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# Stabilization and parameter estimation for an Euler–Bernoulli beam equation with uncertain harmonic disturbance under boundary output feedback control☆

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**Abstract**

This paper is concerned with the boundary stabilization and parameter estimation of an Euler–Bernoulli beam equation with one end fixed, and control and uncertain amplitude of harmonic disturbance at another end. A high-gain adaptive regulator is designed in terms of measured collocated end velocity. The existence and uniqueness of the classical solution as well as smooth solution of the closed-loop system are justified. It is shown that the state of the system approaches the standstill as time goes to infinity and meanwhile the estimated parameter converges to the unknown parameter.


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*Keywords:* Beam equation; Harmonic disturbance rejection; Existence and uniqueness of solution

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## 1. Introduction

The objective of the adaptive control for a class of plants with control input $u(t)$ and measured output $y(t)$ is to synthesize a single control law of the form

$$u(t) = F(t, g(t), y(t)), \quad \dot{g}(t) = H(t, g(t), y(t)), \quad g(0) \in \mathbb{R},$$

which guarantee the resulting closed-loop system with some uncertainty exhibiting some prescribed dynamic behavior, for example, the stability. For finite dimensional systems, certain problems of this kind started from the work of [13]. For a survey of the finite-dimensional theory, we refer to [5,14]. Some attempts that have been made to generalize traditional adaptive control algorithms to infinite-dimensional systems can be found in [11,15]. The first results on high-gain adaptive stabilization of infinite-dimensional systems were obtained by [3,6,11]. A nice review on this aspect can be found in [10]. Some other attempts on high-gain adaptive control can also be found in [7] where a non-identity-based adaptive control is designed and the stabilization of a general collocated first-order system with unbounded input and output is discussed. [4] gives an exponential stabilization of high-gain adaptive direct strain feedback control for an Euler–Bernoulli beam. A Recently progress has been made in [8] where a high-gain adaptive control and regulator are constructed to the stabilization and parameter estimation of a general second order system with unbounded input and output. Unfortunately, to our knowledge, the existence and uniqueness of the solution of the closed-loop system are not available in literature and some of conditions are hard to be verified.

In this paper, we consider an Euler–Bernoulli beam system in the case when one end is fixed, and control and harmonic disturbance with uncertain amplitude are inputted at another end

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad x \in (0,1), \quad t \geqslant 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = 0, \quad t \geqslant 0, \\ y_{xxx}(1,t) = u(t) - \tilde{\theta} \sin t, \quad t \geqslant 0, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \\ y_{\text{out}}(t) = y_t(1,t), \end{cases} \tag{1.1}$$

where and henceforth $y'$ or $y_x$ denote the derivative of $y$ with respect to $x$ and $\dot{y}$ or $y_t$ the derivative with respect to $t$. $u(t): \mathbb{R}^+ \to \mathbb{R}$ is the boundary control force applied at the free end of the beam and $\tilde{\theta}$ is the unknown amplitude of harmonic disturbance. $y_{\text{out}}(t)$ stands for the measured signal of the system at time $t$. $y_0$ and $y_1$ are initial values.

The energy of system (1.1) is defined by

$$E(t) = \frac{1}{2} \int_0^1 [y_t^2(x,t) + y_{xx}^2(x,t)]\, \mathrm{d}x. \tag{1.2}$$

It is well known that if there is no disturbance ($\tilde{\theta} = 0$), then the closed-loop system under the output feedback control $u(t) = -k y_{\text{out}}(t)$, $k > 0$, is exponentially stable, that is to say, there are two positive constants $M$ and $\omega$ such that

$$E(t) \leqslant M \mathrm{e}^{-\omega t} E(0), \quad t \geqslant 0. \tag{1.3}$$

However, when the system is subjected to a disturbance, one cannot guarantee either exponential or even asymptotic stability of the system. The advantage of the adaptive stabilization is that stabilization and good control performance can be automatically achieved even in the presence of various types of uncertainties. In this paper, we consider the following adaptive output feedback regulator for system (1.1) [7]:

$$\begin{cases} u(t)=k(t)y_t(1,t)+\theta(t)\sin t, \\ \dot{k}(t)=ry_t^2(1,t), \quad k(0)=0, \quad r>0, \\ \dot{\theta}(t)=y_t(1,t)\sin t, \quad \theta(0)=\theta_0, \end{cases} \tag{1.4}$$

where $\theta_0$ is the initial condition of the estimator. Under this adaptive controller, the closed-loop system (1.1) becomes

$$\begin{cases} y_{tt}(x,t)+y_{xxxx}(x,t)=0, \quad x\in(0,1), \quad t\geqslant 0, \\ y(0,t)=y_x(0,t)=y_{xx}(1,t)=0, \quad t\geqslant 0, \\ y_{xxx}(1,t)=k(t)y_t(1,t)+[\theta(t)-\tilde{\theta}]\sin t, \quad t\geqslant 0, \\ y(x,0)=y_0(x), \quad y_t(x,0)=y_1(x), \\ \dot{k}(t)=ry_t^2(1,t), \quad k(0)=0, \quad r>0, \\ \dot{\theta}(t)=y_t(1,t)\sin t, \quad \theta(0)=\theta_0. \end{cases} \tag{1.5}$$

It should be pointed out that our regulator is different from the one developed in [8] for general second-order collocated system where combination of displacement and velocity are used in the design.

The paper is organized as follows. In Section 2, using constructive Galerkin approximating scheme, we show the existence and uniqueness of the classical solution. Section 3 is devoted to the smooth solution for smooth initial datum satisfying necessary compatible conditions. The convergence of the solution to standstill and the regulation of the uncertain parameter are presented in Section 4.

## 2. Existence and uniqueness of classical solution

Let $L^2(0,1)$ be the usual Hilbert space with inner product $\langle\cdot,\cdot\rangle_{L^2}$ and the inner product induced norm $\|\cdot\|_{L^2}$. Define operator $A$ in $L^2$ as follows:

$$A\phi=\phi'''', \quad \forall \phi\in D(A)=\{\phi\in L^2|\phi(0)=\phi'(0)=\phi''(1)=\phi'''(1)=0\}. \tag{2.1}$$

Then $A$ is an unbounded self-adjoint positive definite operator in $L^2$ with the eigenpairs $\{(\lambda_n,\phi_n)\}_{n=1}^\infty$:

$$\begin{cases} \lambda_n=\beta_n^4=o(n^4), \, \phi_n(x)=-\frac{1+\gamma}{2}\,\mathrm{e}^{\beta_n x}-\frac{1-\gamma}{2}\,\mathrm{e}^{-\beta_n x}+\gamma\sin(\beta_n x) \\ \quad +\cos(\beta_n x), \\ \gamma=\frac{\mathrm{e}^{\beta_n}-\sin(\beta_n)+\cos(\beta_n)}{\mathrm{e}^{\beta_n}-\sin(\beta_n)+\cos(\beta_n)}\to -1 \text{ as } n\to\infty. \end{cases} \tag{2.2}$$

From general operator theory and Lemma 4.6 of [12], $\{\phi_n\}_{n=1}^\infty$ defined by (2.2) are approximately normalized (i.e., $0<c_1<\|\phi_n\|<c_2$ for some constants $c_1,c_2$ independent of $n$)

and form an orthogonal basis on $L^2$. Let $H_E^2(0,1)=\{\phi\in H^2(0,1)|\phi(0)=0,\ \phi'(1)=0\}$ with the inner product $\langle f,g\rangle_{H_E^2}=\int_0^1 f''(x)g''(x)\,\mathrm{d}x$ be the Hilbert space. It is known from [16] that $[D(A^{1/2})]=H_E^2(0,1)$.

**Theorem 2.1.** *Assume that the initial value* $(y_0,y_1,\theta_0)\in D(A)\times D(A^{1/2})\times\mathbb{R}$. *Then in system* (1.5) *there exists a unique classical solution y in the sense that for any time* $T>0$,

$$\begin{cases} y\in L^\infty(0,T;H^4\cap H_E^3),\quad y_t\in L^\infty(0,T;[D(A^{1/2})]),\quad y_{tt}\in L^\infty(0,T;L^2),\\ k\in C^1[0,T],\quad \theta\in C^1[0,T],\\ y_{tt}(x,t)=y_{xxxx}(x,t)\ in\ L^\infty(0,T;L^2(0,1)),\\ y(0,t)=y_x(0,t)=y_{xx}(1,t)=0,\quad t\geqslant 0,\\ y_{xxx}(1,t)=k(t)y_t(1,t)+[\theta(t)-\tilde{\theta}]\sin t,\quad t\geqslant 0,\\ y(x,0)=y_0(x),\quad y_t(x,0)=y_1(x),\\ \dot{k}(t)=ry_t^2(1,t),\quad k(0)=0,\quad r>0,\\ \dot{\theta}(t)=y_t(1,t)\sin t,\quad \theta(0)=\theta_0, \end{cases}$$

*where* $H_E^3=\{\phi\in H^3(0,1)|\phi(0)=\phi'(0)=\phi''(1)=0\}$. *By the Sobolev embedding theorem, it follows that* $y\in C([0,1]\times[0,T])$.

**Proof.** We start by showing the uniqueness of the classical solution. However, suppose otherwise, there exist two solutions $(y,y_t,k,\theta)$ and $(\hat{y},\hat{y}_t,\hat{k},\hat{\theta})$. Set $p(x,t)=y(x,t)-\hat{y}(x,t)$. Then $p$ satisfies

$$\begin{cases} p_{tt}(x,t)=p_{xxxx}(x,t),\\ p(0,t)=p_x(0,t)=p_{xx}(1,t)=0,\\ p_{xxx}(1,t)=k(t)y_t(1,t)-\hat{k}(t)\hat{y}_t(1,t)-[\hat{\theta}(t)-\theta(t)]\sin t,\\ p(x,0)=0,\quad p_t(x,0)=0,\\ \dot{k}(t)=ry_t^2(1,t),\quad k(0)=0,\\ \dot{\hat{k}}(t)=r\hat{y}_t{}^2(1,t),\quad \hat{k}(0)=0,\\ \dot{\theta}(t)=y_t(1,t)\sin t,\quad \theta(0)=\theta_0,\\ \dot{\hat{\theta}}(t)=\hat{y}_t(1,t)\sin t,\quad \hat{\theta}(0)=\theta_0. \end{cases} \tag{2.3}$$

Define Lyapunov-like functional is as follows:

$$V(t)=\int_0^1 p_t^2(x,t)\,\mathrm{d}x+\int_0^1 p_{xx}^2(x,t)\,\mathrm{d}x+\frac{[k(t)-\hat{k}(t)]^2}{2r}+[\theta(t)-\hat{\theta}(t)]^2. \tag{2.4}$$

A direct computation shows that the time derivative of $V(t)$ along the solution of equation (2.3) satisfies

$$\begin{aligned} \dot{V}(t)=&-2p_t(1,t)p_{xxx}(1,t)+[k(t)-\hat{k}(t)][y_t^2(1,t)-\hat{y}_t^2(1,t)]\\ &+2[\theta(t)-\hat{\theta}(t)][y_t(1,t)-\hat{y}_t(1,t)]\sin t\\ =&-[k(t)+\hat{k}(t)][y_t(1,t)-\hat{y}_t(1,t)]^2\leqslant 0. \end{aligned} \tag{2.5}$$

Hence, $V(t) \equiv V(0) = 0$ or

$$(y, y_t, k, \theta) \equiv (\hat{y}, \hat{y}_t, \hat{k}, \hat{\theta}).$$

Now we turn to existence. Multiply the first equation of (1.5) by $\phi \in D(A^{1/2})$ and integrate over $x \in [0, 1]$ by parts to obtain

$$\langle y_{tt}(\cdot, t), \phi(\cdot)\rangle_{L^2} + \langle y_{xx}(\cdot, t), \phi_{xx}(\cdot)\rangle_{L^2} = -[k(t)y_t(1, t) + [\theta(t) - \tilde{\theta}] \sin t]\phi(1). \tag{2.6}$$

Because system (1.5) is a time-dependent evolution equation, we use standard Galerkin approximating solution to converge the classical one. The process is as follows. Since the sequence $\{\phi_n\}_{n=1}^{\infty}$ defined by (2.2) forms a (orthogonal) Riesz basis for $L^2(0, 1)$, we expand both initial values $y_0$ and $y_1$ into the Fourier series in terms of $\phi_n$ in $L^2$ space, i.e.,

$$y_0 = \sum_{n=1}^{\infty} a_n \phi_n, \quad y_1 = \sum_{n=1}^{\infty} b_n \phi_n. \tag{2.7}$$

For any given positive integer $N$, we find the Galerkin approximating solution $y^N$ to be of the form

$$y^N(x, t) = \sum_{n=1}^{N} g_{nN}(t)\phi_n(x) \tag{2.8}$$

subject to the initial conditions

$$\begin{cases} y^N(x, 0) = \sum\limits_{n=1}^{N} a_n \phi_n \to y_0 \text{ in } [D(A)] \text{ as } N \to \infty, \\ \dot{y}^N(x, 0) = \sum\limits_{n=1}^{N} b_n \phi_n \to y_1 \text{ in } [D(A^{1/2})] \text{ as } N \to \infty, \end{cases} \tag{2.9}$$

where we have used the assumptions on $y_0$ and $y_1$.

The argument is to make

$$\langle y_{tt}^N(\cdot, t) + y_{xxxx}^N(\cdot, t), \phi_n\rangle_{L^2} = 0, \quad n = 1, 2, \ldots, N$$

at the time of regarding $-k_N(t)y^N(1, t) - [\theta_N(t) - \tilde{\theta}] \sin t$ as an approximation of $-k(t)$ $y(1, t) - [\theta(t) - \tilde{\theta}] \sin t$. Hence certain nonlinear ordinary differential equations for $g_{nN}$ must be satisfied. These can be written in the form

$$\begin{cases} \ddot{g}_{nN}(t) + \lambda_n g_{nN}(t) = \left\{ -k_N(t) \sum\limits_{m=1}^{N} \dot{g}_{mN}(t)\phi_m(1) \right. \\ \quad \left. -[\theta_N(t) - \tilde{\theta}] \sin t \right\} \phi_n(1), \quad n = 1, 2, \ldots, N, \\ \dot{k}_N(t) = r\left[ \sum\limits_{m=1}^{N} \dot{g}_{mN}(t)\phi_m(1) \right]^2, \ \ k_N(0) = 0, \\ \dot{\theta}_N(t) = \left[ \sum\limits_{m=1}^{N} \dot{g}_{mN}(t)\phi_m(1) \right] \sin t, \ \ \theta_N(0) = \theta_0, \\ g_{nN}(0) = a_n, \ \ g'_{nN}(0) = b_n, \ \ n = 1, 2, \ldots, N. \end{cases} \tag{2.10}$$

The existence and uniqueness of the solution to (2.10) in some interval $[0, t_N)$, $t_N > 0$ is ensured by local Lipschitz condition. Now we show $t_N = \infty$ by constructing a Lyapunov function for system (2.10). Multiplying the first equation of (2.10) by $\dot{g}_{nN}(t)$ and summating $n$ from 1 to $N$ yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{y}^N(\cdot,t)\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|y_{xx}^N(\cdot,t)\|_{L^2}^2\\ &\quad = 2\{-k_N(t)\dot{y}^N(1,t) - [\theta_N(t) - \tilde{\theta}]\sin t\}\dot{y}^N(1,t).\end{aligned}$$

This shows that the Lyapunov function for system (2.10) defined by

$$V_N(t) = \|y_{xx}^N(\cdot,t)\|_{L^2}^2 + \|\dot{y}^N(\cdot,t)\|_{L^2}^2 + \frac{k_N^2(t)}{2r} + [\theta_N(t) - \tilde{\theta}]^2$$

satisfies

$$\dot{V}_N(t) = -k_N(t)[\dot{y}^N(1,t)]^2 \leqslant 0 \tag{2.11}$$

along the solution of (2.10). Therefore, (2.10) admits a unique global classical solution. □

Next we come to the existence of the solution which shall be split into several lemmas.

**Lemma 2.1.**

$$\begin{cases}\dot{y}^N(1,t) \in L^2(0,\infty),\\ \max\limits_{t\geqslant 0}\sup\limits_N [\|\dot{y}^N(\cdot,t)\|_{L^2} + \|y_{xx}^N(\cdot,t)\|_{L^2} + |\theta_N(t)| + k_N(t)] < \infty.\end{cases} \tag{2.12}$$

**Proof.** The result follows from (2.11) that

$$V_N(t) \leqslant V_N(0) \to \|(y_0, y_1)\|_{[D(A^{1/2})]\times L^2}^2 + [\theta_0 - \tilde{\theta}]^2 \text{ as } N \to \infty. \qquad \square$$

**Lemma 2.2.**

$$\sup_N \|\ddot{y}^N(\cdot,0)\|_{L^2} < \infty. \tag{2.13}$$

**Proof.** From (2.10)

$$\ddot{g}_{nN}(0) + \lambda_n g_{nN}(0) = 0.$$

Multiplying the equality above by $\phi_n(x)$ and summating for $n = 1, 2, \ldots, N$ gives

$$\ddot{y}^N(x,0) = -y_{xxxx}^N(x,0) = -Ay_0^N(x).$$

Hence $\|\ddot{y}^N(\cdot,0)\|_{L^2} = \|Ay_0^N\|_{L^2} = \|y_0^N\|_{[D(A)]}$. (2.13) then follows from the assumption for $y_0$ by (2.9). □

**Lemma 2.3.** *For any $T > 0$*

$$\max_{0\leqslant t\leqslant T}\sup_N [\|\ddot{y}^N(\cdot,t)\|_{L^2} + \|\dot{y}_{xx}^N(\cdot,t)\|_{L^2} + |\dot{y}^N(1,t)|] < \infty. \tag{2.14}$$

**Proof.** We rewrite (2.10) into

$$\begin{cases} \ddot{g}_{nN}(t) + \lambda_n g_{nN}(t) = \{-k_N(t)\dot{y}^N(1,t) - [\theta_N(t) - \tilde{\theta}] \sin t\}\phi_n(1), \\ \quad n = 1, 2, \ldots, N, \\ \dot{k}_N(t) = r[\dot{y}^N(1,t)]^2, \quad k_N(0) = 0, \\ \dot{\theta}_N(t) = \dot{y}^N(1,t) \sin t, \quad \theta_N(0) = \theta_0, \\ g_{nN}(0) = a_n, \quad \dot{g}_{nN}(0) = b_n, \quad n = 1, 2, \ldots, N. \end{cases} \tag{2.15}$$

Now differentiate the first equation of (2.15) with respect to $t$ to obtain

$$\begin{aligned} &g'''_{nN}(t) + \lambda_n \dot{g}_{nN}(t) \\ &\quad = \{-r[\dot{y}^N(1,t)]^3 - k_N(t)\ddot{y}^N(1,t) - \dot{y}^N(1,t)\sin^2 t \\ &\qquad - [\theta_N(t) - \tilde{\theta}] \cos t\}\phi_n(1), \quad n = 1, 2, \ldots, N. \end{aligned} \tag{2.16}$$

And then multiply (2.16) by $\ddot{g}_{nN}(t)$ and sum for $n = 1, 2, \ldots, N$ to produce

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\ddot{y}^N(\cdot,t)\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{y}^N_{xx}(\cdot,t)\|^2_{L^2} \\ &\quad = \{-r[\dot{y}^N(1,t)]^3 - k_N(t)\ddot{y}^N(1,t) - \dot{y}^N(1,t)\sin^2 t \\ &\qquad - [\theta_N(t) - \tilde{\theta}] \cos t\}\ddot{y}^N(1,t). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\ddot{y}^N(\cdot,t)\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{y}^N_{xx}(\cdot,t)\|^2_{L^2} + \frac{r}{4}\frac{\mathrm{d}}{\mathrm{d}t}[\dot{y}^N(1,t)]^4 \\ &\quad = \{-k_N(t)\ddot{y}^N(1,t) - \dot{y}^N(1,t)\sin^2 t - [\theta_N(t) - \tilde{\theta}] \cos t\}\ddot{y}^N(1,t). \end{aligned}$$

Integrating over $[0,t]$ gives

$$\begin{aligned} &\frac{1}{2}\|\ddot{y}^N(\cdot,t)\|^2_{L^2} + \frac{1}{2}\|\dot{y}^N_{xx}(\cdot,t)\|^2_{L^2} + \frac{r}{4}[\dot{y}^N(1,t)]^4 \\ &\quad = -\int_0^t k_N(s)[\ddot{y}^N(1,s)]^2\,\mathrm{d}s - \frac{1}{2}[\dot{y}^N(1,t)]^2\sin^2 t \\ &\qquad + \frac{1}{2}\int_0^t [\dot{y}^N(1,s)]^2 \sin 2s\,\mathrm{d}s \\ &\qquad - \dot{y}^N(1,t)[\theta_N(t) - \tilde{\theta}] \cos t + \dot{y}^N(1,0)[\theta_N(0) - \tilde{\theta}] \\ &\qquad + \int_0^t \dot{y}^N(1,s)\{\dot{y}^N(1,s) \sin s \cos s - [\theta_N(s) - \tilde{\theta}] \sin s\}\,\mathrm{d}s \\ &\qquad + \frac{1}{2}\|\ddot{y}^N(\cdot,0)\|^2_{L^2} + \frac{1}{2}\|\dot{y}^N_{xx}(\cdot,0)\|^2_{L^2} + \frac{r}{4}[\dot{y}^N(1,0)]^4 \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{3}{2}\int_0^t [\dot{y}^N(1,s)]^2\,\mathrm{d}s + \delta_1\|\dot{y}_x^N(\cdot,t)\|^2 \\
&\quad + \frac{1}{4\delta_1}[\theta_N(t)-\tilde{\theta}]^2 + \frac{[\dot{y}^N(1,0)]^2}{2} + (\theta_0-\tilde{\theta})^2 \\
&\quad + \frac{1}{2}\|\ddot{y}^N(\cdot,0)\|_{L^2}^2 + \frac{1}{2}\|\dot{y}_{xx}^N(\cdot,0)\|_{L^2}^2 + \frac{r}{4}[\dot{y}^N(1,0)]^4 \\
&\leqslant \frac{3k_N(t)}{2r} + \delta_1\|\dot{y}_{xx}^N(\cdot,t)\|^2 + \frac{1}{4\delta_1}[\theta_N(t)-\tilde{\theta}]^2 + \frac{[\dot{y}^N(1,0)]^2}{2} + (\theta_0-\tilde{\theta})^2 \\
&\quad + \frac{1}{2}\|\ddot{y}^N(\cdot,0)\|_{L^2}^2 + \frac{1}{2}\|\dot{y}_{xx}^N(\cdot,0)\|_{L^2}^2 + \frac{r}{4}[\dot{y}^N(1,0)]^4. \qquad (2.17)
\end{aligned}
$$

The boundedness of $\|\ddot{y}^N(\cdot,0)\|_{L^2}^2$ follows from Lemma 2.2 and that of $\|\dot{y}_{xx}^N(\cdot,0)\|_{L^2}^2$ and hence $|\dot{y}^N(1,0)|$ follow from (2.9). Taking $\delta_1>0$ so that $1-\delta_1>1/2$ in above inequality proves (2.14). $\square$

**Proof of existence.** Thanks to Lemmas 2.1, 2.3 and Lemma 4 of [1], we may extract a subsequence $N_k$ which is still denoted by $N$, without diffusion, having the properties

$$
\begin{cases}
y^N \rightharpoonup y \text{ in } L^\infty(0,T;[D(A^{1/2})]) \text{ weak}^*;\\
\dot{y}^N \rightharpoonup \dot{y} \text{ in } L^\infty(0,T;[D(A^{1/2})]) \text{ weak}^*;\\
\ddot{y}^N \rightharpoonup \ddot{y} \text{ in } L^\infty(0,T;L^2(0,1)) \text{ weak}^*.
\end{cases} \qquad (2.18)
$$

Since $\dot{y}^N(1,t)=\int_0^1 \dot{y}_x^N(x,t)\,\mathrm{d}x$ and $\dot{y}^N \rightharpoonup \dot{y}$ in $L^\infty(0,T;[D(A^{1/2})])$ weak star topology, and for any fixed $t\in[0,T]$, $\{\dot{y}_x^N\}_{N=1}^\infty$ is a compact subset of $L^2$ due to the boundedness of $\{\dot{y}_{xx}^N\}_{N=1}^\infty$ in $L^2$, where there is a subsequence $\{\dot{y}_x^{Nk}\}_{k=1}^\infty$ of $\{\dot{y}_x^N\}_{N=1}^\infty$ such that $\dot{y}_x^{Nk}(\cdot,t)\to\dot{y}_x(\cdot,t)$in $L^2$ as $k\to\infty$, which implies that $\dot{y}^{Nk}(1,t)\to\dot{y}(1,t)$ for almost every $t\in[0,T]$. By Lebesgue dominant convergence theorem, it follows that $\dot{y}^{Nk}(1,t)\to\dot{y}(1,t)$ as $k\to\infty$ in $L^2[0,T]$. For brevity in notation, the sequence $\{\dot{y}^{Nk}(1,t)\}$ is still denoted by $\{\dot{y}^N(1,t)\}$. Thus,

$$
\begin{aligned}
\theta_N(t) &= \int_0^t \dot{y}^N(1,s)\sin s\,\mathrm{d}s + \theta_0 \to \theta(t)\\
&= \int_0^t \dot{y}(1,s)\sin s\,\mathrm{d}s + \theta_0 \text{ as } N\to\infty. \qquad (2.19)
\end{aligned}
$$

Moreover, since $\dot{y}^N(1,\cdot)\in L^\infty(0,T)$ and

$$\dot{y}^N(1,t)\to\dot{y}(1,t) \text{ in } L^2(0,T)$$

it follows that $[\dot{y}^N(1,t)]^2 \rightharpoonup \dot{y}^2(1,t)$ in $L^2(0,T)$ as $N\to\infty$. Therefore, for any $t\geqslant 0$

$$k_N(t)=r\int_0^t[\dot{y}^N(1,s)]^2\,\mathrm{d}s \to k(t)=r\int_0^t \dot{y}^2(1,s)\,\mathrm{d}s \text{ as } N\to\infty. \qquad (2.20)$$

Now multiplying the first equation of (2.10) by $\phi_n$ and summating for $n = 1, 2, \ldots, N$ and then taking inner product with $\phi \in D(A^{1/2})$, we obtain

$$\langle \ddot{y}^N, \phi \rangle_{L^2} + \langle y^N_{xx}, \phi'' \rangle_{L^2} = \{-k_N(t)\dot{y}^N(1,t) - [\theta_N(t) - \tilde{\theta}] \sin t\} \times \sum_{n=1}^{N} \langle \phi_n, \phi \rangle \phi_n(1) \quad \forall \phi \in D(A^{1/2}).$$

Next, for any distribution $\psi \in \mathscr{D}(0, T)$, taking inner product with $\psi$ on both sides of equality above thus gives

$$\begin{aligned}&\int_0^T \langle \ddot{y}^N, \phi \rangle_{L^2} \psi(t)\,\mathrm{d}t + \int_0^T \langle y^N_{xx}, \phi'' \rangle_{L^2} \psi(t)\,\mathrm{d}t \\ &\quad = \int_0^T \{-k_N(t)\dot{y}^N(1,t) - [\theta_N(t) - \tilde{\theta}] \sin t\} \psi(t)\,\mathrm{d}t \sum_{n=1}^{N} \langle \phi_n, \phi \rangle \phi_n(1) \\ &\quad\quad \forall \phi \in D(A^{1/2}), \psi \in \mathscr{D}(0, T).\end{aligned}$$

Letting $N \to \infty$ produces

$$\begin{aligned}&\int_0^T \langle \ddot{y}, \phi \rangle_{L^2} \psi(t)\,\mathrm{d}t + \int_0^T \langle y_{xx}, \phi'' \rangle_{L^2} \psi(t)\,\mathrm{d}t \\ &\quad = \int_0^T \{-k(t)\dot{y}(1,t) - [\theta(t) - \tilde{\theta}] \sin t\} \psi(t)\,\mathrm{d}t\, \phi(1) \\ &\quad\quad \forall \phi \in D(A^{1/2}), \psi \in \mathscr{D}(0, T).\end{aligned}$$

Hence,

$$\begin{aligned}&\langle \ddot{y}, \phi \rangle_{L^2} + \langle y_{xx}, \phi'' \rangle_{L^2} = \{-k(t)\dot{y}(1,t) - [\theta(t) - \tilde{\theta}] \sin t\} \phi(1) \\ &\quad \forall \phi \in D(A^{1/2}) \text{ and } t \in [0, T] \text{ a.e.}\end{aligned} \tag{2.21}$$

Now, take particularly $\phi$ to be the distribution, i.e., $\phi \in \mathscr{D}(0, 1)$. We have

$$\langle \ddot{y}, \phi \rangle_{L^2} + \langle y_{xx}, \phi'' \rangle_{L^2} = 0 \quad \forall \phi \in \mathscr{D}(0, 1). \tag{2.22}$$

This shows that the generalized derivative $y_{xxxx}$ exists and

$$y_{xxxx}(\cdot, t) = y_{tt}(\cdot, t) \in L^2(0, 1) \quad \text{for } t \in [0, T] \text{ a.e.}$$

Hence $y(\cdot, t) \in H^4(0, 1)$. Since $y_{xxx}(\cdot, t) \in L^2(0, 1)$, integrating (2.1) by parts over $x \in [0, 1]$ yields

$$\begin{aligned}&\langle \ddot{y}, \phi \rangle_{L^2} + y_{xx}(1)\phi_x(1) - y_{xxx}(1)\phi(1) + \langle y_{xxxx}, \phi \rangle_{L^2} \\ &\quad = \{-k(t)\dot{y}(1,t) - [\theta(t) - \tilde{\theta}] \sin t\} \phi(1) \\ &\quad\quad \forall \phi \in D(A^{1/2}) \text{ and } t \in [0, T] \text{ a.e.}\end{aligned} \tag{2.23}$$

Note that $y_{xxxx}(x,t)+y_{tt}(x,t)=0$. It follows from (2.23) that

$$y_{xxx}(1,t)\phi(1)=\{k(t)\dot{y}(1,t)+[\theta(t)-\tilde{\theta}]\sin t\}\phi(1),\quad y_{xx}(1)\phi_x(1)=0$$
$$\forall\phi\in D(A^{1/2}).$$

Hence,

$$y_{xxx}(1,t)=k(t)\dot{y}(1,t)+[\theta(t)-\tilde{\theta}]\sin t,\quad y_{xx}(1)=0.$$

This together with the fact that $y(\cdot,t)\in H^4(0,1)$ shows that $y(\cdot,t)\in H^4\cap H_E^3$.

It remains to show that the initial conditions are satisfied by $y$. As

$$y^N\to y,\quad \dot{y}_N\to\dot{y}\ \text{in}\ L^2(0,T;L^2(0,1)),$$

it follows from Lemma 5 of [1] that $\langle y^N(\cdot,0),\phi\rangle_{L^2}\to\langle y(\cdot,0),\phi\rangle_{L^2}$ for every $\phi\in L^2(0,1)$. By virtue of (2.9), $y(x,0)=y_0(x)$. Similarly, from the fact that

$$\ddot{y}^N\rightharpoonup\ddot{y},\quad \dot{y}^N\rightharpoonup\dot{y}\ \text{weakly in}\ L^2(0,T;L^2(0,1)),$$

we obtain $\dot{y}(x,0)=y_1(x)$. This completes the proof. □

## 3. Smoother solutions

This section presents the smoother solution to (1.5) under the smooth assumption and compatible conditions for initial datum. To do so, we need the following Bihari's inequality [2].

**Lemma 3.1.** *If $L,M\geqslant 0$ and $g(s)>0$ is nondecreasing for $s>0$, then the inequality*

$$u(t)\leqslant L+M\int_a^t v(s)g(u(s))\,\mathrm{d}s,\quad a<t<b$$

*implies that*

$$u(t)\leqslant G^{-1}\left[G(L)+M\int_0^t v(s)\,\mathrm{d}s\right],$$

*where $G(u)=\int_{u_0}^u\frac{1}{g(t)}\,\mathrm{d}t$, $u>u_0>0$.*

**Theorem 3.1.** *For any given $T>0$, let the condition of Theorem 2.1 be satisfied, and assume further that*

$$y_0\in D(A^{3/2})=\{y\in H^6(0,1)|y(0)=y'(1)=y''(1)=y'''(1)$$
$$=y^{(4)}(0)=y^{(5)}(0)=0\},\quad y_1\in D(A)$$

*and compatible condition*

$$-ry_1^3(1)=\theta_0-\tilde{\theta}. \tag{3.1}$$

Then the solution of equation (1.5) has the regularity property: $\ddot{y}\in L^\infty(0,T;[D(A^{1/2})])$.

**Proof.** The proof depends on the estimate of $y'''(0)$. To this end, transform (1.5) into an equivalent problem with zero initial values by the following transformation:

$$v(x,t) = y(x,t) - u(x,t), \tag{3.2}$$

$$u(x,t) = y_0(x) + t y_1(x). \tag{3.3}$$

Then $v$ satisfies

$$\begin{cases} v_{tt}(x,t) + v_{xxxx}(x,t) + u_{xxxx}(x,t) = 0, \quad x \in (0,1), \quad t \geqslant 0, \\ v(0,t) = v_x(0,t) = v_{xx}(1,t) = 0, \quad t \geqslant 0, \\ v_{xxx}(1,t) = k(t)[v_t(1,t) + y_1(1)] + [\theta(t) - \tilde{\theta}] \sin t - u_{xxx}(1,t), \quad t \geqslant 0, \\ v(x,0) = 0, \quad v_t(x,0) = 0, \\ \dot{k}(t) = r[v_t(1,t) + y_1(1)]^2, \quad k(0) = 0, \quad r > 0, \\ \dot{\theta}(t) = [v_t(1,t) + y_1(1)] \sin t, \quad \theta(0) = \theta_0. \end{cases} \tag{3.4}$$

$v$ is a solution of (3.4) if and only if $y = v + u$ is a solution of (1.5). Once again, we adopt the Galerkin approximation. Still choose $\{\phi_n\}_{n=1}^{\infty}$ defined by (2.2) which form an orthogonal basis for $L^2(0,1)$.

Let $V_N$ be the space generated by $\phi_1, \phi_2, \ldots, \phi_N$ and let

$$v^N(x,t) = \sum_{n=1}^{N} f_{nN}(t)\phi_n(x) \tag{3.5}$$

be the Galerkin approximation of $v$, which satisfies the following equation:

$$\begin{cases} \langle \ddot{v}^N(\cdot,t), \phi \rangle_{L^2} + \langle v^N_{xx}(\cdot,t), \phi_{xx} \rangle_{L^2} + \langle u_{xx}(\cdot,t), \phi_{xx} \rangle_{L^2} \\ \quad = \{-k^N(t)[\dot{v}^N(1,t) + y_1(1)] + [\theta^N(t) - \tilde{\theta}] \sin t\}\phi(1) \ \forall \phi \in V_N, \\ v^N(\cdot,0) = \dot{v}^N(\cdot,0) = 0, \\ \dot{k}^N(t) = r\left[\sum_{n=1}^{N} \dot{f}_{nN}(t)\phi_n(1) + y_1(1)\right]^2, \ k^N(0) = 0, \\ \dot{\theta}^N(t) = \left[\sum_{n=1}^{N} \dot{f}_{nN}(t)\phi_n(1) + y_1(1)\right] \sin t, \ \theta^N(0) = \theta_0. \end{cases} \tag{3.6}$$

Once again, there exists a unique classical solution to (3.6) in some $[0, t_N)$, $t_N > 0$ and we split the proof into several lemmas.

The first Lemma 3.2 below shows that one can take $t_N = T$.

**Lemma 3.2.**

$$\sup_N [\|\dot{v}^N(\cdot,t)\|_{L^2} + \|v^N_{xx}(\cdot,t)\|_{L^2} + |\theta^N(t)| + k^N(t)] < \infty, \quad t \in [0,T], \ \text{a.e.} \tag{3.7}$$

**Proof.** Taking $\phi = 2\dot{v}^N(\cdot, t)$ in (3.6) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{v}^N(\cdot,t)\|^2_{L^2} + \frac{\mathrm{d}}{\mathrm{d}t}\|v^N_{xx}(\cdot,t)\|^2_{L^2} + 2\langle u_{xx}(\cdot,t), \dot{v}^N_{xx}(\cdot,t)\rangle_{L^2} \\
&\quad = 2\{-k^N(t)[\dot{v}^N(1,t) + y_1(1)] - [\theta^N(t) - \tilde{\theta}]\sin t\}\dot{v}^N(1,t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|v^N_{xx}(\cdot,t)\|^2_{L^2} + \|\dot{v}^N(\cdot,t)\|^2_{L^2} + \frac{[k^N(t)]^2}{2r} + [\theta^N(t) - \tilde{\theta}]^2 \\
&\quad \leqslant \int_0^t k^N(s) y_1^2(1)\,\mathrm{d}s + 2\int_0^t y_1(1)[\theta^N(s) - \tilde{\theta}]\sin s\,\mathrm{d}s \\
&\qquad - 2\int_0^t \langle u_{xx}(\cdot,s), \dot{v}^N_{xx}(\cdot,s)\rangle_{L^2}\,\mathrm{d}s + (\theta_0 - \tilde{\theta})^2.
\end{aligned} \tag{3.8}$$

From the fact that

$$\langle u_{xx}, \dot{v}^N_{xx}\rangle_{L^2} = \frac{\mathrm{d}}{\mathrm{d}t}\langle u_{xx}, v^N_{xx}\rangle_{L^2} - \langle y_1'', v^N_{xx}\rangle_{L^2}, \tag{3.9}$$

and (3.8), it follows that

$$\begin{aligned}
&\|v^N_{xx}(\cdot,t)\|^2_{L^2} + \|\dot{v}^N(\cdot,t)\|^2_{L^2} + \frac{[k^N(t)]^2}{2r} + [\theta^N(t) - \tilde{\theta}]^2 \\
&\quad \leqslant \int_0^t k^N(s) y_1^2(1)\,\mathrm{d}s + 2\int_0^t y_1(1)[\theta^N(s) - \tilde{\theta}]\sin s\,\mathrm{d}s \\
&\qquad - 2\int_0^t \langle u_{xx}(\cdot,s), \dot{v}^N_{xx}(\cdot,s)\rangle_{L^2}\,\mathrm{d}s + (\theta_0 - \tilde{\theta})^2 \\
&\quad = \int_0^t k^N(s) y_1^2(1)\,\mathrm{d}s + 2\int_0^t y_1(1)[\theta^N(s) - \tilde{\theta}]\sin s\,\mathrm{d}s \\
&\qquad + 2\int_0^t \langle y_1'', v^N_{xx}(\cdot,s)\rangle_{L^2}\,\mathrm{d}s - 2\langle u_{xx}(\cdot,t), v^N_{xx}(\cdot,t)\rangle_{L^2} + (\theta_0 - \tilde{\theta})^2 \\
&\quad \leqslant \int_0^t \left[\frac{[k^N(s)]^2}{2r} + \frac{r}{2}\,y_1^4(1)\right]\mathrm{d}s + \int_0^t [y_1^2(1) + (\theta^N(s) - \tilde{\theta})^2]\,\mathrm{d}s \\
&\qquad + \eta_1\|u_{xx}(\cdot,t)\|^2_{L^2} + \frac{\|v^N_{xx}(\cdot,t)\|^2_{L^2}}{\eta_1} \\
&\qquad + \int_0^t \left[\eta_2\|y_1''\|^2_{L^2} + \frac{\|v^N_{xx}(\cdot,s)\|^2_{L^2}}{\eta_2}\right]\mathrm{d}s + (\theta_0 - \tilde{\theta})^2.
\end{aligned} \tag{3.10}$$

Taking $\eta_1 > 0$ and $\eta_2 > 0$ so that $1 - 1/\eta_1 = 1/\eta_2$ and applying the Grownwall's inequality to (3.10), we obtain

$$\sup_N [\|\dot{v}^N(\cdot,t)\|_{L^2} + \|v^N_{xx}(\cdot,t)\|_{L^2} + |\theta^N(t)| + k^N(t)] < \infty, \quad t \in [0, T], \text{ a.e.}$$

Therefore, (3.6) admits a unique global classical solution in whole $[0, T]$. □

**Lemma 3.3.**

$$\sup_N \|\ddot{v}^N(\cdot,0)\|_{L^2} < \infty. \tag{3.11}$$

**Proof.** Setting $t=0$ in the first equation of (3.6) gives

$$\langle v_{tt}^N(\cdot,0),\phi\rangle_{L^2} + \langle v_{xx}^N(\cdot,0),\phi_{xx}\rangle_{L^2} + \langle u_{xx}(\cdot,0),\phi_{xx}\rangle_{L^2} = 0 \quad \forall\phi\in V_N. \tag{3.12}$$

This together with the fact that $v^N(\cdot,0)=\dot{v}^N(\cdot,0)=0$ leads to

$$\langle v_{tt}^N(\cdot,0),\phi\rangle_{L^2} + \langle y_0'',\phi_{xx}\rangle_{L^2} = 0.$$

Take $\phi = v_{tt}^N(\cdot,0)$ in the above equality thus to obtain $\|\ddot{v}^N(\cdot,0)\|_{L^2} \leqslant \|y_0''''\|_{L^2}$, proving (3.11). □

**Lemma 3.4.**

$$\sup_N [\|\ddot{v}^N(\cdot,t)\|_{L^2} + \|\dot{v}_{xx}^N(\cdot,t)\|_{L^2} + |\dot{v}^N(1,t)|] < \infty \quad \textit{for } t\in[0,T] \text{ a.e.} \tag{3.13}$$

**Proof.** Differentiate the first equation of (3.6) with respect to $t$, to give

$$\begin{aligned}
&\langle v_{ttt}^N(\cdot,t),\phi\rangle_{L^2} + \langle \dot{v}_{xx}^N(\cdot,t),\phi_{xx}\rangle_{L^2} + \langle y_1'',\phi_{xx}\rangle_{L^2} \\
&\quad = \{-r[\dot{v}^N(1,t)+y_1(1)]^3 - k^N(t)\ddot{v}^N(1,t) \\
&\qquad - [\dot{v}^N(1,t)+y_1(1)]\sin^2 t - [\theta^N(t)-\tilde{\theta}]\cos t\}\phi(1).
\end{aligned} \tag{3.14}$$

Substitution of $\phi$ by $v_{tt}^N(\cdot,t)$ in (3.14) results in that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\ddot{v}^N(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + \langle y_1'',\ddot{v}_{xx}^N(\cdot,t)\rangle_{L^2} \\
&\quad = \{-r[\dot{v}^N(1,t)+y_1(1)]^3 - k^N(t)\ddot{v}^N(1,t) - [\dot{v}^N(1,t)+y_1(1)]\sin^2 t \\
&\qquad - [\theta^N(t)-\tilde{\theta}]\cos t\}\ddot{v}^N(1,t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\ddot{v}^N(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + \frac{r}{4}\frac{\mathrm{d}}{\mathrm{d}t}[\dot{v}^N(1,t)+y_1(1)]^4 \\
&\quad + \langle y_1'',\ddot{v}_{xx}^N(\cdot,t)\rangle_{L^2} \\
&\qquad = \{-k^N(t)\ddot{v}^N(1,t) - [\dot{v}^N(1,t)+y_1(1)]\sin^2 t \\
&\qquad\quad - [\theta^N(t)-\tilde{\theta}]\cos t\}\ddot{v}^N(1,t).
\end{aligned}$$

Integrating over $[0, t]$ then gives

$$
\begin{aligned}
&\frac{1}{2}\|\ddot{v}^N(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + \frac{r}{4}\,[\dot{v}^N(1,t)+y_1(1)]^4 \\
&\quad = -\int_0^t k^N(s)[\ddot{v}^N(1,s)]^2\,\mathrm{d}s - \frac{1}{2}\,[\dot{v}^N(1,t)+y_1(1)]^2\sin^2 t \\
&\qquad + \frac{1}{2}\int_0^t [\dot{v}^N(1,s)+y_1(1)]^2\sin 2s\,\mathrm{d}s \\
&\qquad - \dot{v}^N(1,t)[\theta^N(t)-\tilde{\theta}]\cos t + \dot{v}^N(1,0)[\theta^N(0)-\tilde{\theta}] \\
&\qquad + \int_0^t \dot{v}^N(1,s)\{[\dot{v}^N(1,s)+y_1(1)]\sin s\,\cos s - [\theta^N(s)-\tilde{\theta}]\sin s\}\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\|\ddot{v}^N(\cdot,0)\|_{L^2}^2 + \frac{r}{4}\,[y_1(1)]^4 - \langle y_1'', \dot{v}_{xx}^N(\cdot,t)\rangle_{L^2} \\
&\quad \leqslant \frac{3}{2}\int_0^t [\dot{v}^N(1,s)+y_1(1)]^2\,\mathrm{d}s + \xi_1[\dot{v}^N(1,t)]^2 + \frac{1}{4\xi_1}\,[\theta^N(t)-\tilde{\theta}]^2 \\
&\qquad - \int_0^t y_1(1)[\dot{v}^N(1,s)+y_1(1)]\sin s\,\cos s\,\mathrm{d}s \\
&\qquad - \int_0^t \dot{v}^N(1,s)[\theta^N(s)-\tilde{\theta}]\sin s\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\|\ddot{v}^N(\cdot,0)\|_{L^2}^2 + \frac{r}{4}[y_1(1)]^4 + \|y_1''\|_{L^2}\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2} \\
&\quad \leqslant \frac{3k^N(t)}{2r} + \xi_1\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + \frac{1}{4\xi_1}\,[\theta^N(t)-\tilde{\theta}]^2 \\
&\qquad - y_1(1)\cos t[\theta^N(t)-\tilde{\theta}] + y_1(1)(\theta_0-\tilde{\theta}) \\
&\qquad + \frac{1}{2}\,(\theta_0-\tilde{\theta})^2 + \frac{1}{2}\|\ddot{v}^N(\cdot,0)\|_{L^2}^2 + \frac{r}{4}\,[y_1(1)]^4 \\
&\qquad + \frac{1}{4\xi_2}\|y_1''\|_{L^2}^2 + \xi_2\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 \\
&\quad \leqslant \frac{3k^N(t)}{2r} + \xi_1\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + \left(\frac{1}{4\xi_1}+\frac{1}{2}\right)[\theta^N(t)-\tilde{\theta}]^2 + \|y_1'\|_{L^2}^2 \\
&\qquad + (\theta_0-\tilde{\theta})^2 + \frac{1}{2}\|\ddot{v}^N(\cdot,0)\|_{L^2}^2 + \frac{r}{4}\,[y_1(1)]^4 \\
&\qquad + \frac{1}{4\xi_2}\|y_1''\|_{L^2}^2 + \xi_2\|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2.
\end{aligned}
\tag{3.15}
$$

Take $\xi_1 > 0$ and $\xi_2 > 0$ so that $\frac{1}{2} - (\xi_1+\xi_2) > \frac{1}{2}$ in (3.15) and note that Lemmas 3.2 and 3.3, we get

$$
\sup_N [\|\ddot{v}^N(\cdot,t)\|_{L^2} + \|\dot{v}_{xx}^N(\cdot,t)\|_{L^2} + |\dot{v}^N(1,t)|] < \infty \quad \text{for } t \in [0,T] \text{ a.e.}
$$

This gives the required result. □

**Lemma 3.5.**

$$\sup_N \|\ddot{v}^N_{xx}(\cdot, t)\|_{L^2} < \infty \quad \textit{for } t \in [0, T] \text{ a.e.}$$

**Proof.** Differentiating the first equation of (3.14) with respect to $t$ leads to

$$\begin{aligned}&\langle v^N_{tttt}(\cdot, t), \phi\rangle_{L^2} + \langle \ddot{v}^N_{xx}(\cdot, t), \phi_{xx}\rangle_{L^2}\\ &\quad = \bigg\{ -4r[\dot{v}^N(1,t) + y_1(1)]^2 \ddot{v}^N(1,t) - k^N(t)\ddot{v}^N_t(1,t) - \ddot{v}^N(1,t)\sin^2 t\\ &\qquad - \frac{3}{2}[\dot{v}^N(1,t) + y_1(1)]\sin 2t + [\theta^N(t) - \tilde{\theta}]\sin t\bigg\}\phi(1). \qquad (3.16)\end{aligned}$$

Again, substitution of $\phi$ by $v^N_{ttt}(\cdot, t)$ in (3.16) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\ddot{v}^N_t(\cdot,t)\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\ddot{v}^N_{xx}(\cdot,t)\|^2_{L^2}\\ &\quad = \bigg\{ -4r[\dot{v}^N(1,t) + y_1(1)]^2 \ddot{v}^N(1,t) - k^N(t)\ddot{v}^N_t(1,t) - \ddot{v}^N(1,t)\sin^2 t\\ &\qquad - \frac{3}{2}[\dot{v}^N(1,t) + y_1(1)]\sin 2t + [\theta^N(t) - \tilde{\theta}]\sin t\bigg\}\ddot{v}^N_t(1,t).\end{aligned}$$

Integrate over $[0, t]$ thus to give

$$\begin{aligned}\|\ddot{v}^N_t(\cdot,t)\|^2_{L^2} + \|\ddot{v}^N_{xx}(\cdot,t)\|^2_{L^2} = {} & -2\int_0^t k^N(s)[\ddot{v}^N_t(1,s)]^2\,\mathrm{d}s\\ & - 8r\int_0^t \ddot{v}^N_t(1,s)[\dot{v}^N(1,s) + y_1(1)]^2\ddot{v}^N(1,s)\,\mathrm{d}s\\ & - 2\int_0^t \ddot{v}^N_t(1,s)\ddot{v}^N(1,s)\sin^2 s\,\mathrm{d}s\\ & - 3\int_0^t \ddot{v}^N_t(1,s)[\dot{v}^N(1,s) + y_1(1)]\sin 2s\,\mathrm{d}s\\ & + 2\int_0^t \ddot{v}^N_t(1,s)[\theta^N(s) - \tilde{\theta}]\sin s\,\mathrm{d}s\\ & + \|\ddot{v}^N_{xx}(\cdot,0)\|^2_{L^2} + \|\ddot{v}^N_t(\cdot,0)\|^2_{L^2}\\ = {} & \sum_{i=1}^7 I_i. \qquad (3.17)\end{aligned}$$

Now estimate each term on the right-hand side of (3.17).

$$
\begin{aligned}
I_2 = & -8r\int_0^t \ddot{v}_t^N(1,s)[\dot{v}^N(1,s)+y_1(1)]^2\ddot{v}^N(1,s)\,\mathrm{d}s\\
&\leqslant 4r[\ddot{v}^N(1,0)y_1(1)]^2+8r\int_0^t[\ddot{v}^N(1,s)]^3[\dot{v}^N(1,s)+y_1(1)]\,\mathrm{d}s\\
&\leqslant 4r\|\ddot{v}_x^N(\cdot,0)\|_{L^2}^2\|y_1'\|_{L^2}^2+4r\int_0^t\|\ddot{v}_x^N(\cdot,s)\|_{L^2}^6+4r\int_0^t[\dot{v}^N(1,s)+y_1(1)]^2\,\mathrm{d}s\\
&\leqslant 4r\|\ddot{v}_x^N(\cdot,0)\|_{L^2}^2\|y_1'\|_{L^2}^2+4r\int_0^t\|\ddot{v}_{xx}^N(\cdot,s)\|_{L^2}^6+4k^N(t);
\end{aligned}
$$

$$
\begin{aligned}
I_3 = & -2\int_0^t \ddot{v}_t^N(1,s)\ddot{v}^N(1,s)\sin^2 s\,\mathrm{d}s=-[\ddot{v}^N(1,t)\sin t]^2\\
&+\int_0^t[\ddot{v}^N(1,s)]^2\sin 2s\,\mathrm{d}s\\
&\leqslant\int_0^t\|\ddot{v}_x^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s\leqslant\int_0^t\|\ddot{v}_{xx}^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s;
\end{aligned}
$$

$$
\begin{aligned}
I_4 = & -3\int_0^t \ddot{v}_t^N(1,s)[\dot{v}^N(1,s)+y_1(1)]\sin 2s\,\mathrm{d}s\\
= & -3\ddot{v}^N(1,t)[\dot{v}^N(1,t)+y_1(1)]\sin 2t\\
&+3\int_0^t[\ddot{v}^N(1,s)]^2\sin 2s\,\mathrm{d}s+6\int_0^t\ddot{v}^N(1,s)[\dot{v}^N(1,s)+y_1(1)]\cos 2s\,\mathrm{d}s\\
&\leqslant\frac{3}{2}\sigma_1\|\ddot{v}_x^N(\cdot,t)\|_{L^2}^2+\frac{3}{2\sigma_1}\|\dot{v}_x^N(\cdot,t)+y_1'\|_{L^2}^2+6\int_0^t\|\ddot{v}_x^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s\\
&+3\int_0^t[\dot{v}^N(1,s)+y_1(1)]^2\,\mathrm{d}s;\\
&=\frac{3}{2}\sigma_1\|\ddot{v}_{xx}^N(\cdot,t)\|_{L^2}^2+\frac{3}{2\sigma_1}\|\dot{v}_x^N(\cdot,t)+y_1'\|_{L^2}^2\\
&+6\int_0^t\|\ddot{v}_x^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s+\frac{3}{r}k^N(t);
\end{aligned}
$$

$$
\begin{aligned}
I_5 = & 2\int_0^t \ddot{v}_t^N(1,s)[\theta^N(s)-\tilde{\theta}]\sin s\,\mathrm{d}s\\
&=2\ddot{v}^N(1,t)[\theta^N(t)-\tilde{\theta}]\sin t-2\int_0^t\ddot{v}^N(1,s)[\dot{v}^N(1,s)+y_1(1)]\sin^2 s\,\mathrm{d}s\\
&-2\int_0^t\ddot{v}^N(1,s)[\theta^N(s)-\tilde{\theta}]\cos s\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sigma_2 \|\ddot{v}_x^N(\cdot,t)\|_{L^2}^2 + \frac{1}{\sigma_2}[\theta^N(t)-\tilde{\theta}]^2 \\
&\quad + \int_0^t \|\ddot{v}_x^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s + \int_0^t [\dot{v}^N(1,s)+y_1(1)]^2\,\mathrm{d}s \\
&\quad - 2\dot{v}^N(1,t)[\theta^N(t)-\tilde{\theta}]\cos t + 2\dot{v}^N(1,0)[\theta_0-\tilde{\theta}] \\
&\quad + 2\int_0^t \dot{v}^N(1,s)\{[\dot{v}^N(1,s)+y_1(1)]\sin s\,\cos s - [\theta^N(s)-\tilde{\theta}]\sin s\}\,\mathrm{d}s \\
&\leqslant \sigma_2 \|\ddot{v}_x^N(\cdot,t)\|_{L^2}^2 + \frac{1}{\sigma_2}[\theta^N(t)-\tilde{\theta}]^2 + \int_0^t \|\ddot{v}_x^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s \\
&\quad + \int_0^t [\dot{v}^N(1,s)+y_1(1)]^2\,\mathrm{d}s + [\dot{v}^N(1,t)]^2 \\
&\quad + [\theta^N(t)-\tilde{\theta}]^2 + 2\int_0^t [\dot{v}^N(1,s)+y_1(1)]^2 \sin s\,\cos s\,\mathrm{d}s \\
&\quad - 2y_1(1)\int_0^t [\dot{v}^N(1,s)+y_1(1)]\sin s\,\cos s\,\mathrm{d}s \\
&\quad - 2\int_0^t [\dot{v}^N(1,s)+y_1(1)][\theta^N(s)-\tilde{\theta}]\sin s\,\mathrm{d}s \\
&\quad + 2y_1(1)\int_0^t [\theta^N(s)-\tilde{\theta}]\sin s\,\mathrm{d}s \\
&\leqslant \sigma_2 \|\ddot{v}_x^N(\cdot,t)\|_{L^2}^2 + \left(1+\frac{1}{\sigma_2}\right)[\theta^N(t)-\tilde{\theta}]^2 + \int_0^t \|\ddot{v}_x^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s + \frac{3k^N(t)}{r} \\
&\quad + \|\dot{v}_x^N(\cdot,t)\|_{L^2}^2 - 2y_1(1)\cos t[\theta^N(t)-\theta] + 2y_1(1)[\theta_0-\tilde{\theta}] + [\theta_0-\tilde{\theta}]^2 \\
&\leqslant \sigma_2 \|\ddot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + \left(2+\frac{1}{\sigma_2}\right)[\theta^N(t)-\tilde{\theta}]^2 + \int_0^t \|\ddot{v}_{xx}^N(\cdot,s)\|_{L^2}^2\,\mathrm{d}s + \frac{3k^N(t)}{r} \\
&\quad + \|\dot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 + 2\|y_1'\|_{L^2} + 2[\theta_0-\tilde{\theta}]^2;
\end{aligned}
$$

$$
I_6 = \|\ddot{v}_{xx}^N(\cdot,0)\|_{L^2}^2 \leqslant C_1 \|y_0^{(6)}\|_{L^2}^2
$$

where $C_1 > 0$ is a constant independent of $N$. Now it remains to estimate $I_7$. From (3.16), it follows that

$$
\langle v_{ttt}^N(\cdot,0),\phi\rangle_{L^2} + \langle y_1'',\phi_{xx}\rangle_{L^2} = \{-r[y_1(1)]^3 - [\theta_0-\tilde{\theta}]\}\phi(1). \tag{3.18}
$$

This together with the compatible condition (3.1) shows that

$$
I_7 = \|\ddot{v}_t^N(\cdot,0)\|_{L^2} \leqslant \|y_1^{(4)}\|_{L^2}.
$$

By these estimates justified and Lemmas 3.2 and 3.4, we obtain, from (3.17), that

$$
\begin{aligned}
&\|\ddot{v}_t^N(\cdot,t)\|_{L^2}^2 + \|\ddot{v}_{xx}^N(\cdot,t)\|_{L^2}^2 \\
&\quad \leqslant C + \int_0^t [4r\|\ddot{v}_{xx}^N(\cdot,s)\|_{L^2}^6 + 8\|\ddot{v}_{xx}^N(\cdot,s)\|_{L^2}^2]\,\mathrm{d}s + \left(\frac{3\sigma_1}{2}+\sigma_2\right)\|\ddot{v}_{xx}^N(\cdot,t)\|_{L^2}^2,
\end{aligned}
$$

where $C$ is a constant independent of $N$ but depending on initial date. Taking $\sigma_1 > 0$ and $\sigma_2 > 0$ so that $1 - (3\sigma_1/2 + \sigma_2) > 1/2$, we get

$$\|\ddot{v}_{xx}^N(\cdot, t)\|_{L^2}^2 \leqslant 2C + 8\int_0^t [r\|\ddot{v}_{xx}^N(\cdot, s)\|_{L^2}^6 + 2\|\ddot{v}_{xx}^N(\cdot, s)\|_{L^2}^2]\,\mathrm{d}s. \tag{3.19}$$

Apply Lemma 3.1 to (3.19) with $L = 2C,\ M = 8,\ v = 1,\ g(s) = rs^3 + 2s$ to produce $\sup_N \|\ddot{v}_{xx}^N(\cdot, t)\|_{L^2} < \infty$ uniformly for almost every $t \in [0, T]$. The proof is complete. □

## 4. Asymptotic stability

In this section, we establish the convergence of our adaptive regulator system (1.5). To do this, we need the weak solution of (1.5).

**Definition 4.1.** For any initial data $(y_0, y_1, \theta_0) \in D(A^{1/2}) \times L^2(0, 1) \times \mathbb{R}$, the weak solution $(y, y_t, k, \theta)$ of equation (1.5) is defined as the limit of any convergent subsequence of $(y^n, y_t^n, k^n, \theta^n)$ in the space $L^\infty(0, \infty; [D(A^{1/2})] \times L^2(0, 1) \times \mathbb{R}^2)$ where $(y^n, y_t^n, k^n, \theta^n)$ is the classical solution ensured by Theorem 2.1 with the initial condition

$$\begin{aligned}&(y^n(x, 0), y_t^n(x, 0), \theta^n(0)) = (y_{0n}(x), y_{1n}(x), \theta_{0n}) \in D(A^{3/2}) \times D(A) \times \mathbb{R}\\ &\quad \forall x \in (0, 1)\end{aligned}$$

satisfying

$$\lim_{n\to\infty} \|(y_{n0}, y_{1n}, \theta_{0n}) - (y_0, y_1, \theta_0)\|_{[D(A^{1/2})]\times L^2(0,1)\times\mathbb{R}} = 0.$$

Definition 4.1 makes sense because from (2.4) and (2.5) we know that $\{(y^n, y_t^n, k^n, \theta^n)\}$ must be a Cauchy sequence in $L^\infty(0, \infty; [D(A^{1/2})] \times L^2(0, 1) \times \mathbb{R}^2)$ and its limit does not depend on the choice of initial datum.

**Theorem 4.1.** *For any initial condition* $(y_0, y_1) \in D(A^{1/2}) \times L^2(0, 1)$, *the solution of the system* (1.5) *is asymptotically stable, that is to say*

$$\begin{cases} \lim\limits_{t\to+\infty} E(t) = \lim\limits_{t\to+\infty} \frac{1}{2}\int_0^1 [y_t^2(x, t) + y_{xx}^2(x, t)]\,\mathrm{d}x = 0;\\ \sup\limits_{t\geqslant 0} k(t) < \infty;\\ \lim\limits_{t\to+\infty} \theta(t) = \tilde{\theta}. \end{cases} \tag{4.1}$$

**Proof.** In terms of density argument, we may regard without loss of generality that the initial value $(y_0, y_1, \theta_0)$ belongs to $\in D(A^{3/2}) \times D(A) \times \mathbb{R}$. Construct Lyapunov functional $U(t)$ for the system (1.5) as follows:

$$U(t) = \int_0^1 [y_t^2(x, t) + y_{xx}^2(x, t)]\,\mathrm{d}x + \frac{k^2(t)}{2r} + [\theta(t) - \hat{\theta}]^2. \tag{4.2}$$

Then the time derivative of $U(t)$ along the solution of system (1.5) satisfies

$$\begin{aligned}\dot{U}(t) &= -2y_{xxx}(1,t)y_t(1,t) + k(t)y_t^2(1,t) + 2[\theta(t)-\tilde{\theta}]\sin t \cdot y_t(1,t)\\ &= 2\{-k(t)y_t(1,t) - [\theta(t)-\tilde{\theta}]\sin t\}y_t(1,t) + k(t)y_t^2(1,t)\\ &\quad + 2[\theta(t)-\tilde{\theta}]\sin t \cdot y_t(1,t)\\ &= -k(t)y_t^2(1,t).\end{aligned}$$

This shows that $U(t) \leqslant U(0)$ and hence

$$\sup_{t \geqslant 0} [E(t) + |\theta(t)| + k(t)] < \infty, \tag{4.3}$$

which in particular deduces, from the definition of $k(t)$ in (1.5), that

$$y_t(1,\cdot) \in L^2(0,\infty). \tag{4.4}$$

Similarly, let

$$\bar{U}(t) = \frac{1}{2}\int_0^1 [y_{xxxx}^2(x,t) + y_{xxt}^2(x,t)]\,\mathrm{d}x + \frac{r}{4}y_t^4(1,t).$$

It is found that the time derivative of $\bar{U}(t)$ along the solution of system (1.5) can be estimated as

$$\begin{aligned}\dot{\bar{U}}(t) &= y_{tt}(1,t)\{-k(t)y_{tt}(1,t) - y_t(1,t)\sin^2 t - [\theta(t)-\tilde{\theta}]\cos t\}\\ &= -k(t)y_{tt}^2(1,t) - y_{tt}(1,t)y_t(1,t)\sin^2 t - y_{tt}(1,t)[\theta(t)-\tilde{\theta}]\cos t\\ &\leqslant -y_{tt}(1,t)y_t(1,t)\sin^2 t - y_{tt}(1,t)[\theta(t)-\tilde{\theta}]\cos t.\end{aligned}$$

Hence,

$$\begin{aligned}\bar{U}(t) &\leqslant -\int_0^t y_t(1,s)y_{tt}(1,s)\sin^2 s\,\mathrm{d}s - \int_0^t y_{tt}(1,s)[\theta(s)-\tilde{\theta}]\cos s\,\mathrm{d}s + \bar{U}(0)\\ &= -\frac{1}{2}y_t^2(1,t)\sin^2 t + \int_0^t y_t^2(1,s)\sin s\cos s\,\mathrm{d}s - y_t(1,t)[\theta(t)-\tilde{\theta}]\cos t\\ &\quad + y_t(1,0)[\theta_0-\tilde{\theta}]\\ &\quad + \int_0^t y_t(1,s)\{y_t(1,s)\sin s\cos s - [\theta(s)-\tilde{\theta}]\sin s\}\,\mathrm{d}s + \bar{U}(0)\\ &\leqslant 2\int_0^t y_t^2(1,s)\,\mathrm{d}s + \zeta y_t^2(1,t) + \frac{1}{4\zeta}[\theta(t)-\tilde{\theta}]^2\\ &\quad + \frac{1}{2}y_t^2(1,0) + [\theta_0-\tilde{\theta}]^2 + \bar{U}(0)\\ &\leqslant \frac{2k(t)}{r} + \zeta\|y_{xxt}(x,t)\|_{L^2}^2 + \frac{1}{4\zeta}[\theta(t)-\tilde{\theta}]^2 + \frac{1}{2}\|y_1'\|_{L^2}^2 + [\theta_0-\tilde{\theta}]^2 + \bar{U}(0),\end{aligned}$$

where $0 < \zeta < 1/2$ is constant. This together with (4.3) shows that

$$\sup_{t \geqslant 0} \bar{U}(t) < C < \infty \tag{4.5}$$

for some constant $C$ depending on $\|(y_0, y_1)\|_{[D(A)]\times[D(A^{1/2})]}$. Therefore, the trajectory

$$\gamma(y_0, y_1, 0, \theta_0) = \{(y(\cdot, t), \dot{y}(\cdot, t), k(t), \theta(t))|t \geqslant 0\}$$

is precompact in $D(A^{1/2}) \times L^2 \times \mathbb{R}^2$. In light of Lasalle's principle [9], the solution of (1.5) converges asymptotically to the maximal invariant subset of the following set:

$$S = \{(y, \dot{y}, k, \theta) \in [D(A^{1/2})] \times L^2 \times \mathbb{R}^2 | \dot{U} = 0\}. \tag{4.6}$$

The proof will be accomplished if we can show that the set $S$ contains single point $(0, 0, \tilde{k}, \tilde{\theta})$ only. However, when $\dot{U} = 0$ one has $k(t)y_t^2(1, t) = 0$. Since $k(t)$ is nondecreasing and bounded, we may assume that

$$\lim_{t\to\infty} k(t) = \hat{k}.$$

If $\hat{k} = 0$, then it follows from (1.5) that $k(t) \equiv y_t(1, t) \equiv 0$. However, as $\hat{k} \neq 0$, $y_t(1, t) = 0$ for sufficiently large $t$. In any case, there exists a time $t_0 > 0$ such that when $t > t_0$ both $k(t)$ and $\theta(t)$ remain constants: $k(t) = \hat{k}$, $\theta(t) = \hat{\theta}$, and (1.5) reduces to

$$\begin{cases} y_{tt}(x, t) - y_{xxxx}(x, t) = 0, & x \in (0, 1), \quad t > t_0, \\ y(0, t) = y_x(0, t) = y_{xx}(1, t) = y_t(1, t) = 0, \\ y_{xxx}(1, t) = (\hat{\theta} - \tilde{\theta}) \sin t. \end{cases} \tag{4.7}$$

In order to show that system (4.7) admits only zero solution, we assume without loss of generality that the initial values for equation (4.7) are smooth enough. First, we consider the solution of the equation

$$\begin{cases} y_{tt}(x, t) - y_{xxxx}(x, t) = 0, & x \in (0, 1), \\ y(0, t) = y_x(0, t) = y_{xx}(1, t) = y_t(1, t) = 0. \end{cases} \tag{4.8}$$

Introduce a Hilbert space $\mathbf{H} = [D(A^{1/2})] \times L^2(0, 1)$ with the inner product

$$\langle (y_1, z_1), (y_2, z_2) \rangle = \int_0^1 [y_1''(x)\overline{y_2''(x)} + z_1(x)\overline{z_2(x)}]\, dx.$$

Define a linear operator $A_0$ associated to the system (4.8)

$$\begin{cases} A_0(y, z) & = & (z, -y''''), \\ D(A_0) & = & \{(y, z) \in H^4(0, 1) \times H^2(0, 1), y(0) = y'(0) = y''(1) = 0 \\ & & \quad z(0) = z(1) = z'(0) = 0\}. \end{cases} \tag{4.9}$$

The operator $A_0$ is skew-adjoint with compact resolvent on $\mathbf{H}$.

**Lemma 4.1.** *There is a family of eigenvalues $\{\lambda_n = i\omega_n^2, -i\omega_n^2\}$ of $A_0$, which have the following asymptotic expansion*

$$\lambda_n = i\omega_n^2, \quad \omega_n = \left(n + \frac{1}{4}\right)\pi + O(e^{-n}).$$

And the corresponding eigenfunctions have the asymptotic expansion:

$$\begin{aligned} f_n(x) = {} & \mathrm{e}^{-(n+1/4)\pi x} - \cos\left(n+\frac{1}{4}\right)\pi x - \sin\left(n+\frac{1}{4}\right)\pi x \\ & - \cos\left(n+\frac{1}{4}\right)\pi \mathrm{e}^{-(n+1/4)\pi(1-x)} \\ & - \sin\left(n+\frac{1}{4}\pi\right)\mathrm{e}^{-(n+1/4\pi)(1-x)} + \mathrm{O}(n^{-1}) \end{aligned}$$

$$\begin{aligned} \lambda_n^{-1}\omega_n^{-1} f_n''''(x) = {} & -\mathrm{i}\Bigg[-\mathrm{e}^{-(n+1/4)\pi x} - \cos\left(n+\frac{1}{4}\right)\pi x - \sin\left(n+\frac{1}{4}\right)\pi x \\ & - \cos\left(n+\frac{1}{4}\right)\pi \mathrm{e}^{-(n+1/4)\pi(1-x)} \\ & - \sin\left(n+\frac{1}{4}\pi\right)\mathrm{e}^{-(n+1/4\pi)(1-x)}\Bigg] + \mathrm{O}(n^{-1}) \end{aligned} \tag{4.10}$$

which hold uniformly pointwise for $x \in [0, 1]$.

**Proof.** Solving the eigenvalue problem

$$A_0(f, g) = \lambda(f, g)$$

one has $g = \lambda f$ with $f \neq 0$ satisfying

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ f(0,t) = f'(0) = f(1) = f''(1) = 0. \end{cases} \tag{4.11}$$

Let $\lambda = \mathrm{i}\omega^2$. Then the equation

$$\begin{cases} f^{(4)}(x) - \omega^4 / f(x) = 0, \\ f(0) = f'(0) = 0 \end{cases} \tag{4.12}$$

has the general solution

$$f(x) = c_1(\cosh \omega x - \cos \omega x) + c_2(\sinh \omega x - \sin \omega x),$$

where $c_1$ and $c_2$ are arbitrary constants. By $f(1) = f''(1) = 0$, one has

$$\begin{pmatrix} \cosh \omega - \cos \omega & \sinh \omega - \sin \omega \\ \cosh \omega + \cos \omega & \sinh \omega + \sin \omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

In order $f$ to be a nonzero solution of (4.11), it is necessary and sufficient that

$$\cosh \omega \sin \omega - \sinh \omega \cos \omega = 0. \tag{4.13}$$

Therefore, the solution to (4.11) is given by

$$\begin{aligned} f(x) = {} & \sinh \omega(1-x) - \sin \omega(1-x) - \sinh \omega \cos \omega x - \sin \omega \cosh \omega x \\ & + \cosh \omega \sin \omega x + \cos \omega \sinh \omega x. \end{aligned}$$

Only $\omega > 0$ should be considered. From (4.13) one obtains

$$\omega_n = \left(n + \frac{1}{4}\right)\pi + \mathrm{O}(\mathrm{e}^{-n}).$$

Taking $\omega = \omega_n$, we get

$$\begin{aligned} 2\mathrm{e}^{-\omega} f(x) = & -\mathrm{e}^{-\omega x} - \cos\,\omega x + \sin\,\omega x + \cos\,\omega \mathrm{e}^{-\omega(1-x)} \\ & - \sin\,\omega \mathrm{e}^{-\omega(1-x)} + \mathrm{O}(\mathrm{e}^{-n}) \\ = & \mathrm{e}^{-(n+1/4\pi)x} - \cos\left(n + \frac{1}{4}\pi\right)x + \sin\left(n + \frac{1}{4}\pi\right)x \\ & + \cos\left(n + \frac{1}{4}\pi\right)\mathrm{e}^{-(n+1/4\pi)(1-x)} \\ & - \sin\left(n + \frac{1}{4}\pi\right)\mathrm{e}^{-(n+1/4\pi)(1-x)} + \mathrm{O}(\mathrm{e}^{n^{-1}}). \end{aligned}$$

Since

$$\begin{aligned} \omega^{-3} f'''(x) = & -\cosh\omega(1-x) + \cos\,\omega(1-x) - \sinh\omega\,\sin\,\omega x - \sin\,\omega\sinh\omega x \\ & + \cosh\omega\,\cos\,\omega x - \cosh\omega\cos\,\omega x, \end{aligned}$$

we can easily get the required estimate for $f_n'''(x)$ by regarding $2\mathrm{e}^{-\omega} f(x)$ as $f_n(x)$. □

**Continuation of the proof of Theorem 4.1.** Define

$$\begin{cases} \lambda_n = \mathrm{i}\omega_n^2, \quad \lambda_{-n} = -\mathrm{i}\omega_n^2, \\ \Phi_n = (\lambda_n^{-1} f_n, f_n), \quad \Phi_{-n} = (\lambda_{-n}^{-1} f_n, f_n), \ n = 1, 2, \ldots. \end{cases}$$

Then $\{\Phi_n\}_{n\in\mathscr{Z}}$ form a (orthogonal) Riesz basis for $\mathbf{H}$. The solution of (4.7) can then be represented as

$$(y(\cdot,t), y_t(\cdot,t)) = \sum_{n\in\mathscr{Z}} a_n \mathrm{e}^{\lambda_n t}\Phi_n,$$

where $\{a_n\}_{n\in\mathscr{Z}}$ are constants determined by the initial condition. From (4.10), we have

$$|\omega_n^{-3} f_n'''(1)| \leqslant 5$$

for sufficiently large $t$. Thus in this case, the smooth solution satisfies $\sum_{n=1}^{\infty}|a_n\omega_n|^2 < \infty$ and

$$y_{xxx}(1,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{\lambda_n t}\lambda_n^{-1} f_n'''(1) = (\hat{\theta} - \tilde{\theta})\sin\, t$$

holds for all $t \geqslant 0$. By the orthogonality of the system $\{(\sin\,\omega_n^2 t, \cos\,\omega_n^2 t)\}$, we have immediately that all $a_n = 0$ and hence

$$\hat{\theta} = \tilde{\theta}, \quad y \equiv 0.$$

We have thus proved that $S$ contains only single point $(0, 0, \hat{k}, \tilde{\theta})$. The proof is complete. □

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