ D(\mathcal{A}_1) = \{(f, g, \eta) \in (H^4 \cap H_E^2) \times H_E^2 \times \mathbb{C}, f''(1) = 0, \eta = -f'''(1) + \theta g(1)\}. \end{cases}$$

Then Theorem 4.1 holds true for operator $\mathcal{A}_1$ for any reals β^{-1}, α, and θ. Furthermore, let $\mathcal{A}_2$ be defined by setting $\alpha = \beta^{-1} = 0$ in the definition of $\mathcal{A}_1$. We have

$$\begin{cases} \mathcal{A}_2(f, g, \eta) = (g, -f^{(4)}, 0) \quad \forall (f, g, \eta) \in D(\mathcal{A}_2), \\ D(\mathcal{A}_2) = \{(f, g, \eta) \in (H^4 \cap H_E^2) \times H_E^2 \times \mathbb{C}, f''(1) = 0, \eta = -f'''(1) + \theta g(1)\}. \end{cases}$$

And Theorem 4.1 is also true for operator $\mathcal{A}_2$ for any real θ. However,

$$\frac{dY}{dt} = \mathcal{A}_2 Y(t)$$

is equivalent to

$$\begin{cases} y_{tt}(x, t) + y_{xxxx}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \\ y(0, t) = y_x(0, t) = y_{xx}(1, t) = 0, \quad t \geq 0, \\ y_{xxx}(1, t) = \theta y_t(1, t), \quad t \geq 0. \end{cases} \tag{22}$$

That is, system (22) is also a Riesz spectral system. This system is just the cantilevered beam equation considered in Conrad (1990), Conrad and Morgül (1998), and Guo (to appear).

Remark 4.3. The conclusion of Corollary 4.2 was proved in Conrad and Morgül (1998) by energy multiplier method. Theorem 4.1 (i) was shown there in the case of $m = \alpha\beta$ for almost every $\alpha > 0$ with other additional conditions. Theorem 4.1 (iii) was also shown there by complex analysis. The Riesz basis property for (22) was obtained there for almost every $\theta > 0$.

5. Concluding remarks. In this paper, an abstract condition for Riesz basis generation of discrete operators in Hilbert spaces is used to show that a sequence of generalized eigenfunctions of an Euler–Bernoulli beam equation with a tip mass under boundary linear feedback control forms a Riesz basis for the state Hilbert space. The stability of the system is also established. This paper greatly improves the work of Conrad and Morgül (1998), where the same results are obtained but for a very special case where $m = \alpha\beta$. Besides these results, the contributions of this paper lie in providing a very simple method which enables us (a) to obtain the asymptotic expressions of eigenvalues and eigenfunctions; (b) to avoid the usual treatment for "low" eigenfrequencies in applying Bari's theorem; and (c) to study potential applications to other problems of beam equations (see Guo and Chan, 2001).

6. Appendix. Abstract result on Riesz basis property. In this appendix, we present the abstract result together with a simplified proof about Riesz basis generation for discrete operators in the Hilbert spaces. This result is crucial to the establishment of the main results of the present paper.

Let us recall that for a closed linear operator A in a Hilbert space H, a nonzero $x \in H$ is called a generalized eigenvector of A, corresponding to an eigenvalue λ (with finite algebraic multiplicity) of A, if there is a positive integer n such that $(\lambda - A)^n x = 0$. Let $\overline{sp}(A)$, the so-called root subspace of A, be the closed subspace spanned by all generalized eigenvectors of A. The following theorem gives a simple characterization of the completeness of $\overline{sp}(A)$; that is, $\overline{sp}(A) = H$.

LEMMA 6.1. *Let A be a densely defined discrete operator (that is, there is a $\lambda \in \rho(A)$ such that $R(\lambda, A) = (\lambda - A)^{-1}$ is compact) in a Hilbert space H. Then $\overline{sp}(A) = H$ if and only if the codimension of $\overline{sp}(A)$ in H is finite.*

Proof. It is well known that the adjoint operator A^* of a densely defined discrete operator A is also a discrete operator. It follows from Lemma 5 on p. 2355 of Dunford and Schwartz (1971) that the following orthogonal decomposition holds:

$$H = \sigma_\infty(A^*) \oplus \overline{sp}(A),$$

where $\sigma_\infty(A^*) = \{x | E(\lambda)x = 0,\ \forall\lambda \in \sigma(A^*)\}, E(\lambda)$ is the eigen-projector of A^* corresponding to λ. Hence $\overline{sp}(A) = H$ if and only if $\sigma_\infty(A^*) = \{0\}$. On the other hand, Lemma 5 on p. 2295 of Dunford and Schwartz (1971) suggests that $\sigma_\infty(A^*)$ is either $\{0\}$ or infinite dimensional. Therefore the codimension of $\overline{sp}(A)$ is finite if and only if $\sigma_\infty(A^*) = \{0\}$. The proof is complete. □

LEMMA 6.2. *Let $\{\phi_n\}_1^\infty$ be a Riesz basis in a Hilbert space H. Let $\{\psi_n\}_{N+1}^\infty$ ($N \geq 0$) be another sequence in H. If*

$$\sum_{n=N+1}^{\infty} \| \phi_n - \psi_n \|^2 < \infty,$$

then there exists an $M \geq N$ such that $\{\phi_n\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ is a Riesz basis of H. In particular, $\{\psi_n\}_{M+1}^\infty$ is ω-linearly independent.

Proof. The proof can follow from Corollary 11.4 on page 374 of Singer (1970).

THEOREM 6.3. *Let A be a densely defined discrete operator in a Hilbert space H. Let $\{\phi_n\}_1^\infty$ be a Riesz basis of H. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of A such that*

$$\sum_{N+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then the following hold.

(i) *There are a constant $M > N$ and generalized eigenvectors $\{\psi_{n0}\}_1^M$ of $\mathcal{A}$ such that $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis of H.*

(ii) *Let $\{\psi_{n0}\}_1^M \cup \{\psi_n\}_{M+1}^\infty$ correspond to eigenvalues $\{\sigma_n\}_1^\infty$ of A. Then $\sigma(A) = \{\sigma_n\}_1^\infty$, where σ_n is counted according to its algebraic multiplicity.*

(iii) *If there is an* $M_0 > 0$ *such that* $\sigma_n \neq \sigma_m$ *for all* $m, n > M_0$*, then there is an* $N_0 > M_0$ *such that all* σ_n *are algebraically simple if* $n > N_0$. □

Proof. (ii) and (iii) are consequences of (i). Only proof of (i) is needed. By Lemma 6.2, there is an $M > N$ such that $\{\psi_n\}_{M+1}^{\infty}$ is ω-linearly independent. Let $\{\psi_\alpha\}$ be an arbitrary set such that $\{\psi_n\}_{M+1}^{\infty} \cup \{\psi_\alpha\}$ is a "maximal" ω-linearly independent set of generalized eigenvectors of A; that is, $\{\psi_n\}_{M+1}^{\infty} \cup \{\psi_\alpha\}$ is a ω-linearly independent subset of the set of the generalized eigenvectors of A, and for any generalized eigenvector ψ of A, the extended set $\{\psi_n\}_{M+1}^{\infty} \cup \{\psi_\alpha\} \cup \{\psi\}$ must not be ω-linearly independent anymore. Therefore, $\{\psi_n\}_{M+1}^{\infty} \cup \{\psi_\alpha\}$ spans the root subspace $\overline{sp}(A)$. By the assumption and Bari's theorem, the number of such ψ_α's does not exceed M. Let $\{\psi_\alpha\} = \{\psi_{n0}\}_1^L, L \leq M$. It follows from Theorem 3.2 of Rao (1997) that $\{\psi_{n0}\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ forms a Riesz basis of $\overline{sp}(A)$.

On the other hand, by the assumption and Bari's theorem, the number of linearly independent elements in the orthogonal complement of $\overline{sp}(A)$ in H cannot exceed M, and hence the codimension of $\overline{sp}(A)$ is finite. Then from Lemma 6.1, $\overline{sp}(A) = H$.

Therefore, $\{\psi_{n0}\}_1^L \cup \{\psi_n\}_{M+1}^{\infty}$ forms a Riesz basis for the entire space H.

Since a "proper" subset of a basis can not be a basis, it follows from Bari's theorem and the assumption that $L = M$. This is (i). The proof is complete. □

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