State Space Analysis of Boolean Networks

Daizhan Cheng, *Fellow, IEEE*, Hongsheng Qi

***Abstract*—This paper provides a comprehensive framework for the state space approach to Boolean networks. First, it surveys the authors' recent work on the topic: Using semi-tensor product of matrices and the matrix expression of logic, the logical dynamic equations of Boolean (control) networks can be converted into standard discrete-time dynamics. To use the state space approach, the state space and its subspaces of a Boolean network have been carefully defined. The basis of a subspace has been constructed. Particularly, the regular subspace, $\mathcal{Y}$-friendly subspace, and invariant subspace are precisely defined, and the verifying algorithms are presented. As an application, the indistinct rolling gear structure of a Boolean network is revealed.**

***Index Terms*—Boolean (control) network, state space, subspace, basis, indistinct rolling gear structure.**

I. Introduction

BECAUSE of the development of systems biology, the study of Boolean networks becomes a new cross-discipline hot topic. Kauffman is the pioneer on this field [15]. [16] provides a less academic but more intuitive description for the role of Boolean network in cellular regulation.

Using semi-tensor product and the matrix expression of logic, we have developed a new systematic approach to the analysis and control of Boolean (control) networks [4]–[6]. The engine of this new approach is the state space approach of the logical dynamic systems. Summarizing our previous results, this paper intends to build a comprehensive framework for the state space approach to the Boolean networks. Certain new results have been added to make this engine structurally complete.

Denote by

$$\mathcal{D} = \{0 \sim F, 1 \sim T\}$$

the set of logical values. A logical variable is an independent variable which can take any value from $\mathcal{D}$. A logical function f with logical variables $x_1, \cdots, x_n$ as its arguments is a mapping $f : \mathcal{D}^n \to \mathcal{D}$.

Now assume that $x_1, \cdots, x_n$ are a set of time-varying logical variables. Their involvement subjects to the following logical dynamic equations:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \end{cases} \tag{1}$$

where f_i, $i = 1, \cdots, n$ are logical functions. We call (1) a discrete-time logical dynamic system.

This work was supported in part by the National Natural Science Foundation (NNSF) of China under Grants 60674022, 60736022, and 60821091.

The authors are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, P.R.China. (E-mail: dcheng@iss.ac.cn, qihongsh@amss.ac.cn)

For a discrete-time logical dynamic system if there are some additional inputs, called the controls, and some outputs, it becomes a discrete-time logical dynamic control system. Its dynamics can be expressed as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), u_1, \cdots, u_m) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), u_1, \cdots, u_m), \\ y_j(t) = h_j(x_1(t), \cdots, x_n(t)), \quad j = 1, \cdots, p, \end{cases} \tag{2}$$

where f_i, $i = 1, \cdots, n$, h_j, $j = 1, \cdots, p$ are logical functions, u_i, $i = 1, \cdots, m$ are controls, y_j, $j = 1, \cdots, p$ are outputs.

A logical dynamic (control) system is also called a Boolean (correspondingly, control) network, which is firstly introduced by Kauffman [14]. Boolean network has attracted a considerable attention from biologists, physicians, and system scientists, because it has been proved to be a proper tool to describe cellular networks [11], [15].

Physically, a Boolean network consists of n nodes, denoted by $\mathcal{N} = \{1, 2, \cdots, n\}$, and a set of edges, denoted by $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$. $(i, j) \in \mathcal{E}$ means there is a directed side from i to j. Physically, it means in the dynamics of node j is affected by node i directly. Using x_i to describe the i-th node, which can take values from $\mathcal{D}$, (1) is a proper way to describe its dynamics. The dynamics of a Boolean control network is described by (2).

We give two simple examples for a Boolean network and a Boolean control network respectively. (We refer to [12], [17] or any other standard textbook of mathematical logic for the logical operators used in the sequel.)

Example I.1. *1. Consider a Boolean network depicted in Fig. 1. Its dynamics is described as*

$$\begin{cases} A(t+1) = B(t) \bar{\vee} E(t) \\ B(t+1) = C(t) \\ C(t+1) = (A(t) \bar{\vee} D(t)) \wedge C(t) \\ D(t+1) = E(t) \\ E(t+1) = D(t). \end{cases} \tag{3}$$

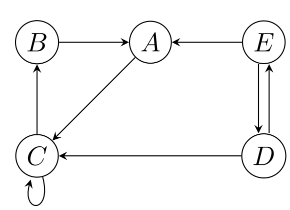

Fig. 1: A Boolean network

2. Consider a Boolean control network depicted in Fig. 2, which is obtained from Fig. 1 by adding two inputs, u_1, u_2, *and one output, y. Its dynamics is described as*

$$\begin{cases} A(t+1) = B(t)\bar{\vee}E(t) \\ B(t+1) = C(t) \vee (D(t) \wedge u_1(t) \\ C(t+1) = (A(t)\bar{\vee}D(t)) \wedge C(t) \\ D(t+1) = E(t) \\ E(t+1) = D(t) \to u_2(t); \\ \\ y(t) = D(t) \leftrightarrow E(t). \end{cases} \tag{4}$$

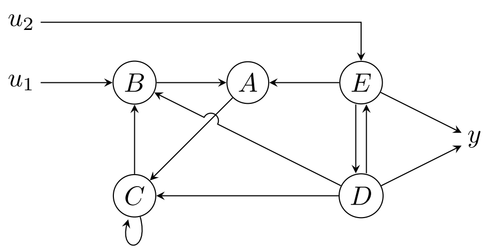

Fig. 2: A Boolean control network

One of the milestones in modern control theory is the state space description of the control systems, proposed by Kalman. Observing equations (1) and (2), one sees that they are formally the same as the state space description of dynamic (control) systems. Unfortunately, they are essentially different from the conventional dynamic (control) systems. Let us first investigate this: Denote $\mathcal{X} = (x_1, \cdots, x_n)$. In linear case, $\mathcal{X}$ is in a linear vector space, say, $\mathbb{R}^n$, and in nonlinear case, $\mathcal{X}$ could be in an n dimensional manifold, which could be $\mathbb{R}^n$ or locally diffeomorphic to an open set of $\mathbb{R}^n$. But in logical case, $\mathcal{X} = \mathcal{D}^n$, which does not have vector space structure such as $\mathbb{R}^n$. So the state space approach seems not directly applicable to the Boolean (control) networks. In our previous series works, we have gradually introduced the concepts of coordinate transformation, regular subspace, invariance subspace etc. to the logical systems, which make it possible to use the state space approach to logical dynamic systems. The purpose of this paper is to systemize what we proposed in previous works with certain necessary new techniques to form a systematic state space approach to logical dynamic (control) systems.

The main tool for this approach is the new matrix product, called the semi-tensor product of matrices (denoted by $A \ltimes B$). It is a generalization of conventional matrix product to the case when the column number of the first factor matrix, A, is not the same as the row number of the second factor matrix, B. Using it, a logical equation can be expressed as an algebraic equation. We refer to [2] or [3] for a systematic introduction to this new matrix product. Throughout this paper the matrix product is assumed to be semi-tensor product. When the dimension matching condition is satisfied for two matrices A and B, the product $A \ltimes B = AB$ becomes the conventional matrix product.

II. Algebraic Form of Logical Dynamics

To use matrix expression of logic, we need some notations.

- δ_n^i: the i-th column of the identity matrix I_n;
- $\Delta_n = \{\delta_n^i \mid i = 1, 2, \cdots, n\}$;
- $\mathrm{Col}(A)$: the set of columns of A;
- A matrix $L \in M_{n\times s}$ is called a logical matrix, if
$$\mathrm{Col}(L) \subset \Delta_n.$$
Denote the set of $n \times s$ logical matrices by $\mathcal{L}_{n\times s}$.
- If $A \in \mathcal{L}_{n\times s}$, A can be expressed as $A = [\delta_n^{i_1}, \cdots, \delta_n^{i_s}]$. For the sake of condense, A is denoted as
$$A = \delta_n[i_1, \cdots, i_s].$$
- Vector form of logical values. We identify
$$1 \sim \delta_2^1; \quad 0 \sim \delta_2^2.$$
Then the vector form of the set of logical values is Δ_2. That is, $\mathcal{D} \sim \Delta_2$.

The following result is one of the key points in our approach.

Theorem II.1 ([2], [3]). *Let f be a logical function of n arguments. Then there exists a unique $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of f, such that*

$$f(x_1, \cdots, x_n) = M_f x_1 x_2 \cdots x_n, \quad x_i \in \Delta_2, \ \forall\, i. \tag{5}$$

It is worth noting that the products on the right hand side of the above equality are semi-tensor product of matrices, and the symbol $\ltimes$ is omitted. In conventional sense, they are not defined.

Denote $x = \ltimes_{i=1}^n x_i$. Then (5) can also be expressed as

$$f(x_1, \cdots, x_n) = M_f x. \tag{6}$$

Note that $\ltimes_{i=1}^n : (\Delta_2)^n \to \Delta_{2^n}$, which maps $(x_1, \cdots, x_n) \mapsto x$ is a bijective mapping. The converting formula was given in [6].

In the following table we list the structure matrices for some basic logical operators (LO), which are used in the sequel.

TABLE I: Structure Matrix of Logical Operators

LO	Structure Matrix	LO	Structure Matrix
$\neg$	$M_n = \delta_2[2\ 1]$	$\vee$	$M_d = \delta_2[1\ 1\ 1\ 2]$
$\to$	$M_i = \delta_2[1\ 2\ 1\ 1]$	$\leftrightarrow$	$M_e = \delta_2[1\ 2\ 2\ 1]$
$\wedge$	$M_c = \delta_2[1\ 2\ 2\ 2]$	$\bar{\vee}$	$M_p = \delta_2[2\ 1\ 1\ 2]$

A logical function can be expressed by some fundamental logical operators. For instance, since $\{\neg, \wedge, \vee\}$ is an adequate set [12], any logical function can be expressed by them. Then the structure matrix M_f of a logical function f can be calculated by using the structure matrices of some fundamental logical operators as in Table I and some properties of semi-tensor product. It was briefly reviewed in [4].

The structure matrix M_f of f can also be calculated directly as follows: Denote $M_f = [c_1\ c_2\ \cdots\ c_{2^n}]$. Let $x = \ltimes_{i=1}^n x_i = \delta_{2^n}^k$. We can uniquely calculate out $\{x_1, \cdots, x_n\}$ from x. Say, we have $x_i = a_i^k \in \Delta_2$, $i = 1, \cdots, n$. Then the k-th column c_k of M_f is

$$c_k = f(a_1^k, \cdots, a_n^k), \quad k = 1, \cdots, 2^n. \tag{7}$$

Using Theorem II.1 and some properties of semi-tensor product of matrices, we can convert a logical dynamic (control) system into its algebraic form.

Theorem II.2 ([6]). *1. Consider system* (1)*. Define* $x = \ltimes_{i=1}^{n} x_i$*. Then there exists a unique* $L \in \mathcal{L}_{2^n \times 2^n}$*, called the transition matrix of the system, such that*

$$x(t+1) = Lx(t). \tag{8}$$

(8) *is called the algebraic form of* (1)*.*

2. Consider system (2)*. Define* $x = \ltimes_{i=1}^{n} x_i$*,* $u = \ltimes_{i=1}^{m} u_i$*,* $y = \ltimes_{i=1}^{p} y_i$*. Then there exist unique* $L \in \mathcal{L}_{2^n \times 2^{n+m}}$*, and unique* $H \in \mathcal{L}_{2^p \times 2^n}$ *such that*

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t). \end{cases} \tag{9}$$

(9) *is called the algebraic form of* (2)*.*

To illustrate this, we recall Example I.1.

Example II.3. *Consider Example I.1.*

1) *Let* $x(t) = A(t)B(t)C(t)D(t)E(t)$*. Then the algebraic form of system* (3) *is*

$$x(t+1) = Lx(t),$$

where

$$\begin{array}{l} L = \delta_{32}[21 \;\; 7 \;\; 18 \;\; 4 \;\; 29 \;\; 15 \;\; 30 \;\; 16 \\ \quad 5 \;\; 23 \;\; 2 \;\; 20 \;\; 13 \;\; 31 \;\; 14 \;\; 32 \\ \quad 17 \;\; 3 \;\; 22 \;\; 8 \;\; 29 \;\; 15 \;\; 30 \;\; 16 \\ \quad 1 \;\; 19 \;\; 6 \;\; 24 \;\; 13 \;\; 31 \;\; 14 \;\; 32]. \end{array}$$

2) *Let* $x(t) = A(t)B(t)C(t)D(t)E(t)$*, and* $u(t) = u_1(t)u_2(t)$*. Then the algebraic form of system* (4) *is*

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases}$$

where

$$\begin{array}{l} L = \delta_{32}[21 \;\; 7 \;\; 17 \;\; 3 \;\; 21 \;\; 7 \;\; 29 \;\; 15 \\ \quad 5 \;\; 23 \;\; 1 \;\; 19 \;\; 5 \;\; 23 \;\; 13 \;\; 31 \\ \quad 17 \;\; 3 \;\; 21 \;\; 7 \;\; 21 \;\; 7 \;\; 29 \;\; 15 \\ \quad 1 \;\; 19 \;\; 5 \;\; 23 \;\; 5 \;\; 23 \;\; 13 \;\; 31 \\ \quad 22 \;\; 8 \;\; 17 \;\; 3 \;\; 22 \;\; 8 \;\; 29 \;\; 15 \\ \quad 6 \;\; 24 \;\; 1 \;\; 19 \;\; 6 \;\; 24 \;\; 13 \;\; 31 \\ \quad 18 \;\; 4 \;\; 21 \;\; 7 \;\; 22 \;\; 8 \;\; 29 \;\; 15 \\ \quad 2 \;\; 20 \;\; 5 \;\; 23 \;\; 6 \;\; 24 \;\; 13 \;\; 31 \\ \quad 21 \;\; 7 \;\; 17 \;\; 3 \;\; 29 \;\; 15 \;\; 29 \;\; 15 \\ \quad 5 \;\; 23 \;\; 1 \;\; 19 \;\; 13 \;\; 31 \;\; 13 \;\; 31 \\ \quad 17 \;\; 3 \;\; 21 \;\; 7 \;\; 29 \;\; 15 \;\; 29 \;\; 15 \\ \quad 1 \;\; 19 \;\; 5 \;\; 23 \;\; 13 \;\; 31 \;\; 13 \;\; 31 \\ \quad 22 \;\; 8 \;\; 17 \;\; 3 \;\; 30 \;\; 16 \;\; 29 \;\; 15 \\ \quad 6 \;\; 24 \;\; 1 \;\; 19 \;\; 14 \;\; 32 \;\; 13 \;\; 31 \\ \quad 18 \;\; 4 \;\; 21 \;\; 7 \;\; 30 \;\; 16 \;\; 29 \;\; 15 \\ \quad 2 \;\; 20 \;\; 5 \;\; 23 \;\; 14 \;\; 32 \;\; 13 \;\; 31], \end{array}$$

and

$$\begin{array}{l} H = \delta_2[1 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \\ \quad 1 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \\ \quad 1 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \\ \quad 1 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 2 \;\; 2 \;\; 1]. \end{array}$$

A logical dynamic (control) system is commonly expressed in its logical form (1) (respectively, (2)). It can be converted into its algebraic form (8) (respectively, (9)) and vise versa. In fact, logical form and algebraic form are equivalent. We refer to [5] for converting algorithms from one to the other. [1]

III. State Space and Subspace

Consider a conventional linear system

$$x(t+1) = Ax(t), \quad x \in \mathbb{R}^n.$$

The state space can be expressed as

$$\mathcal{X} = \{ (x_1, \cdots, x_n) \mid x_i \in \mathbb{R}, \ i = 1, \cdots, n \}. \tag{10}$$

In fact, the state space is spanned by $\{x_1, \cdots, x_n\}$. So we can write

$$\mathcal{X} = \mathrm{Span}\{x_1, \cdots, x_n\}. \tag{11}$$

Now, each x_i spans a one-dimensional subspace of $\mathcal{X}$, denoted as

$$V_i = \mathrm{Span}\{x_i\}, \quad i = 1, 2, \cdots, n. \tag{12}$$

In generally, a subset of k elements, $\{x_{i_1}, \cdots, x_{i_k}\}$ can span a k-dimensional subspace, denoted by

$$W = \mathrm{Span}\{x_{i_1}, \cdots, x_{i_k}\}. \tag{13}$$

Alternatively, we may consider $\{x_1, \cdots, x_n\}$ as a coordinate frame of the state space $\mathcal{X}$. Each $a \in \mathcal{X}$ can be expressed uniquely as

$$a = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \quad \alpha_1, \cdots, \alpha_n \in \mathbb{R}.$$

Under this consideration, the state space is the set of linear functions of $\{x_1, \cdots, x_n\}$. Denote the set of linear functions of $\{x_1, \cdots, x_n\}$ by $F_s\{x_1, \cdots, x_n\}$, then we can express the state space alternatively as

$$\mathcal{X} = F_s\{x_1, \cdots, x_n\}. \tag{14}$$

Similarly, (12) can be expressed alternatively as

$$V_i = F_s\{x_i\}, \quad i = 1, 2, \cdots, n. \tag{15}$$

(13) can be expressed alternatively as

$$W = F_s\{x_{i_1}, \cdots, x_{i_k}\}. \tag{16}$$

In fact, here we define a subspace in "dual" way. That is, a set of linear functions determined a subspace, which is the domain of this set of functions.

Now consider the logical dynamic system (1). Similar to conventional dynamic systems, we can define the state space as

$$\mathcal{X} = \{ (x_1, \cdots, x_n) \mid x_i \in \mathcal{D}, \ i = 1, \cdots, n \} = \mathcal{D}^n. \tag{17}$$

Motivated by (14)-(16), we give the following definition.

Definition III.1. *Denote the state space of system* (1) *by*

$$\mathcal{X} := \mathrm{F}_\ell\{x_1, \cdots, x_n\},$$

[1] A toolbox for all the related computations is available at http://lsc.amss.ac.cn/~dcheng/

which is the set of logical functions of the arguments $x_1, \cdots, x_n$.

Throughout this paper F_ℓ of a set of logical variables means the set of logical functions with the logical variables as their arguments.

Remark III.2. *Since a logical function is compounded from (unary or binary) logical operators, a logical space means a set of logical variables, which is closed under logical operators. Since* $\{\neg, \wedge, \vee\}$ *is a commonly used adequate set, we can also say that a logical space is a set of logical variables, which are closed under* $\{\neg, \wedge, \vee\}$.

Definition III.3. *Let* $\mathcal{Z} \subset \mathcal{X}$. $\mathcal{Z}$ *is called a subspace of* $\mathcal{X}$*, if is closed under* $\{\neg, \wedge, \vee\}$.

Note that both state space $\mathcal{X}$ and subspace $\mathcal{Z}$ are sets. There is no special topological structure, or only the trivial discrete topology is applicable.

Definition III.4. *Let* $\mathcal{Z} \subset \mathcal{X}$ *be a subspace. Consider a finite set* $z_i \in \mathcal{X}$*,* $i = 1, \cdots, k$. $\{z_1, \cdots, z_k\}$ *is called a generator of* $\mathcal{Z}$*, if* $\mathcal{Z} = \mathrm{F}_\ell\{z_1, \cdots, z_k\}$. *A generator with minimum number is called a basis of the subspace.*

Remark III.5. *Let* $\{z_1, \cdots, z_k\}$ *be a set of* k *independent logical variables. Then from Theorem II.1 it is clear that* $\mathrm{F}_\ell(z_1, \cdots, z_k)$ *contains* 2^{2^k} *different elements. Independence means each* z_i *cannot be expressed as a logical function of* z_j*,* $j \neq i$.

For any subspace $\mathcal{Z} \subset \mathcal{X}$, there is at least one generator, because the set of all its elements, which is a finite set, is its generator. Now the following two problems are natural: (i) Given a subspace, how to find its basis? (ii) Is this basis unique in certain equivalent sense?

Let $\{z_1, \cdots, z_k\}$ be a generator of $\mathcal{Z}$. That is, $\mathcal{Z} = \mathrm{F}_\ell\{z_1, \cdots, z_k\}$. Denote $z = \ltimes_{i=1}^k z_i$ and $x = \ltimes_{i=1}^n x_i$. From previous section we know that we can express z as

$$z = T_0 x, \tag{18}$$

where $T_0 \in \mathcal{L}_{2^k \times 2^n}$. We call (18) the algebraic form of the subspace $\mathcal{Z}$ with respect to the generator $\{z_1, \cdots, z_k\}$.

We first seek for a generator with minimum number of elements. Assume $\{\xi_1, \cdots, \xi_s\}$ is another generator of $\mathcal{Z}$ with $s \leq k$. Since $\xi_i \in \mathcal{Z}$, we can find a logical matrix $P \in \mathcal{L}_{2^s \times 2^k}$, such that

$$\xi := \ltimes_{i=1}^s \xi_i = Pz = PT_0 x. \tag{19}$$

Since $\{\xi_1, \cdots, \xi_s\}$ is also a generator of $\mathcal{Z}$, we can find another logical matrix $Q \in \mathcal{L}_{2^k \times 2^s}$, such that

$$z = Q\xi = QPT_0 x. \tag{20}$$

Using above notations, we have

Theorem III.6. *Let*

$$\mathcal{Z} = \mathrm{F}_\ell\{z_1, \cdots, z_k\} \subset \mathcal{X}$$

be a subspace with its algebraic form with respect to the generator $\{z_1, \cdots, z_k\}$ *as in* (18). *Let* $s > 0$ *be an integer, such that*

$$2^{s-1} < \mathrm{rank}(T_0) \leq 2^s. \tag{21}$$

Then there exists at least one generator of s *elements, which is the generator with minimum number of elements, i.e., it is a basis.*

Proof: Comparing (20) with (18), since the coefficient matrix is unique, we have

$$\Phi T_0 = T_0, \quad \text{where } \Phi := QP \in \mathcal{L}_{2^k \times 2^k}. \tag{22}$$

Since $T_0 \in \mathcal{L}_{2^k \times 2^n}$, it can be expressed as

$$T_0 = \delta_{2^k}[i_1, i_2, \cdots, i_{2^n}].$$

It is obvious that $r := \mathrm{rank}(T_0)$ is the number of different entries in $\{i_1, \cdots, i_{2^n}\}$.

We claim the following:

Fact 1: Let $\delta_{2^k}^{i_j} \in \mathrm{Col}(T_0)$. Then to meet (22), we must have that the i_j-th column of Φ is $\delta_{2^k}^{i_j}$.

To see this, let $c_s \in \mathrm{Col}(T_0)$ be the s-th column of T_0 and $c_s = \delta_{2^k}^{i_j}$. It follows from (22) that $\Phi c_s = c_s$, $s = 1, \cdots, 2^n$. So the i_j-th column of Φ must be $\delta_{2^k}^{i_j}$.

According to Fact 1, the r columns of Φ have been determined uniquely. Moreover, these fixed r columns are enough to assure (22). Hence, the other columns of Φ can be chosen freely.

Next, we try to find logical matrices P and Q such that $QP = \Phi$. It is worth noting that

$$\mathrm{rank}(Q) \geq \mathrm{rank}(\Phi) \geq r, \tag{23}$$

which means if a generator has number s of elements, then $2^s \geq r$. Let s be the unique integer satisfying (21). Then if a generator contains exactly s elements, it is a basis.

First, we assume $r = 2^s$. Choosing r different columns from T_0 to form a matrix Q. That is,

$$Q = \begin{bmatrix} \delta_{2^k}^{i_{j_1}} & \cdots & \delta_{2^k}^{i_{j_r}} \end{bmatrix} \in \mathcal{L}_{2^k \times 2^s}.$$

Then we set $P_0 = Q^T$. Note that $j_1, \cdots, j_r$ are r distinguished numbers. It follows that

$$\mathrm{Col}(P_0) \subset \Delta_{2^s} \cup \{0_{2^s}\},$$

where $0_{2^s} \in \mathbb{R}^{2^s}$ is a zero vector. Replacing the zero columns of P_0 by any element in Δ_{2^s} yields a matrix $P \in \mathcal{L}_{2^s \times 2^k}$. Set $\Phi = QP$. Note that by construction it is clear that the i_{j_t} column of P, denoted by $p_{i_{j_t}}$, is

$$p_{i_{j_t}} = \delta_{2_r}^t, \quad t = 1, \cdots, r.$$

Then the i_{j_t}-th column of Φ, denoted by $\phi_{i_{j_t}}$, is

$$\phi_{i_{j_t}} = Qp_{i_{j_t}} = q_t = \delta_{2^k}^{i_{j_t}}, \quad t = 1, \cdots, r, \tag{24}$$

where q_t is the t-th column of Q. (24) shows that Φ satisfies Factor 1.

Note that the (semi-tensor) product of two logical matrices is still a logical matrix. So $\Phi \in \mathcal{L}_{2^k \times 2^k}$.

As for $r < 2^s$, in addition to the r different columns from T_0, we can choose additional $2^s - r$ columns $c_i \in \Delta_{2^k}$ such that the 2^s columns are linearly independent. Then they form the logical matrix $Q \in \mathcal{L}_{2^k \times 2^s}$. Using the same procedure as in the above, we can also construct a logical matrix P such that $\Phi = QP$ satisfied Fact 1.

Let $\xi = \ltimes_{i=1}^s \xi_i$ be determined by

$$\xi = Pz = PT_0 x.$$

By construction, we have

$$z = T_0 x = \Phi T_0 x = QPT_0 x = Q\xi,$$

which means ξ is a generator of $\mathcal{Z}$. Recall (23) and the argument after it, ξ is a basis of $\mathcal{Z}$. ■

In fact the above constructive proof provides an algorithm for constructing a basis. We summarizing it as follows.

Proposition III.7. *Let a subspace $\mathcal{Z} \subset \mathcal{X}$ be given with generator $\{z_1, \cdots, z_k\}$. The following Algorithm III.8 provides a basis of $\mathcal{Z}$.*

Algorithm III.8. *Step 1: Get the algebraic form of $\mathcal{Z}$ with respect to the generator $\{z_1, \cdots, z_k\}$ as*

$$z := \ltimes_{i=1}^k z_i = T_0 x, \tag{25}$$

where $T_0 \in \mathcal{L}_{2^k \times 2^n}$, with $\operatorname{rank}(T_0) = r$. Find s satisfying (21).

Step 2: Choose r distinct columns of T_0, say, $\left\{\delta_{2^k}^{i_{j_1}}, \cdots, \delta_{2^k}^{i_{j_r}}\right\}$ and add $2^s - r$ linearly independent $\delta_{2^k}^{i_{j_{r+1}}}, \cdots, \delta_{2^k}^{i_{j_{2^s}}}$ to form a matrix

$$Q = \left[\delta_{2^k}^{i_{j_1}}, \cdots, \delta_{2^k}^{i_{2^s}}\right] \in \mathcal{L}_{2^k \times 2^s}.$$

Step 3: Set $P_0 = Q^T$ and replace the zero columns of P_0 by any $\delta_{2^s}^t \in \Delta_{2^s}$ to get a matrix $P \in \mathcal{L}_{2^s \times 2^k}$.

Step 4: Set $\xi = PT_0 x$. Then ξ is a basis of $\mathcal{Z}$.

We use the following examples to show how to find a basis.

Example III.9. *Consider $\mathcal{X} = \mathrm{F}_\ell\{x_1, x_2, x_3\}$.*

1) *Assume*

$$\begin{cases} z_1 = x_3 \vee (\neg x_1 \wedge \neg x_2) \\ z_2 = (x_3 \wedge x_1) \vee (x_3 \wedge x_2) \\ z_3 = x_3. \end{cases}$$

We want to find a basis for $\mathcal{Z} = \mathrm{F}_\ell\{z_1, z_2, z_3\}$. Setting $z = \ltimes_{i=1}^3 z_i$, and $x = \ltimes_{i=1}^3 x_i$. Then it is easy to calculate that

$$z = T_0 x, \quad \text{where} \quad T_0 = \delta_8[1, 8, 1, 8, 1, 8, 3, 4].$$

Choosing different columns, we can form Q as

$$Q = \delta_8[1, 8, 3, 4].$$

Then we have

$$P_0 = Q^T = \delta_4[1, 0, 3, 4, 0, 0, 0, 2],$$

where $\delta_k^0 := 0_k \in \mathbb{R}^k$. Replacing δ_4^0 by δ_4^k, for any $1 \le k \le 4$, say, setting $k = 1$ we have

$$P = \delta_4[1, 1, 3, 4, 1, 1, 1, 2].$$

Setting

$$\Phi = QP = \delta_8[1, 1, 3, 4, 1, 1, 1, 8],$$

it is easy to check that $\Phi T_0 = T_0$. Hence

$$\xi = PT_0 x, \quad \text{with} \quad PT_0 = \delta_4[1, 2, 1, 2, 1, 2, 3, 4]$$

is a basis of $\mathcal{Z}$. Back to logical form, setting $\xi = \xi_1 \ltimes \xi_2$, where

$$\xi_i = M_i x, \quad i = 1, 2,$$

then it is easy to calculate that

$$\begin{cases} M_1 = \delta_2[1, 1, 1, 1, 1, 1, 2, 2] \\ M_2 = \delta_2[1, 2, 1, 2, 1, 2, 1, 2]. \end{cases}$$

It follows that

$$\begin{cases} \xi_1 = x_1 \vee x_2 \\ \xi_2 = x_3. \end{cases}$$

2) *Assume*

$$\begin{cases} z_1 = (x_1 \wedge x_2) \vee (\neg x_1 \wedge x_3) \\ z_2 = [x_1 \wedge (\neg x_2 \wedge x_3)] \vee [\neg x_1 \wedge (x_2 \wedge \neg x_3)] \\ z_3 = \neg(x_2 \vee x_3). \end{cases}$$

We want to find a basis for $\mathcal{Z} = \mathrm{F}_\ell\{z_1, z_2, z_3\}$. It is easy to calculate that

$$z = T_0 x, \quad \text{where} \quad T_0 = \delta_8[4, 4, 6, 7, 4, 6, 4, 7].$$

Choosing 3 different columns and add one more linearly independent column, say, δ_8^8, we can form Q as

$$Q = \delta_8[4, 6, 7, 8].$$

Then we have

$$P_0 = Q^T = \delta_4[0, 0, 0, 1, 0, 2, 3, 4]$$

Replacing δ_4^0 by, say, δ_4^1, we have

$$P = \delta_4[1, 1, 1, 1, 1, 2, 3, 4].$$

Setting

$$\Phi = QP = \delta_8[4, 4, 4, 4, 4, 6, 7, 8],$$

it is easy to check that $\Phi T_0 = T_0$. Hence

$$\xi = PT_0 x, \quad \text{with} \quad PT_0 = \delta_4[1, 1, 2, 3, 1, 2, 1, 3]$$

is a basis of $\mathcal{Z}$. Back to logic form, setting $\xi = \xi_1 \ltimes \xi_2$, where

$$\xi_i = M_i x, \quad i = 1, 2,$$

then it is easy to calculate that

$$\begin{cases} M_1 = \delta_2[1, 1, 1, 2, 1, 1, 1, 2] \\ M_2 = \delta_2[1, 1, 2, 1, 1, 2, 1, 1]. \end{cases}$$

It follows that

$$\begin{cases} \xi_1 = x_2 \vee x_3 \\ \xi_2 = [x_1 \wedge (x_3 \to x_2)] \vee [\neg x_1 \wedge (x_2 \to x_3)]. \end{cases}$$

In the above, it was shown that a subspace has at least a basis. In general, it is hard to say whether the basis is unique (under certain equivalent sense). It will be discussed in next section for regular case.

IV. Coordinate Transformation and Regular Subspace

In the previous section, the subspace and its basis have been discussed in detail. There is a special subspace, which plays an important role in analysis (synthesis) of Boolean (control) networks.

Definition IV.1. *Consider system* (1)*. Let*

$$\{x_{i_1},\cdots,x_{i_k}\}\subset\{x_1,\cdots,x_n\}$$

be a subset of the state variables. The set

$$\mathcal{X}_0=\mathrm{F}_\ell\{x_{i_1},\cdots,x_{i_k}\}$$

is called a regular subspace of $\mathcal{X}$ *of dimension* $\dim(\mathcal{X}_0)=k$*.*

From the logical dynamic equation (1), Definition IV.1 is very natural. Here "regular" is used to emphasize that it is not an arbitrary subset of $\mathcal{X}$. It has a subsystem structure in system dynamics. In [4] it was shown that if a regular subspace is invariant, the structure of the network is heavily depending on it. Precisely, it provides a rolling gear structure for the cycles of the system. But the above definition depends on the expression of the system. We need a coordinate-free definition.

In modern control theory, the state space approach is powerful in analysis and control design. Particularly, the invariant subspace, the output kernel space, and the controllable (observable) subspace, etc. are of fundamental importance in the synthesis of control systems. To define and apply similar subspaces of discrete time dynamic (control) systems, the coordinate transformation is also essential. The coordinate transformation of logical dynamic systems was firstly proposed in [7].

Definition IV.2. *Let* $y_1,\cdots,y_n\in\mathcal{X}$*, be defined by*

$$\begin{cases}y_1=g_1(x_1,\cdots,x_n)\\ \vdots\\ y_n=g_n(x_1,\cdots,x_n).\end{cases}\tag{26}$$

The mapping $G:\mathcal{D}^n\to\mathcal{D}^n$*, defined by*

$$G:(x_1,\cdots,x_n)\mapsto(y_1,\cdots,y_n)$$

is called a coordinate transformation (briefly, coordinate change), if G *is one to one and onto.*

Denote $x=\ltimes_{i=1}^n x_i$ and $y=\ltimes_{i=1}^n y_i$. Then we can get the algebraic form of (26) as

$$y=Tx,\tag{27}$$

where $T\in\mathcal{L}_{2^n\times 2^n}$ is called the transfer matrix.

Theorem IV.3 ([7])**.** *Using the above notations,* G *is a coordinate transformation, iff its transfer matrix* T *is nonsingular.*

In fact, the algorithms for converting a logical dynamic system into its algebraic form and vise versa can also be used to construct the coordinate change. We give a simple example to illustrate this.

Example IV.4. *Consider a set of mappings*

$$\begin{cases}y_1=\neg x_2\\ y_2=x_1\leftrightarrow x_3\\ y_3=x_3,\end{cases}\tag{28}$$

whose algebraic form is

$$\begin{cases}y_1=M_nx_2\\ y_2=M_ex_1x_3\\ y_3=x_3.\end{cases}$$

Setting $x=\ltimes_{i=1}^3x_i$*,* $y=\ltimes_{i=1}^3y_i$*, we have*

$$y=Tx,$$

where $T=\delta_8[5,8,1,4,7,6,3,2]$*. Since* T *is nonsingular,* (28) *is a logical coordinate transformation.*

It is easy to check that if $T\in\mathcal{L}_{8\times 8}$ *is invertible, then* $T^{-1}\in\mathcal{L}_{8\times 8}$*. Hence* $x=T^{-1}y$ *is the algebraic form of* x*, which are the logical functions of* y*.*

Using the standard process given in [7], we can get the inverse transformation of (28) *as*

$$\begin{cases}x_1=y_2\leftrightarrow y_3\\ x_2=\neg y_1\\ x_3=y_3.\end{cases}$$

Now we can give a coordinate-free definition of a regular subspace.

Definition IV.5. *Let* $z_1,\cdots,z_k\in\mathcal{X}$*.* $\mathcal{Z}_0=\mathrm{F}_\ell\{z_1,\cdots,z_k\}$ *is called a* k*-dimensional regular subspace of* $\mathcal{X}$*, if there are* $z_{k+1},\cdots,z_n\in\mathcal{X}$*, such that* $G:(x_1,\cdots,x_n)\mapsto(z_1,\cdots,z_n)$ *is a coordinate transformation.*

Definition IV.5 is very general. It will be powerful in the synthesis of logical dynamic control systems, provided we are able to verify it and to construct a new coordinate system, which has the basis of $\mathcal{Z}$ as part of the coordinates. We will briefly describe how to verify it.

Since $z_1,\cdots,z_k\in\mathcal{X}$, they can be expressed as

$$\begin{cases}z_1=g_1(x_1,\cdots,x_n)\\ \vdots\\ z_k=g_k(x_1,\cdots,x_n).\end{cases}\tag{29}$$

Define $z^1=\ltimes_{i=1}^kz_i$, and $x=\ltimes_{i=1}^nx_i$. Then we can easily get the algebraic form of (29) as

$$z^1=T_0x,\tag{30}$$

where $T_0\in\mathcal{L}_{2^k\times 2^n}$, which can be expressed as

$$T_0=\begin{bmatrix}t_{11}&t_{12}&\cdots&t_{1,2^n}\\ \vdots&&&\\ t_{2^k,1}&t_{2^k,2}&\cdots&t_{2^k,2^n}\end{bmatrix}.$$

Using the above notations, we have the following theorem, which is of fundamental importance.

Theorem IV.6 ([7])**.** *Assume that a set of logical variables* $z_1,\cdots,z_k$ *(*$k\le n$*) satisfies* (27)*. Then* $\mathcal{Z}_0=\mathrm{F}_\ell\{z_1,\cdots,z_k\}$

is a k-dimensional regular subspace, iff the corresponding coefficient matrix T_0 satisfies

$$\sum_{i=1}^{2^n} t_{r,i} = 2^{n-k}, \quad r = 1, 2, \cdots, 2^k. \tag{31}$$

If condition (31) is satisfied, [7] provides a mechanical way to construct a new coordinate frame, which has $\mathcal{Z}$ as part of its coordinates.

We give some examples to illustrate the regular subspace.

Example IV.7. *Assume that a state space is given as $\mathcal{X} = \mathrm{F}_\ell\{x_1, x_2, x_3\}$. Set $x = \ltimes_{i=1}^3 x_i$.*

1)

$$\mathcal{Y} = \mathrm{F}_\ell\{x_1 \wedge x_2\}. \tag{32}$$

Let $y = x_1 \wedge x_2$. Then its algebraic form can be expressed as

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} x$$

Since $\sum_{i=1}^8 t_{1i} = 2$, $\sum_{i=1}^8 t_{2i} = 6$, $\mathcal{Y}$ is not a regular subspace.

2)

$$\mathcal{Z} = \mathrm{F}_\ell\{z_1, z_2\}, \tag{33}$$

where

$$\begin{cases} z_1 = x_1 \leftrightarrow x_3 \\ z_2 = \neg x_3. \end{cases}$$

Let $z = z_1 \ltimes z_2$. Then its algebraic from can be expressed as

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} x$$

Since $\sum_{i=1}^8 t_{ri} = 2$, $r = 1, 2, 3, 4$, $\mathcal{Z}$ is a regular subspace. Set $z_3 = x_2$, it is ready to check that $(x_1, x_2, x_3) \mapsto (z_1, z_2, z_3)$ is a coordinate transformation.

A natural question is: if $\mathrm{rank}(T_0) = 2^k$, can we claim that Z is a k-dimensional subspace and using the normal routine in Linear Algebra to construct a basis of $\mathcal{Z}$? The answer is "No". This shows the difference between logical subspace and the linear subspace. As a counter example, space (32) has $\mathrm{rank}(T_0) = 2$, but it is not a one-dimensional subspace.

The concept of regular subspace is very important in constructing controllable/uncontrollable (observable/unobservable) subspaces, which provide the controllable (observable) canonical forms of logical dynamic control systems [5].

Let $\mathcal{X} = \mathrm{F}_\ell\{x_1, \cdots, x_n\}$. Consider a set of functions $\{y_1, \cdots, y_p\} \subset \mathcal{X}$, which may come from the outputs of system (2). Theorem IV.6 tells us how to check whether $\mathcal{Y} = \mathrm{F}_\ell\{y_1, \cdots, y_p\}$ is a regular subspace of $\mathcal{X}$. In case $\mathcal{Y}$ is not a regular subspace, we need to find a regular subspace $\mathcal{Z}$, such that $\mathcal{Y} \subset \mathcal{Z}$. $\mathcal{Z}$ is called the $\mathcal{Y}$-friendly (regular) subspace. It is important in decoupling problems [9]. Let $y = \ltimes_{i=1}^p y_i$ and $x = \ltimes_{i=1}^n x_i$. Assume that the algebraic form of $\mathcal{Y}$ is

$$y = Hx, \tag{34}$$

where $H \in \mathcal{L}_{2^p \times 2^n}$. Set

$$n_j = \left| \left\{ \xi \in \mathrm{Col}(H) \,|\, \xi = \delta_{2^p}^j \right\} \right|, \quad j = 1, \cdots, 2^p,$$

where $|\cdot|$ is the cardinality (number of the elements) of the set. Using above notations and statements, we have the following result. (For statement ease, a factor of the form 2^s is called a 2-type factor.)

Theorem IV.8 ([9]). *Assume that $\mathcal{Y}$ has algebraic form $y = Hx$.*

1) *There is a regular subspace, $\mathcal{Z}$, of dimension r, such that $\mathcal{Y} \subset \mathcal{Z}$, iff n_1, n_2, $\cdots$, n_{2^p} have a common factor 2^{n-r}.*
2) *Assume that the largest 2-type common factor of n_1, n_2, $\cdots$, n_{2^p} is 2^s. Then the smallest regular subspace, containing $\mathcal{Y}$, is of dimension 2^{n-s}.*

Given $\mathcal{Y}$, a detailed algorithm for constructing $\mathcal{Y}$-friendly subspace $\mathcal{Z}$ is given in [9].

V. Invariant Subspace

Consider system (1) again. If it can be expressed (under a suitable coordinate frame) as

$$\begin{cases} z^1(t+1) = F^1(z^1(t)), & z^1 \in \mathcal{D}^s \\ z^2(t+1) = F^2(z(t)), & z^2 \in \mathcal{D}^{n-s}. \end{cases} \tag{35}$$

Then $\mathcal{Z}_1 = \mathrm{F}_\ell\{z^1\} = \mathrm{F}_\ell\{z_1^1, \cdots, z_s^1\}$ is called an invariant subspace of (1).

In general sense, a subspace $\mathcal{Z}$ is invariant with respect to system (1) if starting from a point $z_0 \in \mathcal{Z}$, then the trajectory of (1) will remain on $\mathcal{Z}$.

An invariant subspace is very important in investigating the topological structure of a network [4]. Note that in [4] the invariant subspace was only defined under the original coordinate frame. But, obviously, the invariant subspaces in general sense play the same role in determining the topological structure of the network. Let $z_1, \cdots, z_s \in \mathcal{X}$ and $\mathcal{Z} = \mathrm{F}_\ell\{z_1, \cdots, z_s\}$, and set $z = \ltimes_{i=1}^s z_i$. Then we have the following result.

Theorem V.1. *Consider system* (1) *with its algebraic form* (8)*. Assume that a regular subspace $\mathcal{Z} = \mathrm{F}_\ell\{z_1, \cdots, z_s\}$ with $z = \ltimes_{i=1}^s z_i$ has the following algebraic form*

$$z = Qx, \tag{36}$$

where $Q \in \mathcal{L}_{2^s \times 2^n}$. Then $\mathcal{Z} = \mathrm{F}_\ell\{z_1, \cdots, z_s\}$ is an invariant subspace of system (1)*, iff*

$$\mathrm{Row}(QL) \subset \mathrm{Span}\,\mathrm{Row}(Q), \tag{37}$$

where L is in (8)*, i.e., it is the transition matrix of the algebraic form of system* (1).

Proof: Since $\mathcal{Z}$ is a regular subspace, there is a set $\{w_1, \cdots, w_{n-s}\}$ such that $\{z_1, \cdots, z_s, w_1, \cdots, w_{n-s}\}$ form a new coordinate frame.

(Sufficiency) From (36) we have

$$z(t+1) = Qx(t+1) = QLx(t). \tag{38}$$

Since $\mathrm{Row}\,(QL) \subset \mathrm{Span}\,\mathrm{Row}\,(Q)$, there exists η such that $QL = \eta Q$. Hence

$$z(t+1) = \eta Qx(t) = \eta z(t). \tag{39}$$

Converting the algebraic form (38) back to logical form, say, F^1 is the logical form of η, we have

$$\begin{cases} z(t+1) = F^1(z(t)) \\ w(t+1) = F^2(z(t), w(t)). \end{cases}.$$

(Necessity) Converting $z(t+1) = F^1(z(t))$ into algebraic form, we have

$$z(t+1) = \eta z(t) = \eta Qx(t). \tag{40}$$

Comparing (40) with (38), we have $QL = \eta Q$, which implies (37). ∎

Note that in (37) the Span is the span over $\mathcal{D}$. Precisely, (37) means there exists an H such that

$$QL = HQ. \tag{41}$$

It is easy to check that the product of logical matrices is still a logical matrix. Now, QL is a logical matrix, and hence so is HQ. Note that since $\mathcal{Z}$ is a regular subspace, Q has full row rank, which means $\mathrm{Col}(Q) = \Delta_{2^s}$. Hence $\mathrm{Col}(H) = \mathrm{Col}(HQ) \subset \Delta_{2^s}$. That is, $H \in \mathcal{L}_{2^s \times 2^s}$. Hence we have

Corollary V.2. *Using the notations in Theorem V.1, $\mathcal{Z}$ is an invariant subspace, iff there exists an $H \in \mathcal{L}_{2^s \times 2^s}$, such that* (41) *holds.*

Example V.3. *Consider the following Boolean network*

$$\begin{cases} x_1(t+1) = (x_1(t) \wedge x_2(t) \wedge \neg x_4(t)) \vee (\neg x_1(t) \wedge x_2(t)) \\ x_2(t+1) = x_2(t) \vee (x_3(t) \leftrightarrow x_4(t)) \\ x_3(t+1) = (x_1(t) \wedge \neg x_4(t)) \vee (\neg x_1(t) \wedge x_2(t)) \\ \qquad\qquad \vee (\neg x_1(t) \wedge \neg x_2(t) \wedge x_4(t)) \\ x_4(t+1) = x_1(t) \wedge \neg x_2(t) \wedge x_4(t). \end{cases} \tag{42}$$

Let $\mathcal{Z} = \mathrm{F}_\ell\{z_1, z_2, z_3\}$, where

$$\begin{cases} z_1 = x_1 \bar{\vee} x_4 \\ z_2 = \neg x_2 \\ z_3 = x_3 \leftrightarrow \neg x_4. \end{cases} \tag{43}$$

Set $x = \ltimes_{i=1}^4 x_i$, $z = \ltimes_{i=1}^3 z_i$. Then we have

$$z = Qx,$$

where

$$Q = \delta_8[8,3,7,4,6,1,5,2,4,7,3,8,2,5,1,6]$$

and the algebraic form of (42) *is*

$$x(t+1) = Lx(t),$$

where

$$L = \delta_{16}[11,1,11,1,11,13,15,9,1,2,1,2,9,15,13,11].$$

It is easy to calculate that

$$QL = \delta_8[3,8,3,8,3,2,1,4,8,3,8,3,4,1,2,3],$$

which satisfies (37)*. Hence $\mathcal{Z}$ is an invariant subspace of* (42)*.*

In fact we can choose $z_4 = x_4$ such that

$$\begin{cases} z_1 = x_1 \bar{\vee} x_4 \\ z_2 = \neg x_2 \\ z_3 = x_3 \leftrightarrow \neg x_4 \\ z_4 = x_4 \end{cases} \tag{44}$$

is a coordinate transformation. Moreover, under coordinate frame z, system (42) *can be expressed into the cascading form* (35) *as*

$$\begin{cases} z_1(t+1) = z_1(t) \to z_2(t) \\ z_2(t+1) = z_2(t) \wedge z_3(t) \\ z_3(t+1) = \neg z_1(t) \\ z_4(t+1) = z_1(t) \vee z_2(t) \vee z_4(t). \end{cases} \tag{45}$$

VI. Indistinct Rolling Gear Structure

Consider system (35). Assume its algebraic form (in a decomposed form) is

$$\begin{cases} z^1(t+1) = L_1 z^1(t) \\ z^2(t+1) = L_2 z^1(t) z^2(t). \end{cases} \tag{46}$$

Denote $\mathcal{Z}_1 = \mathrm{F}_\ell(z_1^1, \cdots, z_s^1)$ and $\mathcal{Z}_2 = \mathrm{F}_\ell(z_1^2, \cdots, z_{n-s}^2)$. It was proved in [4] that the cycle of (35) is compounded by the cycle in $\mathcal{Z}_1$ and a "formal cycle" in $\mathcal{Z}_2$. Precisely, let $C_z^k = (z_0, z_1, \cdots, z_k = z_0)$ be a cycle of length k, with $z_i = z_i^1 z_i^2$, $i = 0, \cdots, k$. Then for any $z \in C_z^k$, without loss of generality, say, $z_0 = z_0^1 z_0^2 \in C_z^k$, there exists an $\ell \le k$ as a factor of k, such that

$$C_{z^1}^\ell = \left(z_0^1, z_1^1 = (L_1) z_0^1, z_2^1 = (L_1)^2 z_0^1, \cdots, z_\ell^1 = (L_1)^\ell z_0^1 = z_0^1\right)$$

is a cycle in the $\mathcal{Z}_1$ subspace. Moreover, define

$$\Psi := L_2 z_{\ell-1}^1 L_2 z_{\ell-2}^1 \cdots L_2 z_1^1 L_2 z_0^1.$$

We can construct an auxiliary system

$$z^2(t+1) = \Psi z^2(t). \tag{47}$$

Then

$$C_{z^2}^j = \left(z_0^2, z_1^2 = \Psi z_0^2, \cdots, z_j^2 = \Psi^j z_0^2 = z_0^2\right)$$

is a cycle of (47), where $j = k/\ell$. Finally, the cycle C_z^k is decomposed as

$$\begin{array}{l} z_0 = z_0^1 z_0^2 \to z_1 = z_1^1 L_2 z_0^1 z_0^2 \to z_2 = z_2^1 L_2 z_1^1 L_2 z_0^1 z_0^2 \to \cdots \to \\ z_\ell = z_0^1 z_1^2 \to z_{\ell+1} = z_1^1 L_2 z_0^1 z_1^2 \to z_{\ell+2} = z_2^1 L_2 z_1^1 L_2 z_0^1 z_1^2 \to \cdots \to \\ \vdots \\ z_{(j-1)\ell} = z_0^1 z_{(j-1)}^2 \to z_{(j-1)\ell+1} = z_1^1 L_2 z_0^1 z_{(j-1)}^2 \to \\ z_{(j-1)\ell+2} = z_2^1 L_2 z_1^1 L_2 z_0^1 z_{(j-1)}^2 \to z_{j\ell} = z_0^1 z_j^2 = z_0^1 z_0^2 = z_0. \end{array} \tag{48}$$

We call this C_z^k the compounded cycle of $C_{z^1}^\ell$ and $C_{z^2}^j$, denoted by $C_z^k = C_{z^1}^\ell \circ C_{z^2}^j$.

Remark VI.1. 1) *As long as the dynamics of a Boolean network has a cascading structure as* (35)*, its cycles have such a "compounded structure", which is called the rolling gear structure, described in [4].*

2) $C^{\ell}_{z^1}$ *is a real cycle, which involves only part of nodes (precisely,* s *nodes).* $C^{j}_{z^2}$ *is not a real cycle. It is a cycle of the auxiliary system* (47)*.*

3) *To the best of our knowledge, in current lectures (for instance, [1], [6], [11], [13], [18] and the references therein) only overall node cycles and fixed points are considered. Cycles and fixed points involving part of nodes, such as* $C^{\ell}_{z^1}$*, are ignored. They can be found only in the cascading form.*

If a system is not originally in a cascading form but under a suitable coordinate frame it has cascading form. The the system still has the cycles and/or fixed points involving part of state variables. Moreover, the rolling gear structure still exists, which will be called the indistinct rolling gear structure. We investigate it through the following example.

Example VI.2. *Consider the following Boolean network*

$$\begin{cases} x_1(t+1) = [x_5(t) \wedge (x_3(t) \bar{\vee} x_4(t))] \leftrightarrow (x_5(t) \bar{\vee} x_3(t)) \\ x_2(t+1) = x_5(t) \bar{\vee} x_3(t) \\ x_3(t+1) = (x_3(t) \bar{\vee} x_4(t)) \, \bar{\vee} x_2(t) \\ x_4(t+1) = [\neg (x_1(t) \leftrightarrow x_2(t))] \, \bar{\vee} \, [(x_3(t) \bar{\vee} x_4(t)) \, \bar{\vee} x_2(t)] \\ x_5(t+1) = x_5(t) \vee (x_3(t) \bar{\vee} x_4(t)) \\ x_6(t+1) = [(x_1(t) \leftrightarrow x_2(t)) \leftrightarrow (x_2(t) \bar{\vee} x_6(t))] \\ \qquad\qquad \bar{\vee} \, (x_5(t) \bar{\vee} x_3(t)) \, . \end{cases} \tag{49}$$

Setting $x = \ltimes_{i=1}^{6} x_i$*, the algebraic form of the system* (49) *is*

$$x(t+1) = Lx(t), \tag{50}$$

where

$$\begin{aligned} L = \delta_{64}[&18 \;\; 17 \;\; 35 \;\; 36 \;\; 62 \;\; 61 \;\; 45 \;\; 46 \\ &13 \;\; 14 \;\; 30 \;\; 29 \;\; 33 \;\; 34 \;\; 20 \;\; 19 \\ &26 \;\; 25 \;\; 43 \;\; 44 \;\; 54 \;\; 53 \;\; 37 \;\; 38 \\ &5 \;\; 6 \;\; 22 \;\; 21 \;\; 41 \;\; 42 \;\; 28 \;\; 27 \\ &21 \;\; 22 \;\; 40 \;\; 39 \;\; 57 \;\; 58 \;\; 42 \;\; 41 \\ &10 \;\; 9 \;\; 25 \;\; 26 \;\; 38 \;\; 37 \;\; 23 \;\; 24 \\ &29 \;\; 30 \;\; 48 \;\; 47 \;\; 49 \;\; 50 \;\; 34 \;\; 33 \\ &2 \;\; 1 \;\; 17 \;\; 18 \;\; 46 \;\; 45 \;\; 31 \;\; 32]. \end{aligned}$$

Using the method proposed in [6], it is easy to calculate that the attractive set of (49) *consists of* 4 *cycles of length* 8*. They are:*

$$\begin{aligned} C_1: \;& (111111) \to (101110) \to (100111) \to (111011) \to \\ & (000010) \to (010011) \to (011010) \to (000110) \to \\ & (111111), \end{aligned}$$

$$\begin{aligned} C_2: \;& (111110) \to (101111) \to (100110) \to (111010) \to \\ & (000011) \to (010010) \to (011011) \to (000111) \to \\ & (111110), \end{aligned}$$

$$\begin{aligned} C_3: \;& (110111) \to (110011) \to (011111) \to (101011) \to \\ & (001010) \to (001110) \to (100010) \to (010110) \to \\ & (110111), \end{aligned}$$

$$\begin{aligned} C_4: \;& (110110) \to (110010) \to (011110) \to (101010) \to \\ & (001011) \to (001111) \to (100011) \to (010111) \to \\ & (110110). \end{aligned}$$

Under this coordinate frame, we are not able to find cycles, which contained in smaller invariant subspaces. And therefore, we are not able to reveal the rolling gear structure for the network.

To find tiny cycles and the rolling gear structure of the network, we try to convert (49)*, if possible, into a cascading form to investigate its indistinct rolling gear structure. Note that Theorem V.1 says that* $\mathrm{Span}\{\mathrm{Col}(Q)^T\}$ *is a standard* L^T *invariant subspace. So the standard tools from linear algebra can be used to find the invariant subspaces. We skip the tedious and straightforward computation and consider the following two nested spaces:*

$$\begin{aligned} &\mathcal{Z}_1 = \mathrm{F}_\ell\{z_1 = x_1 \leftrightarrow x_2;\; z_2 = x_5;\; z_3 = x_3 \bar{\vee} x_4\} \\ &\mathcal{Z}_2 = \mathrm{F}_\ell\{z_1 = x_1 \leftrightarrow x_2;\; z_2 = x_5;\; z_3 = x_3 \bar{\vee} x_4;\; z_4 = x_2 \bar{\vee} x_6\} \end{aligned}$$

Set $z^1 = z_1 \ltimes z_2 \ltimes z_3$*. It is easy to calculate that*

$$z^1 = Q_1 x,$$

where

$$\begin{aligned} Q_1 = \delta_8[&2 \;\; 2 \;\; 4 \;\; 4 \;\; 1 \;\; 1 \;\; 3 \;\; 3 \\ &1 \;\; 1 \;\; 3 \;\; 3 \;\; 2 \;\; 2 \;\; 4 \;\; 4 \\ &6 \;\; 6 \;\; 8 \;\; 8 \;\; 5 \;\; 5 \;\; 7 \;\; 7 \\ &5 \;\; 5 \;\; 7 \;\; 7 \;\; 6 \;\; 6 \;\; 8 \;\; 8 \\ &6 \;\; 6 \;\; 8 \;\; 8 \;\; 5 \;\; 5 \;\; 7 \;\; 7 \\ &5 \;\; 5 \;\; 7 \;\; 7 \;\; 6 \;\; 6 \;\; 8 \;\; 8 \\ &2 \;\; 2 \;\; 4 \;\; 4 \;\; 1 \;\; 1 \;\; 3 \;\; 3 \\ &1 \;\; 1 \;\; 3 \;\; 3 \;\; 2 \;\; 2 \;\; 4 \;\; 4]. \end{aligned}$$

Similarly, set $z^2 = z_1 \ltimes z_2 \ltimes z_3 \ltimes z_4$*. We have*

$$z^2 = Q_2 x,$$

where

$$\begin{aligned} Q_2 = \delta_{16}[\,&4 \;\; 3 \;\; 8 \;\; 7 \;\; 2 \;\; 1 \;\; 6 \;\; 5 \\ &2 \;\; 1 \;\; 6 \;\; 5 \;\; 4 \;\; 3 \;\; 8 \;\; 7 \\ &11 \;\; 12 \;\; 15 \;\; 16 \;\; 9 \;\; 10 \;\; 13 \;\; 14 \\ &9 \;\; 10 \;\; 13 \;\; 14 \;\; 11 \;\; 12 \;\; 15 \;\; 16 \\ &12 \;\; 11 \;\; 16 \;\; 15 \;\; 10 \;\; 9 \;\; 14 \;\; 13 \\ &10 \;\; 9 \;\; 14 \;\; 13 \;\; 12 \;\; 11 \;\; 16 \;\; 15 \\ &3 \;\; 4 \;\; 7 \;\; 8 \;\; 1 \;\; 2 \;\; 5 \;\; 6 \\ &1 \;\; 2 \;\; 5 \;\; 6 \;\; 3 \;\; 4 \;\; 7 \;\; 8\,]. \end{aligned}$$

Using Theorem IV.6, it is easy to check that $\mathcal{Z}_1 \subset \mathcal{Z}_2$ *are two nested regular subspaces.*

To see they are invariant subspaces of system (49)*, it suffices to find* H_i*,* $i = 1, 2$*, such that* (38) *holds. That is,* $Q_i L = H_i Q_i$*. The* H_i *can be calculated as*

$$\begin{aligned} &H_1 = \delta_8[2, 6, 6, 8, 1, 5, 5, 7]; \\ &H_2 = \delta_{16}[3, 4, 11, 12, 11, 12, 15, 16, 2, 1, 10, 9, 10, 9, 14, 13]. \end{aligned}$$

It is not difficult to find $z_5 = x_2$ and $z_6 = x_3$, such that $\Psi : (x_1, \cdots, x_6) \mapsto (z_1, \cdots, z_6)$ is a coordinate transformation:

$$\Psi : \begin{cases} z_1 = x_1 \leftrightarrow x_2 \\ z_2 = x_5 \\ z_3 = x_3 \bar{\vee} x_4 \\ z_4 = x_2 \bar{\vee} x_6 \\ z_5 = x_2 \\ z_6 = x_3. \end{cases}$$

(We refer to [9] for the mechanical procedure of finding additional coordinate variables to make a basis of a regular subspace into a coordinate transformation.)

The algebraic form of Ψ is

$$z = \ltimes_{i=1}^{6} z_i = Tx, \tag{51}$$

where

$$T = \delta_{64}[\begin{array}{cccccccc} 13 & 9 & 29 & 25 & 5 & 1 & 21 & 17 \\ 6 & 2 & 22 & 18 & 14 & 10 & 30 & 26 \\ 43 & 47 & 59 & 63 & 35 & 39 & 51 & 55 \\ 36 & 40 & 52 & 56 & 44 & 48 & 60 & 64 \\ 45 & 41 & 61 & 57 & 37 & 33 & 53 & 49 \\ 38 & 34 & 54 & 50 & 46 & 42 & 62 & 58 \\ 11 & 15 & 27 & 31 & 3 & 7 & 19 & 23 \\ 4 & 8 & 20 & 24 & 12 & 16 & 28 & 32 \end{array}].$$

Now under the coordinate frame $z = Tx$ we have the algebraic form of system (49) *as*

$$z(t+1) = Tx(t+1) = TLx(t) = TLT^{-1}z(t) := \tilde{L}z(t), \tag{52}$$

where

$$\tilde{L} = \delta_{64}[\begin{array}{cccccccc} 12 & 10 & 11 & 9 & 16 & 14 & 15 & 13 \\ 43 & 41 & 44 & 42 & 47 & 45 & 48 & 46 \\ 42 & 44 & 41 & 43 & 46 & 48 & 45 & 47 \\ 57 & 59 & 58 & 60 & 61 & 63 & 62 & 64 \\ 8 & 6 & 7 & 5 & 4 & 2 & 3 & 1 \\ 39 & 37 & 40 & 38 & 35 & 33 & 36 & 34 \\ 38 & 40 & 37 & 39 & 34 & 36 & 33 & 35 \\ 53 & 55 & 54 & 56 & 49 & 51 & 50 & 52 \end{array}].$$

A mechanical procedure was provided in [6] to convert the algebraic form of a Boolean network back to logic form. Using it, we can convert (52) *into a logical form as (omitting the mechanical procedure)*

$$\begin{cases} z_1(t+1) = z_2(t) \wedge z_3(t) \\ z_2(t+1) = z_2(t) \vee z_3(t) \\ z_3(t+1) = \neg z_1(t) \\ z_4(t+1) = z_1(t) \leftrightarrow z_4(t) \\ z_5(t+1) = z_2(t) \bar{\vee} z_6(t) \\ z_6(t+1) = z_3(t) \bar{\vee} z_5(t). \end{cases} \tag{53}$$

From this cascading form one sees easily that $\mathcal{Z}_1 = \mathrm{F}_\ell\{z_1, z_2, z_3\}$ and $\mathcal{Z}_2 = \mathrm{F}_\ell\{z_1, z_2, z_3, z_4\}$ are invariant subspaces.

The subsystem with respect to $\mathcal{Z}_1$ has 1 cycle of length 4, which is

$$(111) \to (110) \to (010) \to (011) \to (111),$$

and the sub-system with respect to $\mathcal{Z}_2$ has 2 cycles of length 4, which are

$$(\underline{111}1) \to (\underline{110}1) \to (\underline{010}1) \to (\underline{011}0) \to (\underline{111}1),$$
$$(\underline{111}0) \to (\underline{110}0) \to (\underline{010}0) \to (\underline{011}1) \to (\underline{111}0).$$

The corresponding cycles of system (49) *become*

$\tilde{C}_1 :$ $(\underline{110}011) \to (\underline{010}001) \to (\underline{011}100) \to (\underline{111}011) \to (\underline{110}000) \to (\underline{010}010) \to (\underline{011}111) \to (\underline{111}000) \to (\underline{110}011)$,

$\tilde{C}_2 :$ $(\underline{110}111) \to (\underline{010}101) \to (\underline{011}000) \to (\underline{111}111) \to (\underline{110}100) \to (\underline{010}110) \to (\underline{011}011) \to (\underline{111}100) \to (\underline{110}111)$,

$\tilde{C}_3 :$ $(\underline{111}010) \to (\underline{110}010) \to (\underline{010}011) \to (\underline{011}101) \to (\underline{111}001) \to (\underline{110}001) \to (\underline{010}000) \to (\underline{011}110) \to (\underline{111}010)$,

$\tilde{C}_4 :$ $(\underline{111}110) \to (\underline{110}110) \to (\underline{010}111) \to (\underline{011}001) \to (\underline{111}101) \to (\underline{110}101) \to (\underline{010}100) \to (\underline{011}010) \to (\underline{111}110)$.

It is easily seen that the cycle of $\mathcal{Z}_1$ is implicitly contained in the cycles of $\mathcal{Z}_2$ (marked with underline), and similarly, the cycles of $\mathcal{Z}_2$ are implicitly contained in the cycles of (49)*. They form several groups of three assembled gears, which form the so called indistinct rolling gear structure.*

Note that cycles C_i and $\tilde{C}_i$, $i = 1, 2, 3, 4$ are exactly the same. (We have put them in a point-point corresponding way. The only difference is caused by the different coordinate frames.)

VII. Conclusion

Recently, the authors have developed a systematic new approach to the analysis and control of logical dynamic (control) systems, by using the semi-tensor product of matrices and the matrix expression of logic, proposed by the authors. A key point in this new approach is to convert a logical dynamic (control) system into a discrete-time dynamic system. It makes the state space technique applicable to logical dynamic (control) systems.

Since in logical systems the state space is not a vector space, some additional techniques have to be developed to deal with "state space" and "subspaces". Defining a space by a set of logical functions, we introduced some new concepts such as "regular subspace", "$\mathcal{Y}$-friendly subspace", "invariant subspace" etc. They have both clear physical meanings and neat verifying formulas.

Using the well defined different subspaces, the controllability and observability [5], stability and stabilization [8], disturbance decoupling and other decoupling problems [9], [10], etc. have been investigated.

As another interesting application, the tool of invariant subspace has been used to convert a Boolean network into a cascading form, if possible. Then the indistinct rolling gear structure of a Boolean network under arbitrary coordinates is revealed.

REFERENCES

[1] Aracena, J., J. Demongeot, E. Goles, On limit cycles of monotone functions with symmetric connection graph, *Theoretical Computer Science*, Vol. 322, 237-244, 2004.

[2] Cheng, D., H. Qi, *Semi-tensor product of matrices: Theory and applications*, Science Press, Beijing, 2007. (in Chinese)

[3] Cheng, D., Semi-tensor product of matrices and its applications – A survey, Proc. ICCA 2007, Vol. 3, 641-668, 2007.

[4] Cheng, D. Input-state approach to Boolean networks, *IEEE Trans. Neural Networks*, Vol. 20, No. 3, 512-521, 2009.

[5] Cheng, D., H. Qi, Controllability and observability of Boolean control networks, *Automatica*, Vol. 45, No. 7, 1659-1667, 2009.

[6] Cheng, D., H. Qi, A linear representation of dynamics of Boolean networks, *IEEE Trans. Aut. Contr.*, (Provisionally accepted).

[7] Cheng, D., Z. Li, H. Qi, Realization of Boolean control networks, submitted to *Automatica*, (to appear).

[8] Cheng, D., Stabilization of Boolean control networks, submitted to *IEEE Trans. Aut. Contr.*. (Preprint: http://lsc.amss.ac.cn/~dcheng/preprint)

[9] Cheng, D., Disturbance Decoupling of Boolean Control Networks, submitted to *IEEE Trans. Aut. Contr.*

[10] Cheng, D., Z. Li, H. Qi, Canalizing Boolean mapping and its application to disturbance decoupling of Boolean control networks, Proc. ICCA 2009, Christchurch, New Zealand, (to appear).

[11] Farrow, C., J. Heidel, H. Maloney, J. Rogers, Scalar equations for synchronous Boolean networks with biological applications, *IEEE Trans. Neural Networks*, Vol. 15, No. 2, 348-354, 2004.

[12] Hamilton, A.G., *Logic for Mathematicians*, Revised Ed., Cambridge Univ. Press, Cambridge, 1988.

[13] J. Heidel, J. Maloney, J. Farrow, and J. Rogers, Finding cycles in synchronous Boolean networks with applications to biochemical systems, *Int. J. Bifurcat. Chaos*, Vol. 13, No. 3, 535-552, 2003.

[14] Kauffman, S.A., Metabolic stability and epigenesis in randomly constructed genetic nets, *J. Theoretical Biology*, Vol. 22, 437-467, 1969.

[15] Kauffman, S.A., *The Origins of Order: Self-organization and selsction in Evolution*, Oxford Univ. Press, Inc., New York, 1993.

[16] Kauffman, S.A., *At Home in the Universe*, Oxford Univ. Press, Inc., New York, 1995.

[17] Rade, L., B. Westergren, *Mathematics Handbook for Science and Engineering*, 4th Ed., Studentlitteratur, Lund, 1998.

[18] Shih, M.-H., J.-L. Dong, A combinatorial analogue of the Jacobian problem in automata networks, *Advances in Applied Mathematics*, Vol. 34, 30-46, 2005.