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Systems & Control Letters 51 (2004) 79–88
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Stabilization of planar switched systems☆

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Received 7 July 2001; received in revised form 25 February 2003; accepted 1 July 2003

Abstract

This paper considers the problem of stabilization of single-input planar switched systems. We assume the switching law is observable, a formula is presented, which provides a necessary and sufficient condition for the system to be quadratically stabilizable. A set of linear inequalities are given to describe the set of all quadratic Lyapunov functions. The solvability and the control design technique are clearly described in a straightforward computation algorithm.

Keywords: Switching system; Quadratic stabilization; Common quadratic Lyapunov function; Brunovskey canonical form

1. Introduction

In the last decade, the switched systems have been investigated by a number of researchers [2,4,8,15]. This problem is not only theoretically interesting but also practically important. Such control systems appear in robot manipulators [5], traffic management [16], etc. Many human behaviors, the animals to overcome obstacles, or the mutual actions between preys and predators can be described as a switching pattern between its various models. Power systems provide another useful example [14,17]. While the switching follows certain stochastic process, adaptive stabilization of switched systems is investigated [6,18].

One way to solve the problem is to find a quadratic Lyapunov function. In some literatures it is called the quadratic stabilization [13]. For a switched system with several linear models, it means to find a common quadratic Lyapunov function for a set of matrices. Finding a common quadratic Lyapunov function is still an open problem, even though several progresses have been done [1,7,9–12]. Recently, we propose a new approach technique to find a common quadratic Lyapunov function [3].

Even though quadratic Lyapunov function is not necessary for exponential stability [4], it is still a powerful tool in considering stability and stabilization of switched systems.

A switched control system can be described as [15]

$$\dot{x} = A_{\sigma(x,t)}x + B_{\sigma(x,t)}u_{\sigma(x,t)}, \quad x \in R^n, \ u \in R^m, \quad (1.1)$$

where $\sigma(x,t): R^n \times [0,\infty) \to \Lambda$ is an arbitrary mapping unless elsewhere specified. In this paper, we assume $\Lambda = \{1,2,\ldots,N\}$. But some of the results can be extended to infinity case.

In this paper we consider the quadratic stabilization of the switched system (1.1). That is, we are looking for a state feedback control and a quadratic Lyapunov function, $x^{\mathrm{T}}Mx$, with $M > 0$, such that the feedback

☆ Supported partly by National 973 Project G1998020308 of China.
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0167-6911/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0167-6911(03)00208-1

switching models share $x^{\mathrm{T}}Mx$ as a common quadratic Lyapunov function.

The paper is organized as follows: Section 2 considers the planar switched systems. A necessary and sufficient condition is presented. Section 3 expresses the necessary and sufficient condition as the solution of a set of linear inequalities. The construction of controls is also precisely described. Section 4 is a brief summary.

2. Quadratic stabilization of planar switched systems

First, we give a definition to formulate the problem concerned in this section clearly.

Definition 2.1. System (1.1) is said to be quadratically stabilizable with observable $\sigma(x,t)$, if there exists a common quadratic Lyapunov function, $x^{\mathrm{T}}Mx$, with $M>0$, a set of state feedback controls $u_\lambda = K_\lambda x$ such that $A_\lambda + B_\lambda K_\lambda$, $\lambda = 1,\dots,N$ share a common quadratic Lyapunov function, $x^{\mathrm{T}}Mx$.

We need some preparations. First of all, a necessary condition for system (1.1) to be stabilizable is that every model is stabilizable. So a reasonable assumption is

A1. All the models are stabilizable.

The following lemma shows that without loss of generality we can replace A1 by

A2. All the models are controllable.

Lemma 2.2. *If a single-input planar linear system (A,b) is stabilizable but not controllable, then for any positive-definite matrix, M, there exists a suitable state feedback such that $x^{\mathrm{T}}Mx$ is a quadratic Lyapunov function of the closed-loop system.*

The proof of Lemma 2.2 is in the Appendix. All the proofs of lemmas or theorems, etc., if not right after them, are collected in the Appendix.

Now if a switching model, (A_λ, b_λ), is stabilizable but not controllable, we can ignore it. After finding a common quadratic Lyapunov function for all controllable models, the stabilizable but not controllable models can use it as their quadratic Lyapunov function. A2 is assumed hereafter.

The following lemma claims that the transformation of a single-input system to its canonical form is unique. It is simple but essential for the later approach and computer coding.

Lemma 2.3. *Let (A,b) be a single-input linear system. There exists a unique state transformation matrix, T, which converts the system into the Brunovsky canonical form as*

$$T^{-1}AT = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix},$$

$$T^{-1}b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \qquad (2.1)$$

Moreover, the parameters a_i are

$$\begin{pmatrix} a_1 \\ a_2 \\ a_n \end{pmatrix} = (b \quad Ab \quad \cdots \quad A^{n-1}b)^{-1}A^n b;$$

and the unique state transformation matrix, $T = (T_1, T_2, \dots, T_n)$, can be inductively determined column by column as

$$T_n = b,$$

$$T_{k-1} = AT_k - a_k b, \quad k = n, n-1, \dots, 2.$$

Lemma 2.4. *Given a positive definite matrix*

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0.$$

There exists a feedback $u = Kx = (k_1, k_2)x$, such that the closed-loop system of the canonical single-input planar system

$$\tilde{A} = A + bK = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 \quad k_2) := \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \qquad (2.2)$$

has M as its quadratic Lyapunov function, iff $m_2 > 0$.

Based on the previous lemma, we give the following definition:

Definition 2.5. A matrix

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0$$

is said to be canonical-friend (to the canonical controllable form), if $m_2 > 0$.

Lemma 2.6. *Given a nonsingular matrix*

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.$$

There exists a canonical-friend matrix $M > 0$, such that $T^{\mathrm{T}}MT$ is also canonical-friend, iff the following quadratic equation:

$$t_{21}t_{22}x^2 + (t_{12}t_{21} + t_{11}t_{22})x + t_{11}t_{12} > 0 \tag{2.3}$$

has a positive solution $x > 0$.

Note that for a given state transformation matrix, T, the solution of (2.3) consists of one or two open intervals. It might be empty. We denote the set of solutions as I_T.

Consider the single-input planar switched system (1.1) and assume $\Lambda = (1, \dots, N)$ be a finite set. For each switching model we denote the state transformation matrix, which converts it to the canonical controllable form, by C_i, $i = 1, \dots, N$. That is, let

$$z_i = C_i x, \quad i = 1, \dots, N.$$

Then the ith model $\dot{x} = A_i x + b_i u$, when expressed into z_i coordinates, is a Brunovsky canonical form (2.1). According to Lemma 2.3, C_i, $i = 1, \dots, N$ are uniquely determined.

Set $T_i = C_1 C_{i+1}^{-1}$, $i = 1, \dots, N-1$. T_i is the state transformation matrix from z_{i+1} to z_1. That is,

$$z_1 = C_1 x = C_1 C_{i+1}^{-1} z_{i+1} = T_i z_{i+1}.$$

Next, we classify $T_i = (t^i_{j,k})$ into three categories:

$$S_p = \{i \in \Lambda \mid t^i_{21}t^i_{22} > 0\}, \quad S_n = \{i \in \Lambda \mid t^i_{21}t^i_{22} < 0\},$$

$$S_z = \{i \in \Lambda \mid t^i_{21}t^i_{22} = 0\}.$$

Then $\Lambda = S_p \cup S_n \cup S_z$. Next, for $i \in S_z$, we define

$$c_i = t^i_{12}t^i_{21} + t^i_{11}t^i_{22},$$
$$d_i = t^i_{11}t^i_{12}, \quad i \in S_z. \tag{2.4}$$

For the other cases, we define

$$a_i = \frac{t^i_{11}}{t^i_{21}},$$
$$b_i = \frac{t^i_{12}}{t^i_{22}}, \quad i \in S_p \cup S_n. \tag{2.5}$$

For each $i \in S_z$, we define a linear form as

$$L_i = c_i x + d_i, \quad i \in S_z; \tag{2.6}$$

and for each $i \in S_p$ or $i \in S_n$, we define a quadratic form as

$$Q_i = x^2 + (a_i + b_i)x + a_i b_i, \quad i \in S_p \cup S_n. \tag{2.7}$$

According to the Lemma 2.6, we may solve x from

$$Q_i(x) < 0, \quad i \in S_n,$$
$$Q_i(x) > 0, \quad i \in S_p,$$
$$L_i(x) > 0, \quad i \in S_z.$$

Note that Q_i in (2.7) can be factorized, the roots are $\{-a_i, -b_i\}$. So we can simply define the solution set as follows:

If $i \in S_p$, we define an open set consists of two semi-lines as

$$I_i = (-\infty, \min(-a_i, -b_i)) \cup (\max(-a_i, -b_i), \infty).$$

If $i \in S_n$, we define an open set as

$$I_i = (\min(-a_i, -b_i), \max(-a_i, -b_i)).$$

If $i \in S_z$, since T_i is non-singular, it follows that $c_i \neq 0$. Hence we can define an open set as

$$I_i = \begin{cases} (-d_i/c_i, \infty), & c_i > 0, \\ (-\infty, -d_i/c_i), & c_i < 0. \end{cases}$$

As an immediate consequence of Lemma 2.6, we have the following simple conclusion:

Theorem 2.7. (1) *A sufficient condition for the switched system* (1.1) *to be quadratically stabilizable is*

$$I = \bigcap_{k=1}^{N-1} I_k \neq \varnothing. \tag{2.8}$$

(2) *If all* $i \in S_p$, $i = 1, \dots, N-1$, *then the switched system* (1.1) *is always stabilizable.*

(3) *If all* $i \in S_n$, $i = 1, \dots, N-1$, (2.8) *is also necessary.*

Theorem 2.7 provides a very convenient condition for the existence of a common quadratic Lyapunov function. We may expect that it is also necessary. When $N = 2$ Lemma 2.6 says that it is. But in general it is not. We have the following counter-example:

Example 2.8. Consider a system

$$\dot{x} = A_{\sigma(x,t)}x + b_{\sigma(x,t)}u_{\sigma(x,t)}, \tag{2.9}$$

where $\sigma(x,t) : R^2 \times [0,\infty) \to \{1,2,3,4\}$, with four switching models as

$$A_1 = \begin{pmatrix} -4 & -2 \\ 9 & 5 \end{pmatrix}, \quad b_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 2 & -\frac{1}{3} \\ -3 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -7 & -6 \\ 9.5 & 8 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 6 \\ -7 \end{pmatrix};$$

$$A_4 = \begin{pmatrix} -12 & -11 \\ 13 & 12 \end{pmatrix}, \quad b_4 = \begin{pmatrix} -7 \\ 8 \end{pmatrix}.$$

We investigate whether it is quadratically stabilizable. Suppose z_1, z_2, z_3, and z_4 are the canonical coordinates of models 1,2,3,4, respectively. That is, under z_i model i has Brunovsky canonical form. Let

$$z_i = C_i x, \quad i = 1,2,3,4.$$

Using Lemma 2.3, we can get C_i as (note that C_i is T^{-1} of Lemma 2.3)

$$C_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad C_2 = \begin{pmatrix} -2 & -1\frac{2}{3} \\ 1 & \frac{2}{3} \end{pmatrix};$$

$$C_3 = \begin{pmatrix} 1\frac{1}{6} & 1 \\ 1\frac{1}{3} & 1 \end{pmatrix}; \quad C_4 = \begin{pmatrix} 2\frac{2}{3} & 2\frac{1}{3} \\ -1\frac{2}{3} & -1\frac{1}{3} \end{pmatrix}.$$

Then the state transformation matrices between z_1 and z_i, $i = 2,3,4$, determined by $T_i = C_1 C_{i+1}^{-1}$, $=1,2,3$, are obtained as

$$T_1 = \begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix}; \quad T_2 = \begin{pmatrix} -4 & 5 \\ 2 & -1 \end{pmatrix};$$

$$T_3 = \begin{pmatrix} -3 & -6 \\ 1 & 1 \end{pmatrix}.$$

Now for T_1, since $t_{21}t_{22} = 1$, $1 \in S_p$, and $a_1 = t_{11}/t_{21} = -1$, $b_1 = t_{21}/t_{22} = -4$, so $I_1 = (-\infty, 1) \cup (4, \infty)$. For T_2, since $t_{21}t_{22} = -2$, $2 \in S_n$, and $a_2 = t_{11}/t_{21} = -2$, $b_2 = t_{21}/t_{22} = -5$, so $I_2 = (2,5)$. For T_3, since $t_{21}t_{22} = 1$, $3 \in S_p$, and $a_3 = t_{11}/t_{21} = -3$, $b_3 = t_{21}/t_{22} = -6$, so $I_3 = (-\infty, 3) \cup (6, \infty)$. We conclude that $I = I_1 \cap I_2 \cap I_3 = \varnothing$. But we may choose the controls as $u = K_i x$ with

$$K_1 = (-11.2143 \quad -8.2143),$$

$$K_2 = (-36 \quad -23.3333),$$

$$K_3 = (-18 \quad -13.5),$$

$$K_4 = (58 \quad 46)$$

and let the quadratic Lyapunov function $x^{\mathrm{T}}Mx$ be constructed by

$$M = \begin{pmatrix} 31.75 & 26.25 \\ 26.25 & 21.75 \end{pmatrix} > 0.$$

Then it is ready to verify that

$$M(A_1 + b_1K_1) + (A_1 + b_1K_1)^{\mathrm{T}}M = \begin{pmatrix} -246.8929 & -205.3929 \\ -205.3929 & -170.8929 \end{pmatrix} < 0;$$

$$M(A_2 + b_2K_2) + (A_2 + b_2K_2)^{\mathrm{T}}M = \begin{pmatrix} -120.5000 & -79.5000 \\ -79.5000 & -52.5000 \end{pmatrix} < 0;$$

$$M(A_3 + b_3K_3) + (A_3 + b_3K_3)^{\mathrm{T}}M = \begin{pmatrix} -188.7500 & -143.2500 \\ -143.2500 & -108.7500 \end{pmatrix} < 0;$$

$$M(A_4 + b_4K_4) + (A_4 + b_4K_4)^{\mathrm{T}}M = 10^3 \begin{pmatrix} -1.5005 & -1.1955 \\ -1.1955 & -0.9525 \end{pmatrix} < 0.$$

Hence system (2.13) does not satisfy the condition of Theorem 2.7. But it is quadratically stabilizable with arbitrary but known switching rule.

We will go back to this simple condition of Theorem 2.7 again, and also will re-visit Example 2.8 to see how to design the controls. Next, we investigate the necessary and sufficient condition for system (1.1) to be quadratically stabilizable. Carefully go through the proof of Lemma 2.6 and the followed argument, we can find the following result:

Theorem 2.9. *Let* $\Lambda=(1,\dots,N)$. *Then system* (1.1) *is quadratically stabilizable, iff there exists a positive* x *such that*

$$\max_{i\in S_n} Q_i(x)<0,$$

$$\min_{i\in S_p} Q_i(x)>\max_{i\in S_n} Q_i(x),$$

$$L_i(x)>0,\quad I\in S_z. \tag{2.10}$$

Proof. Assume there is a quadratic Lyapunov function in z_1 coordinates, which is expressed as

$$M_1=\begin{pmatrix}1 & m_2\\ m_2 & m_3\end{pmatrix}.$$

According to Lemma 2.4, $m_2>0$. It is easy to see from the proof of Lemma 2.6 that M_1 is a common quadratic Lyapunov function for the other models, iff $H_i(1,m_2,m_3)>0$, $i=1,2,\dots,N-1$, which leads to

$$m_3+(a_i+b_i)m_2+a_ib_i>0,\quad i\in S_p,$$

$$m_3+(a_i+b_i)m_2+a_ib_i<0,\quad i\in S_n,$$

$$c_im_2+d_i>0,\quad i\in S_z. \tag{2.11}$$

Since $m_3>m_2^2$ we may rewrite the first two equations as

$$e+m_2^2+(a_i+b_i)m_2+a_ib_i>0,\quad i\in S_p,$$

$$e+m_2^2+(a_i+b_i)m_2+a_ib_i<0,\quad i\in S_n, \tag{2.12}$$

where $e>0$. The necessity of (2.10) is obvious. As for the sufficiency, assume there is a solution x such that $\min_{i\in S_p} Q_i(x)>0$, then we can choose $m_2=x$ and $m_3=x^2+\varepsilon$. As $\varepsilon>0$ small enough, (2.11) is satisfied. Otherwise, denote $w=\min_{i\in S_p} Q_i(x)\leqslant 0$. We choose $m_2=x$ and

$$m_3=x^2+\tfrac{1}{2}\,(\min_{i\in S_p} Q_i(x)-\max_{i\in S_n} Q_i(x))-w. \tag{2.13}$$

It is easy to see that for such a choice (2.11) holds. The matrix, therefore, constructed as

$$M=\begin{pmatrix}1 & m_2\\ m_2 & m_3\end{pmatrix}$$

meets the requirement. □

Next, we go back to Theorem 2.7. In fact, it was proved in Theorem 2.7 that if all $i\in S_p$, or all $i\in S_n$, $i=1,\dots,N-1$, then (2.8) is also necessary. Since I is simply computable, it is a very convenient condition. Example 2.8 shows that it isn't necessary in general. But when $N\leqslant 3$ we can prove the following proposition:

Corollary 2.10. *If* $N\leqslant 3$, (2.8) *is also necessity.*

3. Design of the controls

This section provides the numerical details for solving the problem of stabilizing single-input planar systems. First we claim that (2.10) is equivalent to a set of linear inequalities.

Consider the first inequality in (2.10). It is ready to see that this inequality is equivalent to

$$\min\{a_j,b_j\}<x<\max\{a_j,b_j\},\quad j\in S_n.$$

For the second inequality, it is equivalent to that each Q_i, $i\in S_p$ is greater than each Q_j, $j\in S_n$. Hence, it is equivalent to

$$(a_i+b_i-a_j-b_j)x+a_ib_i-a_jb_j>0,$$

$$i\in S_p,\ \ j\in S_n.$$

Since we are looking for a positive solution, then we have

Corollary 3.1. *System* (1.1) *is quadratically stabilizable, iff the following set of linear inequalities have a solution:*

$$\min\{a_i,b_j\}<x<\max\{a_j,b_j\},\quad j\in S_n,$$

$$(a_i+b_i-a_j-b_j)x+a_ib_i-a_jb_j>0,$$

$i \in S_p,\ \ j \in S_n,$

$c_i x + d_i > 0, \quad i \in S_z,$

$$x > 0. \tag{3.1}$$

Recall the proof of Lemma 2.4, a stabilizing control is easily constructible. We state it formally as

Proposition 3.2. *Let (A,b) be a canonical planar system, i.e.,*

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \tag{3.2}$$

and

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0$$

be canonical-friend (i.e., $m_2 > 0$). To make $x^T M x$ a quadratic Lyapunov function of the closed-loop system, a feedback control can be chosen as

$$u = kx = \left((\alpha - a_{21}) \quad \left(\frac{\alpha m_3 - m_1}{m_2} - a_{22} \right) \right) x, \tag{3.3}$$

where $\alpha < 0$ can be any negative real number.

Remark. Of course (3.3) is only a possible choice. But we always choose (3.3) as our control in later use. Because from the proof of Lemma 2.4 we know that the (α, β) chosen in this way makes the determination of Q to reach the maximum. In some sense, it makes Q most negative.

Next, we give a step-by-step algorithm for solving the quadratic stabilization problem of the single-input planar systems:

Algorithm 3.3.

Step 1. Using Lemma 2.3 to find the state transformation matrices C_i, $i = 1, \dots, N$, $z_i = C_i x$, such that in coordinate frame z_i the ith switching model is in Brunovskey canonical form.

Step 2. Define another set of state transformation matrices: $T_i = C_1 C_{i+1}^{-1}$, $i = 1, \dots, N-1$, such that

$$z_1 = T_i z_{i+1}, \quad i = 1, \dots, N-1.$$

Step 3. Calculate a_i, b_i by (2.5) if $i \in S_p \cup S_n$, c_i, d_i by (2.4) if $i \in S_z$.

Step 4. Construct the system of inequalities (3.1). Find any one solution, $x = x_0$. (If there is no solution, the quadratic stabilization problem has no solution.)

Step 5. Using inequalities (2.12), setting $m_2 = x_0$, to find a positive solution $e > 0$. Set $m_3 = m_2^2 + e$. Construct a positive definite matrix

$$M_1 = \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0,$$

which is a common quadratic Lyapunov function for all switching models with certain feedback controls. (Note that if Step 4 has a solution then there exist solutions for the inequalities of (2.12).)

Step 6. Convert M_1 to each canonical coordinate frames as

$$M_{i+1} = T_i^T M_1 T_i, \quad i = 1, \dots, N-1. \tag{3.4}$$

Convert model (A_i, b_i) into its canonical coordinate chart as

$$\tilde{A}_i = C_i^{-1} A C_i, \quad \tilde{b}_i = (0,1)^T, \quad i = 1, \dots, N. \tag{3.5}$$

Using formula (3.3) to construct the feedback controls: k_i, $i = 1, \dots, N$.

Step 7. Back to the original coordinate frame x. The controls should be

$$K_i = (k_i) C_i, \quad i = 1, \dots, N. \tag{3.6}$$

The common quadratic Lyapunov function for all closed-loop switching models is

$$M = C_1^T M_1 C_1. \tag{3.7}$$

Example 3.4. Re-visit Example 2.8, we will follow Algorithm 3.3 to construct the controls.

Steps 1–3 have already been done in Example 2.8. So we can start from Step 4 to construct the inequalities.

Since $2 \in S_n$, $a_2 = -2$ and $b_2 = -5$, the corresponding inequality is

$$2 = \min\{-a_2, -b_2\} < x < \max\{-a_2, -b_2\} = 5.$$

Now we consider $i = 1 \in S_p$ and $j = 2 \in S_n$, since $a_1 = -1$ and $b_1 = -4$, the corresponding inequality is

$$(a_1 + b_1 - a_2 - b_2)x + a_1 b_1 - a_2 b_2 = 2x - 6 > 0.$$

Finally we consider $i = 3 \in S_p$ and $j = 2 \in S_n$, since $a_3 = -3$ and $b_3 = -6$, the corresponding inequality is

$$(a_3 + b_3 - a_2 - b_2)x + a_3 b_3 - a_2 b_2 = -2x + 8 > 0.$$

Now we have whole system of the inequalities (3.1) as

$$2 < x < 5,$$

$$2x - 6 > 0,$$

$$-2x + 8 > 0,$$

$$x > 0.$$

The solution is $3 < x < 4$. We know any solution x can be used to construct a common quadratic Lyapunov function.

For instance, we may choose $x_0 = 3.5$ and go to Step 5. Then (2.12) becomes

$$e + 12.5 + (-1 - 4) \times 3.5 + 4 > 0,$$

$$e + 12.5 + (-3 - 6) \times 3.5 + 18 > 0,$$

$$e + 12.5 + (-2 - 5) \times 3.5 + 10 < 0,$$

$$e > 0.$$

The solution is $1 < e < 2$. Any solution e can be used to construct a common quadratic Lyapunov function. We may naturally choose $e = 1.5$. Then setting $m_2 = x_0$, $m_3 = x_0^2 + e$, we have

$$M_1 = \begin{pmatrix} 1 & 3.5 \\ 3.5 & 14 \end{pmatrix}.$$

Now for Steps 6 and 7, convert (A_1, b_1) to z_1 coordinates, we have $\tilde{A}_1 = C_1 A C_1^{-1}$, which is

$$\tilde{A}_1 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad \tilde{b}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using (3.3), say we choose $\alpha = -1 < 0$, then

$$\tilde{k}_1 = \left((\alpha - a_{21}) \quad \left(\frac{\alpha m_3 - m_1}{m_2} - a_{22} \right) \right)$$

$$= \left((-1 - 2) \quad \left(\frac{-1(14) - 1}{3.5} - 1 \right) \right)$$

$$= (-3 \quad -5\tfrac{2}{7}).$$

Then in the original coordinate frame x we have

$$K_1 = \tilde{k}_1 C_1 = (-11.2857 \quad -8.2857).$$

Next, for (A_2, b_2) to z_2 coordinates, we have

$$\tilde{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \quad \tilde{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To get the feedback control law, we need to convert M_1 into z_2 frame, which is

$$M_2 = T_1^{\mathrm{T}} M_1 T_1 = \begin{pmatrix} 8.0000 & 0.5000 \\ 0.5000 & 2.0000 \end{pmatrix}.$$

Then we can get the feedback law as

$$\tilde{k}_2 = (-2.0000 \quad -22.0000);$$

$$K_2 = (-18.0000 \quad -11.3333).$$

For (A_3, b_3) to z_3 coordinates, we have

$$\tilde{A}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \tilde{b}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$M_3 = T_2^{\mathrm{T}} M_1 T_2 = \begin{pmatrix} 16.0000 & 1.0000 \\ 1.0000 & 4.0000 \end{pmatrix}.$$

The feedback law as

$$\tilde{k}_3 = (0 \quad -21.0000);$$

$$K_3 = (-28.0000 \quad -21.0000).$$

Finally, for (A_4, b_4) to z_4 coordinates, we have

$$\tilde{A}_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{b}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$M_4 = T_3^{\mathrm{T}} M_1 T_3 = \begin{pmatrix} 2.0000 & 0.5000 \\ 0.5000 & 8.0000 \end{pmatrix}.$$

The feedback law as

$$\tilde{k}_4 = (-2.0000 \quad -20.0000);$$

$$K_4 = (28.0000 \quad 22.0000).$$

To check whether the result is correct, we do the following verification:

Get the quadratic form back to the original coordinates x

$$M_0 = C_1^{\mathrm{T}} M_1 C_1 = \begin{pmatrix} 32.0000 & 26.5000 \\ 26.5000 & 22.0000 \end{pmatrix}.$$

We can check whether it is really a common quadratic Lyapunov function of the four different models.

$$M_0(A_1+B_1K_1)+(A_1+B_1K_1)^{\mathrm{T}}M_0 = \begin{pmatrix} -253.0000 & -211.0000 \\ -211.0000 & -176.0000 \end{pmatrix} < 0;$$

$$M_0(A_2+B_2K_2)+(A_2+B_2K_2)^{\mathrm{T}}M_0 = \begin{pmatrix} -67.0000 & -44.0000 \\ -44.0000 & -29.0000 \end{pmatrix} < 0;$$

$$M_0(A_3+B_3K_3)+(A_3+B_3K_3)^{\mathrm{T}}M_0 = \begin{pmatrix} -308.5000 & -233.0000 \\ -233.0000 & -176.0000 \end{pmatrix} < 0;$$

$$M_0(A_4+B_4K_4)+(A_4+B_4K_4)^{\mathrm{T}}M_0 = \begin{pmatrix} -751.0000 & -596.0000 \\ -596.0000 & -473.0000 \end{pmatrix} < 0.$$

The verification is completed successfully.

One of the advantages in this approach is that it provides not only one solution as in most of the control design problems. It provides all the possible quadratic Lyapunov functions. There is no exception at all.

4. Conclusion

The stabilization problem of switched systems was discussed in the paper. A necessary and sufficient condition for the stabilizability of a single-input planar switched system with observable switching law was obtained. The condition provides a set of linear inequalities. Solving them we get the designed common quadratic Lyapunov function. Then a formula was obtained to design the stabilizing control.

The general problem of the stabilization for a switched system of dimension greater than two remains for further study.

Appendix

Proof of Lemma 2.2. Without loss of generality, we can assume $b=(0,1)$. Then the system becomes

$$\dot{x} = Ax+bu = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

Since it is stabilizable, $a_{11}<0$. Without loss of generality, we can assume $a_{11}=-1$. Then the closed-loop matrix, $\tilde{A}=A+bK$, is

$$\tilde{A} = \begin{pmatrix} -1 & 0 \\ \alpha & \beta \end{pmatrix},$$

where α and β can be chosen arbitrary.

Assume

$$M = \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0.$$

We have only to prove that we can choose α and β such that $M\tilde{A}+(\tilde{A})^{\mathrm{T}}M<0$. A straightforward computation shows

$$Q := M\tilde{A}+(\tilde{A})^{\mathrm{T}}M = \begin{pmatrix} 2(\alpha m_2-1) & \alpha m_3+(\beta-1)m_2 \\ \alpha m_3+(\beta-1)m_2 & 2\beta m_3 \end{pmatrix}.$$

Now since $m_3>0$, to get $Q<0$ we have only to choose α and $\beta<0$ such that $\det(Q)>0$. $\det(Q)$ is calculated as $\det(Q)=-(\alpha m_3-\beta m_2)^2+2\alpha m_2 m_3+2\beta m_2^2-m_2^2-4\beta m_3$. Choosing $\alpha=\beta m_2/m_3$ yields

$$\det(Q) = -4\beta(m_3-m_2^2)-m_2^2 = -4\beta\det(M)-m_2^2.$$

Choosing

$$\beta < -\frac{m_2^2}{4\det(M)} < 0$$

yields $\det(Q)>0$, which completes the proof. □

Proof of Lemma 2.3. We have only to prove the uniqueness of T. From (2.1) it is easy to get the following equation:

$$AT_1 = a_1 b,$$

$$AT_2 = T_1 + a_2 b,$$

$$\cdots$$

$$AT_{n-1} = T_{n-2} + a_{n-1} b,$$

$$Ab = T_{n-1} + a_n b.$$

Then a straightforward computation shows the required formulas. □

Proof of Lemma 2.4. Without loss of generality, we assume $m_1 = 1$. Then for $M > 0$ to be the quadratic Lyapunov function of $\tilde{A}$, iff,

$$M\tilde{A} + \tilde{A}^{\mathrm{T}}M = \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix} < 0, \tag{A.1}$$

which leads to

$$Q := \begin{pmatrix} 2\alpha m_2 & 1 + \alpha m_3 + \beta m_2 \\ 1 + \alpha m_3 + \beta m_2 & 2(m_2 + \beta m_3) \end{pmatrix} < 0. \tag{A.2}$$

Note that β is the trace of matrix of the closed-loop system, hence $\beta < 0$ is a necessary condition for the closed-loop system to be stable. It is assumed in the following discussion.

It is obvious that $m_2 = 0$ is not allowed. In fact, $\alpha m_2 < 0$ is necessary. Then we have

Case 1: $m_2 < 0$ and $\alpha > 0$: Now for (A.2) to hold, it is necessary and sufficient to have the determinant of the left-hand side of (A.2), denoted by $D(\alpha, \beta)$, to be positive. After a simple computation, we have

$$D(\alpha, \beta) = -(\beta m_2 - \alpha m_3)^2 - 2(\beta m_2 + \alpha m_3) + 4\alpha m_2^2 - 1.$$

Then setting

$$\frac{\partial D(\alpha, \beta)}{\partial \beta} = -2m_2(\beta m_2 - \alpha m_3) - 2m_2 = 0,$$

yields

$$\beta m_2 = \alpha m_3 - 1.$$

So the maximum $D(\alpha, \beta)$ is

$$D_{\max} = -1 - 2(2\alpha m_3 - 1) + 4\alpha m_2^2 - 1 = -4\alpha(m_3 - m_2^2) = -4\alpha \det(M) < 0.$$

Hence $m_2 > 0$ is necessary for the matrix Q in (A.2) to be negative definite, which proves the necessity.

Case 2: $m_2 > 0$ and $\alpha < 0$: In this case we can simply choose any $\alpha < 0$ and let

$$\beta = \frac{\alpha m_3 - 1}{m_2}. \tag{A.3}$$

Then

$$D(\alpha, \beta) = -4\alpha(m_3 - m_2^2) > 0.$$

The sufficiency is proved. □

Proof of Lemma 2.6. (Sufficiency) Consider a matrix

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0$$

with $m_2 > 0$. Calculating $\tilde{M} = T^{\mathrm{T}}MT$ shows that

$$H(m_1, m_2, m_3) := \tilde{m}_{12} = t_{21}t_{22}m_3 + (t_{12}t_{21} + t_{11}t_{22})m_2 + t_{11}t_{12}m_1.$$

From (2.3), $H(1, x, x^2) > 0$. By continuity, there exists $\varepsilon > 0$ small enough, such that $H(1, x, x^2 + \varepsilon) > 0$. Set $m_1 = 1$, $m_2 = x$ and $m_3 = x^2 + \varepsilon$, the matrix M meets our requirement.

(Necessity) Without loss of generality, we can assume there exists

$$M = \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 \end{pmatrix} > 0$$

such that both M and $T^{\mathrm{T}}MT$ are canonical-friend. That is: $m_2 > 0$ and $H(1, m_2, m_3) > 0$. Now, if $t_{21}t_{22} > 0$, then it is trivial that (2.3) has a positive solution. As $t_{21}t_{22} \leqslant 0$, since $m_3 > m_2^2$

$$H(1, m_2, m_2^2) \geqslant H(1, m_2, m_3) > 0. \tag{A.4}$$

So m_2 is a positive solution of (2.3). □

Proof of Theorem 2.7. (1) Choose $m_2 \in I$ and $m_3 = m_2^2 + \varepsilon$. It is easy to see that when $\varepsilon > 0$ small enough, the corresponding matrix

$$M = \begin{pmatrix} 1 & m_2 \\ m_2 & m_3 + \varepsilon \end{pmatrix}$$

(in z_1 coordinates) becomes the common quadratic Lyapunov function (for suitably chosen controls).

(2) Choose $m_2 > 0$ large enough, then $m_2 \in I$.

(3) From the proof of Lemma 2.6, inequality (A.4) shows that $m_2 \in I$. So $I \neq \emptyset$. □

Proof of Corollary 2.10. When $N=2$ it was proved in Lemma 2.6. We have only to prove it for $N=3$. We have T_1 and T_2. If both 1 and 2 are in S_p (or S_n), it has been proved in the Theorem 2.7. Without loss of generality we may assume $1\in S_p$ and $2\in S_n$. To see the necessity, we assume $I_1\cap I_2=\emptyset$, it turns out that

$$a_1 \geqslant a_2 \geqslant b_2 \geqslant b_1.$$

Here without loss of generality, we assume $a_1 \geqslant b_1$ and $a_2 \geqslant b_2$. Now it is obvious that

$$Q_2(x) \geqslant Q_1(x), \quad x\in(b_2,a_2). \tag{A.5}$$

A rigorous proof of (A.5) is tedious. But think about the relative position of the two parabola of the same shape, the conclusion is obvious.

Hence the second inequality in (2.10) failed. □

Proof of Proposition 3.2. It is from the proof of Lemma 2.4. In fact, (3.3) is from (A.3) with an obvious modification for $m_1 \neq 1$. □

References

[1] S. Boyd, L.E. Ghaoui, E. Feron, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Application Mathematics, SIAM, Philadelphia, 1994.

[2] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched hybrid systems, IEEE Trans. Automat. Control. 43 (4) (1998) 475–482.

[3] D. Cheng, L. Guo, J. Huang, On quadratic Lyapunov functions, IEEE Trans. Automat. Control 48 (5) (2003) 885–890.

[4] W.P. Dayawansa, C.F. Martin, A converse Lyapunov theorem for a class of dynamical systems which undergo switching, IEEE Trans. Automat. Control 44 (4) (1999) 751–760.

[5] D. Jeon, M. Tomizuka, Learning hybrid force and position control of robot manipulators, IEEE Trans. Robotic. Automat. 9 (1996) 423–431.

[6] Y.D. Ji, H.J. Chizeck, Jump linear quadratic Gaussian control: steady-state solution and testable conditions, Control Theory Adv. Tech. 6 (3) (1990) 289–319.

[7] D. Liberzon, J.P. Hespanha, A.S. Morse, Stability of switched systems: a Lie-algebraic condition, Systems Control Lett. 37 (3) (1999) 117–122.

[8] D. Liberzon, A.S. Morse, Basic problems on stability and design of switched systems, IEEE Control Systems Mag. 34 (6) (1999) 1914–1946.

[9] T. Ooba, Y. Funahashi, Two conditions concerning common QLFs for linear systems, IEEE Trans. Automat. Control 42 (5) (1997) 719–721.

[10] T. Ooba, Y. Funahashi, Stability robustness for linear state space models—a Lyapunov mapping approach, Systems Control Lett. 29 (1997) 191–196.

[11] R.N. Shortenm, K.S. Narendra, A sufficient condition for the existence of a common Lyapunov function for two second-order linear systems, Proceedings of the 36th Conference on Decision and Control, San Diego, CA, 1997, pp. 3521–3522.

[12] R.N. Shortenm, K.S. Narendra, On the existence of a common Lyapunov function for linear stable switched systems, Proceedings of the 10th Yale Workshop on Adaptive and Learning Systems, Tampa, FL, 1998, pp. 130–140.

[13] R.N. Shortenm, K.S. Narendra, Necessary and sufficient condition for the existence of a common quadratic Lyapunov function for M-stable linear second-order systems, Proceedings of 2000 ACC, Chicago, IL, 2000, pp.359–363.

[14] H. Sira-Ranirez, Nonlinear P-I controller design for switch mode DC-to-DC power converters, IEEE Trans. Circuits Systems 38 (4) (1991) 410–417.

[15] Z. Sun, D. Zheng, On reachability and stabilization of switched linear systems, IEEE Trans. Automat. Control. 46 (2) (2001) 291–295.

[16] P.P. Varaiya, Smart car on smart roads: problems of control, IEEE Trans. Automat. Control. 38 (2) (1993) 195–207.

[17] S.M. Williams, R.G. Hoft, Adaptive frequency domain control of PM switched power line conditioner, IEEE Trans. Power Electron. 6 (10) (1991) 665–670.

[18] F. Xue, L. Guo, Necessary and Sufficient conditions for adaptive stabilizability of jump linear systems, Comm. Inform. Systems 1 (2) (2001) 205–224.