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# Regularity of a Schrödinger equation with Dirichlet control and colocated observation☆

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**Abstract**

This paper studies the basic properties of the Schrödinger equation defined on a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$ with partial Dirichlet control and colocated observation. It is shown that the system is not only wellposed in the sense of D. Salamon but also regular in the sense of G. Weiss. It is also shown that the corresponding feedthrough operator is zero.


*Keywords:* Schrödinger equation; Transfer function; Well-posed and regular system

## 1. Introduction and main result

In the last 15 years, extensive studies have been devoted to a wide class of linear infinite-dimensional systems called *well-posed and regular linear systems* (see the surveys [7,24]). Not only does this general framework cover many partial differential equations with actuators and sensors supported on isolated points, on sub-domains, or on a part of the boundary of the spatial regions. More importantly, the studies have shown that this class of infinite-dimensional systems possesses many properties that are parallel in many ways to those of finite-dimensional (see for instance [8,22]). The concept of "*regularity*" while very useful in this framework, rarely appears in the literature on control of partial differential equation systems [6]. In [2], the well-posedness of a wave equation with Dirichlet input and colocated output on the 2-D disk was proved by the direct method. The well-posedness of the same equation on a bounded open domain of $\mathbb{R}^n$, $n \geqslant 2$ with smooth boundary was proved in [1] by the microlocal analysis. The well-posedness and regularity of a multi-dimensional heat equation with both Dirichlet

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and Neumann-type boundary controls is derived in [5]. Other examples on the well-posedness and regularity of some multi-dimensional partial differential equation systems can be found occasionally in [16,25,3,4,23]. The regularity of a multi-dimensional wave equation with Dirichlet control and colocated observation was recently proved in [11].

The objective of this paper is to prove the regularity of a multi-dimensional Schrödinger equation with partial Dirichlet control and colocated observation described by

$$\begin{aligned} &w_t(x,t)+\mathrm{i}\Delta w(x,t)=0, \quad x\in\Omega,\ t>0,\\ &w(x,t)=0, \quad x\in\Gamma_1,\ t\geqslant 0,\\ &w(x,t)=u(x,t), \quad x\in\Gamma_0,\ t\geqslant 0,\\ &y(x,t)=\mathrm{i}\frac{\partial(\Delta^{-1}w)}{\partial\nu}, \quad x\in\Gamma_0,\ t\geqslant 0, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$, $n\geqslant 2$ is an open bounded region with smooth $C^3$-boundary $\partial\Omega=\overline{\Gamma_0}\cup\overline{\Gamma_1}$. $\Gamma_0,\Gamma_1$ are disjoint parts of the boundary relatively open in $\partial\Omega$ and $\mathrm{int}(\Gamma_0)\neq\emptyset$. $\nu$ is the unit normal vector of $\Gamma_0$ pointing towards the exterior of $\Omega$. $u$ is the input function (or control) and $y$ is the output function (or output).

Let $H=H^{-1}(\Omega)$ be the state space and $U=L^2(\Gamma_0)$ be the control (input) or observation (output) space. The following well-posedness (see the definition for instance in [15]) result follows from Proposition 4.2 of [13]. A slightly different proof (without using "lifting theorem") based on Proposition 4.2 of [13] is presented as an Appendix at the end of this article.

**Theorem 1.1.** *System* (1.1) *is well-posed. More precisely, let $T>0$ be any constant and $C_T$ be some positive constant depending only on T. Let $w(\cdot,0)=w_0\in H$ be any initial state. Then for any control input $u\in L^2(0,T;U)$, there exists a unique solution to Eq.* (1.1) *such that $w\in C(0,T;H)$ and*

$$\|w(\cdot,T)\|_H^2+\|y\|_{L^2(0,T;U)}^2 \leqslant C_T\left[\|w_0\|_H^2+\|u\|_{L^2(0,T;U)}^2\right]. \tag{1.2}$$

The well-posedness claimed by Theorem 1.1 is a very important property, which implies that the exact controllability of system (1.1) in some finite time interval is equivalent to the exponential stability of the closed-loop system under the output feedback control $u=-ky$ with $k>0$ (see [10]). The main result of this paper is that system (1.1) is also regular.

**Theorem 1.2.** *System* (1.1) *is regular, with feedthrough operator zero. More precisely, if the initial state $w(\cdot,0)=0$ and $u(\cdot,t)=u(\cdot)\in U$ is a step control input, then the corresponding output satisfies*

$$\lim_{\sigma\to 0}\int_{\Gamma_0}\left|\frac{1}{\sigma}\int_0^\sigma y(x,t)\,\mathrm{d}t\right|^2\mathrm{d}x=0. \tag{1.3}$$

Theorems 1.1 and 1.2 ensure that system (1.1) is a well-posed regular linear system in the sense of [14,19]. The definition in [15] is not the standard one given by [19,7] but it is equivalent to Weiss's definition. We recall that well-posed and regular infinite-dimensional systems may have unbounded input and output operators, they resemble linear finite-dimensional systems in many ways (see e.g. [8]).

In Section 2, we formulate system (1.1) into a colocated abstract setting. The proof of Theorem 1.2 will be presented in Section 3. The proof of Theorem 1.1, which follows from Proposition 4.2 of [13], is given in the Appendix.

## 2. Colocated formulation of system (1.1)

Many papers have been published in the last 15 years on infinite-dimensional well-posed regular systems. For this material, we refer to [7,15] and [17–21].

It is well known that $H=H^{-1}(\Omega)$ is the dual space of the Sobolev space $H_0^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$. Let $A$ be the positive self-adjoint operator in $H$ produced from the bilinear form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$ defined by

$$\langle Af,g\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}=a(f,g)=\int_\Omega\nabla f(x)\overline{\nabla g(x)}\,\mathrm{d}x,\quad \forall f,\ g\in H_0^1(\Omega). \tag{2.1}$$

According to the Lax–Milgram theorem, $A$ is an canonical isomorphism from $D(A)=H_0^1(\Omega)$ onto $H$. Considering the Laplacian $-\Delta: H^2(\Omega)\cap H_0^1(\Omega)\to L^2(\Omega)$, it is easy to show that $Af=-\Delta f$ whenever $f\in H^2(\Omega)\cap H_0^1(\Omega)$ and that $A^{-1}g=(-\Delta)^{-1}g$ for any $g\in L^2(\Omega)$. Hence $A$ is an extension of the usual Laplacian in $L^2(\Omega)$ to $H$.

It is easily shown (see e.g. [12]) that $D(A^{1/2}) = L^2(\Omega)$ and $A^{1/2}$ is an canonical isomorphism of $L^2(\Omega)$ onto $H$. Define the Dirichlet map $\Upsilon \in \mathscr{L}(L^2(\Gamma_0), L^2(\Omega))$ ([2]), i.e., $\Upsilon u = v$ that is unique if and only if

$$\Delta v = 0 \quad \text{in } \Omega,$$
$$v|_{\Gamma_1} = 0, \quad v|_{\Gamma_0} = u. \tag{2.2}$$

Using the Dirichlet map, one can write (1.1) as

$$\dot{w} - \mathrm{i}A(w - \Upsilon u) = 0. \tag{2.3}$$

Identifying $H$ with its dual $H'$, we have the following diagram:

$$[D(A)] \subset [D(A^{1/2})] \hookrightarrow H$$
$$= H' \hookrightarrow [D(A^{1/2})]' \subset [D(A)]'.$$

An extension $\tilde{A} \in \mathscr{L}([D(A^{1/2})], [D(A^{1/2})]')$ of $A$ is defined by

$$\langle \tilde{A} f, g\rangle_{[D(A^{1/2})]' \times [D(A^{1/2})]}$$
$$= \langle A^{1/2} f, A^{1/2} g\rangle_{H\times H}, \quad \forall f,\ g \in D(A^{1/2})$$

and $\mathrm{i}\tilde{A}$ also generates a $C_0$-group on $[D(A^{1/2})]'$. Hence (2.3) can be written on $[D(A)]'$ as

$$\dot{w} = \mathrm{i}\tilde{A}w + Bu, \tag{2.4}$$

where $B \in \mathscr{L}(U, [D(A^{1/2})]')$ is given by

$$Bu = -\mathrm{i}\tilde{A}\Upsilon u, \quad \forall u \in U. \tag{2.5}$$

Define $B^* \in \mathscr{L}([D(A^{1/2})], U)$, the adjoint of $B^*$, by

$$\langle B^* f, u\rangle_{U\times U} = \langle f, Bu\rangle_{[D(A^{1/2})]\times[D(A^{1/2})]'},$$
$$\forall f \in D(A^{1/2}), \quad u \in U.$$

Then for any $f \in D(A)$ and $u \in C_0^\infty(\Gamma_0)$, we have

$$\langle f, Bu\rangle_{[D(A^{1/2})]\times[D(A^{1/2})]'}$$
$$= \langle Af,\ \tilde{A}^{-1}Bu\rangle_{H\times H} = \mathrm{i}\langle Af, \Upsilon u\rangle_{H\times H}$$
$$= \mathrm{i}\langle A^{1/2} f, A^{-1/2}\Upsilon u\rangle_{L^2(\Omega)\times L^2(\Omega)}$$
$$= \mathrm{i}\langle AA^{-1} f,\ \Upsilon u\rangle_{L^2(\Omega)\times L^2(\Omega)}$$
$$= \left\langle \mathrm{i}\frac{\partial(\Delta^{-1} f)}{\partial \nu}, u\right\rangle_{U\times U}.$$

In the last step, we have used the fact that

$$\int_\Omega \nabla v \nabla\phi = 0, \quad \forall \phi \in H_0^1(\Omega)$$

for any classical solution $v$ of (2.2). Since $C_0^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, we obtain

$$B^* = \mathrm{i}\left.\frac{\partial\Delta^{-1}}{\partial \nu}\right|_{\Gamma_0}. \tag{2.6}$$

Now, we can formulate system (1.1) as an abstract form of a first-order system in the state space $H$ as follows:

$$\dot{w}(t) = \mathrm{i}Aw(t) + Bu(t),$$
$$y(t) = B^* w, \tag{2.7}$$

where $B$ and $B^*$ are defined by (2.5) and (2.6), respectively.

The main contribution of this paper is to show that system (2.7) is regular with feedthrough operator $D = 0$.

## 3. Proof of Theorem 1.2

Since (according to Theorem 1.1) system (2.7) is well-posed, it follows from the Appendix of [10] that the transfer function of system (2.7) is

$$H(\lambda) = B^*(\lambda - \mathrm{i}A)^{-1}B, \tag{3.1}$$

where $A$, $B$ and $B^*$ are given by (2.1), (2.5) and (2.6), respectively. Moreover, from the well-posedness claimed by Theorem 1.1, it follows that there exists a positive number $\alpha > 0$ such that (see [9])

$$\sup_{\mathrm{Re}\,\lambda \geqslant \alpha} \|H(\lambda)\|_{\mathscr{L}(U)} = M < \infty. \tag{3.2}$$

**Proposition 2.1.** *Theorem* 1.2 *is valid if for any* $u \in C_0^\infty(\Gamma_0)$ *and any* $\varepsilon > 0$, *the solution* $u_\varepsilon$ *of*

$$\varepsilon^{-1}u_\varepsilon(x) + \mathrm{i}\Delta u_\varepsilon(x) = 0, \quad x \in \Omega,$$
$$u_\varepsilon(x) = 0, \quad x \in \Gamma_1,$$
$$u_\varepsilon(x) = u(x), \quad x \in \Gamma_0 \tag{3.3}$$

*satisfies*

$$\lim_{\varepsilon\to 0}\int_{\Gamma_0}\left|\varepsilon\frac{\partial u_\varepsilon(x)}{\partial \nu}\right|^2 \mathrm{d}x = 0. \tag{3.4}$$

**Proof.** We only need to show that $H(\lambda)u$ converges to zero in the strong topology of $U$ along the positive real axis (see [21]), that is,

$$\lim_{\lambda\to+\infty} H(\lambda)u = 0 \tag{3.5}$$

for any $u \in L^2(\Gamma_0) = U$. Due to (3.2) and a density argument, it suffices to show that (3.5) is satisfied for all $u \in C_0^\infty(\Gamma_0)$. For any $u \in C_0^\infty(\Gamma_0)$ and $\lambda > 0$, let

$$w_\lambda = (\lambda - \mathrm{i}A)^{-1}Bu.$$

Then along the line from (1.1) to (2.4), we find that $w_\lambda$ satisfies

$$\begin{aligned} &\lambda w_\lambda(x) + \mathrm{i}\Delta w_\lambda(x) = 0, \quad x \in \Omega,\\ &w_\lambda(x) = 0, \quad x \in \Gamma_1,\\ &w_\lambda(x) = u(x), \quad x \in \Gamma_0. \end{aligned} \tag{3.6}$$

Along the line from (1.1) to (2.7), it has

$$(H(\lambda)u)(x) = \mathrm{i}\frac{\partial(\Delta^{-1}w_\lambda(x))}{\partial\nu}, \quad \forall x \in \Gamma_0. \tag{3.7}$$

Since $u \in C_0^\infty(\Gamma_0)$, there exists a unique classical solution to (3.6). Take a function $v \in H^2(\Omega)$ such that

$$\begin{aligned} &-\Delta v(x) = 0, \quad x \in \Omega,\\ &v(x) = 0, \quad x \in \Gamma_1,\\ &v(x) = u(x), \quad x \in \Gamma_0. \end{aligned} \tag{3.8}$$

Then (3.6) can be written as

$$\begin{aligned} &\mathrm{i}\lambda w_\lambda(x) - \Delta(w_\lambda(x) - v(x)) = 0, \quad x \in \Omega,\\ &(w_\lambda - v)|_{\partial\Omega} = 0 \end{aligned} \tag{3.9}$$

or

$$\lambda \mathrm{i}\Delta^{-1}w_\lambda(x) = w_\lambda(x) - v(x).$$

Hence (3.7) becomes

$$(H(\lambda)u)(x) = \frac{1}{\lambda}\frac{\partial w_\lambda(x)}{\partial\nu} - \frac{1}{\lambda}\frac{\partial v(x)}{\partial\nu}. \tag{3.10}$$

Letting $u_\varepsilon(x) = w_\lambda(x)$ with $\varepsilon = \lambda^{-1}$, we conclude the required result. □

**Proof of Theorem 1.2.** Let us denote by $\tau = \tau(x)$ the tangential vector at $x \in \partial\Omega$. Since $\partial\Omega$ is of class $C^3$, it follows from Lemma 2.1 on p. 18 of [12] that there exists a vector field $h = (h_1, h_2, \ldots, h_n) : \overline{\Omega} \to \mathbb{R}^n$ of class $C^2$ (for the regularity, we need only that $h$ is of class $C^1$) such that

$$h(x) = \nu(x) \quad \text{on } \partial\Omega \text{ and } |h| \leqslant 1,$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

Now, multiply both sides of the first equation of (3.3) by $h\cdot\nabla\overline{u_\varepsilon}$ and integrate over $\Omega$. After using the Green's formula and the fact that $\partial u_\varepsilon/\partial\nu = \nabla u_\varepsilon \cdot \nu$ on $\partial\Omega$, we obtain

$$\begin{aligned} 0 &= \int_\Omega \varepsilon^{-1}u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x + \mathrm{i}\int_\Omega \Delta u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x\\ &= \varepsilon^{-1}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x + \mathrm{i}\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial\nu}(h\cdot\nabla\overline{u_\varepsilon})\,\mathrm{d}\Gamma\\ &\quad - \mathrm{i}\int_\Omega \nabla u_\varepsilon\cdot\nabla(h\cdot\nabla\overline{u_\varepsilon})\,\mathrm{d}x\\ &= \varepsilon^{-1}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x + \mathrm{i}\int_{\partial\Omega}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2\mathrm{d}\Gamma\\ &\quad - \mathrm{i}\int_\Omega \nabla u_\varepsilon\cdot\nabla(h\cdot\nabla\overline{u_\varepsilon})\,\mathrm{d}x. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\partial\Omega}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2\mathrm{d}\Gamma &= \mathrm{Re}\int_\Omega \nabla u_\varepsilon\cdot\nabla(h\cdot\nabla\overline{u_\varepsilon})\,\mathrm{d}x\\ &\quad - \mathrm{Im}\,\varepsilon^{-1}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x. \end{aligned} \tag{3.11}$$

A simple computation shows that

$$\begin{aligned} &\mathrm{Re}\,(\nabla u_\varepsilon\cdot\nabla(h\cdot\nabla\overline{u_\varepsilon}))\\ &\quad = \mathrm{Re}\sum_{i,j=1}^n \partial_{x_i}h_j\partial_{x_i}u_\varepsilon\partial_{x_j}\overline{u_\varepsilon}\\ &\qquad + \frac{1}{2}\,\mathrm{div}(h|\nabla u_\varepsilon|^2) - \tfrac{1}{2}\,\mathrm{div}(h)|\nabla u_\varepsilon|^2. \end{aligned} \tag{3.12}$$

Substitute (3.12) into (3.11) and make use of the divergence theorem to produce

$$\begin{aligned} \int_{\partial\Omega}\left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2\mathrm{d}\Gamma &= \mathrm{Re}\sum_{i,j=1}^n\int_\Omega \partial_{x_i}h_j\partial_{x_i}u_\varepsilon\partial_{x_j}\overline{u_\varepsilon}\,\mathrm{d}x\\ &\quad + \frac{1}{2}\int_{\partial\Omega}|\nabla u_\varepsilon|^2\,\mathrm{d}\Gamma\\ &\quad - \frac{1}{2}\int_\Omega \mathrm{div}(h)|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ &\quad - \varepsilon^{-1}\,\mathrm{Im}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Since $|\nabla u_\varepsilon|^2 = |\frac{\partial u_\varepsilon}{\partial \nu}|^2 + |\frac{\partial u_\varepsilon}{\partial \tau}|^2 = |\frac{\partial u_\varepsilon}{\partial \nu}|^2 + |\frac{\partial u}{\partial \tau}|^2$ on $\Gamma_0$ and $|\nabla u_\varepsilon|^2 = |\frac{\partial u_\varepsilon}{\partial \nu}|^2$ on $\Gamma_1$, it follows from (3.13) that

$$\begin{aligned}\int_{\Gamma_0}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2 \mathrm{d}\Gamma &= 2\,\mathrm{Re}\sum_{i,j=1}^{n}\int_\Omega \partial_{x_i}h_j\partial_{x_i}u_\varepsilon\partial_{x_j}\overline{u_\varepsilon}\,\mathrm{d}x\\ &\quad+\int_{\Gamma_0}\left|\frac{\partial u}{\partial \tau}\right|^2\mathrm{d}\Gamma-\int_{\Gamma_1}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2\mathrm{d}\Gamma\\ &\quad-\int_\Omega \mathrm{div}(h)|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ &\quad-2\varepsilon^{-1}\,\mathrm{Im}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x\\ &\leqslant C\left(\int_\Omega|\nabla u_\varepsilon|^2\,\mathrm{d}x+\|u\|^2_{H^1(\Gamma_0)}\right)\\ &\quad-2\varepsilon^{-1}\,\mathrm{Im}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x, \qquad (3.14)\end{aligned}$$

where $C>0$ is a constant independent of $\varepsilon$.

Next, multiply both sides of the first equation of (3.3) by $\overline{u_\varepsilon}$ and integrate over $\Omega$ by parts, to obtain

$$\begin{aligned}&\varepsilon^{-1}\int_\Omega|u_\varepsilon|^2\,\mathrm{d}x-\mathrm{i}\int_\Omega|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ &\quad+\mathrm{i}\int_{\Gamma_0}\overline{u}\frac{\partial u_\varepsilon}{\partial \nu}\,\mathrm{d}\Gamma=0. \qquad (3.15)\end{aligned}$$

Compare the real part of Eq. (3.15) and multiply by $\varepsilon^{3/2}$ to give

$$\begin{aligned}&\varepsilon^{1/2}\int_\Omega|u_\varepsilon|^2\,\mathrm{d}x\\ &\quad=-\varepsilon^{3/2}\,\mathrm{Re}\left(\mathrm{i}\int_{\Gamma_0}\overline{u}\frac{\partial u_\varepsilon}{\partial \nu}\,\mathrm{d}\Gamma\right)\\ &\quad\leqslant\frac{\varepsilon^{1/2}}{2}\int_{\Gamma_0}|u|^2\,\mathrm{d}\Gamma+\frac{\varepsilon^{5/2}}{2}\int_{\Gamma_0}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2\mathrm{d}\Gamma. \qquad (3.16)\end{aligned}$$

The same treatment to the imaginary part of Eq. (3.15) gives

$$\begin{aligned}&\varepsilon^{3/2}\int_\Omega|\nabla u_\varepsilon|^2\,\mathrm{d}x\\ &\quad=\varepsilon^{3/2}\,\mathrm{Im}\left(\mathrm{i}\int_{\Gamma_0}\overline{u}\frac{\partial u_\varepsilon}{\partial \nu}\,\mathrm{d}\Gamma\right)\leqslant\frac{\varepsilon^{1/2}}{2}\int_{\Gamma_0}|u|^2\,\mathrm{d}\Gamma\\ &\qquad+\frac{\varepsilon^{5/2}}{2}\int_{\Gamma_0}\left|\frac{\partial u_\varepsilon}{\partial \nu}\right|^2\mathrm{d}\Gamma. \qquad (3.17)\end{aligned}$$

Finally, since $|h|\leqslant 1$, we have

$$\begin{aligned}&-2\varepsilon\,\mathrm{Im}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x\\ &\quad\leqslant\varepsilon^{1/2}\int_\Omega|u_\varepsilon|^2\,\mathrm{d}x+\varepsilon^{3/2}\int_\Omega|h\cdot\nabla\overline{u_\varepsilon}|^2\,\mathrm{d}x\\ &\quad\leqslant\varepsilon^{1/2}\int_\Omega|u_\varepsilon|^2\,\mathrm{d}x+\varepsilon^{3/2}\int_\Omega|\nabla u_\varepsilon|^2\,\mathrm{d}x. \qquad (3.18)\end{aligned}$$

Combining (3.14) and (3.16)–(3.18), we obtain

$$\begin{aligned}&\int_{\Gamma_0}\left|\varepsilon\frac{\partial u_\varepsilon}{\partial \nu}\right|^2\mathrm{d}\Gamma\\ &\quad\leqslant C\left(\varepsilon^2\int_\Omega|\nabla u_\varepsilon|^2\,\mathrm{d}x+\varepsilon^2\|u\|^2_{H^1(\Gamma_0)}\right)\\ &\qquad-2\varepsilon\,\mathrm{Im}\int_\Omega u_\varepsilon h\cdot\nabla\overline{u_\varepsilon}\,\mathrm{d}x\\ &\quad\leqslant\left(\varepsilon^{1/2}+\frac{C}{2}\varepsilon\right)\int_{\Gamma_0}|u|^2\,\mathrm{d}\Gamma+C\varepsilon^2\|u\|^2_{H^1(\Gamma_0)}\\ &\qquad+\left(1+\frac{C}{2}\varepsilon^{1/2}\right)\varepsilon^{1/2}\int_{\Gamma_0}\left|\varepsilon\frac{\partial u_\varepsilon}{\partial \nu}\right|^2\mathrm{d}\Gamma. \qquad (3.19)\end{aligned}$$

This shows that

$$\lim_{\varepsilon\to 0^+}\int_{\Gamma_0}\left|\varepsilon\frac{\partial u_\varepsilon}{\partial \nu}\right|^2\mathrm{d}\Gamma=0.$$

The result follows. □

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## Appendix. The proof of Theorem 1.1

We use the notations of Section 2. For brevity, we shall denote by $C_T$ a positive constant depending on time $T$ only, which may change its value from line to line even though it is denoted by the same symbol.

First, we need to show that $B$ is admissible for $e^{iA\cdot}$, the $C_0$-group generated by i$A$ on $H$. Since the system (2.7) is colocated, $B$ is admissible for $e^{iA\cdot}$ if and only if $B^*$ is admissible for $e^{-iA^*\cdot}$ (see [18]). Therefore, the admissibility means that

$$\int_0^T \int_\Omega |B^* e^{(iA)^* t} w_0|^2 \,dx\,dt \leqslant C_T \|w_0\|^2, \quad \forall w_0 \in D(A) = H_0^1(\Omega)$$

for some (and hence for all) $T > 0$. Since $e^{iA\cdot}$ is a $C_0$-group, the above is also equivalent to saying that

$$\int_0^T \int_\Omega |B^* e^{iAt} w_0|^2 \,dx\,dt \leqslant C_T \|w_0\|^2, \quad \forall w_0 \in D(A) = H_0^1(\Omega). \tag{A.1}$$

Let

$$z = A^{-1} w.$$

Then instead of (2.7), we may consider equation of $z$ in the space $H_0^1(\Omega)$, which is derived from (2.7), (2.5) and (2.6):

$$\begin{aligned} &z_t(x,t) = -i\triangle z(x,t) - i(\Upsilon u(\cdot,t))(x), \quad x \in \Omega, \\ &z(x,0) = z_0(x), \quad x \in \Omega, \\ &z(x,t) = 0, \quad x \in \partial\Omega, \\ &y(x,t) = B^* w = B^* A A^{-1} w \\ &\qquad = B^* A z = -i \frac{\partial z(x,t)}{\partial \nu}, \quad x \in \Gamma_0. \end{aligned} \tag{A.2}$$

Let $f = i\Upsilon u$. Then by definition of the Dirichlet map, we have

$$\int_0^T \int_\Omega |f|^2 \,dx\,dt \leqslant C_T \int_0^T \int_{\Gamma_0} |u|^2 \,d\Gamma\,dt. \tag{A.3}$$

Similar to (3.13), we have

$$\begin{aligned} &\int_{\Gamma_0} \left|\frac{\partial z}{\partial \nu}\right|^2 d\Gamma \\ &\quad = 2\,\mathrm{Re} \sum_{i,j=1}^n \int_\Omega \partial_{x_i} h_j \partial_{x_i} z \partial_{x_j} \bar{z} \,dx - \int_{\Gamma_1} \left|\frac{\partial z}{\partial \nu}\right|^2 d\Gamma \\ &\qquad - \int_\Omega \mathrm{div}(h) |\nabla z|^2 \,dx - 2\,\mathrm{Im} \int_\Omega z_t h \cdot \nabla \bar{z} \,dx \\ &\qquad - 2\,\mathrm{Im} \int_\Omega f h \cdot \nabla \bar{z} \,dx \\ &\quad \leqslant C \left( \int_\Omega |\nabla z|^2 \,dx + \int_\Omega |f|^2 \,dx \right) \\ &\qquad - 2\,\mathrm{Im} \int_\Omega z_t h \cdot \nabla \bar{z} \,dx. \end{aligned} \tag{A.4}$$

Let us look at the term $\mathrm{Im} \int_\Omega z_t h \cdot \nabla \bar{z} \,dx$. From (4.2), it follows that

$$\begin{aligned} \mathrm{div}(z_t \bar{z} h) &= z_t \bar{z}\, \mathrm{div}(h) + z_t h \cdot \nabla \bar{z} + \bar{z} h \cdot \nabla z_t \\ &= (-i\triangle z - f) \bar{z}\, \mathrm{div}(h) + z_t h \cdot \nabla \bar{z} \\ &\quad + \frac{d}{dt}(\bar{z} h \cdot \nabla z) - \overline{z_t} h \cdot \nabla z \\ &= (-i\triangle z - f) \bar{z}\, \mathrm{div}(h) + \frac{d}{dt}(\bar{z} h \cdot \nabla z) \\ &\quad + 2i\,\mathrm{Im}(z_t h \cdot \nabla \bar{z}). \end{aligned}$$

Hence

$$\begin{aligned} 2i\,\mathrm{Im}(z_t h \cdot \nabla \bar{z}) &= \mathrm{div}(z_t \bar{z} h) + (i\triangle z + f) \bar{z}\, \mathrm{div}(h) \\ &\quad - \frac{d}{dt}(\bar{z} h \cdot \nabla z) \end{aligned}$$

Using the divergence theorem and Green's formula again, we obtain

$$\begin{aligned} &2i\,\mathrm{Im} \int_\Omega z_t h \cdot \nabla \bar{z} \,dx \\ &\quad = \int_\Omega (i\triangle z + f) \bar{z}\, \mathrm{div}(h) \,dx - \frac{d}{dt} \int_\Omega \bar{z} h \cdot \nabla z \,dx \\ &\quad = -i \int_\Omega \nabla z \cdot \nabla(\bar{z}\, \mathrm{div}(h)) \,dx + \int_\Omega f \bar{z}\, \mathrm{div}(h) \,dx \\ &\qquad - \frac{d}{dt} \int_\Omega \bar{z} h \cdot \nabla z \,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &2\,\mathrm{Im} \int_0^T \int_\Omega z_t h \cdot \nabla \bar{z} \,dx\,dt \\ &\quad = - \int_0^T \int_\Omega \nabla z \cdot \nabla(\bar{z}\, \mathrm{div}(h)) \,dx\,dt \\ &\qquad - i \int_0^T \int_\Omega f \bar{z}\, \mathrm{div}(h) \,dx\,dt \\ &\qquad + i \int_\Omega \bar{z} h \cdot \nabla z \,dx \Big|_0^T. \end{aligned} \tag{A.5}$$

This together with (A.4) gives

$$\begin{aligned} &\int_0^T \int_{\Gamma_0} \left|\frac{\partial z}{\partial \nu}\right|^2 d\Gamma\,dt \\ &\quad \leqslant C_T \left( \|z\|^2_{L^2(0,T;H^1(\Omega))} + \|f\|_{L^2(\Omega\times(0,T))} \right. \\ &\qquad \left. + \|z\|^2_{L_\infty(0,T;H^1(\Omega))} \right). \end{aligned} \tag{A.6}$$

Let $f = \mathrm{i}\Upsilon u = 0$ in (A.2). Then since for any $z_0 \in D(A)$, $\mathrm{e}^{\mathrm{i}At} z_0 \in C^1(0, T; D(A))$, we have, particularly from (A.6), that

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial(\mathrm{e}^{\mathrm{i}At} z_0)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \, \mathrm{d}t \leqslant C_T \|z_0\|^2_{[D(A)]},$$
$$\forall z_0 \in D(A) = H_0^1(\Omega). \tag{A.7}$$

This gives in turn

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial(\mathrm{e}^{\mathrm{i}At} A^{-1} w_0)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \, \mathrm{d}t \leqslant C_T \|w_0\|^2,$$
$$\forall w_0 = A^{-1} z_0 \in D(A). \tag{A.8}$$

(A.8) is the same as (A.1) by the definition of $B^*$. The admissibility follows.

Now we are in a position to show the boundedness of the input–output map, that is, for some (and hence for all) $T > 0$, the solution to (A.2) with $z_0 = 0$ satisfies

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial z(x,t)}{\partial \nu} \right|^2 \mathrm{d}\Gamma \, \mathrm{d}t \leqslant C_T \int_0^T \int_{\Gamma_0} |u(x,t)|^2 \, \mathrm{d}\Gamma \, \mathrm{d}t,$$
$$\forall u \in L^2(0, T; U). \tag{A.9}$$

Notice that the solution to (A.2) with $z_0 = 0$ is given by

$$z(x,t) = -\int_0^t [\mathrm{e}^{\mathrm{i}\tilde{A}(t-s)} f(\cdot, s)](x) \, \mathrm{d}s = -\mathrm{i} \int_0^t [\mathrm{e}^{\mathrm{i}\tilde{A}(t-s)} \Upsilon u(\cdot, s)](x) \, \mathrm{d}s.$$

By the admissibility just verified, we have ([17])

$$\tilde{A} z(x,t) = -\mathrm{i} \int_0^t [\mathrm{e}^{\mathrm{i}\tilde{A}(t-s)} \tilde{A} \Upsilon u(\cdot, s)](x) \, \mathrm{d}s = \int_0^t [\mathrm{e}^{\mathrm{i}\tilde{A}(t-s)} B u(\cdot, s)](x) \, \mathrm{d}s \in C(0, T; H).$$

Hence

$$z \in C(0, T; H_0^1(\Omega)). \tag{A.10}$$

This together with (A.3) and (A.6) shows that the solution to (A.2) with $z_0 = 0$ indeed satisfies (A.9). The proof is complete. □

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