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OBSERVABILITY OF SYSTEMS ON LIE GROUPS AND COSET SPACES*

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Abstract. The purpose of this paper is to study the observability of a class of systems for which the state space is a Lie group and the output space is a coset space. The study of this type of system was initiated by Brockett [*SIAM J. Control*, 10 (1972), pp. 265–284]. In this paper, Brockett's observability results are generalized and necessary and sufficient conditions for observability are obtained. Effective algorithms are established to verify such conditions. Finally, as an application, some disturbance decoupling problems are considered.

Key words. observability, global observability, Lie groups, analytic systems

AMS(MOS) subject classification. 93B

1. Introduction. In this paper, we study the observability properties of systems that are described by a state equation that evolves on a Lie group G and an output equation that takes values in a coset space of G. These equations are assumed to be of the form

$$\dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)u_i, \qquad x \in G, \tag{1.1}$$

$$y = Cx \tag{1.2}$$

where $A(x)$, $B_1(x), \cdots, B_m(x)$ are right-invariant vector fields on G, C is a closed subgroup of G, and the notation Cx is to be interpreted as the right coset of C in G that contains x.

This system model has been studied by Brockett [1] where G was assumed to be a group of matrices. Brockett has shown [1] that there are many important applications in engineering and in physics that have models of this form. Jurdjevic and Sussmann [2] have extended (1.1) to an abstract Lie group G and have obtained a set of basic controllability properties of (1.1). Our work is related to and extends the work of [1].

The observability properties are discussed by Brockett [1]. To describe Brockett's observability result, we need a preliminary definition.

DEFINITION 1.1. Two points x_1 and x_2 are distinguishable if there exists some control that gives rise to different outputs for the two starting points.

Let S be a subset of G. We denote by $\{S\}_G$ the subgroup generated by S, i.e., the smallest subgroup of G containing S. Let $\mathscr{K}$ be a set of right invariant vector fields of G. We denote by $\{\mathscr{K}\}_{\text{LA}}$ the Lie subalgebra generated by $\mathscr{K}$, and

$$\exp(\{\mathscr{K}\}_{\text{LA}}) = \{\exp X \mid X \in \{\mathscr{K}\}_{\text{LA}}\}.$$

The main observability result of [1] is Theorem 1.1.

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THEOREM 1.1 [1]. *Let $\mathcal{H}$ and $\mathcal{L}$ be Lie algebras in* gl$\{n, \mathbb{R}\}$, *and suppose that all the points reachable from the identity for matrix system*

$$\dot{x}=\left(A+\sum_{i=1}^{m} u_i B_i\right)x, \qquad y=(\{\exp \mathcal{H}\}_G)x$$

are $\{\exp \mathcal{L}\}_G$. Then the set of initial states $\mathcal{S}$, which are indistinguishable from the identity, contains $\{\exp \mathcal{K}\}_G$ if and only if $\{\mathrm{ad}_{\mathcal{L}} \mathcal{K}\}_{\mathrm{LA}} \subset \mathcal{H}$. Therefore a necessary condition for all states to be distinguishable from the identity is that $\mathcal{H}$ contains no subalgebra $\mathcal{K}$ such that $\{\mathrm{ad}_{\mathcal{L}} \mathcal{K}\}_{\mathrm{LA}} \subset \mathcal{H}$.

It is shown by example in Brockett [1] that the preceding theorem is not sufficient. An important point in this theorem is that the "unobservable" part is related to an $\mathcal{L}$-invariant subalgebra $\{\mathrm{ad}_{\mathcal{L}} \mathcal{K}\}_{\mathrm{LA}}$, which is contained in the Lie subalgebra of the output subgroup.

Motivated by this fact, we investigate the "unobservable" part in more detail. The results of Jurdjevic and Sussmann [2] enable us to describe the controllable set, which corresponds to $\{\exp \mathcal{L}\}_G$ of Theorem 1.1. Based on [1] and [2], we give necessary and sufficient conditions for the system (1.1), (1.2) to be observable.

The paper is organized in the following way. Section 2 contains two main results—local observability conditions and global observability conditions. In § 3, we develop algorithms that are useful for studying groups of matrices. In § 4, we give some examples. Finally, in § 5 the input-output decoupling problem is discussed as an application.

2. Observability results. To avoid unnecessary complexity, we assume throughout this paper that the controls are piecewise constant. In fact, this is not essential. For instance, if we replace the set of piecewise constant functions by the set of the piecewise continuous functions, all of the arguments remain valid.

Let $R(x)$ be the reachable set starting from x, i.e., $R(x)$ is the set of points y such that there exist a piecewise constant control u and a time $T \geqq 0$, such that the solution of (1.1) satisfies $x(0)=x$, $x(T)=y$. We denote by $R(x, t)$ the reachable set at time t, starting from x.

It is proved in Jurdjevic and Sussmann [2] that for the right-invariant system (1.1), the reachable set of x is related to the reachable set of the identity e by

$$R(x)=R(e)x. \tag{2.1}$$

Using this fact, we prove the following elementary result, which shows that distinguishing two arbitrary points is equivalent to distinguishing a point from the identity.

LEMMA 2.1. *Two points p and q are indistinguishable if and only if for each $r \in R(e)$*

$$\mathrm{Ad}\,(r)pq^{-1} \in C. \tag{2.2}$$

Proof. By the structure of the output (1.2) it is clear that p and q are indistinguishable if and only if for all t, $R(p, t)$ and $R(q, t)$ are in the same coset of C. From (2.1), it follows that

$$Crp = Crq \quad \text{for all } r \in R(e), \tag{2.3}$$

that is,

$$rpq^{-1}r^{-1} = \mathrm{Ad}\,(r)pq^{-1} \in C. \qquad \square$$

Now we may define an unobservable state as follows. (It is similar to the linear case: x_1 and x_2 are indistinguishable if and only if $x_1 - x_2$ belongs to an unobservable subspace.)

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DEFINITION 2.1. A point h is called unobservable if there exist p and q such that $pq^{-1}=h$ and p and q are indistinguishable.

Remark 1. Let h be unobservable. Then it follows from Lemma 2.1 that for any pair (p^1, q^1) if $p^1(q^1)^{-1}=h$, then p^1 and q^1 are indistinguishable.

Let

$$H=\{h\in G \mid h \text{ is unobservable}\}.$$

By definition of unobservable state and equation (2.2), it is clear that

$$H\subset C. \tag{2.4}$$

In fact, H has a subgroup structure that is shown in the following lemma.

LEMMA 2.2. *Assume C is closed. Then the unobservable set H is a closed Lie subgroup of G.*

Proof. By definition and Lemma 2.1,

$$H=\{h\in G \mid rhr^{-1}\in C \text{ for all } r\in R(e)\}. \tag{2.5}$$

Let $h_1, h_2\in H$. Then,

$$rh_1h_2^{-1}r^{-1}=rh_1r^{-1}rh_2^{-1}r^{-1}=(rh_1r^{-1})(rh_2r^{-1})^{-1}\in C.$$

Thus, H is a subgroup of G.

Since C is closed, if for a sequence $\{h_n\}\subset H$, $h_n\to h$, as $n\to\infty$, then

$$rh_nr^{-1}\to rhr^{-1}\in C.$$

Thus, $h\in H$, and hence H is closed. Now the result follows from the well-known fact (see for example, Hausner and Schwartz [4]) that a closed subgroup of a Lie group is a Lie subgroup. □

If C is closed, the output mapping has an analytic structure that is described by the following well-known theorem.

THEOREM 2.1 [3]. *Let G be a Lie group and C a closed subgroup of G. Then the quotient space $C\backslash G$ admits the structure of real analytic manifold in such a way that the action of G on $C\backslash G$ is real analytic, that is, the mapping $G\times C\backslash G\to C\backslash G$, which maps (p, Cq) into Cpq, is real analytic. In particular, the projection $G\to C\backslash G$ is real analytic.*

Let $\{R(e)\}_G$ be the subgroup of G generated by $R(e)$ and let $\overline{\{R(e)\}_G}$ denote the closure of $\{R(e)\}_G$. For convenience denote the vector fields $A(x), B_1(x), \cdots, B_m(x)$ by $A, B_1, \cdots, B_m$, respectively, where A and B_i are elements in $\mathcal{G}(G)$, the Lie algebra of G. Then we have the following lemma.

LEMMA 2.3. *Assume $h\in H$. Then*

$$\operatorname{Ad}(r)h\in H \quad \textit{for all } r\in\overline{\{G(e)\}_G}. \tag{2.6}$$

Proof. First, we claim that

$$\operatorname{Ad}(r)h\in H \quad \text{for all } r\in G(e). \tag{2.7}$$

Since $R(e)$ is a semigroup [2], for any $\tilde r\in R(e)$ we have $\tilde r r\in R(e)$. Thus,

$$(\tilde r r)h(\tilde r r)^{-1}=\tilde r(rhr^{-1})\tilde r^{-1}\in C \quad \text{for all } \tilde r\in R(e).$$

It follows that $rhr^{-1}\in H$.

From its defining properties, it is clear that

$$\{R(e)\}_G=\Big\{\exp(t_sX_s)\cdots\exp(t_1X_1) \,\Big|\, t_i\in\mathbb{R},\ s\in Z^+,$$
$$X_i\in\Big\{A+\sum_{j=1}^m u_jB_j \,\Big|\, u_j\in\mathbb{R}\Big\},\ i=1,\cdots,s\Big\}. \tag{2.8}$$

Set

$$E_s = \{(t_1, \cdots, t_s) \in \mathbb{R}^s \,|\, \mathrm{Ad}\,(\exp(t_sX_s)\cdots\exp(t_1X_1))h \in C\}.$$

Then, to prove (2.6) for $r \in \{R(e)\}_G$ it is enough to show that $E_s = \mathbb{R}^s$, $s = 1, 2, \cdots$. We proceed by induction. For $s = 1$, if $\mathrm{Ad}\,(\exp t_1X_1)h \not\subset C$, then there exists $\tilde{t}_1$ such that

$$\left.\frac{d}{dt_1}\right|_{\tilde{t}_1} \mathrm{Ad}\,(\exp t_1X_1)h \notin (R_p)_* \mathscr{G}(C)$$

where $p = \mathrm{Ad}\,(\exp \tilde{t}_1X_1)h$, $\mathscr{G}(C)$ is the Lie algebra of C and R_p is the right translation, i.e., $R_p: G \to G$ is defined as $x \to xp$. In other words, there exists a right-invariant one-form $w(x)$ generated by $w \in (\mathscr{G}(C))^{\perp}$ such that

$$\left\langle w_{(p)}, \left.\frac{d}{dt_1}\right|_{\tilde{t}_1} \mathrm{Ad}\,(\exp t_1X_1)h \right\rangle \neq 0. \tag{2.9}$$

By analyticity, (2.9) holds in an open dense subset of $\mathbb{R}$. But according to (2.7), for $\tilde{t}_1 \in \mathbb{R}_+ = \{t \in \mathbb{R}, t \geqq 0\}$ the left-hand side of (2.9) is zero; this leads to a contradiction. Now, assume that

$$\mathrm{Ad}\,(\exp(t_{s-1}X_{s-1})\cdots\exp(t_1X_1))h \in C, \qquad t_i \in \mathbb{R}$$

and

$$\{\mathrm{Ad}\,(\exp(t_sX_s)\cdots\exp(t_1X_1))h \,|\, (t_1, \cdots, t_s) \in \mathbb{R}^s\} \not\subset C.$$

Then there exists $\tilde{t} = (\tilde{t}_1, \cdots, \tilde{t}_s)$ such that

$$\left.\frac{d}{dt}\right|_{\tilde{t}_s} \mathrm{Ad}\,(\exp(tX_s)\exp(\tilde{t}_{s-1}X_{s-1})\cdots\exp(\tilde{t}_1X_1))h \in \mathscr{G}(C).$$

Similar to the case when $s = 1$, we have a contradiction.

Thus, we have shown that (2.6) holds for all $r \in \{R(e)\}_G$. By continuity, it holds for all $r \in \overline{\{R(e)\}_G}$. □

Remark 2.[1] It is clear by (2.8) that $\{R(e)\}_G$ is a path-connected group, hence a Lie subgroup [5]. Now since $\{R(e)\}_G$ is a connected Lie group, and $A, B_1, B_2, \cdots, B_m$ generate $\mathscr{G}(\{R(e)\}_G)$, then [2, Lemma 6.2]

$$\begin{aligned}\{R(e)\}_G = \{\exp(t_xX_s)\cdots\exp(t_iX_1) \,|\, t_i \in \mathbb{R}, s \in Z^+,\\ X_i \in \{A, B_1, \cdots, B_m\}, i = 1, \cdots, s\}.\end{aligned} \tag{2.10}$$ □

Next, we investigate the relations among the Lie algebras $\mathscr{G}(H)$, $\mathscr{G}(C)$, and $\mathscr{G}(G)$, which are the Lie algebras of H, C, and G, respectively.

Let $\{X_1(x), \cdots, X_s(x)\}$ be a set of right-invariant vector fields generated by $X_i \in \mathscr{G}(G)$, $i = 1, \cdots, s$, respectively. Let Δ denote the subspace of $\mathscr{G}(G)$ spanned by $\{x_1, \cdots, x_5\}$. A subspace Δ of $\mathscr{G}(G)$ is called $Y \in \mathscr{G}(G)$ invariant if

$$\{[Y, X] \,|\, X \in \Delta\} \subset \Delta.$$

Likewise, for right-invariant we form $w_1(x), \cdots, w_s(x)$ generated by $w_i \in \mathscr{G}^*(G)$, the cotangent space of G at the identity e, we have a right-invariant subspace

$$\Omega = \mathrm{span}\,\{w_1, \cdots, w_s\}.$$

[1] This remark was suggested by an anonymous reviewer.

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Ω is Y invariant if

$$\{L_Y w \mid w \in \Omega\} \subset \Omega.$$

The following two lemmas are generalizations of Theorem 1.1.

LEMMA 2.4. *$\mathcal{G}(H)$ is A and B_i, $i=1, \cdots, m$, invariant.*

Proof. Let $X = A$ or B_i, $t \in \mathbb{R}$, $p = \exp(tX)$. According to Lemma 2.3 and (2.10), $(\mathrm{Ad}\exp(tX))_* \mathcal{G}(H) \subset \mathcal{G}(H)$. Now let $Y \in \mathcal{G}(H)$. Then

$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}\exp(tX)_* Y \in \mathcal{G}(H). \qquad \square$$

LEMMA 2.5. *$\mathcal{G}(H)$ is the largest A and B_i, $i=1, \cdots, m$, invariant Lie subalgebra contained in $\mathcal{G}(C)$.*

Proof. We claim that

$$\mathcal{G}(H) = \bigcap_{\substack{X_1, \cdots, X_p \in \{A, B_1, \cdots, B_m\} \\ p \in Z^+}} \mathrm{ad}_{X_1}^{-1} \cdots \mathrm{ad}_{X_p}^{-1} \mathcal{G}(C). \tag{2.11}$$

First, we show that (2.11) implies $\mathcal{G}(H)$ is the largest A and B_i invariant Lie subalgebra contained in $\mathcal{G}(C)$. Assume $\mathcal{G}(\tilde{H}) \subset \mathcal{G}(C)$ is also A and B_i invariant. Then, for any $X_1, \cdots, X_p \in \{A, B_1, \cdots, B_m\}$,

$$\mathrm{ad}_{X_1} \cdots \mathrm{ad}_{X_p} \mathcal{G}(\tilde{H}) \subset \mathcal{G}(\tilde{H}) \subset \mathcal{G}(C).$$

Thus,

$$\mathcal{G}(\tilde{H}) \subset \mathrm{ad}_{X_1}^{-1} \cdots \mathrm{ad}_{X_p}^{-1} \mathcal{G}(C).$$

Since $X_1, \cdots, X_p$ are chosen arbitrarily, we have that

$$\mathcal{G}(\tilde{H}) \subset \mathcal{G}(H).$$

Next, we prove (2.11).

($\subseteq$) Lemma 2.4 shows that $\mathcal{G}(H)$ is A and B_i invariant. The inclusion follows by an argument similar to the above.

($\supseteq$) Let

$$Y \in \bigcap_{\substack{X_1, \cdots, X_p \in \{A, B_1, \cdots, B_m\} \\ P \in Z^+}} \mathrm{ad}_{X_1}^{-1} \cdots \mathrm{ad}_{X_p}^{-1} \mathcal{G}(C).$$

To show that $Y \in \mathcal{G}(H)$, it is enough to show that

$$\exp(\tau Y) \in H \quad \text{for all } \tau \in \mathbb{R}.$$

Using (2.10), it suffices to show that for any

$$\begin{aligned} &X_1, \cdots, X_p \in \{A, B_1, \cdots, B_m\}, \qquad (t_1, \cdots, t_p) \in \mathbb{R}^p, \\ &\mathrm{Ad}(\exp(t_p X_p) \cdots \exp(t_1 X_1)) \exp \tau Y \in C. \end{aligned} \tag{2.12}$$

Since $\mathrm{Ad}(\exp(t_p X_p) \cdots \exp(t_1 X_1))$ is a diffeomorphism, we have

$$\mathrm{Ad}(\exp(t_p X_p) \cdots \exp(t_1 X_1)) \exp \tau Y = \exp(\mathrm{Ad}(\exp(t_p X_p) \cdots \exp(t_1 X_1) \tau Y)).$$

Now to prove (2.12), it suffices to show that

$$\mathrm{Ad}(\exp(t_p X_p) \cdots \exp(t_1 X_1)) \tau Y \in \mathcal{G}(C). \tag{2.13}$$

Let us denote the right-hand side of (2.11) by $\mathcal{J}$. Now since

$$\begin{aligned} &\mathrm{Ad}(\exp(t_p X_p) \cdots \exp(t_1 X_1) \tau Y) \\ &\quad = \mathrm{Ad}(\exp t_p X_p)\, \mathrm{Ad}(\exp t_{p-1} X_{p-1}) \cdots \mathrm{Ad}(\exp(t_1 X_1)) \tau Y, \end{aligned}$$

it suffices to prove that

$$\text{Ad}\,(\exp\,(t_1X_1)\tau Y)\in\mathcal{J}.$$

But

$$\begin{aligned}\text{Ad}\,(\exp\,(t_1x_1)\tau y)&=\exp\,(\text{ad}\,(t_1x_1))\tau y\\ &=\sum_{n=0}^{\infty}\frac{(\text{ad}\,(t_1x_1))^n}{n!}(\tau y).\end{aligned}$$

Therefore, obviously, $\text{Ad}\,(\exp\,(t_1x_1)\tau y)\in\mathcal{J}$. □

We are now ready to discuss the observability properties of (1.1), (1.2).

DEFINITION 2.2. System (1.1), (1.2) is locally observable at x if there exists a neighborhood V_x of x such that

$$I_x\cap V_x=\{x\},$$

where I_x is the set of points that are indistinguishable from x. System (1.1), (1.2) is locally observable if it is locally observable everywhere. System (1.1), (1.2) is (globally) observable if

$$I_x=\{x\}.$$

In fact, Lemma 2.5 leads to the following local observability result, which is now obvious.

THEOREM 2.2. *System* (1.1), (1.2) *is locally observable if and only if the largest A and B_i, $i=1,\cdots,m$, invariant subalgebra contained in $\mathcal{G}(C)$ is zero. Moreover, if V_e is a neighborhood of e such that $I_e\cap V_e=\{e\}$, then $V_x=R_x(V_e)$ is a neighborhood of x such that $I_x\cap V_x=\{x\}$.*

Let S be the centralizer of $\{R(e)\}_G$, i.e.,

$$S=\{x\in G\,|\,rx=xr \text{ for all } r\in\{R(e)\}_G\}. \tag{2.14}$$

According to (2.10), we may express S in an easily verifiable form as

$$S=\{x\in G\,|\,x\exp\,(tX)=\exp\,(tX)x,\ X\in\{A,B_1,\cdots,B_m\}\}. \tag{2.15}$$

We will use S to establish a global result.

It is obvious that S is a closed subgroup of G, and hence is a Lie subgroup. Moreover, by the construction of $\{R(e)\}_G$ we see that to verify that $x\in S$ it is enough to verify that

$$\text{Ad}\,(x)\exp\,(tY)=\exp tY$$

for

$$Y\in\{A,B_1,\cdots,B_m\},\qquad t\in\mathbb{R}.$$

Now we state our global observability theorem.

THEOREM 2.3. *System* (1.1), (1.2) *is globally observable if and only if the following two conditions are satisfied*:

(a) $\mathcal{G}(H)=\{0\}$,

(b) $S\cap C=\{e\}$.

Proof. Necessity. The necessity of (a) has been proved in Theorem 2.2. The necessity of (b) is obvious, because if $e\neq h\in S\cap C$, then $h\in H$, i.e., h is indistinguishable from e.

Sufficiency. From (a) we see that H is a discrete subgroup of G. Now for each $h \in H$, we define a mapping $\phi: \{R(e)\}_G \to H$ as

$$\phi(r) = \mathrm{Ad}\,(r)h.$$

According to Lemma 2.3, ϕ maps $\{R(e)\}_G$ into H. Now, since $\{R(e)\}_G$ is connected and ϕ is continuous, $\{\mathrm{Ad}\,(r)h \mid r \in \{R(e)\}_G\} \subset H$ is connected, but since

$$h \in \{\mathrm{Ad}\,(r)h \mid r \in \{R(e)\}_G\}$$

it follows that

$$\{h\} = \{\mathrm{Ad}\,(r)h \mid r \in \{R(e)\}_G\},$$

i.e.,

$$\mathrm{Ad}\,(r)h = h \quad \text{for all } r \in \{R(e)\}_G.$$

In other words, $h \in S$. Now using condition (b), we see that $h = e$, i.e., $H = \{e\}$. □

3. Algorithm. In the previous section, we saw that the Lie subalgebra $\mathscr{G}(H)$ of the unobservable Lie group H plays an important role in investigating the observability of the system (1.1), (1.2). The following algorithm gives a method to compute it.

ALGORITHM 3.1.

$$\Omega_0 \triangleq \mathscr{G}(C)^{\perp},$$

$$\Omega_{k+1} \triangleq \Omega_k + L_A\Omega_k + \sum_{i=1}^{m} L_{B_i}\Omega_k, \qquad k \geqq 1.$$

Algorithm 3.1 produces an increasing sequence of right-invariant subspaces. To see that it provides $\mathscr{G}(H)$, we need the following theorem. The proof may be found in Isidori [6].

THEOREM 3.1. *In Algorithm* 3.1 *if* $\Omega_{k^*+1} = \Omega_{k^*}$ *then*

$$\mathscr{G}(H) = \Omega_{k^*}^{\perp}. \tag{3.1}$$

Note that the algorithm converges since the sequence of subspace $\{\Omega_K\}$ is increasing.

Since every Lie algebra over the field of real numbers $\mathbb{R}$ is isomorphic to some matrix algebra, we may consider further algorithmic details for the Lie algebras of groups of matrices.

First, let $\omega(x) \in V^*(G)$ be a right-invariant covector field (one-form) generated by $\omega \in \mathscr{G}^*(G)$, and let $A(x)$, $B(x) \in V(G)$ be the right-invariant vector fields generated by $A, B \in \mathscr{G}(G)$, respectively. Then,

$$\begin{aligned} \langle L_A\omega, B\rangle &= \langle L_{A(x)}\omega(x), B(x)\rangle \\ &= L_{A(x)}\langle \omega(x), B(x)\rangle - \langle \omega(x), [A(x), B(x)]\rangle. \end{aligned} \tag{3.2}$$

Since $\langle \omega(x), B(x)\rangle$ is constant, the first term of the right-hand side of (3.2) is zero. Thus, we have

$$\langle L_A\omega, B\rangle = -\langle \omega, [A, B]\rangle. \tag{3.3}$$

Now we consider a group of matrices. Assume the group considered is $\mathrm{GL}\,(n, \mathbb{R})$ (or a subgroup of it). Then, $A, B \in \mathrm{gl}\,(n, \mathbb{R})$ may be considered as matrices $A = (a_{ij})$

and $B=(b_{ij})$, respectively. Let $\omega \in \mathrm{gl}^*(n,\mathbb{R})$. We may assume ω is also expressed as a matrix $\omega=(\omega_{ij})$ and define

$$\langle \omega, A\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_{ij}a_{ij}. \tag{3.4}$$

Now,

$$\begin{aligned}\langle L_A\omega, B\rangle &= -\langle \omega, [A, B]\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} (a_{kj}\omega_{ij}b_{ik} - a_{ik}\omega_{ij}b_{kj}) \\ &= \sum_{p=1}^{n}\sum_{q=1}^{n}\left(\sum_{k=1}^{n} \omega_{pk}a_{qk} - a_{kp}\omega_{kq}\right) b_{pq}.\end{aligned}$$

Thus

$$L_A\omega = [\omega, A^T] = \omega A^T - A^T\omega, \tag{3.5}$$

where T stands for transpose. To apply Algorithm 3.1, formula (3.5) is helpful.

Remark 3. As shown in Brockett [1], a right-invariant vector field on $\mathcal{G}(n,\mathbb{R})$ may be written as

$$A(x) = Ax,$$

where $A = A(e)$ and $A(x) = (R_x)_* A(e) = Ax$. Similarly, a right-invariant covector field may be written as

$$\omega(x) = \omega(x^T)^{-1}$$

where $\omega = \omega(e)$ and $\omega(x) = (R_{x^{-1}})^*\omega(e) = \omega(x^T)^{-1}$.

To see this, we only have to show that

$$\langle \omega(x), A(x)\rangle = \langle \omega, A\rangle.$$

In fact, if we denote $y = x^{-1}$, $x=(x_{ij})$, and $y=(y_{ij})$, then

$$\begin{aligned}\langle \omega(x), A(x)\rangle &= \sum_i\sum_j\left(\sum_p \omega_{ip}y_{jp}\right)\left(\sum_q a_{iq}x_{qj}\right) \\ &= \sum_i\sum_p\sum_q \omega_{ip}a_{iq}\left(\sum_j x_{qj}y_{jp}\right) \\ &= \sum_i\sum_p\sum_q \omega_{ip}a_{iq}\delta_{qp} \\ &= \sum_i\sum_p \omega_{ip}a_{ip} = \langle \omega, A\rangle.\end{aligned}$$

In fact, if we rewrite $A(x)$ in the "usual fashion" as a vector

$$A(e) = (a_{11}, \cdots, a_{1n}, a_{21}, \cdots, a_{2n}, \cdots, a_{n1}, \cdots, a_{nn})^T,$$

then

$$A(x) = \underbrace{(x^T \dotplus x^T \dotplus \cdots \dotplus x^T)}A(e) \quad n \text{ terms}.$$

Similarly,

$$\omega(x) = \omega(e)((x^T)^{-1} \dotplus (x^T)^{-1} \dotplus \cdots \dotplus (x^T)^{-1})$$

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where "$\dot{+}$" denotes the direct sum of matrices, and $(x^T \dot{+} x^T \dot{+} \cdots \dot{+} x^T)$ and $((x^T)^{-1} \dot{+} (x^T)^{-1} \dot{+} \cdots \dot{+} (x^T)^{-1})$ are the Jacobian matrices of R_x and $R_{x^{-1}}$, respectively.

4. Examples. In this section, we present some examples to demonstrate our results and algorithms.

Example 4.1. Consider a system

$$\dot{x} = uBx, \tag{4.1}$$

$$y = Cx \tag{4.2}$$

where $x \in \mathrm{GL}(3, \mathbb{R})$, $C = \mathrm{SO}(3)$, and

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now $\mathscr{G}(C)$ is the following set of skew-symmetric matrices:

$$\mathscr{G}(C) = \mathrm{span}\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$$

According to Algorithm 3.1, we set

$$\Omega_0 = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

$$\triangleq \mathrm{span}\,\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}.$$

Using formula (3.5), we see that

$$L_B\omega_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_B\omega_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_B\omega_6 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\Omega_1 = \mathscr{G}^*(G)$ and $k_* = 1$. Therefore,

$$\mathscr{G}(H) = \Omega_1^\perp = \{0\}.$$

Next, let us consider

$$S \cap C = \{x \in C \mid x \exp tB = \exp(tB)x, \text{ for all } t \in \mathbb{R}\}.$$

Let $x = (x_{ij}) \in C$. Since

$$\exp tB = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we set

$$x \exp tB = \begin{bmatrix} x_{11} + x_{12} & x_{12} & x_{13} \\ x_{21} + tx_{22} & x_{22} & x_{23} \\ x_{31} + tx_{32} & x_{32} & x_{33} \end{bmatrix} = \exp(tB)x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ tx_{11} + x_{21} & tx_{12} + x_{22} & tx_{13} + x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}.$$

It follows that

$$x_{12} = 0, \qquad x_{11} = x_{22}, \quad x_{13} = 0, \qquad x_{32} = 0. \tag{4.3}$$

Since $x \in C = \mathrm{SO}(3)$, the only solutions of (4.3) are

$$x_1 = e, \qquad x_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.4}$$

According to Theorem 2.3, system (4.1), (4.2) is not globally observable. □

Example 4.2. Consider the following system:

$$\dot{x} = Ax + uBx, \tag{4.5}$$

$$y = Cx \tag{4.6}$$

where B and C are as in Example 4.1, and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As in the previous example, we see that $\mathcal{G}(H) = \{0\}$. So the system is locally observable. Now

$$e^{At} = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

According to (2.15), we have only to check the commutativity of both x_1 and x_2 of (4.4) with $\exp(tA)$. For x_2 the answer is "no." Therefore, $x_1 = e = I_3$ is the only element in $S \cap C$. It follows that system (4.5), (4.6) is globally observable.

Remark 4. In Example 4.2, if we consider e^{At}, e^{Bt}, $e^{-Bt}e^{-At}e^{Bt}e^{At}$ and their products, it is easy to see that

$$\{R_{(e)}\}_G = \left\{ \begin{pmatrix} 1 & 0 & a \\ b & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}.$$

It follows that

$$S = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \middle| \; x, y \in \mathbb{R}, x \neq 0 \right\}$$

and therefore,

$$S \cap C = I_3.$$

But in general, it is difficult to calculate $\{R(e)\}_G$ and S. In fact, Example 4.2 shows that to use Theorem 2.3 it is not necessary to construct $\{R(e)\}_G$ and S directly. We may check the global observability by the following rule, which may be considered as a corollary of Theorem 2.3.

COROLLARY 4.1. *System* (1.1), (1.2) *is globally observable, if and only if,*

(a) $\mathcal{G}(H) = \{0\}$,

(b) $\{x \in C \mid \exp(tX)x = x\exp(tX),\ X \in \{A, B_1, \cdots, B_m\},\ t \in \mathbb{R}\} = \{e\}$.

Exhibit 3-33

5. Decoupling problems. As an application, we consider a decoupling problem. To keep the right-invariance of $A(x)$ and $B_i(x)$, we consider only a constant feedback

$$u = \alpha + \beta u \tag{5.1}$$

where $\alpha \in \mathbb{R}^m$ and $\beta \in GL(m, \mathbb{R})$.

Now assume

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) + \omega W(x), \tag{5.2}$$

$$y = Cx \tag{5.3}$$

where ω is a disturbance.

LEMMA 5.1. *The disturbance ω does not affect the output y if and only if*

$$W \in \mathcal{G}(H). \tag{5.4}$$

Proof. In fact, we may choose a local coordinate chart (ϕ, U) around e, say $x = (x^1, x^2)$, such that

$$C \cap U = \{p \in U \mid x_p^2 = 0\}.$$

Thus,

$$y = x^2(q), \qquad q \in U.$$

Now, it is easy to see that on U, $\mathcal{G}(H)$ is the largest A and B_i invariant distribution contained in the $\ker(y_*)$. Note that constant feedback does not affect $\mathcal{G}(H)$. Thus, the canonical decoupling result shows that (Isidori et al. [7]) (5.4) is a necessary and sufficient condition that ω does not affect y on V. By the analyticity, it is also true globally. $\square$

Next, we consider the input-output decoupling problem. Assume $C_1, \cdots, C_k$ are Lie subgroups of G. Let $C = C_1 \cap \cdots \cap C_k$. Then it is easy to see that

$$y = Cx \tag{5.5}$$

is equivalent to

$$\begin{aligned} y_1 &= C_1 x \\ &\vdots \\ y_k &= C_k x \end{aligned} \tag{5.6}$$

in the sense that p and q are indistinguishable in (5.5) if and only if they are indistinguishable in (5.6).

Let $\mathcal{G}(H^i)$ be the largest A and B_i invariant Lie subalgebra contained in $\mathcal{G}(C_1 \cap \cdots C_{i-1} \cap C_{i+1} \cdots \cap C_k)$. Consider the system

$$\begin{aligned} \dot{x} &= A(x) + \sum_{i=1}^{m} u_i B_i(x), \\ y_j &= C_j x, \qquad j = 1, \cdots, k. \end{aligned} \tag{5.7}$$

We say that the input-output decoupling problem is solvable if there exists $\beta = (\beta_{ij}) \in GL(m, \mathbb{R})$, such that for

$$u = v\beta$$

there exists a partition of v, namely $v = (v^1, \cdots, v^k)$, such that v^i affects only the corresponding y_i, $i = 1, \cdots, k$.

THEOREM 5.1. *For the system described by* (5.7) *the input-output decoupling problem is solvable if and only if*

$$B = B \cap \mathcal{G}(H^1) + \cdots + B \cap \mathcal{G}(H^k)$$

where $B = \text{span}\{B_1, \cdots, B_m\}$. *Moreover, if the system* (5.7) *satisfies the controllability rank condition* (*i.e.,* $\mathcal{G}(\{R_e\}_G) = \mathcal{G}(G)$), *then* v^i *controls* y^i *completely.*

Proof. The proof is immediate from Lemma 5.1 and the well-known decoupling results of Nijmeijer and Schumacher [8] and Cheng [9].

6. Conclusion. We have considered a system defined on a Lie group with outputs in a coset space as described in Brockett [1]. The main results of this paper are two observability theorems, Theorems 2.2 and 2.3, that give necessary and sufficient conditions for local and global observability, respectively. Algorithm 3.1 calculates the A and B_i invariant Lie subalgebra contained in a given Lie subalgebra, which makes the condition in the above two theorems computably verifiable. Some examples are included. Finally, we have briefly discussed the input-output decoupling problem of a system on a Lie group with output in a coset space.

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