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LINEARIZATION WITH DYNAMIC COMPENSATION

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1. INTRODUCTION

Consider an affine nonlinear system which is defined in an n-dimensional manifold M and expressed on each coordinate chart as

$$\Sigma: \dot{x} = f(x) + \Sigma_{i=1}^{m} g_i(x)u_i, \tag{1}$$

where $x \in M$, $f(x)$, $g_i(x)$, $i = 1, \cdots, m$, are C^∞ vector fields. The static state feedback linearization problem is defined as that of finding a feedback law

$$u = \alpha(x) + \beta(x)v, \tag{2}$$

where $\alpha(x)$ is an $m \times 1$ C^∞ vector function, $\beta(x)$ is an $m \times m$ nonsingular C^∞ function matrix, and a diffeomorphism $z = z(x)$, such that the new system

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \tag{3}$$

can be expressed in the coordinate frame z as a completely controllable linear system

$$\dot{z} = Az + Bv. \tag{4}$$

In this paper we treat only the local linearization problem.

In (3), $g(x) := (g_1(x), \cdots, g_m(x))$. We give some more notations and conventions as follows.

Let $\{X_i, i \in I\}$ be a set of vector fields. $\{X_i, i \in I\}_{LA}$ denotes the smallest Lie subalgebra of $T(M)$ containing $\{X_i, i \in I\}$; $\mathrm{Sp}\{X_i, i \in I\}$ is the submodule of $T(M)$ generated by $\{X_i, i \in I\}$ over the ring of $C^\infty(M)$. $G := \mathrm{Sp}\{g_i, i = 1, \cdots, m\}$.

The main result for local static state feedback linearization is given by Jakubczyk and Respondek[1] as follows.

Theorem 1. *System (1) is locally static state feedback linearizable (around x_0), if and only if there exists a neighborhood U of x_0, such that on U*

i) $\Delta_k := \mathrm{Sp}\,\{L_f^i g_j,\ i = 0, 1, \cdots, k; j = 1, \cdots, m\}, k = 0, 1, \cdots, n-1$ *are nonsingular and involutive,*

ii) $\dim(\Delta_{n-1}) = n$.

Some improvements of Theorem 1 may be found in [2—4]. There are some algorithms in [5].

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Very recently, in investigating affine nonlinear systems, the dynamic compensation method has been used for searching into the invertible property and decoupling form[6] and the linear matching problem[7], etc.

The dynamic compensation method is that of putting integrators in some input terminals, such that a new extended dynamic system is obtained as

$$\Sigma^e: \begin{cases} \dot{x} = f(x) + \bar{g}^1 z^1 + g^1 v^1, \\ \dot{z}^1 = v^2, \end{cases} \tag{5}$$

where $\bar{g}^1$ and g^1 are $r \times n$ and $(m-r) \times n$ matrices respectively and $(\bar{g}^1, g^1) = g\beta^0$. System Σ^e will be called one-fold extension of the original system Σ.

In this paper the dynamic compensation method will be used to investigate the linearization problem. The idea is that: An affine nonlinear system fails to be linearizable, but by choosing a suitable subdistribution G^1 of G, the extended system Σ^e of Σ may be linearizable. A simple example demonstrates that.

Example 1. Consider the following system

$$\Sigma: \begin{cases} \dot{x}_1 = x_2 + (x_1 + 1)u_1 + x_2 u_2, \\ \dot{x}_2 = x_3 + 1 + x_2 u_2, \\ \dot{x}_3 = \cos(x_1 + x_3) u_2. \end{cases} \tag{6}$$

Since it is easy to check that G is not involutive, system (6) is not static state feedback linearizable. Choosing $\bar{g}^1 = g_1, g^1 = g_2$, one may prove that the extended system is linearizable.

2. Dynamic State Feedback Iinearization

An affine nonlinear system Σ is called one-fold dynamic state feedback linearizable if there exists an extension Σ^e of Σ, such that Σ^e is static state feedback linearizable.

Let G^1 be a subdistribution of G. We define a sequence of nested distributions inductively as

$$\begin{cases} D_0 := G^1, \\ D_i := \Delta_{i-1} + L_f^i G^1, \ i \geqslant 1, \end{cases} \tag{7}$$

where $\Delta_k, k \geqslant 0$, are defined as in Theorem 1. Then we have the following theorem.

Theorem 2. *System (1) is one-fold dynamic state feedback linearizable, if and only if there exists a subdistribution G^1 of G, such that*

i) $D_i, i = 0, 1, \cdots, n-1$, *are nonsingular and involutive*;

ii) $\dim (D_{n-1}) = n$.

Proof. Choosing G^1, the extended system (5) may be built. According to Theorem 1, Σ^e is static state linearizable if and only if, Δ_i^e, $i = 0, 1, \cdots, n+r-1$ are nonsingular and involutive, and $\dim(\Delta_{n+r-1}^e) = n + r$. Hence the proof is completed by the following lemma.

Lemma 1. *The following two sets of conditions are equivalent*:

i) *the distributions* $D_i, i = 0, 1, \cdots, n-1$, *are nonsingular and involutive,*

ii) $\dim(D_{n-1}) = n$;

i)′ *the distributions* $\Delta_k^e, k = 0, 1, \cdots, n+r-1$, *are nonsingular and involutive,*

ii)′ $\dim(\Delta_{n+r-1}^e) = n + r$.

Proof. We claim that under either i) and ii) or i)′ and ii)′, the following relation holds:

$$\Delta_k^e = \mathrm{Sp}\left\{\begin{bmatrix} X \\ 0 \end{bmatrix}, X \in D_k\right\} + \mathrm{Sp}\left\{\begin{bmatrix} 0 \\ I_{m-r} \end{bmatrix}\right\}. \tag{8}$$

Assuming i) and ii), we prove (8) inductively. It is obviously true for $k = 0$. Suppose it is true for k. Note that

$$\Delta_{k+1}^e = \Delta_k^e + L_{f^e}\Delta_k^e.$$

A straightforward computation shows that

$$L_{f^e}\begin{bmatrix} L_f^i g_j \\ 0 \end{bmatrix} = \begin{bmatrix} L_f^{i+1} g_j + \Sigma_s z_s^1 [L_f^i g_j, g_s^1] \\ 0 \end{bmatrix}. \tag{9}$$

By the involutivity of D_k, we have $[L_f^i g_j, g_s^1] \in D_k$. Thus (9) implies (8).

Assuming i)′ and ii)′, equation (8) can be proved in a similar way. Using (8), the equivalence of i), ii) and i)′, ii)′ is obvious. □

Comparing Theorem 2 with Theorem 1, it is clear that Theorem 1 is a special case of Theorem 2 with $G^1 = G$.

3. Multi-fold Case

Sometimes it is possible that the one-fold integrator is not enough. Thus a natural development is putting more integrators on some terminals. Precisely, let

$$G^s \subset G^{s-1} \cdots \subset G^1 \subset G$$

be a sequence of nested subdistributions of G. t-fold integrators are attached to each terminal of G^t. Hence there exists a nonsingular matrix $\beta^t(x) = (p^t(x), q^t(x))$, with $p^t(x)$ and $q^t(x)$ as $n_t \times n_{t+1}$ and $n_t \times (n_t - n_{t+1})$ matrices respectively, such that $G^t = G^{t-1}p^t(x), t = 1, \cdots, s$, where $G^0 = G$. When s integrators are attached to G^s, the extended system is called s-fold extension. If the s-fold extension of Σ is static state feedback linearizable, we say that the original system is s-fold dynamic state feedback linearizable.

To facilitate the presentation of our result, we define a sequence of distributions by using a given nested sequence of subdistributions $H_1 \subset H_2 \cdots \subset H_s \subset G$ of G as follows:

$$\begin{cases} D_0 := H_1, \\ D_k := H_{k+1} + L_f D_{k-1}, \quad k = 1, 2, \cdots, \end{cases} \tag{10}$$

where $H_t := G$ for $t > s$.

Considering the linearization around 0, we have the following conclusion.

Theorem 3. *System* (1) *is s-fold linearizable around* 0, *if and only if there exists* *a sequence of nested subdistributions* $H_1 \subset H_2 \cdots \subset H_s \subset G$ *of* G, *such that the corresponding distributions* $D_0, D_1, \cdots, D_{n-1}$ *constructed by* (10) *satisfy the following conditions:*

i) $D_i, i = 0, 1, \cdots, n+s-m$, *are nonsingular and involutive;*

ii) $\dim (D_{n+s-m}) = n$;

iii) $[H_{i-1}, G] \subset D_i, \quad i = 1, \cdots, n+s-m$.

Proof. Necessity: We first construct $H_i, i = 1, \cdots, s$, inductively as

$$H_1 := \mathrm{Sp}\{gq^1\},$$

$$H_t := H_{t-1} + \mathrm{Sp}\ \{gp^1 \cdots p^{t-1} q^t\}, \quad t = 2, 3, \cdots, s.$$

It will be shown that such $H_i, i = 1, 2, \cdots, s$, are as required.

Now the extended system can be described as

$$\Sigma^e: \begin{cases} \dot{x} = f(x) + \bar{g}^1(x) z^1 + g^1(x) v^1, \\ \dot{z}^1 = p^2(x) z^2 + q^2(x) v^2, \\ \cdots \\ \dot{z}^{s-1} = p^s(x) z^s + q^s(x) v^s, \\ \dot{z}^s = v^{s+1}, \end{cases} \tag{11}$$

where $(\bar{g}^1(x), g^1(x)) = g(p^1(x), q^1(x))$.

Without loss of generality we can assume that $p^i(x)$ and $q^i(x), i=2, \cdots, s$, are constant, because it is fairly easy to construct a local diffeomorphism as follows. Let $\bar{z}^s = z^s$ and assume $\bar{z}^s, \bar{z}^{s-1}, \cdots, \bar{z}^j$ are well defined and $\bar{z}^j = A^j(x) z^j$. Then we set

$$\bar{z}^{j-1} = (p^j(x) A^j(x), q^j(x))^{-1} z^{j-1}, \ j = s, s-1, \cdots, 2,$$

and let $\bar{x} = x$. That under this new coordinate frame system (11) has constant p^i and q^i follows quickly.

Notice that system (11) is static state feedback linearizable, if and only if the corresponding $\Delta_i^e, i = 0, 1 \cdots$, satisfy the conditions of Theorem 1. Since $\Delta_i^e = \mathrm{Sp}\ \{g^e\}$, Δ_0^e is nonsingular and involutive, if and only if so is D_0. A simple computation shows that

$$\begin{aligned} \Delta_1^e = \Delta_0^e + \mathrm{Sp}\{&((L_f g_j^1 + [\bar{g}^1, g_j^1] z^1)^T, 0, \cdots, 0)^T, j = 1, \cdots, m - n_1; \\ &((\bar{g}^1 q^2)^T, 0, \cdots, 0)^T; (0, (p^2 q^3, q^2)^T, 0, \cdots, 0)^T; \cdots; (0, \cdots, \\ &(p^{s-1} q^s, q^{s-1})^T, 0, 0)^T; (0, \cdots, I_{s-1}, 0)^T; (0, \cdots, I_s)^T\}, \end{aligned} \tag{12}$$

where I_{s-1} and I_s are identity matrices with dimensions $n_{s-2} - n_{s-1}$ and $n_{s-1} - n_s$ respectively.

For the nonsingularity of Δ_1^e (around 0), we should have

$$\mathrm{Sp}\{[\bar{g}^1, g^1]\} \subset \mathrm{Sp}\{L_f g^1, g^1, g^{-1} q^2\}.$$

Thus in (12) $L_f g_j^1 + [\bar{g}^1, g_j^1] z^1$ can be replaced by $L_f g_j^1$. Hence the nonsingularity and involutivity of Δ_1^e and those of D_1 are equivalent.

Now since $g^1 \in D_1$ and D_1 is involutive, $[\bar{g}^1, g^1] \in D_1$ if and only if $[G, H_1] \subset D_1$.

Continuing this procedure, one can prove i) and iii) inductively.

It can also be proved by induction that $(0, \cdots, 0, I_{n_{s-t-1} - n_{s-t}}, 0, \cdots, 0)^T \in \Delta_i^e$. It is also true that $(G^T, 0, \cdots, 0)^T \in \Delta_s^e$.

Hence Δ^e_{n+s-m} must span the whole tangent bundle of $U \subset M \times \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_s}$. That yields ii).

Sufficiency: When conditions i), ii) and iii) are satisfied, it is ready to prove that system (11) is static state feedback linearizable. □

Using Taylor series expansion of coefficients, one may show

Corollary 1. If $s=2$, and system (1) is analytic, Theorem 3 holds locally everywhere.

Finally, we give an example which shows that some systems need multi-fold extension for their linearization.

Example 2. Consider the following system

$$\Sigma: \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3,$$

where $x \in \mathbb{R}^6, f(x) = (0, 0, x_6^4 + x_1, 0, x, \sin x_3)^T$, $g_1 = (0, x_1, 0, x_2 + 1, 0, x_4^2 - x_2)^T$, $g_2 = (1, \cos x_5, 0, 0, 0, 0)^T, g_3 = (x_3^2 + 1, 0, 0, 0, 0, 0)^T$.

Since $[g_1, g_2]$ is not in $\mathrm{Sp}\{g_1, g_2, g_3, L_f g_1, L_f g_2, L_f g_3\}$, the system is neither static state feedback linearizable no one-fold linearizable. But if we choose $H_1 = \mathrm{Sp}\{g_3\}$, $H_2 = \mathrm{Sp}\{g_2, g_3\}$ and $H_3 = G$, the extended system follows as

$$\Sigma^e: \begin{cases} \dot{x} = f(x) + g_1(x)z_1 + g_2(x)z_2 + g_3(x)v_1, \\ \dot{z}_1 = z_3, \quad \dot{z}_2 = v_2, \quad \dot{z}_3 = v_3. \end{cases}$$

It is readily verified that Σ^e satisfies the conditions of Theorem 1, so it is static state feedback linearizable.

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动 态 补 偿 线 性 化

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摘 要

本文讨论仿射非线性系统带有动态补偿的线性化问题。我们对单重及多重动态补偿，分别给出动态反馈线性化的充要条件。这里给出的结果可以看作对原有的状态反馈线性化结果的一个推广。