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Consensus of Discrete-Time Multi-Agent Systems with Nonlinear Local Rules and Time-Varying Delays

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Abstract— **In a multi-agent system (MAS), the agents are often considered to be autonomous entities, such as robots or software programs, each under the influence of a local rule, representing its interaction with other agents. Over the past few years, most research in the study of discrete-time MAS's concentrates on linear local rules. However, local interactions between agents are more likely to be governed by nonlinear rules with time-varying delays. This paper investigates the consensus of discrete-time MAS's with nonlinear local rules and time-varying delays. Based on a representative model, we obtain some basic criteria for the consensus of such MAS's. These results cover several existing results as special cases. Moreover, the above criteria are applied to the consensus of the classical Vicsek model with time-varying delays. Simulation results are presented to validate the obtained criteria.**

I. INTRODUCTION

A multi-agent system (MAS) is a system that is composed of multiple interacting intelligent agents ([1]-[4]). Here, the agents are often considered to be autonomous entities, such as humans, robots, and software programs ([5]-[8]). Their interactions can be either cooperative or selfish. These agents may share a common goal, such as in a flock of nightingales, or they may pursue their own interests, such as in the community with free market economy. MAS's can be used to solve problems which are difficult or impossible for an individual agent or a monolithic system to solve, such as disaster response and Internet structure modeling. Topics of research on MAS's include cooperation and coordination, distributed problem solving, multi-agent learning, and communication ([9]-[16]).

An MAS may often manifest self-organization and complex collective behaviors even when the individual strategies of all its agents are very simple ([1], [4], [5], [14], [15]). Consensus or synchronization is one of typical collective behaviors in an MAS. In fact, consensus is a fundamental nature phenomenon. Hereafter, by consensus we mean a general agreement among all members of a given group or community, each of which exercises some discretion in decision making and its interactions other agents. A typical example is that a number of autonomous vehicles align their headings by using only the local information. To reveal the inherent mechanism of consensus in an MAS, many mathematical models have been introduced, including the Vicsek model [1] and Couzin-Levin model [14]. Based on these models, many interesting results have been obtained on the consensus of MAS's (see, for example, [2], [12], and [16]). In particular, the constructive approach in [12] laid a solid foundation for the theoretical analysis of consensus in many recent works ([2], [7], [11]). The method in [12] can also be used to establish the exponential convergence of an MAS. There exist some other ways to reach a consensus for all agents in an MAS (see, for example, [3] and [8]).

We observe that most of the above results are based on the linear local rules that govern the interactions among agents (see, for example, [2]-[4], [6], [7], and [10]). However, in real-world applications, the local interactions among agents are very often represented by nonlinear rules with time-varying delays. Based on the work of [3] and [8], in this paper, we aim to further study the consensus in a discrete-time MAS with nonlinear local rules and time-varying delays. In particular, we will deduce several basic consensus criteria for such an MAS. These criteria are then extended to establish the consensus of the classical Vicsek model [1] with time-varying delays. Finally, simple simulation results are given to validate the proposed criteria.

This paper is organized as follows. Section II describes the background of consensus for the discrete-time MAS's with nonlinear local rules and time-varying delays. The main theorems are then presented and proven in Section III. In Section IV, the above criteria are applied to the consensus of the classical Vicsek model with time-varying delays. Simulation results are given in Section V. Finally, Section VI concludes this paper.

II. DESCRIPTION OF THE PROBLEM

Consider a discrete-time MAS consisting of n autonomous agents, labeled 1 to n. Denote all the agents by the set $V = \{1, 2, \cdots, n\}$. Let the state of agent $i \in V$ be denoted by $x_i(t)$ $(t \geq 0)$. There exist some communication connections among these n agents. If agent i has access to the information of agent j, then agent j is said to be a neighbor of agent i and the set of all neighbors of agent i at time t is denoted by $N_i(t)$. Consequently, $i \in N_i(t)$ if agent i has access to the information of itself. Let graph $G(t) = (V, E(t))$ be the communication topology at time t, where $E(t) \subseteq V \times V$ is the set of edges and $(i, j) \in E(t)$ if and only if agent j is

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one of the neighbors of agent i. For several different graphs $G_k = \{V, E_k\}$ with $1 \le k \le \Gamma$, the union of these Γ graphs is given by $\bigcup_{k=1}^{\Gamma} G_k = \{V, \bigcup_{k=1}^{\Gamma} E_k\}$.

For two nodes i, j in a graph $G = \{V, E\}$, if there are k different nodes $i_s (1 \le s \le k)$ in V such that $(i, i_1), (i_s, i_{s+1}), (i_k, j) \in V$ for $1 \le s \le k-1$, then there is a path from i to j. If there are paths from any node $i \in V$ to any node $j \in V (i \ne j)$, this graph is said to be strongly connected.

Assume that time-delays in the channels of communication among agents are not negligible. In this case, the evolution of the states of these n agents will comply with the following rules:

$$y_j^i(t) = f(x_j(t - \tau_j^i(t))), \tag{1}$$

$$x_i(t+1) = F\left(\frac{1}{n_i(t)} \sum_{j \in N_i(t)} y_j^i(t)\right), \tag{2}$$

where $n_i(t)$ is the number of elements in set $N_i(t)$ and $i \in V$. Here, the updating rule (1) is a temporary process which means that agent i can get information $y_j^i(t)$ directly (no time-delay, *i.e.*, $\tau_j^i(t) = 0$) or indirectly (with time-delay, *i.e.*, $\tau_j^i(t) > 0$) from agent j. The updating rule (2) indicates that there exists a process that transforms the information of the neighboring agents into the state of agent i in the next time after collecting the information from all its neighbors.

According to the updating rules (1) and (2), the overall updating rule is described by

$$x_i(t+1) = F\left(\frac{1}{n_i(t)} \sum_{j \in N_i(t)} f\left(x_j\left(t - \tau_j^i(t)\right)\right)\right), \tag{3}$$

for all $i \in V$.

Note that $\{i\} \ne N_i(t)$ if $G(t)$ is strongly connected. In this case, each agent must update its state at each time instant.

For the local interactions (3), the question of interest is what kinds of functions f and F and communication topology $G(t)$ will guarantee the consensus of all the agents, that is, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for any $i, j \in V$.

Usually, we consider $G(t)$ to be an undirected graph. That is, neighbors exchange their information between each other at any time instant, which is said to be agreeing synchronously. However, in real-world applications, the reachability of information channel from agent i to agent j only guarantees that agent j gets the information from agent i but does not guarantee that agent i gets the information from j at the same time. This asymmetry in the communication topology is said to be asynchronous. An asynchronous communication topology can be described by a directed graph $G(t)$. In this paper, we will deal with the situation when $G(t)$ is a directed graph.

III. MAIN RESULTS

To begin with, let V, $G(t)$, Π denote the set of agents, the topology of communication at time t, and the local updating rules, respectively. Then the triple $(V, G(t), \Pi)$ describes an MAS.

Let $V = \{1, 2, \cdots, n\}$ and $\Pi =$ Equation (3). Then, the MAS considered in this paper is denoted by $(V, G(t), (3))$. In (3), let F be an invertible function and $g = F^{-1}$, the following assumptions are necessary in the discussions that follow.

(A1) f and g are both continuous functions defined on $[a, b]$ and $f(b) = g(b)$.
(A2) f is monotonically increasing and g is strictly monotonically increasing.
(A3) $f(x) \ge g(x)$ for $x \in [a, b]$.
(A4) There exists an integer $\Gamma > 0$ such that $\cup_{s=1}^{\Gamma} G(t+s)$ is strongly connected for any $t \ge 0$.
(A5) There exists an integer $B > 0$ such that $0 \le \tau_j^i(t) < B$ for any $i \ne j$.
(A6) $\tau_i^i(t) = 0$ for $\forall t \ge 0$ and $i \in V$.

Let

$$\begin{aligned}
M_i(t) &= \max\{x_i(t), x_i(t-1), \cdots, x_i(t-B+1)\}, \\
m_i(t) &= \min\{x_i(t), x_i(t-1), \cdots, x_i(t-B+1)\}, \\
M_i &= \overline{\lim}_{t\to\infty} M_i(t), \\
m_i &= \underline{\lim}_{t\to\infty} m_i(t), \\
m &= \min_{i=1}^{n} m_i, \\
m(t) &= \min_{i=1}^{n} m_i(t).
\end{aligned}$$

In what follows, we will state several lemmas, some of which generalize the corresponding results in the absence of time delays in [3]. The proofs of these lemmas are omitted due to space limitation.

Lemma 1: Suppose that Assumptions (A1)-(A3), (A5), and (A6) hold for the given MAS $(V, G(t), (3))$. If the initial states $x_i(t) \in [a, b]$ with $i \in V$ and $-B < t \le 0$, then $x_i(t) \in [a, b]$ for any $t > 0$ and $i \in V$.

Lemma 1 indicates the validity of local rule (3).

Lemma 2: Suppose that Assumption (A1)-(A3), (A5) and (A6) hold for the given MAS $(V, G(t), (3))$. Then there exists an m' such that $\lim_{t\to\infty} m(t) = m'$ and $f(m') = g(m')$.

Lemma 3: Suppose that Assumptions (A1)-(A3), (A5) and (A6) hold for the given MAS $(V, G(t), (3))$. Then $m' = m$, where m' is as defined in Lemma 2.

Lemma 4: Assume that functions f and g satisfy Assumptions (A2) and (A3). Then, for any given $l > m$ and any positive integer N, there exist some $\varepsilon > 0$ and a sequence $\{l_p\}$ with $l_0 = l$ such that

$$\begin{aligned}
g(l_{p+1} - \varepsilon) &< \frac{1}{k}((k-1)f(m-\varepsilon) + f(l_p - \varepsilon)), \\
m + \varepsilon &< l_p - \varepsilon, \\
l_{p+1} &< l_p,
\end{aligned}$$

hold for all $k \in \{1, 2, \cdots, n\}$ and $p \in \{0, 1, 2, \cdots, N\}$.

Lemma 5: For the sequence of natural numbers $\{t\} = \{1, 2, \cdots\}$, there exists a subsequence $\{t_k\}$ of $\{t\}$ satisfying $\lim_{t_k\to\infty} m_i(t_k) = r_i$ and $m = \min_{i=1}^{n} m_i = \min_{i=1}^{n} r_i$.

For $a' < b'$, denote

$$\begin{aligned}
V_t(a') &= \{i \in V : a' < m_i(t)\}, \\
\Lambda_t(a', b') &= \{i \in V : a' < m_i(t) < b'\}.
\end{aligned}$$

Theorem 1: Suppose that Assumptions (A1)-(A6) hold for the given MAS $(V, G(t), (3))$. Then, for any given initial states $x_i(t) \in [a, b]$ with $i \in V$ and $-B < t \leq 0$, the states of all agents reach consensus.

Proof. According to Lemma 5, for the natural number series $\{t\} = \{1, 2, \cdots\}$, there exists a subsequence $\{t_k\}$ satisfying $\lim_{k\to\infty} m_i(t_k) = r_i$ and $m = \min_{i=1}^n r_i$. Let $R = \max_{i=1}^n r_i$.

If $R \neq m$, let $l = \min\{r_i | r_i > m\}$. By Lemma 4, there exists a sequence $\{l_p\}$ with $l_0 = l$ and $\varepsilon > 0$ such that

$$\begin{aligned} g(l_{p+1} - \varepsilon) &< \frac{1}{k}((k-1)f(m-\varepsilon) + f(l_p - \varepsilon)), \\ m + \varepsilon &< l_p - \varepsilon, \\ l_{p+1} &< l_p, \end{aligned}$$

holds for all $k \in \{1, 2, \cdots, n\}$, $p \in \{0, 1, 2, \cdots, n\cdot(B+\Gamma)\}$.

By the definition of $\{t_k\}$, there exists a $t_{k^*} \in \{t_k\}$ such that $V_{t_k}(l - \varepsilon) = \{i \in V : r_i > m\} \neq V$ and $\Lambda_{t_k}(m - \varepsilon, m + \varepsilon) \neq \emptyset$ with $t_k > t_{k^*}$. Select an interval $[t_k, t_{k+1})$ satisfying $t_k > t_{k^*}$ and $t_{k+1} - t_k = T > n(B + \Gamma)$. (If $t_{k+1} - t_k \leq n(B + \Gamma)$, then one can use some t_{k+s} to replace t_{k+1} to obtain $t_{k+s} - t_k > n(B + \Gamma)$).

Let

$$A_p = V_{t_k+p}(l_p - \varepsilon), \quad (4)$$

$$C_p = V - A_p. \quad (5)$$

Since $\cup_{s=1}^{\Gamma} G(t + s)$ is strongly connected, if A_{p+t} and C_{p+t} are nonempty and unchanged for $0 \leq t < \Gamma$, then there exists at least one edge from A_{p+t} to C_{p+t} at some time $0 \leq t < \Gamma$.

By using an induction procedure, one can prove that $A_p \subseteq A_{p+1}$ for $p \in \{0, 1, 2, \cdots, n(B + \Gamma)\}$.

i) $p = 0$:

It is obvious that $A_0 \neq \emptyset$ and $C_0 \neq \emptyset$. If $i \in A_0$, then $m_i(t_k) > l - \varepsilon$. That is,

$$\begin{aligned} x_i(t_k - B + 2) &> l - \varepsilon > l_1 - \varepsilon, \\ x_i(t_k - B + 3) &> l - \varepsilon > l_1 - \varepsilon, \\ &\vdots \\ x_i(t_k) &> l - \varepsilon > l_1 - \varepsilon. \end{aligned}$$

For $\tau_i^i = 0$, by the definition of ε, one gets

$$\begin{aligned} x_i(t_k + 1) &= g^{-1}\left(\frac{1}{n_i(t_k)} \sum_{j \in N_i(t_k)} f\left(x_j(t_k - \tau_j^i(t_k))\right)\right) \\ &\geq g^{-1}\left(\frac{1}{n_i(t_k)}((n_i(t_k) - 1)f(m - \varepsilon) \right. \\ &\quad \left. + f(l - \varepsilon))\right) \\ &> l_1 - \varepsilon \\ &> m + \varepsilon. \end{aligned}$$

Hence,

$$m_i(t_k + 1) \geq l_1 - \varepsilon$$

and $i \in A_1$. Therefore, $A_0 \subseteq A_1$.

ii) Suppose that the case of $p - 1$ holds, then for the case of p:

If $i \in A_p$, similarly, one has $i \in A_{p+1}$. Thus, $A_p \subseteq A_{p+1}$ for $p \in \{0, 1, 2, \cdots, n \cdot (B + \Gamma)\}$. Therefore, $C_{p+1} \subseteq C_p$.

If $A_0 = V$, then $\Lambda_{t_k}(m - \varepsilon, m + \varepsilon) = \emptyset$ and $C_0 = \emptyset$. Obviously, this is a contradiction. Therefore, $A_0 \neq V$.

In what follows, we prove that $A_0 \subsetneq A_{B+\Gamma}$:

Suppose that $A_0 = A_{B+\Gamma}$, then $A_0 = A_t = A_{B+\Gamma}$ for any $0 \leq t \leq B + \Gamma$. For $A_0 \neq V$ and $C_0 \neq \emptyset$, if $i \in C_s \subseteq C_0$ and $N_i(t_k + s) \cap A_s \neq \emptyset$ for some $0 \leq s < \Gamma$, then one gets

$$\begin{aligned} & x_i(t_k + s + 1) \\ &= g^{-1}\left(\frac{1}{n_i(t_k + s)} \sum_{j \in N_i(t_k+s)} f\left(x_j(t_k + s - \tau_j^i(t_k + s))\right)\right) \\ &\geq g^{-1}\left(\frac{1}{n_i(t_k + s)}\Big((n_i(t_k + s) - 1)f(m - \varepsilon) \right. \\ &\qquad \left. + f(l_s - \varepsilon)\Big)\right) \\ &> l_{s+1} - \varepsilon \\ &> m + \varepsilon\,. \end{aligned}$$

Here, the existence of i is guaranteed by the fact that $\cup_{s=1}^{\Gamma} G(t + s)$ is strongly connected and $A_s \cap C_s = \emptyset$.

Similarly, one has

$$\begin{aligned} x_i(t_k + s + 2) &> l_{s+2} - \varepsilon > m + \varepsilon, \\ x_i(t_k + s + 3) &> l_{s+3} - \varepsilon > m + \varepsilon, \\ &\vdots \\ x_i(t_k + s + B) &> l_{s+B} - \varepsilon > m + \varepsilon. \end{aligned}$$

For $l_p > l_{p+1}$, one gets $m_i(t_k + s + B) > l_{s+B} - \varepsilon$ and $i \in A_{s+B}$. However, according to $i \in C_s \subseteq C_0$ and $A_0 \cap C_0 = \emptyset$, then $i \notin A_0$. Since $A_p \subseteq A_{p+1}$, then $A_0 \neq A_{B+\Gamma}$. Thus, $A_0 \subsetneq A_{B+\Gamma}$.

Repeat the above process, there exists some Δ satisfying $1 \leq \Delta \leq n - 1$ and $A_0 \subsetneq A_{B+\Gamma} \subsetneq A_{2(B+\Gamma)} \subsetneq \cdots \subsetneq A_{\Delta(B+\Gamma)} = V$.

Hence, $C_{\Delta(B+\Gamma)} = V - A_{\Delta(B+\Gamma)} = \emptyset$. For $t > t_k + \Delta(B + \Gamma)$, one has

$$x_i(t) \geq g^{-1} \circ f(l_{\Delta(B+\Gamma)} - \varepsilon) \geq l_{\Delta(B+\Gamma)} - \varepsilon > m + \varepsilon.$$

Combining the above inequality with $t_{k+1} > t_k + n(B+\Gamma)$, one gets $\Lambda_{t_{k+1}}(m-\varepsilon, m+\varepsilon) = \emptyset$. Obviously, it is contradicts the fact of $\Lambda_{t_{k+1}}(m - \varepsilon, m + \varepsilon) \neq \emptyset$.

Hence, the above reasoning shows that it is impossible that $R \neq m$.

If $m = R$, then one has $r_i = m_i = m$.

Moreover, if $m_i = M_i$ for any $i \in \{1, 2, \cdots, n\}$, then $\lim_{t\to\infty} x_i(t) = m$. Therefore, the states of all agents can reach consensus.

If there exists some $i \in V$ satisfying $\overline{\lim}_{t\to\infty} m_i(t) \neq m$, then there exists a sequence $\{t'_k\}$ satisfying $m_i(t'_k) \to r_i \neq m$ and $m_j(t'_k) \to r_j$ for $\forall j \in V \backslash \{i\}$. Thus the proof of this case is the same as that of the case $m \neq R$ and is thus omitted here.

Beside the above two cases, one need to consider the following case: there exists some $j \in V$ satisfying $m_j \neq M_j$ and $\lim_{t\to\infty} m_i(t) = m$ for any $i \in V$.

For some i satisfying $m_i \neq M_i$, then one can select some accumulation point η_0 of $\{x_i(t)\}$ with $\eta_0 > m$. According to Lemma 4, there exists a sequence $\{\eta_p\}$ and some $\varepsilon > 0$ such that

$$\begin{array}{rcl} \frac{1}{k}((k-1)f(m-\varepsilon)+f(\eta_p-\varepsilon)) & > & g(\eta_{p+1}-\varepsilon), \\ \eta_p - \varepsilon & > & m+\varepsilon, \end{array}$$

where $m < \eta_{p+1} < \eta_p$, $k \in \{1, 2, \cdots, n\}$, and $p \in \{0, 1, 2, \cdots, B\}$.

Therefore, for the above ε, there exists a sufficiently large positive integer ξ satisfying $x_i(\xi) > \eta_0 - \varepsilon$ and $m - \varepsilon < m_j(t) < m + \varepsilon$ for $\forall j \in V$ and $t \geq \xi$.

For $\tau_i^i = 0$, by the definition of ε, one has

$$\begin{array}{rcl} x_i(\xi+1) & = & g^{-1}\left(\frac{1}{n_i(\xi)}\sum_{j\in N_i(\xi)} f\left(x_j\left(\xi-\tau_j^i(\xi)\right)\right)\right) \\ & \geq & g^{-1}\left(\frac{1}{n_i(\xi)}((n_i(\xi)-1)f(m-\varepsilon)\right. \\ & & \left. + f(\eta_0-\varepsilon))\right) \\ & > & \eta_1 - \varepsilon \\ & > & m+\varepsilon, \\ x_i(\xi+2) & > & \eta_2 - \varepsilon \quad > \quad m+\varepsilon, \\ & \vdots & \\ x_i(\xi+B) & > & \eta_B - \varepsilon \quad > \quad m+\varepsilon. \end{array}$$

Then, one gets

$$m_i(\xi+B) > \min\{\eta_0, \eta_1, \cdots, \eta_B\} - \varepsilon = \eta_B - \varepsilon > m+\varepsilon.$$

Obviously, it contradicts the fact that $m-\varepsilon < m_j(t) < m+\varepsilon$ holds for any $j \in V$ and $t \geq \xi$. Hence this case is impossible.

Therefore, the states of all agents can reach consensus. ■

Remark 1: In Assumptions (A1) and (A3), if $f(a) = g(a)$ and $f(x) \leq g(x)$ in $[a, b]$, similar result can also be obtained for the MAS $(V, G(t), (3))$.

Remark 2: If the MAS can reach consensus for the local rule (3), then one gets $f(x_{ss}) = g(x_{ss})$, where x_{ss} is the ultimate state. In fact, Assumption (A1) can naturally guarantee the nonempty of the set $\{x : f(x) = g(x)\}$. By Lemmas 1-3, one gets $f(b) = g(b)$ and $f(m) = g(m)$. If b is the unique point satisfying $f(x) = g(x)$, then $m = b = x_{ss}$.

Remark 3: Consider the situation when a leader, denoted by agent 0, is present within an MAS. In the neighborhood of the leader, denoted by $N_0(t)$, agents can get the information of the leader, but the leader always keeps its state at a fixed location, denoted by x_0, with $f(x_0) = g(x_0)$. For all the follower agents, they update their states according to the local rules (3). It is thus interesting to know whether the MAS can always reach consensus if the topology among the followers is strongly connected and there always exist edges from the leader to the followers. In general, for the MAS with linear local rules, if the topology among the leader and followers exists a spanning tree rooted at the leader, then the states of all agents can reach consensus. However, it is not always true for the MAS with nonlinear local rules, as the following simple example shows.

Let $f(x) = x$ and

$$g(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \ \leq 1, \\ 2x - \frac{3}{2} & 1 \leq x \leq \frac{3}{2}. \end{cases}$$

Construct an MAS with two agents, denoted by 0 and 1. Suppose that 0 is the group leader and 1 is the follower. Moreover, there always exists an edge from the leader to the follower. The initial states are given by $x_0(0) = 0$, $x_1(0) = 1$, and $\tau_j^i(t) = 0$. Then one has $x_1(1) = g^{(-1)}(\frac{1}{2}(f(0) + f(1))) = g^{(-1)}(\frac{1}{2}) = 1$ and $x_1(t) = 1$ for any $t \geq 1$. That is, the MAS cannot reach consensus.

Remark 4: If $f(x) = g(x) = x$ and $\tau_j^i = 0$, then the updating local rules (3) will degenerate into the following linearized model ([2], [3], [7], [8]):

$$x_i(t+1) \quad = \quad \frac{1}{n_i(t)}\sum_{j\in N_i(t)} x_j(t).$$

Let

$$\underline{\Omega}_m(t) = \left\{i : m_i(t) = \min_{j\in\{1,2,\cdots,n\}} m_j(t)\right\},$$
$$\overline{\Omega}_m(t) = \left\{i : m_i(t) = \max_{j\in\{1,2,\cdots,n\}} m_j(t)\right\}.$$

Then, according to the proof of Theorem 1, one can easily deduce the following result.

Theorem 2: Suppose that Assumptions (A1)-(A3), (A5) and (A6) hold for a given MAS $(V, G(t), (3))$. If there exists a sequence $\{t_k\}$ and an integer $\lambda_m > 0$ such that there exists a directed path from $\overline{\Omega}_m(t_k)$ to $\underline{\Omega}_m(t_k)$ and $0 < t_{k+1} - t_k < \lambda_m$ for $\forall k > 0$, then the MAS $(V, G(t), (3))$ can reach consensus.

From Theorem 2, it is easy to see that the agents in $\underline{\Omega}_m(t)$ play an important role for the consensus of the MAS. If there is a directed path from $\overline{\Omega}_m(t_k)$ to $\underline{\Omega}_m(t_k)$, then t_k is called the effective time. Generally speaking, if Assumptions (A1)-(A3), (A5) and (A6) hold and there exist sufficiently many effective times t_k, then the MAS can reach consensus. It indicates us that the MAS can reach consensus even if one greatly reduces the other redundant edges among agents. Therefore, it is very important for us to find all the effective time t_k in an MAS since it can help us to reduce the redundant edges among agents.

Since $G(t)$ is not necessary connected at all time, $\{i\} \subsetneq N_i(t)$ does not always hold. If $\{i\} \subsetneq N_i(t)$, then the updating rule of $x_i(t)$ should follow (3). If $\{i\} = N_i(t)$, according to (3), then one has $x_i(t+1) = g^{-1} \circ f(x_i(t))$. That is, even if an agent does not get any information from the other agents except itself, its state should be updated at the same time. In fact, for the case $\{i\} = N_i(t)$, if the state $x_i(t)$ does not follow the updating rule (3) but keep its state $x_i(t+1) = x_i(t)$ unchanged, then the MAS can also reach consensus under the conditions of Theorems 1 and 2.

Moreover, Theorems 1 and 2 also hold even if the updating rules (3) are generalized to the following form:

$$x_i(t+1) = F\left(\sum_{j\in N_i(t)} a_{ij}(t) f\left(x_j\left(t-\tau_j^i(t)\right)\right)\right), \qquad (6)$$

where $\sum_{j\in N_i(t)} a_{ij}(t) = 1$ and $\inf_{j\in N_i(t)} a_{ij}(t) \geq \alpha$ for some $\alpha \in (0,1)$.

IV. APPLICATION TO THE VICSEK MODEL

The Vicsek model is a typical discrete-time MAS model of collective behaviors [1]. Consider n autonomous agents moving on the plane. Denote the heading of agent i at time t by $\theta_i(t)$. Also, assume that the velocity of each agent is a constant, denoted by v. Then, the dynamics of agent i can be described by the sequence $\{x_i(t), y_i(t), \theta_i(t)\}$, where $(x_i(t), y_i(t))$ are the coordinates of agent i at time t, $x_i(t), y_i(t) \in \mathbf{R}$, $\theta_i(t) \in [0, 2\pi)$. If time delays are included, then the dynamics of each agent can be updated according to the following rules:

$$\begin{aligned} \theta_i(t+1) &= \arctan\left(\frac{\sum_{j\in N_i(t)} \sin\left(\theta_j(t-\tau_j^i(t))\right)}{\sum_{j\in N_i(t)} \cos\left(\theta_j(t-\tau_j^i(t))\right)}\right), && (7)\\ x_i(t+1) &= x_i(t) + v\cos(\theta_i(t+1)), && (8)\\ y_i(t+1) &= y_i(t) + v\sin(\theta_i(t+1)), && (9) \end{aligned}$$

where $t \in \{0, 1, 2, \cdots\}$, and

$$N_i(t) = \left\{ j\in V : \sqrt{(x_i(t)-x_j(t))^2+(y_i(t)-y_j(t))^2} < r \right\},$$

with $r > 0$.

In each time t, agent i will exchange its information with its neighboring agents in $N_i(t)$. Here, $i \in N_i(t)$ by the definition of $N_i(t)$. If two agents are both in some $N_i(t)$, then these two agents are neighbors. And the relation of neighbors at time t can define the topology of communication, denoted by $G(t)$. It is obvious that $G(t)$ is an undirected graph.

Denote

$$G_t = \bigcup_{s=1}^{T} G(t+s). \qquad (10)$$

Theorem 3: Suppose that Assumptions (A5) and (A6) hold for the classical Vicsek model with time-varying delays in the updating rules (7). Also, assume that the graph G_t is connected for all $t \geq 0$. If the initial headings $\theta_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $-B < t \leq 0$, then the headings of agents can reach consensus.

Proof. The updating rules (7) can be rewritten as follows:

$$\tan\theta_i(t+1) = \sum_{j\in N_i(t)} \frac{\cos\theta_j(t-\tau_j^i(t))\tan\theta_j(t-\tau_j^i(t))}{\sum_{k\in N_i(t)}\cos\theta_k(t-\tau_k^i(t))}.$$

Let

$$\begin{aligned} a_{ij}(t) &= \frac{\cos\theta_j(t-\tau_j^i(t))}{\sum_{k\in N_i(t)}\cos\theta_k(t-\tau_k^i(t))},\\ x_i(t) &= \tan\theta_i(t), \end{aligned}$$

then (7) can be recast in the following form:

$$x_i(t+1) = \sum_{j\in N_i(t)} a_{ij}(t) x_j(t-\tau_j^i(t)). \qquad (11)$$

For a given set of initial headings $\theta_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $-B < t \leq 0$, one can prove that

$$\min_{-B<t'\leq 0, 1\leq j\leq n} x_j(t') \leq x_i(t) \leq \max_{-B<t'\leq 0, 1\leq j\leq n} x_j(t'),$$

for any $t > 0$. Hence,

$$\min_{-B<t'\leq 0, 1\leq j\leq n} \theta_j(t') \leq \theta_i(t) \leq \max_{-B<t'\leq 0, 1\leq j\leq n} \theta_j(t'),$$

for any $t > 0$. As a result, from the basic property of $\cos\theta$ on $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there exists an $\alpha \in (0,1)$ satisfying $a_{ij}(t) \in [\alpha, 1]$ or $a_{ij}(t) = 0$. This indicates that (11) is a special case of Theorem 1, in view of (6) with $f(x) = F(x) = x$. ■

V. SIMULATION RESULTS

To verify the effectiveness of the proposed consensus criteria in Sections III and IV, a typical numerical simulation is presented in this section to verify Theorem 2.

Let f and g be two piecewise linear functions, described respectively by

$$f(x) = \begin{cases} x+1 & 0 \leq x \leq 1,\\ 2x & 1 \leq x \leq 2,\\ x+2 & 2 \leq x \leq 3,\\ \frac{1}{3}x+4 & 3 \leq x \leq 6, \end{cases}$$

and

$$g(x) = \begin{cases} 2x & 0 \leq x \leq 2,\\ \frac{1}{2}x+3 & 2 \leq x \leq 6. \end{cases}$$

It is easy to verify that

$$S_a = \{x : f(x) = g(x)\} = \{x : 1 \leq x \leq 2, x = 6\}$$

and the inverse function $F(x)$ of $g(x)$ is given by

$$F(x) = g^{-1}(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 4,\\ 2x-6 & 4 \leq x \leq 6. \end{cases}$$

The functions f and F are shown in Fig. 1.

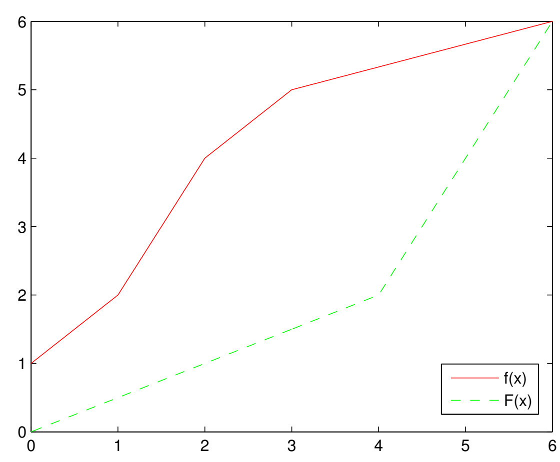

Fig. 1. The functions $f(x)$ and $F(x)$.

Consider a discrete-time MAS $(V, G(t), (3))$ consisting of 8 agents. For simplicity, let $\lambda_m = 2$ and $0 \le \tau_j^i < B = 2$. Then, randomly generate two sets of initial states as follows. The first set of initial states are described by

$$\begin{aligned} x(-1) &= (1.12, 1.52, 0.58, 1.65, 0.17, 0.17, 0.76, 2.06)^T, \\ x(0) &= (0.60, 0.98, 0.16, 2.08, 0.00, 1.59, 0.65, 2.39)^T. \end{aligned}$$

Fig. 2 shows the evolving phase trajectories from the these initial states.

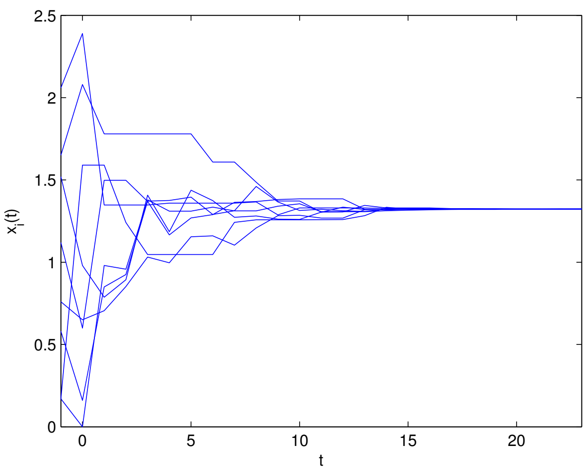

Fig. 2. Phase trajectories from the first set of initial states.

The second set of initial states are given by

$$\begin{aligned} x(-1) &= (0.30, 5.87, 0.71, 3.70, 0.94, 0.05, 4.45, 4.66)^T, \\ x(0) &= (4.82, 0.04, 0.59, 4.91, 1.17, 1.75, 5.63, 0.82)^T. \end{aligned}$$

The evolving phase trajectories from the second set of initial states are shown in Fig. 3.

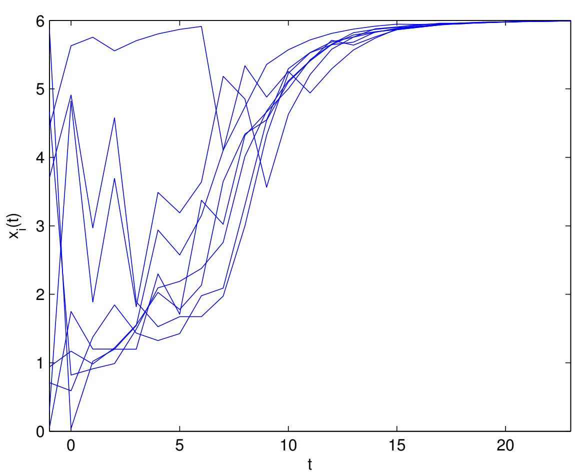

Fig. 3. Phase trajectories from the second set of initial states.

To satisfy the conditions in Theorem 2, $G(t)$ is generated in all simulations in the following way: For each time instant t, choose two agents $\underline{x}_m$ and $\overline{x}_m$ from $\underline{\Omega}_m(t)$ and $\overline{\Omega}_m(t)$, respectively, then generate a directed path from $\overline{x}_m$ to $\underline{x}_m$ in $G(t)$, and finally, some random edges are added to $G(t)$. With $0 \le \tau_j^i < B = 2$, then $\tau_j^i(t) = 0$ or 1 in all simulations. Moreover, all values of τ_j^i are randomly chosen from the set $\{0, 1\}$ except $\tau_i^i = 0$.

From Figs. 2 and 3, it is obvious that all ultimate states x_{ss} will certainly converge to the set $S_1 = \{x : 1 \le x \le 2\}$ if the initial states are not too large. However, consensus of the MAS will certainly be reached at the point $S_2 = \{6\}$ if the initial states are too large. Here, $S_a = S_1 \cup S_2$.

VI. CONCLUDING REMARKS

In this paper, we have further explored the consensus of discrete-time MAS's with nonlinear local rules and time-varying delays. Some basic consensus criteria are obtained for such MAS's. Our results include several well-known results as special cases. Also, these consensus criteria are applied to the consensus of the classical Vicsek model with time-varying delays. Simulation results were presented to validate the proposed criteria. The rate of convergence for the discrete-time MAS with time-varying delays will be further investigated in the near future.

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