

Bi-decomposition of Logical Mappings via Semi-tensor Product of Matrices^{*}

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Abstract. The decomposition of logical mappings, including disjoint and non-disjoint cases, is considered. First, we consider the Boolean functions, then the results are extended to multi-valued logical functions and further to mix-valued logical mappings, which contain multi-input and multi-output mapping as its particular case. Using semi-tensor product, straightforward verifiable necessary and sufficient conditions are provided for each case. The constructive proof provide algorithms for constructing the decompositions. Finally, the general result is applied to convert a dynamic-static Boolean network into its normal form. Examples for each cases are provided to illustrate the corresponding results.

Key Words. Boolean function, multi-(mix-)valued logical function, decomposition, semi-tensor product, dynamic-static Boolean network.

1 Introduction

Denote by $\mathcal{D} := \{0, 1\}$ the domain of Boolean variables. Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be a Boolean function. In general, it is assumed to be realized by a logical circuit (also called a network) (see Fig. 1 (a)).

1. Let $\{X_1, X_2\}$ be a partition of $X = \{x_1, x_2, \dots, x_n\}$. If f can be expressed as

$$f(X) = F(\phi(X_1), \psi(X_2)), \quad (1)$$

then f can be realized by a bi-disjoint decomposed circuit as in Fig. 1 (b).

2. Let $\{X_1, X_2, X_3\}$ be a partition of X . If f can be expressed as

$$f(X) = F(\phi(X_1, X_2), \psi(X_2, X_3)), \quad (2)$$

then f can be realized by a non-disjoint bi-decomposed circuit as in Fig. 1 (c).

Here, f, F, ϕ, ψ are all Boolean functions.

Figure 1: (a) Boolean function (b) Disjoint decomposition (c) Non-disjoint decomposition

Decomposition is one of the most efficient way to realize networks economically. If the decomposition exists, it can significantly reduce the area, delay, and power for logical synthesis [14]. Therefore, it becomes a long standing research topic since 1950s. There are some interesting and useful results. For instance, when the number of inputs of a switching circuit is small, the Quine-McCluskey procedure is widely used

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for designing two-stage network[13]. When a large number of inputs is involved, decomposition chart method to multi-level minimization was proposed by Ashenurstis [1], and was later further discussed and developed by Curtis [12] and Roth and Karp [20]. There are many articles devoted to developing efficient algorithms for decomposition of switching functions, as well as multi-valued logic functions. We refer to [11], [22], [23], [24], [19], [18], [2] and the references therein for further references.

The purpose of this paper is threefold: (i) providing easily verifiable necessary and sufficient conditions for both disjoint and non-disjoint bi-decompositions of Boolean functions; (ii) extending the result to multi-valued logical functions and mix-valued logical mappings; (iii) applying the result to converting a dynamic-static network into a standard dynamic network.

The basic tool for our analysis is the so called the semi-tensor product of matrices and the matrix expression of logic [4]. It has been successfully applied to the analysis of topological structure of Boolean networks [6], [7], and the synthesis of Boolean networks [5], [8], [9], [10], [17], [16]. The key technique of this approach is converting a Boolean function into an algebraic form, which allows the application of matrix and analyzing tools for discrete-time systems to Boolean functions. It is obvious that this approach is also applicable to the analysis of Boolean functions, particularly, their decompositions.

The rest of this paper is organized as follows. Section 2 is a preliminary, which introduces the basic tools and framework of semi-tensor product approach. Sections 3 and 4 consider the disjoint and non-disjoint bi-decomposition of Boolean functions respectively. Necessary and sufficient conditions are presented for two cases respectively in these two sections. Brief comparisons with existing results are also included. Sections 5 and 6 extend the results obtained in Sections 3 and 4 to the multi-valued logical functions and to the mix-valued logical mappings respectively. As an application, Section 7 considers how to convert a dynamic-static Boolean network into its standard form. The necessary and sufficient conditions for convertibility is also obtained. Section 8 is a brief concluding remark.

2 Matrix Expression of Logic

For statement convenience, we first introduce some notations.

- $\mathcal{M}_{m \times n}$: the set of $m \times n$ real matrices.

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$$\mathcal{D}_k := \left\{ 0, \frac{1}{k-1}, \frac{2}{k-1}, \dots, 1 \right\}, \quad k \geq 2; \quad \mathcal{D}_2 := \mathcal{D} = \{1, 0\}.$$

- δ_n^k is the k -th column of the identity matrix I_n .

- $\Delta_n := \{\delta_n^1, \dots, \delta_n^n\}$. For compactness, $\Delta := \Delta_2$.

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$$diag(A_1 \ A_2 \ \dots \ A_k) = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & A_k \end{bmatrix}.$$

- A matrix $L \in \mathcal{M}_{n \times m}$ is called a logical matrix if its columns $\text{Col}(L) \subset \Delta_n$.

The set of $n \times m$ logical matrices is denoted by $\mathcal{L}_{n \times m}$.

- Let $L \in \mathcal{L}_{n \times m}$. Then it can be expressed as

$$L = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_m}].$$

For the sake of brevity, it is denoted as

$$L = \delta_n[i_1, i_2, \dots, i_m].$$

- A matrix $A = (a_{i,j}) \in \mathcal{M}_{m \times n}$ is called a Boolean matrix if its entries $a_{i,j} \in \mathcal{D}$. The set of $m \times n$ Boolean matrices is denoted by $\mathcal{B}_{m \times n}$.
- Let $A = (a_{i,j}), B = (b_{i,j}) \in \mathcal{B}_{m \times n}$. Then the logical operators can be applied to $\mathcal{B}_{m \times n}$. For instance, $\neg A = (-a_{i,j})$; $A \wedge B = (a_{i,j} \wedge b_{i,j})$; and $A \vee B = (a_{i,j} \vee b_{i,j})$, etc.

Definition 2.1 [8, 10] Let $M \in \mathcal{M}_{m \times n}$ and $N \in \mathcal{M}_{p \times q}$. The semi-tensor product of matrices, denoted by $M \ltimes N$, is defined as

$$M \ltimes N := (M \otimes I_{s/n}) (N \otimes I_{s/p}), \quad (3)$$

where $s = \text{lcm}\{n, p\}$ is the least common multiple of n and p .

Remark 2.2 Throughout this paper, unless else product symbol is used, the matrix product is assumed to be the semi-tensor product, which contains the conventional matrix product as its particular case when $n = p$. Hence, the symbol \ltimes can be omitted. We do this in the sequel. Since all the product properties of the conventional matrix product remain correct, we can perform the semi-tensor product as conventional product without worrying about the dimensions.

Definition 2.3 A swap matrix $W_{[m,n]} \in \mathcal{M}_{mn \times mn}$ is the unique matrix, such that for any two column vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$

$$W_{[m,n]}xy = yx. \quad (4)$$

We refer to [10] for the structure of $W_{[m,n]}$.

Some basic properties, which are used in the sequel, are listed in the following Proposition.

Proposition 2.4 [10]

1. Let $x \in \mathbb{R}^t$ and A is a given matrix. Then

$$xA = (I_t \otimes A)x. \quad (5)$$

2. Let $x = \Delta_k$. Then

$$x^2 = M_r^k x, \quad (6)$$

where

$$M_r^k := \text{diag}[\delta_k^1, \delta_k^2, \dots, \delta_k^n].$$

Identifying $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, a Boolean function $f : \mathcal{D}^n \rightarrow \mathcal{D}$ can be expressed as a mapping $f : \Delta^n \rightarrow \Delta$, which is called the vector form of f .

Theorem 2.5 [10] *Given a Boolean function $f : \mathcal{D}^n \rightarrow \mathcal{D}$. Then there exists a unique matrix $M_f \in \mathcal{L}_{2 \times 2^n}$, called the structure matrix of f , such that the vector form of f can be expressed as*

$$f(x_1, \dots, x_n) = M_f x, \quad (7)$$

where $x = \times_{i=1}^n x_i$.

The matrix product form (7) is also called the algebraic form of f .

Remark 2.6 *For a Boolean function f , its structure matrix*

$$M_f = \begin{bmatrix} \text{Row}_1(M_f) \\ \text{Row}_2(M_f) \end{bmatrix}.$$

Then $\text{Row}_1^T(M_f)$ is the truth table of f , and $\text{Row}_2(M_f) = \neg(\text{Row}_1(M_f))$. So the structure matrix and the truth table of a Boolean function are essentially the same. Precisely speaking, there is a straightforward one-to-one correspondence. But in theoretical deduction, the structure matrix is much convenient. Hence, in the following investigation we using only structure matrices. In most cases we may easily convert the conclusions back to the truth table of f .

3 Disjoint Bi-decomposition

Definition 3.1 *Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be a Boolean function, $\Gamma \cup \Lambda$ be a partition of $\{1, 2, \dots, n\}$. f is said to be bi-decomposable with respect to Γ and Λ if there exist three Boolean functions $F : \mathcal{D}^2 \rightarrow \mathcal{D}$, $\phi : \{x_\gamma | \gamma \in \Gamma\} \rightarrow \mathcal{D}$, and $\psi : \{x_\lambda | \lambda \in \Lambda\} \rightarrow \mathcal{D}$, such that*

$$f(x_1, \dots, x_n) = F(\phi(x_\gamma | \gamma \in \Gamma), \psi(x_\lambda | \lambda \in \Lambda)). \quad (8)$$

First, we assume

$$\Gamma = \{1, 2, \dots, k\}, \quad \text{and} \quad \Lambda = \{k+1, k+2, \dots, n\}. \quad (9)$$

Definition 3.2 *Let $M = \delta_2[i_1 \ i_2 \ \dots \ i_{2^k}] \in \mathcal{L}_{2 \times 2^k}$.*

1. M is called a constant function matrix, if

$$i_1 = i_2 = \dots = i_{2^k}.$$

That is, it is the structure matrix of a constant function.

- 2.

$$\neg M := \delta_2[1 - i_1 \ 1 - i_2 \ \dots \ 1 - i_{2^k}] \in \mathcal{L}_{2 \times 2^k}$$

is called the compliment of M . M and $\neg M$ are called the two complimented matrices.

Theorem 3.3 Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be a Boolean function with its structure matrix M_f , being split into 2^k equal blocks as

$$M_f = [M_1 \ M_2 \ \cdots \ M_{2^k}], \quad (10)$$

where each $M_i \in \mathcal{L}_{2 \times 2^{n-k}}$.

f is bi-decomposable with respect to the partition in (9), if and only if, the set $\{M_i | i = 1, \dots, 2^k\}$ consists of one of the following four possible cases:

- (i) two constant matrices; or
- (ii) one constant matrix and one non-constant matrix; or
- (iii) one non-constant matrix; or
- (iv) two complemented non-constant matrices.

Proof. (Necessity) Assume there are three functions F , ϕ , and ψ , such that (8) holds. Denote the structure matrix of f by $M_f \in \mathcal{L}_{2 \times 2^n}$, and it is split as in (10). Assume the structure matrix of F is

$$M_F = \delta_2[i_1 \ i_2 \ i_3 \ i_4];$$

the structure matrix of ϕ is

$$M_\phi = \delta_2[j_1 \ j_2 \ \cdots \ j_{2^k}];$$

and the structure matrix of ψ is $M_\psi \in \mathcal{L}_{2 \times 2^{n-k}}$. Then we have

$$M_f x = M_F M_\phi x^1 M_\psi x^2, \quad (11)$$

where $x = \times_{i=1}^n x_i$, $x^1 = \times_{i=1}^k x_i$, and $x^2 = \times_{i=k+1}^n x_i$.

Using the Proposition 2.4, we have that

$$M_f = M_F M_\phi (I_{2^k} \otimes M_\psi). \quad (12)$$

We first calculate $M_F M_\phi$, which is denoted by

$$M_F M_\phi := [N_1 \ N_2 \ \cdots \ N_{2^k}].$$

Then a straightforward computation shows that

$$N_s = \begin{cases} \delta_2[i_1 \ i_2], & j_s = 1 \\ \delta_2[i_3 \ i_4], & j_s = 2, \end{cases} \quad s = 1, 2, \dots, 2^k.$$

It follows that if we denote

$$M_F M_\phi (I_{2^k} \otimes M_\psi) = [W_1 \ W_2 \ \cdots \ W_{2^k}],$$

Then

$$W_s = \begin{cases} \delta_2[i_1 \ i_2] M_\psi, & j_s = 1 \\ \delta_2[i_3 \ i_4] M_\psi, & j_s = 2, \end{cases} \quad s = 1, 2, \dots, 2^k.$$

For (12) to be true, we need

$$M_i = W_i, \quad i = 1, \dots, 2^k. \quad (13)$$

Now if $i_1 = i_2$ and $i_3 = i_4$, we have case (i); if either $i_1 = i_2$ or $i_3 = i_4$ but not both, we have case (ii); if $i_1 = i_3$ and $i_2 = i_4$, but $i_1 \neq i_2$, we have case (iii); and if $i_1 \neq i_2$, $i_3 \neq i_4$, and $i_1 \neq i_3$, then we have either $\delta_2[i_1 \ i_2] = I_2$ and $\delta_2[i_1 \ i_2] = \neg I_2$, or $\delta_2[i_1 \ i_2] = \neg I_2$ and $\delta_2[i_1 \ i_2] = I_2$, then we have case (iv).

(sufficiency) Since i_1, i_2, i_3, i_4 and j_1, \dots, j_{2^k} are completely free, if the structure matrix M_f of f satisfies one of the above four cases, we can first choose i_1, i_2, i_3, i_4 , according to the type of M_f . (Please refer to the proof of the necessity to see how to choose this.) Then to choose $j_s, s = 1, \dots, 2^s$, according to the type of M_s . Finally, M_ψ can be determined automatically. \square

Remark 3.4 1. We can state Theorem 3.3 alternatively as: f is bi-decomposed with respect to Γ and Λ (as aforementioned), if and only if the structure matrix of f can be expressed as

$$M_f = [\mu_1 M_\psi \ \mu_2 M_\psi \ \cdots \ \mu_{2^k} M_\psi], \quad (14)$$

where

$$M_\psi \in \mathcal{L}_{2 \times 2^{n-k}};$$

$\mu_i \in S, \forall i$, where S can be one of the following types:

- **Type 1:**

$$S = S_1 = \{\delta_2[1 \ 1], \delta_2[2 \ 2]\};$$

- **Type 2:**

$$S = S_2 = \{\delta_2[1 \ 1], \delta_2[1 \ 2]\} \text{ or } \{\delta_2[2 \ 2], \delta_2[1 \ 2]\};$$

- **Type 3:**

$$S = S_3 = \{\delta_2[1 \ 2]\} \text{ or } \{\delta_2[2 \ 1]\};$$

- **Type 4:**

$$S = S_4 = \{\delta_2[1 \ 2], \delta_2[2 \ 1]\}.$$

2. We ignore the case when the type consists of only one constant matrix, say $\delta_2[1 \ 1]$ (or $\delta_2[2 \ 2]$). Because in this case $f \equiv 1$ (or $f \equiv 0$).
3. Actually, Type 2 may have two other cases $S_2 = \{\delta_2[1 \ 1], \delta_2[2 \ 1]\}$ or $S_2 = \{\delta_2[2 \ 2], \delta_2[2 \ 1]\}$. But they can be realized by using the above S and replacing ψ by $\neg\psi$.
4. Type 3 ($S = S_3$) is a trivial case, because it means f is independent of ϕ . So we may ignore this trivial case.

Remark 3.5 A theory about disjoint bi-decomposition of Boolean functions, given in [23], says that “ f has a disjoint bi-decomposition of form $f(X_1, X_2) = h(g_1(X_1), g_2(X_2))$, if and only if $\mu(f : X_1, X_2) \leq 2$ and $\mu(f : X_2, X_1) \leq 2$ ”. Here, $\mu(f : X_1, X_2)$ denotes the column multiplicities for f with respect to X_1 and X_2 , and $\mu(f : X_2, X_1)$ denotes the row multiplicities for f with respect to X_1 and X_2 , where the number of distinct column(row) patterns in the decomposition chart is called column(row) multiplicities (we refer [23] for the detail). Comparing it with Theorem 3.3, one can see that they are essentially the same. In fact, it is easy to check that

- if $\mu(f : X_1, X_2) = 1$ and $\mu(f : X_2, X_1) = 1$, f is constant.
- if $\mu(f : X_1, X_2) = 2$ and $\mu(f : X_2, X_1) = 1$, it is of Type 1, with $S = \{\delta_2[1\ 1], \delta_2[2\ 2]\}$;
- if $\mu(f : X_1, X_2) = 1$ and $\mu(f : X_2, X_1) = 2$, it is of Type 3, with $S = \{\delta_2[1\ 2]\}$ or $\{\delta_2[2\ 1]\}$;
- if $\mu(f : X_1, X_2) = 2$ and $\mu(f : X_2, X_1) = 2$, it is of Type 2 with $S = \{\delta_2[1\ 1], \delta_2[1\ 2]\}$ or $\{\delta_2[2\ 2], \delta_2[1\ 2]\}$; or Type 4 with $S = \{\delta_2[1\ 2], \delta_2[2\ 1]\}$.

Note that from the structure matrix M_f of f it is easy to figure out the set $\{\mu_1, \dots, \mu_{2^k}\}$ if the conditions of Theorem 3.3 are satisfied. Because we can first find constant function matrices (CFM). If there are two CFMs, we are done. If there is only one CFM, then there is only one non-constant function matrix, and we can choose another μ as $\mu = \delta_2[1\ 2]$. If there is no CFM, we should have $\mu_1 = \delta_2[1\ 2]$ and $\mu_2 = \delta_2[2\ 1]$.

The following corollary gives the way to construct the decomposition.

Corollary 3.6 *Assume the structure matrix M_f of f , as in (14), satisfies the conditions of Theorem 3.3. Then the structure matrices of F , ϕ , and ψ can be figured out by the following process:*

1. If the set $\{\mu_1, \dots, \mu_{2^k}\}$ contains only one element $\delta_2[p, q]$, then

$$M_F = [\delta_2[p, q] \ \delta_2[p, q]]; \quad (15)$$

otherwise the set contains two elements $\delta_2[p_1, q_1], \delta_2[p_2, q_2]$, then

$$M_F = [\delta_2[p_1, q_1] \ \delta_2[p_2, q_2]]. \quad (16)$$

2. Say,

$$\mu_i = M_F \delta_2^{t_i}, \quad i = 1, \dots, 2^k,$$

then

$$M_\phi = \delta_2 [t_1 \ t_2 \ \dots \ t_{2^k}] \quad (17)$$

3. M_ψ equals to the M_ψ in (14).

Using these M_F , M_ϕ , and M_ψ , we can construct the decomposition.

Next, we discuss the general case where (9) is not true. That is, $\{\Gamma, \Lambda\}$ is an arbitrary partition of $\{1, 2, \dots, n\}$ (with $\Gamma \neq \emptyset$ and $\Lambda \neq \emptyset$). First, the order of ϕ and ψ does not matter, because, say,

$$f(X) = F(\phi(X_1), \psi(X_2))$$

has its algebraic form as $f(x) = M_f x$, then

$$M_f x = M_F M_\phi x^1 M_\psi x^2 = M_F W_{[2,2]} M_\psi x^2 M_\phi x^1 = \tilde{F}(\phi(X_2), \psi(X_1)),$$

where \tilde{F} has its structure matrix as $M_{\tilde{F}} := M_F W_{[2,2]}$. Now since we consider all possible M_F , and $M_{\tilde{F}} := M_F W_{[2,2]}$ is another possible M_F , with this $M_{\tilde{F}}$, the order of ϕ and ψ has been reversed. Based on this consideration we can choose $k \leq n/2$ variables as the arguments of the second function ψ . We conclude that

Proposition 3.7 Let $n_0 = \lfloor \frac{n}{2} \rfloor$, where $\lfloor r \rfloor$ denotes the largest integer $s \leq r$. Then there are

$$\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n_0} \quad (18)$$

possible Λ , which contain $1, 2, \dots, n_0$ possible arguments of ψ , corresponding to each term in (18).

Let $\Lambda = \{j_1, j_2, \dots, j_s\}$ be the selected variables, where $s \leq n_0$ and $j_1 < j_2 < \dots < j_s$, and $\Gamma = \Lambda^c$. Denote

$$x^1 := \times_{i \in \Gamma} x_i; \quad \text{and} \quad x^2 = \times_{i=1}^s x_{j_i}.$$

Then we have

$$\begin{aligned} f(x_1, \dots, x_n) &= M_f \times_{i=1}^n x_i \\ &= M_f W_{[2, 2^{j_s-1}]} x_{j_s} \times_{i \neq j_s} x_i \\ &= M_f W_{[2, 2^{j_s-1}]} W_{[2, 2^{j_s-1}]} x_{j_{s-1}} x_{j_s} \times_{i \notin \{j_s, j_{s-1}\}} x_i \\ &= \dots \\ &= M_f W_{[2, 2^{j_s-1}]} W_{[2, 2^{j_{s-1}-1}]} \cdots W_{[2, 2^{j_1+(s-2)}]} \times_{i=1}^s x_{j_i} \times_{i \in \Gamma} x_i \\ &= M_f W_{[2, 2^{j_s-1}]} W_{[2, 2^{j_{s-1}-1}]} \cdots W_{[2, 2^{j_1+(s-2)}]} W_{[2^{n-s}, 2^s]} x^1 x^2. \end{aligned}$$

From the above argument we have the following result, which tells us when the arguments for ψ are chosen, how to use Theorem 3.3 (or Remark 3.4) to check possible decomposition.

Theorem 3.8 Using above notations, when $\{x_{j_1}, \dots, x_{j_s}\}$ are chosen as the arguments for possible ψ , the structure matrix, corresponding to $x^1 = \times_{i \in \Gamma} x_i$ and $x^2 = \times_{i \in \Lambda} x_i$, is

$$\tilde{M}_f = M_f \times_{k=s}^1 W_{[2, 2^{j_k+(s-1)-k}]} \times W_{[2^{n-s}, 2^s]}. \quad (19)$$

Example 3.9 1. Assume

$$f(x_1, x_2, x_3, x_4) = (x_1 \leftrightarrow x_2) \vee (x_3 \wedge x_4).$$

Then we have

$$M_f = \delta_2[1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 1 \ 2 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1].$$

It is of the type of $S = S_2$. Choosing $\delta_2[i_1 \ i_2] = \delta_2[1 \ 1]$ and $\delta_2[i_3 \ i_4] = \delta_2[1 \ 2]$, then $M_F = \delta_2[1 \ 1 \ 1 \ 2]$. Let $M_\phi = \delta_2[j_1 \ j_2 \ j_3 \ j_4]$, it is clear that $j_1 = j_4 = 1$ and $j_2 = j_3 = 2$; and for M_ψ we have $M_\psi = [1 \ 2 \ 2 \ 2]$.

Note that we can also choose $\delta_2[i_1 \ i_2] = \delta_2[1 \ 2]$ and $\delta_2[i_3 \ i_4] = \delta_2[1 \ 1]$. Then new $\tilde{M}_\phi = \neg M_\phi$.

2. Assume a Boolean function $f(x_1, x_2, x_3, x_4)$ has its structure matrix as

$$M_f = \delta_2[1 \ 2 \ 2 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 1 \ 2 \ 2 \ 2].$$

Obviously, it is of type $S = S_4$. Choosing $\delta_2[i_1 \ i_2] = \delta_2[1 \ 2]$ and $\delta_2[i_3 \ i_4] = \delta_2[2 \ 1]$, then $M_\phi = \delta_2[1 \ 2 \ 1 \ 1]$ and $M_\psi = [1 \ 2 \ 2 \ 2]$. It follows that f can be decomposed as

$$f(x_1, x_2, x_3, x_4) = [(x_1 \wedge x_2) \vee (\neg x_1)] \leftrightarrow (x_3 \wedge x_4).$$

3. Assume a Boolean function $f(x_1, x_2, x_3, x_4, x_5)$ has its structure matrix as

$$M_f = \delta_2[1\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1].$$

It is obvious that M_f does not satisfy the requirements of Theorem 3.3.

Next, we try to choose proper variable(s) for the argument(s) of ψ . By trial-and-error, we choose $\{x_1, x_4\}$. Using (19), we have

$$\begin{aligned}\tilde{M}_f &= M_f W_{[2,2^3]} W_{[2,2]} W_{[2^3,2^2]} \\ &= \delta_2[1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 2\ 1].\end{aligned}$$

It is clear that \tilde{M}_f is of Type 4, and it can be easily constructed as

$$\begin{aligned}f(x) &= \tilde{M}_f x_2 x_3 x_5 x_1 x_4 = M_f M_\phi x_2 x_3 x_5 M_\psi x_1 x_4 \\ &= \delta_2[1\ 2\ 2\ 1] \delta_2[1\ 1\ 1\ 2\ 1\ 1\ 1\ 1] x_2 x_3 x_5 \delta_2[1\ 2\ 2\ 1] x_1 x_4.\end{aligned}$$

Since $M_F = \delta_2[1\ 2\ 2\ 1]$, we have

$$f(x) = \phi(x_2, x_3, x_5) \leftrightarrow \psi(x_1, x_4).$$

Since $M_\phi = \delta_2[1\ 1\ 1\ 2\ 1\ 1\ 1\ 1]$,

$$\phi(x_2, x_3, x_5) = [x_2 \wedge (x_3 \vee x_5)] \vee \neg x_2.$$

Since $M_\psi = \delta_2[1\ 2\ 2\ 1]$,

$$\psi(x_1, x_4) = x_1 \leftrightarrow x_4.$$

Finally, $f(x)$ has the decomposed form as

$$f(x) = \{[x_2 \wedge (x_3 \vee x_5)] \vee \neg x_2\} \leftrightarrow \{x_1 \leftrightarrow x_4\}.$$

4 Non-disjoint Bi-decomposition

Definition 4.1 Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be a Boolean function, $\Gamma \cup \Theta \cup \Lambda$ be a partition of $\{1, 2, \dots, n\}$. f is said to be bi-decomposed with respect to $\Gamma \cup \Theta$ and $\Lambda \cup \Theta$ if there exist three Boolean functions $F : \mathcal{D}^2 \rightarrow \mathcal{D}$, $\phi : \{x_\gamma | \gamma \in \Gamma \cup \Theta\} \rightarrow \mathcal{D}$, and $\psi : \{x_\lambda | \lambda \in \Theta \cup \Lambda\} \rightarrow \mathcal{D}$, such that

$$f(x_1, \dots, x_n) = F(\phi(x_\gamma | \gamma \in \Gamma \cup \Theta), \psi(x_\lambda | \lambda \in \Theta \cup \Lambda)). \quad (20)$$

For statement ease, let

$$\begin{aligned}X^1 &= \{x_1^1, \dots, x_{k_1}^1\} = \{x_i | i \in \Gamma\}; \\ X^2 &= \{x_1^2, \dots, x_{k_2}^2\} = \{x_i | i \in \Theta\}; \\ X^3 &= \{x_1^3, \dots, x_{k_3}^3\} = \{x_i | i \in \Lambda\}.\end{aligned} \quad (21)$$

Theorem 4.2 Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be a Boolean function with its structure matrix M_f . f can be decomposed as in (20), if and only if its structure matrix can be expressed as

$$M_f = \begin{bmatrix} \mu_{1,1}M_\psi^1 & \mu_{1,2}M_\psi^2 & \cdots & \mu_{1,2^{k_2}}M_\psi^{2^{k_2}} \\ \mu_{2,1}M_\psi^1 & \mu_{2,2}M_\psi^2 & \cdots & \mu_{2,2^{k_2}}M_\psi^{2^{k_2}} \\ \vdots & & & \\ \mu_{2^{k_1},1}M_\psi^1 & \mu_{2^{k_1},2}M_\psi^2 & \cdots & \mu_{2^{k_1},2^{k_2}}M_\psi^{2^{k_2}} \end{bmatrix} \quad (22)$$

where each

$$\begin{aligned} M_\psi^s &\in \mathcal{L}_{2 \times 2^{k_3}}, \quad s = 1, \dots, 2^{k_2}; \\ \mu_{i,j} &\in S, \quad i = 1, \dots, 2^{k_1}, \quad j = 1, \dots, 2^{k_2}, \end{aligned}$$

S equals to one of the S_1, S_2, S_3 , or S_4 , which are defined in Remark 3.4.

Proof. (Necessity) Assume there are three functions F, ϕ , and ψ , such that (20) holds. Assume the structure matrix of F is

$$M_F = \delta_2[i_1 \ i_2 \ i_3 \ i_4];$$

the structure matrix of ϕ is

$$M_\phi = \delta_2[j_1 \ j_2 \ \cdots \ j_{2^{k_1+k_2}}];$$

and the structure matrix of ψ is expressed as

$$M_\psi = [M_\psi^1 \ M_\psi^2 \ \cdots \ M_\psi^{2^{k_2}}] \in \mathcal{L}_{2 \times 2^{k_2+k_3}},$$

where

$$M_\psi^i \in \mathcal{L}_{2 \times 2^{k_3}}, \quad i = 1, \dots, 2^{k_2}.$$

Then we have

$$M_f x = M_F M_\phi x^1 x^2 M_\psi x^2 x^3, \quad (23)$$

where $x = \times_{i=1}^n x_i$, $x^j = \times_{i=1}^{k_j} x_i^j$, $j = 1, 2, 3$.

Using Proposition 2.4, we have that

$$M_f = M_F M_\phi (I_{2^{k_1+k_2}} \otimes M_\psi) (I_{2^{k_1}} \otimes M_r^{2^{k_2}}). \quad (24)$$

We first calculate $M_F M_\phi$, which is denoted as

$$M_F M_\phi := [N_1 \ N_2 \ \cdots \ N_{2^{k_1+k_2}}]. \quad (25)$$

Similar to the disjoint case, we have

$$N_s = \begin{cases} \delta_2[i_1 \ i_2], & j_s = 1 \\ \delta_2[i_3 \ i_4], & j_s = 2, \end{cases} \quad s = 1, 2, \dots, 2^{k_1+k_2}. \quad (26)$$

Next, we calculate $(I_{2^{k_1+k_2}} \otimes M_\psi) (I_{2^{k_1}} \otimes M_r^{2^{k_2}})$:

$$(I_{2^{k_1+k_2}} \otimes M_\psi) (I_{2^{k_1}} \otimes M_r^{2^{k_2}}) = I_{2^{k_1}} \otimes [(I_{2^{k_2}} \otimes M_\psi) M_r^{2^{k_2}}]. \quad (27)$$

We simplify $(I_{2^{k_2}} \otimes M_\psi) M_r^{2^{k_2}}$ first. Note that $(I_{2^{k_2}} \otimes M_\psi) \in \mathcal{L}_{2^{k_2+1} \times 2^{2k_2+k_3}}$ and $M_r^{2^{k_2}} \in \mathcal{L}_{2^{2k_2} \times 2^{k_2}}$, converting them back to conventional matrix product we have

$$(I_{2^{k_2}} \otimes M_\psi) M_r^{2^{k_2}} = (I_{2^{k_2}} \otimes M_\psi) \left(M_r^{2^{k_2}} \otimes I_{2^{k_3}} \right), \quad (28)$$

and

$$I_{2^{k_2}} \otimes M_\psi = \left[\begin{array}{cccc} M_\psi & 0 & \cdots & 0 \\ 0 & M_\psi & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & M_\psi \end{array} \right] \Bigg\} 2^{k_2};$$

$$M_r^{2^{k_2}} \otimes I_{2^{k_3}} = \left[\begin{array}{cccc} \left[\begin{array}{c} I_{2^{k_3}} \\ 0 \\ \vdots \\ 0 \end{array} \right] \Bigg\} 2^{k_2} & 0 & \cdots & 0 \\ 0 & \left[\begin{array}{c} 0 \\ I_{2^{k_3}} \\ \vdots \\ 0 \end{array} \right] \Bigg\} 2^{k_2} & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \right] \Bigg\} 2^{k_2} \end{array} \right] \Bigg\} I_{2^{k_3}}.$$

It follows that

$$(I_{2^{k_2}} \otimes M_\psi) M_r^{2^{k_2}} = \left[\begin{array}{cccc} M_\psi^1 & 0 & \cdots & 0 \\ 0 & M_\psi^2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & M_\psi^{2^{k_2}} \end{array} \right]. \quad (29)$$

Putting (27), (28), and (29) together, (22) follows immediately.

(sufficiency) Using

$$M_\psi = \left[M_\psi^1 \ M_\psi^2 \ \cdots \ M_\psi^{2^{k_2}} \right]$$

as the structure matrix of ψ yields ψ . Denote

$$M_\phi = \left[M_\phi^{1,1} \ \cdots \ M_\phi^{1,k_2} \ \cdots \ M_\phi^{2^{k_1},1} \ \cdots \ M_\phi^{2^{k_1},2^{k_2}} \right].$$

According to $\mu_{\alpha,\beta}$ we can uniquely determine $M_\phi^{\alpha,\beta}$. Precisely, we set

$$M_\phi^{\alpha,\beta} = \begin{cases} \delta_2^1, & \mu_{\alpha,\beta} = \delta_2[i_1, i_2] \\ \delta_2^2, & \mu_{\alpha,\beta} = \delta_2[i_3, i_4]. \end{cases}$$

Using this pair of $\{\phi, \psi\}$, it is easy to check the factorization (20) holds. \square

Remark 4.3 Observe Theorem 4.2. Note that $\forall X^2 \in \{0, 1\}^{k_2}$, say $X^2 = \mathbf{a}$, with t the decimal number of \mathbf{a} , then the structure matrix of $f(X^1, \mathbf{a}, X^3)$ is the combination of t -th, $(2^{k_2} + t)$ -th, $(2^{k_2+1} + t)$ -th, \dots , $(2^{k_2+(2^{k_1-1})} + t)$ -th blocks of M_f (if we consider (22) as an array, structure matrix of $f(X^1, \mathbf{a}, X^3)$ is the combination of blocks in the t -th column).

By remark 3.4, $f(X^1, \mathbf{a}, X^3)$ can be disjoint decomposed, if and only if it's structure matrix is of the form $[\mu_1 M_\psi \ \mu_2 M_\psi \ \dots \ \mu_{2^k} M_\psi]$. Thus, if f can be decomposed as in (20), then $\forall X^2 \in \{0, 1\}^{k_2}$ (with k as the decimal number of \mathbf{a}), $f(X^1, \mathbf{a}, X^3)$ is disjoint decomposable with respect to X^1 and X^3 ; if $\forall X^2 \in \{0, 1\}^{k_2}$, $f(X^1, \mathbf{a}, X^3)$ is disjoint decomposable with respect to X^1 and X^3 , then the structure matrix can be expressed as (22). In other words, f can be decomposed as in (20), if and only if $\forall X^2 \in \{0, 1\}^{k_2}$ (with k a the decimal number of \mathbf{a}), $f(X^1, \mathbf{a}, X^3)$ is disjoint decomposable with respect to X^1 and X^3 , which is exactly the Theorem 3.2 in [23].

Our explicit expression in Theorem 4.2 is an improvement of the known implicit form in [23], because ours is not only straightforward verifiable but also easily used to construct the decomposition.

The following corollary gives the way to construct the decomposition.

Corollary 4.4 Assume the structure matrix M_f of f as in (22) satisfies the conditions of Theorem 4.2. Then we have the following:

1. If the set $\{\mu_{1,1}, \dots, \mu_{2^{k_1}, 2^{k_2}}\}$ contains only one element $\delta_2[p, q]$, then

$$M_F = [\delta_2[p, q] \ \delta_2[p, q]]; \quad (30)$$

otherwise the set contains two elements $\delta_2[p_1, q_1], \delta_2[p_2, q_2]$, then

$$M_F = [\delta_2[p_1, q_1] \ \delta_2[p_2, q_2]]. \quad (31)$$

2. Consider $\mu_{i,j}$. If

$$\mu_{i,j} = M_F \delta_2^{t_{i,j}}, \quad i = 1, \dots, 2^{k_1}; \ j = 1, \dots, 2^{k_2},$$

then

$$M_\phi = \delta_2 [t_{1,1} \ \dots \ t_{1,2^{k_2}} \ \dots \ t_{2^{k_1},1} \ \dots \ t_{2^{k_1},2^{k_2}}]. \quad (32)$$

3.

$$M_\psi = [M_\psi^1 \ M_\psi^2 \ \dots \ M_\psi^{2^{k_2}}]. \quad (33)$$

Using these M_F , M_ϕ , and M_ψ , we can construct the decomposition.

Remark 4.5 As for arbitrary order variables, the basic idea of Theorem 3.8 remains applicable. When $\{x_{j_1}, \dots, x_{j_{k_2}}, x_{j_{k_2+1}}, \dots, x_{j_{k_3}}\}$ are chosen as the arguments for possible ψ , the structure matrix corresponding to $x^1 = \times_{i \in \Gamma} x_i$, $x^2 = \times_{i \in \Theta} x_i$ and $x^3 = \times_{i \in \Lambda} x_i$, is

$$\tilde{M}_f = M_f \times_{i=k_3}^1 W_{[2, 2^{j_i+(k_3-1)-i}]} \times W_{[2^{k_1+k_2}, 2^{k_3}]} \times_{i=k_2}^1 W_{[2, 2^{j_i+(k_2-1)-i}]} \times W_{[2^{k_1}, 2^{k_2}]}. \quad (34)$$

We give two examples to depict Theorem 4.2 and Remark 4.5 .

Let $f(x_1, x_2, \dots, x_n)$ be an r -valued logical function. Identifying

$$\delta_r^i \sim \frac{r-i}{r-1}, \quad i = 1, 2, \dots, r,$$

we can express $f : \Delta_{r^n} \rightarrow \Delta_r$, which is called the vector form of f . Similar to Boolean case, we know that in vector form there exists a unique $M_f \in \mathcal{L}_{r \times r^n}$, called the structure matrix of f , such that[10]

$$f(x_1, \dots, x_n) = M_f x, \quad (35)$$

where $x = \times_{i=1}^n x_i$.

Definition 5.1 Let $f : \mathcal{D}_r^n \rightarrow \mathcal{D}_r$ be an r -valued logical function.

1. Assume $\Gamma \cup \Lambda$ is a partition of $\{1, 2, \dots, n\}$. f is said to be bi-decomposed with respect to Γ and Λ if there exist three r -valued logical functions $F : \mathcal{D}_r^2 \rightarrow \mathcal{D}_r$, $\phi : \{x_\gamma | \gamma \in \Gamma\} \rightarrow \mathcal{D}_r$, and $\psi : \{x_\lambda | \lambda \in \Lambda\} \rightarrow \mathcal{D}_r$, such that

$$f(x_1, \dots, x_n) = F(\phi(x_\gamma | \gamma \in \Gamma), \psi(x_\lambda | \lambda \in \Lambda)). \quad (36)$$

2. Assume $\Gamma \cup \Theta \cup \Lambda$ is a partition of $\{1, 2, \dots, n\}$. f is said to be bi-decomposed with respect to $\Gamma \cup \Theta$ and $\Lambda \cup \Theta$ if there exist three Boolean functions $F : \mathcal{D}_r^2 \rightarrow \mathcal{D}_r$, $\phi : \{x_\gamma | \gamma \in \Gamma \cup \Theta\} \rightarrow \mathcal{D}_r$, and $\psi : \{x_\lambda | \lambda \in \Theta \cup \Lambda\} \rightarrow \mathcal{D}_r$, such that

$$f(x_1, \dots, x_n) = F(\phi(x_\gamma | \gamma \in \Gamma \cup \Theta), \psi(x_\lambda | \lambda \in \Theta \cup \Lambda)). \quad (37)$$

First we consider the disjoint case. It is clear that there are r^r mappings from $\mathcal{D}_r \rightarrow \mathcal{D}_r$. In vector form they can be expressed as

$$b_i = T_i x, \quad i = 1, 2, \dots, r^r,$$

where $x, b_i \in \Delta_r$ and $T_i \in \mathcal{L}_{r \times r}$.

We use $\{T_i\}$ to describe F . Choosing r elements from $\mathcal{L}_{r \times r}$, say,

$$\mathcal{T} = \{T_1, T_2, \dots, T_r\} \subset \mathcal{L}_{r \times r},$$

then we say that F has Type \mathcal{T} , if the structure matrix of F is

$$M_F = [T_1 \ T_2 \ \dots \ T_r].$$

As we see in Boolean case, the order of $\{T_i | i = 1, \dots, r^2\}$ does not affect the decomposition.

Similar to Boolean case, we can prove the following:

Theorem 5.2 Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be a Boolean function with its structure matrix M_f , being split into 2^k equal blocks as in (10). Assume Γ and Λ form a partition as in (9). f is decomposable with respect to the partition in (9), if and only if, there exist

(i) a type $\mathcal{T} = \{T_1, T_2, \dots, T_r\} \subset \mathcal{L}_{r \times r}$,

(ii) a logical matrix $M_\psi \in \mathcal{L}_{r \times r^{n-k}}$,

If we define the 3-valued logical operators \rightarrow and \leftrightarrow by using the corresponding formulas of Boolean functions as

$$\begin{aligned} A \rightarrow B &:= (A \wedge B) \vee \neg A \\ A \leftrightarrow B &:= (A \rightarrow B) \wedge (B \rightarrow A), \end{aligned}$$

then it is easy to verify that

$$M_{\rightarrow} = M_{\phi}, \quad \text{and } M_{\leftrightarrow} = M_{\psi}.$$

Eventually, we have the decomposed f as

$$f(x_1, x_2, x_3, x_4) = (x_1 \rightarrow x_2) \vee (x_3 \leftrightarrow x_4).$$

Next, we consider non-disjoint case.

Definition 5.5 Let $f : \mathcal{D}_r^n \rightarrow \mathcal{D}_r$ be an r -valued logical function, $\Gamma \cup \Theta \cup \Lambda$ be a partition of $\{1, 2, \dots, n\}$. f is decomposable with respect to $\Gamma \cup \Theta$ and $\Lambda \cup \Theta$ (as in (20)), if there exist three r -valued logical functions $F : \mathcal{D}_r^2 \rightarrow \mathcal{D}_r$, $\phi : \{x_\gamma | \gamma \in \Gamma \cup \Theta\} \rightarrow \mathcal{D}_r$, and $\psi : \{x_\lambda | \lambda \in \Theta \cup \Lambda\} \rightarrow \mathcal{D}_r$, such that

$$f(x_1, \dots, x_n) = F(\phi(x_\gamma | \gamma \in \Gamma \cup \Theta), \psi(x_\lambda | \lambda \in \Theta \cup \Lambda)). \quad (42)$$

Theorem 5.6 Let $f : \mathcal{D}^n \rightarrow \mathcal{D}$ be an r -valued logical function with its structure matrix M_f . f can be decomposed as in (42) with respect to the partition as in (21), if and only if

(i) there exists a type $\mathcal{T} \subset \mathcal{L}_{r \times r}$,

(ii) there exist

$$M_\psi^i \in \mathcal{L}_{r \times r^{k_3}}, \quad i = 1, \dots, r^{k_2}, \quad (43)$$

such that the structure matrix of f can be expressed as

$$\begin{aligned} M_f &= \begin{bmatrix} \mu_{1,1} M_\psi^1 & \mu_{1,2} M_\psi^2 & \cdots & \mu_{1,r^{k_2}} M_\psi^{r^{k_2}} \\ \mu_{2,1} M_\psi^1 & \mu_{2,2} M_\psi^2 & \cdots & \mu_{2,r^{k_2}} M_\psi^{r^{k_2}} \\ \vdots & & & \\ \mu_{r^{k_1},1} M_\psi^1 & \mu_{r^{k_1},2} M_\psi^2 & \cdots & \mu_{r^{k_1},r^{k_2}} M_\psi^{r^{k_2}} \end{bmatrix} \end{aligned} \quad (44)$$

where

$$\mu_{i,j} \in \mathcal{T}, \quad i = 1, \dots, r^{k_1}, \quad j = 1, \dots, r^{k_2}.$$

Remark 5.7 Similar to Corollary 3.6 for disjoint case (Corollary 4.4 for non-disjoint case), when the conditions in part 1 (part 2) of Theorem 5.6 are satisfied the corresponding decomposition can be easily constructed by using the structure matrices M_F , M_ϕ , and M_ψ .

Example 5.8 Let $f(x_1, x_2, x_3, x_4) : \mathcal{D}_3^4 \rightarrow \mathcal{D}_3$ be a 3-valued logical function with its structure matrix as

$$\begin{aligned} M_f &= \delta_3 [1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 2 \ 2 \ 1 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 3 \ 2 \ 1 \ 3 \ 2 \ 2 \ 3 \ 3 \ 3 \\ &\quad 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \\ &\quad 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 1 \ 2 \ 3 \ 1 \ 2 \ 2 \ 1 \ 1 \ 1]. \end{aligned}$$

If we define the 3-valued logical operators \leftrightarrow as in example 5.4 above, and we try $\mathcal{T} = \mathcal{T}_{\leftrightarrow}$ as in (39), that is,

$$\mathcal{T} = \{T_1 = \delta_3[1\ 2\ 3], T_2 = \delta_3[2\ 2\ 2], T_3 = \delta_3[3\ 2\ 1]\};$$

Choosing

$$M_\psi^1 = \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3],$$

it is easy to check that

$$\begin{aligned} T_1 M_\psi^1 &= [1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3] \\ T_2 M_\psi^1 &= [2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \\ T_3 M_\psi^1 &= [3\ 3\ 3\ 2\ 2\ 2\ 1\ 1\ 1]. \end{aligned}$$

Similarly, choosing

$$M_\psi^2 = \delta_3[2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3]$$

yields

$$\begin{aligned} T_1 M_\psi^2 &= [2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3] \\ T_2 M_\psi^2 &= [2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \\ T_3 M_\psi^2 &= [2\ 2\ 3\ 2\ 2\ 2\ 1\ 1\ 1]; \end{aligned}$$

and choosing

$$M_\psi^3 = \delta_3[3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3],$$

yields

$$\begin{aligned} T_1 M_\psi^3 &= [3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3] \\ T_2 M_\psi^3 &= [2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \\ T_3 M_\psi^3 &= [1\ 2\ 3\ 1\ 2\ 2\ 1\ 1\ 1]. \end{aligned}$$

Comparing each block of M_f with the above product forms, it is easy to see that (44) is satisfied. It follows immediately that

$$M_\psi = [M_\psi^1\ M_\psi^2\ M_\psi^3] = \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ 3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3]$$

$$M_\phi = \delta_3[1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 3].$$

Thus,

$$f(x_1, x_2, x_3, x_4) = M_f x_1 x_2 x_3 x_4 = M_{\leftrightarrow} M_\phi x_1 x_2 M_\psi x_2 x_3 x_4.$$

Back to logical form, we have the decomposed f as

$$f(x_1, x_2, x_3, x_4) = (x_1 \vee x_2) \leftrightarrow [(x_2 \vee \neg x_4) \wedge x_3].$$

6 Decomposition of Mix-Valued Logical Functions

Definition 6.1 1. Let $f : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \rightarrow \mathcal{D}_{r_0}$ be a mix-valued logical function. f is said to be decomposable with respect to \mathcal{D}_{r_1} and \mathcal{D}_{r_2} , if there exist $F : \mathcal{D}_{r_0} \rightarrow \mathcal{D}_{r_0}$, $\phi : \mathcal{D}_{r_1} \rightarrow \mathcal{D}_{r_0}$, and $\psi : \mathcal{D}_{r_2} \rightarrow \mathcal{D}_{r_0}$, such that

$$f(x_1, x_2) = F(\phi(x_1), \psi(x_2)), \quad x_1 \in \mathcal{D}_{r_1}, \quad x_2 \in \mathcal{D}_{r_2}. \quad (45)$$

2. Let $f : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \times \mathcal{D}_{r_3} \rightarrow \mathcal{D}_{r_0}$ be a mix-valued logical function. f is said to be decomposable with respect to $\mathcal{D}_{r_1} \times \mathcal{D}_{r_2}$ and $\mathcal{D}_{r_2} \times \mathcal{D}_{r_3}$, if there exist $F : \mathcal{D}_{r_0} \rightarrow \mathcal{D}_{r_0}$, $\phi : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \rightarrow \mathcal{D}_{r_0}$, and $\psi : \mathcal{D}_{r_2} \times \mathcal{D}_{r_3} \rightarrow \mathcal{D}_{r_0}$, such that

$$f(x_1, x_2, x_3) = F(\phi(x_1, x_2), \psi(x_2, x_3)), \quad x_1 \in \mathcal{D}_{r_1}, x_2 \in \mathcal{D}_{r_2}, x_3 \in \mathcal{D}_{r_3}. \quad (46)$$

Remark 6.2 To see this is a most general case, we consider (in vector form) $x^1 = \times_{i=1}^k x_i$, $x^2 = \times_{i=1}^{n-k} x_i$.

- (i) Let $x_i \in \Delta_2$, $\forall i$ (i.e., $r_1 = 2^k$, $r_2 = 2^{n-k}$), and choose $r_0 = 2$. Then we have the bi-decomposition of Boolean functions.
- (ii) Let $x_i \in \Delta_r$, $\forall i$ (i.e., $r_1 = r^k$, $r_2 = r^{n-k}$), and choose $r_0 = r$. Then we have the bi-decomposition of r -valued logical functions.
- (iii) Let x_i as in the above case 1 (case 2), and choose $r_0 = 2^s$ ($r_0 = r^s$). Then we have the bi-decomposition of Boolean (r -valued) multi-input multi-output (MIMO) mappings.

Using the argument for Boolean or multi-valued case, we can have the following general result immediately.

Theorem 6.3 1. Let $f : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \rightarrow \mathcal{D}_{r_0}$ with its structure matrix as

$$M_f = [M_1 \ M_2 \ \cdots \ M_{r_1}], \quad (47)$$

where $M_i \in \mathcal{L}_{r_0 \times r_2}$. f has a decomposed form with respect to \mathcal{D}_{r_1} and \mathcal{D}_{r_2} , if and only if, there exist

- (i) a type $\mathcal{T} = \{T_1, T_2, \dots, T_{r_0}\} \subset \mathcal{L}_{r_0 \times r_0}$,
- (ii) a logical matrix $M_\psi \in \mathcal{L}_{r_0 \times r_2}$,

such that

$$M_i = T_{s_i} M_\psi, \quad \text{where } T_{s_i} \in \mathcal{T}, \quad i = 1, \dots, r_1. \quad (48)$$

2. Let $f : \mathcal{D}_{r_1} \times \mathcal{D}_{r_2} \times \mathcal{D}_{r_3} \rightarrow \mathcal{D}_{r_0}$ be a mix-valued logical function. f is decomposable with respect to $\mathcal{D}_{r_1} \times \mathcal{D}_{r_2}$ and $\mathcal{D}_{r_2} \times \mathcal{D}_{r_3}$, if and only if,

- (i) there exists a type $\mathcal{T} \subset \mathcal{L}_{r_0 \times r_0}$,
- (ii) there exist

$$M_\psi^i \in \mathcal{L}_{r_0 \times r_3}, \quad i = 1, \dots, r_2, \quad (49)$$

such that the structure matrix of f can be expressed as

$$M_f = \begin{bmatrix} \mu_{1,1} M_\psi^1 & \mu_{1,2} M_\psi^2 & \cdots & \mu_{1,r_2} M_\psi^{r_2} \\ \mu_{2,1} M_\psi^1 & \mu_{2,2} M_\psi^2 & \cdots & \mu_{2,r_2} M_\psi^{r_2} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{r_1,1} M_\psi^1 & \mu_{r_1,2} M_\psi^2 & \cdots & \mu_{r_1,r_2} M_\psi^{r_2} \end{bmatrix} \quad (50)$$

where

$$\mu_{i,j} \in \mathcal{T}, \quad i = 1, \dots, r_1, \quad j = 1, \dots, r_2.$$

7 Normalization of Dynamic-Static Boolean Networks

As an application we consider the dynamic-static Boolean networks. Consider a Boolean network of n nodes. Assume there are $n - k$ nodes, which satisfy Boolean dynamic models as

$$x_i(t+1) = f_i(x_1, \dots, x_n), \quad i = 1, \dots, n - k, \quad (51)$$

and the other k nodes are determined by certain static equations as

$$g_j(x_1, \dots, x_n) = 1, \quad j = 1, \dots, k. \quad (52)$$

Note that the right hand side of (52) can be either 0 or 1. Without loss of generality we can set them to be 1, because for $g_j = 0$ we can use $\neg g_j$ to replace g_j .

In vector form set $x^1 = \times_{i=1}^{n-k} x_i$ and $x^2 = \times_{i=n-k+1}^n x_i$. The system (51)-(52) is said to have a normal form, if (52) can be expressed as

$$x_j = \phi_j(x_1, \dots, x_{n-k}), \quad j = n - k + 1, \dots, n. \quad (53)$$

It is obvious that (53) is very convenient in use, because we can plug (53) into (51) to get a standard Boolean network, and its properties can be analyzed easily. Hence the problem “when (52) can be converted to (53)” is interesting. This section is devoted to solving this problem.

(52) can be expressed in vector form as

$$M_G x^1 x^2 = \delta_{2^k}^1, \quad (54)$$

where $M_G \in \mathcal{L}_{2^k \times 2^n}$.

For any positive integer $s > 1$ define a set of matrices, Ξ_i , as

$$\Xi_i = \{ E_i \in \mathcal{L}_{s \times s} \mid \text{Col}_i(E_i) = \delta_s^1; \text{Col}_j(E_i) \neq \delta_s^1, j \neq i \}, \quad i = 1, 2, \dots, s. \quad (55)$$

Using Ξ_i , we construct a set of types as

$$\mathcal{E}_s := [E_1 \ E_2 \ \dots \ E_s], \quad E_i \in \Xi_i, \quad i = 1, 2, \dots, s. \quad (56)$$

As in previous sections, each type $E \in \mathcal{E}_s$ corresponds to a unique logical mapping $F : \mathcal{D}_s \times \mathcal{D}_s \rightarrow \mathcal{D}_s$, which has E as its structure matrix, that is, $M_f = E$.

Then we have the following result:

Lemma 7.1 *Let $X, Y \in \Delta_s$. $X = Y$, if and only if there exists a $E \in \mathcal{E}_s$ such that*

$$EXY = \delta_s^1. \quad (57)$$

Proof. Denote

$$E = [E_1 \ E_2 \ \dots \ E_s],$$

and assume $X = \delta_s^p$ and $Y = \delta_s^q$. A straightforward computation shows that

$$EXY = \text{Col}_q(E_p).$$

Hence (57) holds, if and only if, $p = q$. \square

Now we are ready to present the main result for normalization. We first express (52) into its algebraic form as

$$M_G x = \delta_{2^k}^1, \quad (58)$$

where M_G is the structure matrix of $G = (g_1, \dots, g_k) : \mathcal{D}^n \rightarrow \mathcal{D}^k$.

Theorem 7.2 x_j can be solved as (53) from (52), if and only if, There exists a

$$E = [E_1 \ E_2 \ \dots \ E_{2^k}] \in \mathcal{E}_{2^k},$$

such that the structure matrix of G can be expressed as

$$M_G = [M_1 \ M_2, \dots, M_{2^{n-k}}], \quad (59)$$

and

$$M_i \in \{E_1 \ E_2 \ \dots \ E_{2^k}\}, \quad i = 1, \dots, 2^{n-k}.$$

Proof. (53) can be expressed into vector form as $x^2 = M_\phi x^1$. According to Lemma 7.1, (52) can be expressed into (53), if and only if there exists an $M_F \in \mathcal{E}_{2^k}$ such that $M_F M_\phi x^1 x^2 = \delta_{2^k}^1$. Comparing with (58), it is clear that the necessary and sufficient condition becomes that M_G can be expressed as

$$M_G x = M_F M_\phi x^1 x^2, \quad (60)$$

where $M_F \in \mathcal{E}_{2^k}$. Now formally consider $x^2 = M_\psi x^2$ with $M_\psi = I_{2^k}$, the conclusion follows from Theorem 6.3 immediately. \square

Example 7.3 Consider the follow dynamic-static Boolean network

$$\begin{cases} x_1(t+1) = x_2(t) \rightarrow x_4(t) \\ x_2(t+1) = x_1(t) \wedge x_3(t) \\ 1 = (x_3(t) \bar{\vee} x_4(t)) \leftrightarrow (x_1(t) \bar{\vee} x_2(t)) \\ 0 = x_4(t) \bar{\vee} (x_1(t) \vee x_2(t)). \end{cases} \quad (61)$$

We intend to solve x_3 and x_4 out from the last two equations. First, we convert them to

$$\begin{cases} g_1(x_1, x_2, x_3, x_4) := (x_3(t) \bar{\vee} x_4(t)) \leftrightarrow (x_1(t) \bar{\vee} x_2(t)) = 1 \\ g_2(x_1, x_2, x_3, x_4) := x_4(t) \leftrightarrow (x_1(t) \vee x_2(t)) = 1 \end{cases} \quad (62)$$

It is easy to calculate that in vector form we have

$$\begin{cases} g_1(x_1, x_2, x_3, x_4) = M_{g_1} x = \delta_2 [1 \ 2 \ 2 \ 1 \ 2 \ 1 \ 1 \ 2 \ 2 \ 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1] x \\ g_2(x_1, x_2, x_3, x_4) = M_{g_2} x = \delta_2 [1 \ 2 \ 1 \ 2 \ 1 \ 2 \ 1 \ 2 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 1] x. \end{cases} \quad (63)$$

Then the structure matrix of $G = (g_1, g_2)$ can be easily calculated as

$$M_G = \delta_4 [1 \ 4 \ 3 \ 2 \ 3 \ 2 \ 1 \ 4 \ 3 \ 2 \ 1 \ 4 \ 2 \ 3 \ 4 \ 1]. \quad (64)$$

Now we can construct the structure matrix $M_F \in \mathcal{E}_4$ as

$$M_F = \delta_4[1 \ 4 \ 3 \ 2 \ * \ 1 \ * \ * \ 3 \ 2 \ 1 \ 4 \ 2 \ 3 \ 4 \ 1], \quad (65)$$

where $2 \leq * \leq 4$ can be arbitrary. Comparing (64) with (65) yields that

$$M_\phi = \delta_4[1 \ 3 \ 3 \ 4], \quad (66)$$

which means

$$x_3(t)x_4(t) = \delta_4[1 \ 3 \ 3 \ 4]x_1(t)x_2(t).$$

It follows that $x_3(t)$ and $x_4(t)$ can be solved from (63) uniquely as

$$\begin{cases} x_3(t) = x_1(t) \wedge x_2(t) \\ x_4(t) = x_1(t) \vee x_2(t). \end{cases} \quad (67)$$

plugging (67) into (61) yields

$$\begin{cases} x_1(t+1) = x_2(t) \rightarrow (x_1(t) \vee x_2(t)) \\ x_2(t+1) = x_1(t) \wedge x_2(t). \end{cases} \quad (68)$$

Then the dynamics of the dynamic-static Boolean network (61) is determined by (68) (with algebraic equation (67) for the other two state variables).

Remark 7.4 1. The calculations involved in Example 7.3, such as converting a logical mapping into its algebraic form and back from an algebraic form to a set of logical functions etc. are standard, we refer to [10] for detail.*

2. The method provided above can also be used for dynamic-static Boolean control networks. You have only to replace x^1 by $\{x^1, u\}$ and then use exactly the aforementioned technique.

8 Conclusion

This paper first considered the bi-decomposition of Boolean functions. Necessary and sufficient conditions for both disjoint and non-disjoint cases are obtained. The conditions are easily verifiable and they provide a natural way to construct the decompositions. Then the results were extended first to multi-valued logical functions and then to mix-valued logical mappings to get the corresponding necessary and sufficient conditions and decomposition algorithms. Finally, as an application, the normalization of dynamic-static Boolean networks was considered. Several examples have been presented to illustrate the corresponding theoretical results.

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*An STP toolbox is provided in <http://lsc.amss.ac.cn/dcheng> for the related computations.

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