Periodic Solutions for p-Laplacian Differential Equation With Multiple Deviating Arguments

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Abstract—By employing Mawhin's continuation theorem, the existence of periodic solutions of the p-Laplacian differential equation with multiple deviating arguments

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t)g(x(t-\gamma_j(t))) = e(t)$$

under various assumptions are obtained.

Keywords—periodic solution, Mawhin's continuation theorem, deviating argument.

1. INTRODUCTION

In recent years, there have been a number of results on the existence of periodic solutions for delay differential equations. For example, in [8, 9, 10, 11, 13], the following types of second-order scalar differential equations with delay

$$x''(t) + g(x(t-\tau)) = p(t), \tag{1.1}$$

$$x''(t) + m^2 x(t) + g(x(t-\tau)) = p(t), \tag{1.2}$$

$$x''(t) + f(x(t))x'(t) + g(x(t-\tau(t))) = p(t), \tag{1.3}$$

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$$x''(t) + f(t, x(t), x(t - \tau_0(t)))x'(t) + \beta(t)g(x(t - \tau_1(t))) = p(t), \tag{1.4}$$

and

$$x''(t) + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t)g(x(t - \gamma_j(t))) = p(t) \tag{1.5}$$

have been studied. The main technique used in these works is to convert the problem into the abstract form $Lx = Nx$, with L being a non-invertible linear operator. Here, the crux is that the leading terms of these equations, that is, the 1-dimensional Laplacian $x''(t)$, are linear in the unknown function x so that Mawhin's continuation theorem [6] applies and existence of solutions follows.

Now as the p-Laplacian of a function comes frequently into play in many practical situations (for example, it is used to describe fluid mechanical and nonlinear elastic mechanical phenomena), it is natural to try and consider the existence of solutions of p-Laplacian equations, that is, differential equations with leading term being a p-Laplacian $(\varphi_p(x'(t)))'$, where $\varphi_p(u) = |u|^{p-2}u$. Since $x''(t) = \varphi_2(x'(t))'$, p-Laplacians cover the usual Laplacian as a special case. So it should be interesting to consider the aforesaid equations with $x''(t)$ being replaced by $(\varphi_p(x'(t)))'$ and in fact there have already been a few results for p-Laplacian equations, for example, see [4, 14] and references cited there. But for the existence of solutions of p-Laplacian boundary value problems at resonance or p-Laplacian differential equations with delay (or deviating argument), as far as we are aware of, there have been little results until the very recent works in [2, 3, 7]. The major difficulty in this direction is that except for $p = 2$, $(\varphi_p(x'(t)))'$ is no longer linear and so the usual technique of using Mawhin's continuation theorem does not apply directly. In order to get around with this difficulty, Ge and Ren [7] obtained an extension of Mawhin's continuation theorem and applied it to boundary value problems with a $p-$Laplacian. At the same time, Cheung and Ren [2, 3] designed a new technique of tackling the problem, namely, to translate the p-Laplacian equation into a two-dimensional system for which Mawhin's continuation theorem can be applied.

On the other hand, as multi-delays exist naturally in most non-simple situations, such phenomena are worth investigating. Recent results in this direction include [4, 5, 10, 12, 15]. In this paper, following the line of Cheung-Ren in [2, 3] we consider the p-Laplacian differential equation with multiple deviating arguments

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t)g(x(t - \gamma_j(t))) = e(t)\ , \tag{1.6}$$

where $p > 1$ is a constant; $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$ is a one-dimensional p-Laplacian; $f,\ g,\ e,\ \beta_j \in C(\mathbb{R}, \mathbb{R})$, $j = 1, 2, \cdots, n$, are periodic with period $T > 0$, $\int_0^T e(s)ds = 0$; and $\gamma_j \in C^1(\mathbb{R}, \mathbb{R})$ is periodic with period $T > 0$ for $j = 1, 2, \cdots, n$. Observe that (1.6) covers equations (1.1), (1.2), (1.3), and (1.5) as special cases. By translating equation (1.6) into a 2-dimensional system on which Mawhin's continuation theorem applies, sufficient conditions for the existence of periodic solutions of (1.6) are obtained. Note that this generalizes and improves some existing results in [2], [8], and [10].

2. MAIN LEMMAS

In the sequel, let $T > 0$ be fixed. For $k = 0, 1$, let $C^k_T(\mathbb{R}, \mathbb{R})$ be the space of periodic C^k functions of $\mathbb{R}$ into $\mathbb{R}$ with period T. Equip $C^k_T(\mathbb{R}, \mathbb{R})$ with norm $|\ \ |_0$ by

$$|\phi|_0 := \max_{i\le k} \max_{t\in[0,T]} |\phi^{(i)}(t)| \quad \text{for all} \quad \phi \in C^k_T(\mathbb{R}, \mathbb{R}) \ .$$

Let $C_T(\mathbb{R}, \mathbb{R}^2)$ be the space of periodic C functions of $\mathbb{R}$ into $\mathbb{R}^2$ with period T. Equip $C_T(\mathbb{R}, \mathbb{R}^2)$ with norm $||\ \ ||$ by

$$||x|| = \max\{|x_1|_0, |x_2|_0\} \quad \text{for all} \quad x = (x_1, x_2) \in C_T(\mathbb{R}, \mathbb{R}^2) \ .$$

Clearly, both $C^k_T(\mathbb{R}, \mathbb{R})$ and $C_T(\mathbb{R}, \mathbb{R}^2)$ are Banach spaces with these prescribed norms.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\, L$ is closed in Y and $\dim Ker\, L = \dim(Y/Im\, L) < +\infty$. Consider the complementary subspaces X_1 and Y_1 such that $X = Ker\ L \oplus X_1$ and $Y = Im\, L \oplus Y_1$, and let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $Ker\, L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

The following preliminary results are needed in the derivation of our main theorems.

LEMMA 2.1 (Mawhin's continuation theorem) [6] Let X and Y be Banach spaces, and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. If

(1)$Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(2)$Nx \notin Im\, L, \forall x \in \partial\Omega \cap Ker\, L$; and

(3)$deg\{JQN, \Omega \cap Ker\, L, 0\} \neq 0$, where $J : Im\ Q \to Ker\, L$ is an isomorphism,

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \bigcap D(L)$.

LEMMA 2.2 [11] Let $0 \le \alpha \le T$ be a constant and $s \in C_T(\mathbb{R}, \mathbb{R})$ with $\max\limits_{t\in[0,T]} |s(t)| \le \alpha$. Then for any $u \in C^1_T(\mathbb{R}, \mathbb{R})$, we have

$$\int_0^T |u(t) - u(t - s(t))|^2 dt \le 2\alpha^2 \int_0^T |u'(t)|^2 dt \ .$$

LEMMA 2.3 [10] Let $g \in C_T(\mathbb{R}, \mathbb{R})$ and $\tau \in C^1_T(\mathbb{R}, \mathbb{R})$ with $\tau' < 1$. Then $g(\mu(t)) \in C_T(\mathbb{R}, \mathbb{R})$, where $\mu(t)$ is the inverse function of $t - \tau(t)$.

In order to apply Mawhin's continuation theorem to study the existence of T-periodic solutions of equation (1.6), we denote by $x_1(t) = x(t)$ and $x_2(t) = \varphi_p(x_1'(t))$. Then we can rewrite the equation into the following form

$$\begin{cases} x_1'(t) &= \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\ x_2'(t) &= -\sum\limits_{j=1}^{n} \beta_j(t)g(x_1(t-\gamma_j(t))) - f(x_1(t))\varphi_q(x_2(t)) + e(t) \ , \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to equations (2.1), $x_1(t)$ must be a T-periodic solution to equation (1.6). Thus, the problem of finding a T-periodic solution for equation (1.6) reduces to finding one for equation (2.1).

Now, once and for all we set $X = Y = C_T(\mathbb{R}, \mathbb{R}^2)$ and define operators

$$L : D(L) \subset X \to Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$N : X \to Y, \quad Nx = \begin{pmatrix} \varphi_q(x_2) \\ -\sum\limits_{j=1}^{n} \beta_j(t)g(x_1(t-\gamma_j(t))) - f(x_1(t))\varphi_q(x_2(t)) + e(t) \end{pmatrix} .$$

It is easy to see that $Ker\, L = \mathbb{R}^2$ and $Im\, L = \{y \in Y : \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to Ker\, L$ and $Q : Y \to Im\ Q \subset \mathbb{R}^2$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds \ ; \quad Qy = \frac{1}{T}\int_0^T y(s)ds \ ,$$

and let K denote the inverse of $L|_{Ker\ P \cap D(L)}$. Obviously, $Ker\, L = Im\, Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)ds \ , \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{s}{T}, & 0 \le s < t \le T \\ \dfrac{s-T}{T}, & 0 \le t \le s \le T \ . \end{cases}$$

From (2.2), one easily sees that for any bounded open subset Ω of X, N is $L-$compact on $\overline{\Omega}$.

3. MAIN RESULTS

Throughout the paper, we assume that $\gamma_j \in C_T^1$ with $\gamma_j' < 1$ $(j = 1, 2, \cdots, n)$. So for every $j = 1, 2, \cdots, n$, the function $t - \gamma_j(t)$ has a unique inverse which we shall denote by $\mu_j(t)$. Moreover, in order to simplify the presentation, we shall adopt the following notations.

$$I_n := \{1, 2, \cdots, n\}, \qquad \Lambda_m \subset I_n \ \text{ a subset with } m \text{ elements} \ ,$$

$$G(x) := \int_0^x g(s)ds \ ,$$

$$\bar{h} := \frac{1}{T}\int_0^T h(s)ds, \quad h \in C_T(\mathbb{R}, \mathbb{R}) \ ,$$

$$|h|_p := (\int_0^T |h(s)|^p ds)^{\frac{1}{p}}, \qquad p \ge 1, \ \ h \in C_T(\mathbb{R}, \mathbb{R}) \ ,$$

$$\Gamma(t) := \sum_{j=1}^{n} \frac{\beta_j(\mu_j(t))}{1-\gamma_j'(\mu_j(t))} \ ,$$

$$\Psi(t) := \sum_{j\in\Lambda_m} |\frac{\beta_j(\mu_j(t))}{1-\gamma_j'(\mu_j(t))}| \ .$$

Note that Γ, $\Psi \in C_T(\mathbb{R},\mathbb{R})$.

Various combinations of the following hypotheses will be useful in the study of the existence of T-periodic solutions to equation (1.6):

[$\mathbf{H_1}$] Either $\Gamma(t) \geq 0$ for all $t \in \mathbb{R}$ or $\Gamma(t) \leq 0$ for all $t \in \mathbb{R}$; and $\bar{\Gamma} \neq 0$.

[$\mathbf{H_2}$] There is a constant $d > 0$ such that either $ug(u) > 0 \ \forall \ |u| > d$ or $ug(u) < 0 \ \forall \ |u| > d$.

[$\mathbf{H_3}$] There is a constant $l > 0$ such that

$$|g(u_1) - g(u_2)| \leq l|u_1 - u_2| \qquad \forall u_1, u_2 \in \mathbb{R} \ .$$

[$\mathbf{H_4}$] There is a constant $r \geq 0$ such that $\lim\limits_{|u|\to\infty} \dfrac{|g(u)|}{|u|^{p-1}} \leq r$.

THEOREM 3.1 Suppose $[H_1] - [H_3]$ hold and $\beta_j \in C_T^1(\mathbb{R},\mathbb{R})$, $j \in I_n$. Moreover, if

(1) there is a constant $\sigma > 0$ such that $|f(s)| \geq \sigma$ for all $s \in \mathbb{R}$;

(2) there is an integer m_j such that $\gamma_j(t) \in [m_jT - \alpha_j, m_jT + \alpha_j]$ for all $t \in [0,T]$, $j \in I_n$, where α_j is a constant satisfying $0 \leq \alpha_j \leq T$ and $\sigma > \frac{1}{2}Tl\Big|\sum\limits_{j=1}^{n}\beta_j'\Big|_1 + \sqrt{2}l\sum\limits_{j=1}^{n}|\beta_j|_0\alpha_j$,

then equation (1.6) has at least one T-periodic solution.

Proof: Consider the following operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1) \ . \tag{3.1}$$

Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \ \lambda \in (0,1)\}$. If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1$, then from (3.1) we have

$$\begin{cases} x_1'(t) &= \lambda\varphi_q(x_2(t)) \ = \ \lambda|x_2(t)|^{q-2}x_2(t) \\ x_2'(t) &= -\lambda\sum\limits_{j=1}^{n}\beta_j(t)g(x_1(t-\gamma_j(t))) - \lambda f(x_1(t))\varphi_q(x_2(t)) + \lambda e(t) \ . \end{cases} \tag{3.2}$$

From the first equation of (3.2), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}x_1'(t))$, which together with the second equation of (3.2) yields

$$\Big[\varphi_p\Big(\frac{1}{\lambda}x_1'(t)\Big)\Big]' + f(x_1(t))x_1'(t) + \lambda\sum_{j=1}^{n}\beta_j(t)g(x_1(t-\gamma_j(t))) = \lambda e(t) \ ,$$

i.e.,

$$\big[\varphi_p\big(x_1'(t)\big)\big]' + \lambda^{p-1}f(x_1(t))x_1'(t) + \lambda^p\sum_{j=1}^{n}\beta_j(t)g(x_1(t-\gamma_j(t))) = \lambda^p e(t) \ . \tag{3.3}$$

Integrating both sides of (3.3) over $[0,T]$, we get

$$\sum_{j=1}^{n}\int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))dt = 0 \ . \tag{3.4}$$

By Lemma 2.3, $\beta_j(\mu_j(t))/\{1-\gamma_j'(\mu_j(t))\}$ is a periodic function with period T. So

$$\int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))dt = \int_{-\gamma_j(0)}^{T-\gamma_j(T)} \frac{\beta_j(\mu_j(t))}{1-\gamma_j'(\mu_j(t))}g(x_1(t))dt = \int_0^T \frac{\beta_j(\mu_j(t))}{1-\gamma_j'(\mu_j(t))}g(x_1(t))dt \ . \tag{3.5}$$

Thus (3.4) can be reduced to

$$\int_0^T \Gamma(t)g(x_1(t))dt = 0$$

which means that there is a constant $\xi \in [0,T]$ such that

$$g(x_1(\xi))\bar{\Gamma}T = 0 \ .$$

In view of $\bar{\Gamma} \neq 0$, we get $g(x_1(\xi)) = 0$. By $[H_2]$, it is easy to see that $|x_1(\xi)| \leq d$. Hence

$$|x_1|_0 \leq d + \int_0^T |x_1'(s)|ds \ . \tag{3.6}$$

On the other hand, multiplying both sides of equation (3.3) by $x_1'(t)$ and integrating over $[0,T]$, we have

$$\begin{aligned} & \int_0^T [\varphi_p(x_1'(t))]'x_1'(t)dt + \lambda^{p-1}\int_0^T f(x_1(t))[x_1'(t)]^2dt + \lambda^p\sum_{j=1}^n\int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1'(t)dt \\ = \ & \lambda^p\int_0^T e(t)x_1'(t)dt \ . \end{aligned} \tag{3.7}$$

If we write $w(t) = \varphi_p(x_1'(t))$, then

$$\int_0^T [\varphi_p(x_1'(t))]'x_1'(t)dt = \int_0^T \varphi_q(w(t))dw(t) = 0 \ ,$$

which together with (3.7) and the fact that $\lambda \in (0,1)$ yields

$$\Big|\int_0^T f(x_1(t))[x_1'(t)]^2dt\Big| < \Big|\sum_{j=1}^n\int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big| \ . \tag{3.8}$$

Furthermore, from condition (1), we see that

$$\sigma\int_0^T |x_1'(t)|^2dt \leq \int_0^T |f(x_1(t))||x_1'(t)|^2dt = \Big|\int_0^T f(x_1(t))[x_1'(t)]^2dt\Big| \ .$$

So we have from (3.8) that

$$\sigma\int_0^T |x_1'(t)|^2dt \leq \Big|\sum_{j=1}^n\int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big| \ . \tag{3.9}$$

In view of $[H_3]$ and using integration by parts, we find from (3.9) that

$$\begin{aligned} & \sigma\int_0^T |x_1'(t)|^2dt \\ \leq \ & \Big|-\sum_{j=1}^n\int_0^T \beta_j(t)g(x_1(t))x_1'(t)dt\Big| + \Big|\sum_{j=1}^n\int_0^T \beta_j(t)[g(x_1(t)) - g(x_1(t-\tau(t)))]x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big| \\ \leq \ & \Big|\int_0^T \sum_{j=1}^n \beta_j'(t)G(x_1(t))dt\Big| + \sum_{j=1}^n |\beta_j|_0 l\int_0^T |x_1'(t)||x_1(t) - x_1(t-\gamma_j(t))|dt + |e|_0\int_0^T |x_1'(t)|dt \ . \end{aligned}$$

(3.10)

By $[H_2]$, there is a constant $\zeta \in [-d, d]$ such that $g(\zeta) = 0$. Then, by the definition of $G(x)$, we have

$$\begin{aligned} |G(x_1(t))|_0 &= \Big|\int_0^{x_1(t)} g(s)ds\Big|_0 \\ &\le \Big|\int_0^{x_1(t)} |g(s) - g(\zeta)|ds\Big|_0 \\ &\le l\Big|\int_0^{x_1(t)} |s-\zeta|ds\Big|_0 \\ &\le l\int_0^{|x_1|_0} (s + |\zeta|)ds \\ &\le \frac{1}{2}l|x_1|_0^2 + ld|x_1|_0 \ . \end{aligned}$$

It follows that

$$\Big|\int_0^T \sum_{j=1}^n \beta_j'(t)G(x_1(t))dt\Big| \le \Big|\sum_{j=1}^n \beta_j'\Big|_1 (\frac{1}{2}l|x_1|_0^2 + ld|x_1|_0) \ . \tag{3.11}$$

On the other hand, as $\gamma_j(t) \in [m_jT - \alpha_j, m_jT + \alpha_j]$ for all $t \in [0, T]$, if we denote by $s_j = \gamma_j(t) - m_jT$, we have $|s_j| \le \alpha_j$ for all $j \in I_n$. Hence by Lemma 2.2, we have

$$\begin{aligned} & \textstyle\int_0^T |x_1'(t)||x_1(t) - x_1(t-\gamma_j(t))|dt \\ = & \textstyle\int_0^T |x_1'(t)||x_1(t) - x_1(t-s_j(t))|dt \\ \le & \textstyle(\int_0^T |x_1'(t)|^2dt)^{\frac{1}{2}}(\int_0^T |x_1(t) - x_1(t-s_j(t))|^2dt)^{\frac{1}{2}} \\ \le & \textstyle\sqrt{2}\alpha_j \int_0^T |x_1'(t)|^2dt \ . \end{aligned}$$

Therefore,

$$\sum_{j=1}^n |\beta_j|_0 l\int_0^T |x_1'(t)||x_1(t) - x_1(t-\gamma_j(t))|dt \le \sqrt{2}l\sum_{j=1}^n |\beta_j|_0\alpha_j \int_0^T |x_1'(t)|^2dt \ . \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we have

$$\sigma\int_0^T |x_1'(t)|^2dt \le \frac{1}{2}l|x_1|_0^2\Big|\sum_{j=1}^n \beta_j'\Big|_1 + |e|_0\int_0^T |x_1'(t)|dt + \sqrt{2}l\sum_{j=1}^n |\beta_j|_0\alpha_j\int_0^T |x_1'(t)|^2dt + ld\Big|\sum_{j=1}^n \beta_j'\Big|_1|x_1|_0 \ . \tag{3.13}$$

Substituting (3.6) into (3.13) and by Hölder's inequality, we get

$$\begin{aligned} &\textstyle\sigma\int_0^T |x_1'(t)|^2dt \\ &\textstyle\le l\Big(\frac{1}{2}\Big|\sum\limits_{j=1}^n \beta_j'\Big|_1 T + \sqrt{2}\sum\limits_{j=1}^n |\beta_j|_0\alpha_j\Big)\int_0^T |x_1'(t)|^2dt + T^{\frac{1}{2}}\Big(2ld\Big|\sum\limits_{j=1}^n \beta_j'\Big|_1 + |e|_0\Big)\Big(\int_0^T |x_1'(t)|^2dt\Big)^{\frac{1}{2}} + \frac{3}{2}ld^2\Big|\sum\limits_{j=1}^n \beta_j'\Big|_1 \ . \end{aligned} \tag{3.14}$$

By assumption (2), it is easy to see from above that there is a constant $R_0 > 0$, independent of λ, such that

$$\int_0^T |x_1'(t)|^2dt \le R_0 \ .$$

It follows from (3.6) and Hölder's inequality that

$$|x_1|_0 \le d + T^{1/2}R_0^{\frac{1}{2}} =: M_1 \ . \tag{3.15}$$

By the first equation of (3.2), we have

$$\int_0^T |x_2(s)|^{q-2}x_2(s)ds = 0 \ ,$$

which implies that there is a constant $t_2 \in [0,T]$ such that $x_2(t_2) = 0$. So

$$|x_2|_0 \le \int_0^{t_2} |x_2'(s)|ds \le \int_0^T |x_2'(s)|ds \ .$$

By the two equations of (3.2) and Hölder's inequality, we obtain

$$\begin{aligned}\int_0^T |x_2'(s)|ds &\le \lambda \sum_{j=1}^n |\beta_j|_0 g_{M_1} T + \lambda \int_0^T |f(x_1(s))||x_2(s)|^{q-1}ds + \lambda |e|_1 \\ &= \lambda \sum_{j=1}^n |\beta_j|_0 g_{M_1} T + \int_0^T |f(x_1(s))||x_1'(s)|ds + \lambda |e|_1 \\ &< \sum_{j=1}^n |\beta_j|_0 g_{M_1} T + |f|_0 T^{1/2} R_0^{1/2} + |e|_1 \ ,\end{aligned}$$

where $g_{M_1} := \max_{|s|\le M_1} |g(s)|$. So we have

$$|x_2|_0 \le \sum_{j=1}^n |\beta_j|_0 g_{M_1} T + |f|_0 T^{1/2} R_0^{1/2} + |e|_1 =: M_2 \ . \tag{3.16}$$

Let $\Omega_2 = \{x \in Ker\, L : Nx \in Im\, L\}$. If $x \in \Omega_2$, then $x \in Ker\, L$ and $QNx = 0$. By the assumption that $\int_0^T e(t)dt = 0$, we see that

$$\begin{cases} |x_2|^{q-2}x_2 = 0 \\ \sum\limits_{j=1}^n \bar\beta_j g(x_1) = 0 \ . \end{cases} \tag{3.17}$$

So $x_2 = 0 \le M_2$. At the same time, from the proof of Lemma 2.3, we find that $\mu_j(T) = T + \mu_j(0),\ j \in I_n$. So

$$\int_0^T \frac{\beta_j(\mu_j(s))}{1-\gamma_j'(\mu_j(s))}ds = \int_{\mu_j(0)}^{\mu_j(T)} \frac{\beta_j(t)}{1-\gamma_j'(t)}(1-\gamma_j'(t))dt = T\bar\beta_j \ ,$$

which yields $\sum\limits_{j=1}^n \bar\beta_j = \bar\Gamma \ne 0$ by $[H_1]$. So by (3.17), we have

$$g(x_1) = 0 \ .$$

By $[H_2]$,

$$|x_1| \le d \le M_1 \ .$$

Now let $\Omega = \{x = (x_1,x_2)^\top \in X : |x_1|_0 < N_1, |x_2|_0 < N_2\}$, where N_1 and N_2 are constants with $N_1 > M_1$, $N_2 > M_2$ and $(N_2)^q > dg_d$, where $g_d := \max_{|u|\le d} |g(u)|$. Then $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega$. From (3.10), (3.11) and the above, it is easy to see that conditions (1) and (2) of Lemma 2.1 are satisfied.

Next, we claim that condition (3) of Lemma 2.1 is also satisfied. For this, define the isomorphism $J : Im\ Q \to Ker\, L$ by

$$J(x_1,x_2) = \begin{cases} (x_2,x_1) & \text{if } \sum\limits_{j=1}^n \bar\beta_j u g(u) < 0 \text{ for } |u| > d \\ (-x_2,x_1) & \text{if } \sum\limits_{j=1}^n \bar\beta_j u g(u) > 0 \text{ for } |u| > d, \end{cases}$$

and let $H(v,\mu) := \mu v + \frac{1-\mu}{T} JQNv,\ (v,\mu) \in \Omega \times [0,1]$. By simple calculation, we obtain, for $(x,\mu) \in \partial(\Omega \cap KerL) \times [0,1]$,

$$x^\top H(x,\mu) = \begin{cases} \mu(x_1^2+x_2^2) + \frac{1-\mu}{T}(-\sum\limits_{j=1}^n \bar{\beta}_j x_1 g(x_1) + |x_2|^q) > 0 & \text{if } \sum\limits_{j=1}^n \bar{\beta}_j u g(u) < 0 \text{ for } |u| > d \\ \mu(x_1^2+x_2^2) + \frac{1-\mu}{T}(\sum\limits_{j=1}^n \bar{\beta}_j x_1 g(x_1) + |x_2|^q) > 0 & \text{if } \sum\limits_{j=1}^n \bar{\beta}_j u g(u) > 0 \text{ for } |u| > d\ . \end{cases}$$

Obviously, $x^\top H(x,\mu) \neq 0$ for $(x,\mu) \in \partial(\Omega \cap KerL) \times [0,1]$. Hence

$$\begin{aligned} & deg\{JQN, \Omega \cap KerL, 0\} = deg\{H(x,0), \Omega \cap KerL, 0\} \\ = \ & deg\{H(x,1), \Omega \cap KerL, 0\} = deg\{I, \Omega \cap KerL, 0\} \\ \neq \ & 0, \end{aligned}$$

and so condition (3) of Lemma 2.1 is satisfied.

Therefore, by Lemma 2.1, we conclude that equation

$$Lx = Nx$$

has a solution $x(t) = (x_1(t), x_2(t))^\top$ on $\overline{\Omega}$, i.e., equation (1.6) has a T-periodic solution $x_1(t)$ with $|x_1|_0 \leq M_1$. This completes the proof of Theorem 3.1. □

REMARK 3.1 (i) In case $n = 1$ and $\beta_1 \equiv 1$, Theorem 3.1 reduces to Theorem 3.1 in [2]. (ii) Similarly, Theorem 3.1 is a generalization of Theorem 1 in [10]. Moreover, assumption (2) in the former is considerably weaker than the corresponding one in the latter, which reads "there is an integer m_j such that $\gamma_j(t) \in [m_jT - \alpha_j, m_jT + \alpha_j]$ for all $t \in [0,T]$, $j \in I_n$, where α_j is a constant satisfying $0 \leq \alpha_j \leq T$ and $\sigma > \frac{1}{2}Tl\Big|\sum\limits_{j=1}^n \beta_j'\Big|_1 + \sqrt{2}l \sum\limits_{j=1}^n |\beta_j|_0 (1+\frac{\alpha_j}{T})^{\frac{1}{2}}\alpha_j$".

THEOREM 3.2 If $[H_1]$, $[H_2]$ and $[H_4]$ hold, and if

(1) $\bar{\Psi} T^p r < 1$;

(2) $\gamma_j(t) = k_jT$ for $j \in I_n - \Lambda_m$, and $\sum\limits_{j \in I_n - \Lambda_m} \beta_j(t) u g(u) \leq 0$ for $|u| > d$,

then equation (1.6) has at least one T-periodic solution.

PROOF: Let Ω_1 be defined as in Theorem 3.1. If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1$, then from the proof of Theorem 3.1 we see that

$$[\varphi_p(x_1'(t))]' + \lambda^{p-1} f(x_1(t)) x_1'(t) + \lambda^p \sum_{j=1}^n \beta_j(t) g(x_1(t-\gamma_j(t))) = \lambda^p e(t)\ , \tag{3.18}$$

and

$$|x_1|_0 \leq d + \int_0^T |x_1'(s)| ds\ . \tag{3.19}$$

Multiplying both sides of equation (3.18) by $x_1(t)$ and integrating over $[0,T]$, we have

$$\begin{aligned} & \int_0^T \big[\varphi_p(x_1'(t))\big]' x_1(t) dt + \lambda^{p-1} \int_0^T x_1(t) f(x_1(t)) x_1'(t) dt + \lambda^p \sum_{j=1}^n \int_0^T \beta_j(t) g(x_1(t-\gamma_j(t)))\big) x_1(t) dt \\ = \ & \lambda^p \int_0^T x_1(t) e(t) dt\ . \end{aligned} \tag{3.20}$$

In view of $\int_0^T x_1(t)f(x_1(t))x_1'(t)dt = 0$ and $\int_0^T [\varphi_p(x_1'(t))]'x_1(t)dt = -\int_0^T |x_1'(t)|^p dt$, it follows from (3.20) that

$$\begin{aligned}
& \int_0^T |x_1'(t)|^p dt \\
= \;& \lambda^p \sum_{j=1}^n \int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1(t)dt - \lambda^p \int_0^T x_1(t)e(t)dt \\
< \;& \sum_{j=1}^n \int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1(t)dt - \int_0^T x_1(t)e(t)dt \\
= \;& \sum_{j\in\Lambda_m} \int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1(t)dt - \int_0^T x_1(t)e(t)dt + \sum_{j\in I_n-\Lambda_m} \int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1(t)dt \\
= \;& \sum_{j\in\Lambda_m} \int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1(t)dt - \int_0^T x_1(t)e(t)dt + \sum_{j\in I_n-\Lambda_m} \int_0^T \beta_j(t)x_1(t)g(x_1(t))dt \;.
\end{aligned} \tag{3.21}$$

Let $E_1 = \{t \in [0,T] : \; |u(t)| > d\}$, $E_2 = \{t \in [0,T] : \; |u(t)| \le d\}$. By assumption (2),

$$\int_{E_1} \sum_{j\in I_n-\Lambda_m} \beta_j(t)x_1(t)g(x_1(t))dt \le 0 \;.$$

Hence (3.21) can be reduced into

$$\begin{aligned}
& \int_0^T |x_1'(t)|^p dt \\
\le \;& \sum_{j\in\Lambda_m} \int_0^T \beta_j(t)g(x_1(t-\gamma_j(t)))x_1(t)dt - \int_0^T x_1(t)e(t)dt \\
& + \int_{E_1} \sum_{j\in I_n-\Lambda_m} \beta_j(t)x_1(t)g(x_1(t))dt + \int_{E_2} \sum_{j\in I_n-\Lambda_m} \beta_j(t)x_1(t)g(x_1(t))dt \\
\le \;& |x_1|_0 \int_0^T \sum_{j\in\Lambda_m} \tfrac{\beta(\mu_j(t))}{1-\gamma_j'(\mu_j(t))} g(x_1(t))dt + \int_{E_2} \sum_{j\in I_n-\Lambda_m} \beta_j(t)x_1(t)g(x_1(t))dt - \int_0^T x_1(t)e(t)dt \\
\le \;& |x_1|_0 \int_0^T |\Psi(t)||g(x_1(t))|dt + T \sum_{j\in I_n-\Lambda_m} |\beta_j|_0 dg_d + |x_1|_0|e|_1 \;.
\end{aligned} \tag{3.22}$$

By assumption (1), we easily see that there is a constant $\varepsilon > 0$, independent of λ, such that

$$\bar{\Psi}T^p(r+\varepsilon) < 1 \;. \tag{3.23}$$

For such a constant ε, we have by assumption $[H_4]$ that there is a constant $\rho > d$, independent of λ, such that

$$|g(u)| \le (r+\varepsilon)|u|^{p-1} \quad \text{for } |u| > \rho \;. \tag{3.24}$$

Let $E_3 = \{t \in [0,T] : \; |u(t)| \le \rho\}$ and $E_4 = \{t \in [0,T] : \; |u(t)| > \rho\}$. Then by (3.22), (3.24), and (3.19),

$$\begin{aligned}
& \int_0^T |x_1'(t)|^p dt \\
\le \;& |x_1|_0 \int_{E_3} |\Psi(t)||g(x_1(t))|dt + |x_1|_0 \int_{E_4} |\Psi(t)||g(x_1(t))|dt + T \sum_{j\in I_n-\Lambda_m} |\beta_j|_0 dg_d + |x_1|_0|e|_1 \\
\le \;& g_\rho|\Psi|_1|x_1|_0 + (r+\varepsilon)T\bar{\Psi}|x_1|_0^p + T \sum_{j\in I_n-\Lambda_m} |\beta_j|_0 dg_d + |x_1|_0|e|_1 \\
= \;& (r+\varepsilon)T\bar{\Psi}|x_1|_0^p + (g_\rho|\Psi|_1 + |e|_1)|x_1|_0 + T \sum_{j\in I_n-\Lambda_m} |\beta_j|_0 dg_d \\
\le \;& (r+\varepsilon)T\bar{\Psi}(d + \int_0^T |x_1'(t)|dt)^p + (g_\rho|\Psi|_1 + |e|_1) \int_0^T |x_1'(t)|dt + g_\rho|\Psi|_1 d + |e|_1 d + T \sum_{j\in I_n-\Lambda_m} |\beta_j|_0 dg_d \;.
\end{aligned} \tag{3.25}$$

We claim that there exists a constant $M_1 > 0$ such that

$$|x_1|_0 \le M_1 \;. \tag{3.26}$$

Case 1. If $\int_0^T |x_1'(s)|ds = 0$, then by (3.19), $|x_1|_0 \le d$.

Case 2. If $\int_0^T |x_1'(s)|ds > 0$, then by (3.19),

$$\Big[d + \int_0^T |x_1'(s)|ds\Big]^p = \Big(\int_0^T |x_1'(s)|ds\Big)^p \Big[1 + \frac{d}{\int_0^T |x_1'(s)|ds}\Big]^p . \tag{3.27}$$

By elementary analysis, there is a constant $h > 0$, independent of λ, such that

$$(1+x)^p < 1 + (1+p)x \quad \forall x \in (0, h] . \tag{3.28}$$

If $\dfrac{d}{\int_0^T |x_1'(s)|ds} \ge h$, then

$$\int_0^T |x_1'(s)|ds \le d/h ,$$

and so by (3.19),

$$|x_1|_0 \le d + d/h . \tag{3.29}$$

If $\dfrac{d}{\int_0^T |x_1'(s)|ds} < h$, by (3.27) and (3.28), we have

$$\begin{aligned} & \Big[d + \int_0^T |x_1'(s)|ds\Big]^p \\ \le\ & \Big(\int_0^T |x_1'(s)|ds\Big)^p \Big[1 + \frac{(p+1)d}{\int_0^T |x_1'(s)|ds}\Big] \\ =\ & \Big(\int_0^T |x_1'(s)|ds\Big)^p + (p+1)d\Big(\int_0^T |x_1'(s)|ds\Big)^{p-1} \\ \le\ & T^{p/q}\int_0^T |x_1'(s)|^p ds + (p+1)dT^{(p-1)/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/q}. \end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.25), we obtain

$$\begin{aligned} & \int_0^T |x_1'(t)|^p dt \\ \le\ & (r+\varepsilon)T^{1+p/q}\bar\Psi \int_0^T |x_1'(s)|^p ds + (r+\varepsilon)(p+1)dT^{(p+q-1)/q}\bar\Psi\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/q} + \\ & \big[g_\rho|\Psi|_1 + |e|_1\big]T^{1/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/p} + g_\rho|\Psi|_1 d + |e|_1 d + T\sum_{j\in I_n-\Lambda_m}|\beta_j|_0 d g_d \\ =\ & (r+\varepsilon)T^p\bar\Psi \int_0^T |x_1'(s)|^p ds + (r+\varepsilon)(p+1)dT^{(p+q-1)/q}\bar\Psi\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/q} + \\ & \big[g_\rho|\Psi|_1 + |e|_1\big]T^{1/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/p} + g_\rho|\Psi|_1 d + |e|_1 d + T\sum_{j\in I_n-\Lambda_m}|\beta_j|_0 d g_d . \end{aligned} \tag{3.31}$$

In view of $\frac{1}{q} < 1$, $\frac{1}{p} < 1$, and $\bar\Psi T^p(r+\varepsilon) < 1$, it follows that there is a constant $M_0 > 0$, independent of λ, such that $\int_0^T |x_1'(t)|^p dt \le M_0$, which together with (3.19) yields

$$|x_1|_0 \le d + T^{1/q}(M_0)^{1/p} . \tag{3.32}$$

This completes the proof of the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1. $\square$

REMARK 3.2 In case $p = 2$, Theorem 3.2 reduces to Theorem 3 in [10].

We conclude by applying the above results to a simple p-Laplacian equation with 2 delays.

EXAMPLE 3.2 Consider the equation

$$(\varphi_p(x'(t)))' + (\delta + x^2(t))x'(t) + (1 + \frac{1}{2}\sin t)x(t - \frac{1}{2}\cos t) - \frac{1}{2}\sin tu(t - 2\pi) = 2\sin t, \tag{3.33}$$

where $p,\ \delta$ are constants with $p > 1$ and $\delta > 0$.

Comparing with Theorem 3.1, we have $n = 2$, $\sigma = \delta$, $\beta_1(t) = 1 + \frac{1}{2}\sin t$, $\beta_2(t) = -\frac{1}{2}\sin t$, $\gamma_1(t) = \frac{1}{2}\cos t$, $\gamma_2(t) \equiv 2\pi$, $e(t) = 2\sin t$, $l = 1$, $T = 2\pi$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 0$. Let $\mu(t)$ be the inverse of function $t - \frac{1}{2}\cos t$, we have

$$\Gamma(t) = -\frac{1}{2}\sin t + \frac{1 + \frac{1}{2}\sin\mu(t)}{1 + \frac{1}{2}\sin\mu(t)} = 1 - \frac{1}{2}\sin t > 0$$

and

$$\begin{aligned} & \tfrac{1}{2}Tl\Big|\sum_{j=1}^{n}\beta_j'\Big|_1 + \sqrt{2}l\sum_{j=1}^{n}|\beta_j|_0\alpha_j \\ = & \tfrac{1}{2}Tl\Big|\beta_1' + \beta_2'\Big|_1 + \sqrt{2}\Big|\beta_1\Big|_0\alpha_1 + \sqrt{2}\Big|\beta_2\Big|_0\alpha_2 \\ = & \sqrt{2}\Big|\beta_1\Big|_0\alpha_1 \\ = & \tfrac{3\sqrt{2}}{4}. \end{aligned}$$

In view of Theorem 3.1, for every $p > 1$, (3.33) has at least one 2π-periodic solution when $\delta > \frac{3\sqrt{2}}{4}$.

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